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Let $\mathcal{V}$ be the category of vector bundles and W^q the $q^{\underline{th}}$ Stiefel-Whitney class, regarded as a function on $\mathcal{V}$. We consider the category $\mathcal{V}_{p.g.}$ whose objects are pairs $(\mathcal{B}, J)$, where $\mathcal{B}$ is a vector bundle and J a pseudo-group of local bundle maps on $\mathcal{B}$, and whose morphisms are equivariant bundle maps. Clearly $\mathcal{V}_{p.g.}$ contains $\mathcal{V}$ as a full subcategory.

In this paper we construct a maximal extension W_e^q of W^q to a full subcategory of $\mathcal{V}_{p.g.}$. The extended function W_e^q is natural with respect to morphisms in $\mathcal{V}_{p.g.}$ and constitutes the primary obstruction to equivariant cross-sections in the associated frame bundle of appropriate dimension.

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EQUIVARIANT STIEFEL-WHITNEY CLASSES

I. INTRODUCTION

Let $\mathcal{B} = (B, X, \pi)$ be an n-dimensional vector bundle, B its total space (bundle space), X its base space, and $\pi : B \to X$ the projection. For each $q \geq 0$, <u>the q^{th} Stiefel-Whitney class $W^q(\mathcal{B})$</u> is an element of $H^q(X;Z_2)$, the singular q-dimensional cohomology group of X with coefficients the integers modulo two.

Let $\mathcal{V}$ be the category of vector bundles. The functions W^q on $\mathcal{V}$ satisfy the following properties (4, p. 281):

1) $W^0(\mathcal{B})$ is the unit class of $H^0(X;Z_2)$;

2) $W^q(\mathcal{B}) = 0$ for $q > n$;

3) If $f = (\check{f}, \hat{f}) : \mathcal{B} \to \mathcal{B}'$ is a bundle map, then $\check{f}^*(W^q(\mathcal{B}')) = W^q(\mathcal{B})$;

4) If $\mathcal{B} \oplus \mathcal{B}'$ is the Whitney sum of the bundles $\mathcal{B}$ and $\mathcal{B}'$, then $W^q(\mathcal{B} \oplus \mathcal{B}') = \sum_{i+j=q} W^i(\mathcal{B}) \cup W^j(\mathcal{B}')$;

5) If $\mathcal{B}$ is the one dimensional vector bundle over real projective 1-space, then $W^1(\mathcal{B}) \neq 0$.

If the base space X of $\mathcal{B}$ is paracompact, then these five properties characterize the functions W^q (2, p. 6-7).

In (3), J. W. Smith extends the category $\mathcal{V}$ to a <u>category of generalized vector bundles</u>. He then defines an extended Euler class on a full subcategory of this new category. The classical Euler class of an oriented n-dimensional vector bundle, reduced modulo two, is equal to the $n^{\underline{th}}$ Stiefel-Whitney class of the bundle (2, p. 41). This leads one to consider the possibility of defining the Stiefel-Whitney classes of generalized vector bundles.

A generalized vector bundle is essentially represented by a vector bundle together with a set of local bundle maps. In the following chapters we consider vector bundles together with pseudo-groups of local bundle maps and thereby get a category $\mathcal{V}_{p.g.}$, the category of vector bundles with pseudo-groups. Using an obvious identification, we observe that $\mathcal{V}$ can be considered as a full subcategory of $\mathcal{V}_{p.g.}$

The functions W^q on $\mathcal{V}$ can be defined in essentially two ways. One method, discussed in (2, p. 33-35) and (4, p. 281), uses the Thom isomorphism and the Steenrod squaring operations. The second method is based upon obstruction theory, and assumes that the base space X is a finite complex (5, p. 148-204). For each q, $0 \leq q \leq n - 1$, let $\mathcal{B}^q$ be the associated bundle of $\mathcal{B}$ (5, p. 43) with fibre $V_{n, n-q}$, the Stiefel manifold of (n-q)-frames in n-space. The primary obstruction to a cross-section of $\mathcal{B}^q$ is an element $O^{q+1}(\mathcal{B}) \in H^{q+1}(X;\pi_q(V_{n, n-q}))$. The homotopy group $\pi_q(V_{n, n-q})$ is isomorphic to the group Z of integers if q is even or $q=n-1$, and isomorphic to the group Z_2 of integers modulo two if q is odd and $q < n - 1$ (5, p. 132). The $(q+1)^{st}$ Stiefel-Whitney class

$W^{q+1}(\mathcal{B})$ is then equal to $O^{q+1}(\mathcal{B})$ reduced modulo two (2, p. 57-59).

Our <u>equivariant Stiefel-Whitney classes W_e^{q+1}</u>, defined on a certain subcategory of $\mathcal{V}_{p.g.}$, shall be obtained by an adaptation of the obstruction theoretic approach. For $0 \leq q \leq n - 1$, W_e^{q+1} shall constitute the primary obstruction to equivariant cross-sections in the appropriate generalized bundle.

II. PRELIMINARIES

Let Δ_q be the standard q-simplex with vertices $(p_o, p_1, \ldots, p_q)$ and let $e^i_{q+1} : \Delta_q \to \Delta_{q+1}$ be the linear map defined by the vertex map

$$e^i_{q+1}(p_j) = \begin{cases} p_j & \text{for } j < i \\ p_{j+1} & \text{for } j \geq i. \end{cases}$$

The image of Δ_q under e^i_{q+1} is denoted by $\Delta^{(i)}_{q+1}$ and is called the $i^{\underline{th}}$ q-face of Δ_{q+1}. The image in Δ_{q+1} of the composition $e^i_{q+1} \circ e^j_q : \Delta_{q-1} \to \Delta_{q+1}$ is denoted by $\Delta^{(i)(j)}_{q+1}$ and is the $j^{\underline{th}}$ (q-1)-face of $\Delta^{(i)}_{q+1}$.

The singular simplexes of a topological space Y, i. e., the continuous maps of Δ_q into Y, are denoted by small Greek letters, e. g., $\sigma : \Delta_q \to Y$. We shall let $S_q(Y)$ denote the set of all singular q-simplexes of Y. The $i^{\underline{th}}$ (q-1)-face of the singular q-simplex σ is denoted by $\sigma^{(i)}$ and is defined to be the composition $\sigma^{(i)} = \sigma \circ e^i_q : \Delta_{q-1} \to Y$. The $j^{\underline{th}}$ (q-2)-face of $\sigma^{(i)}$ is denoted by $\sigma^{(i)(j)}$. We let $C_*(Y;G) = \{C_q(Y;G), \partial_q\}$ be the singular chain complex of Y with coefficient group G. The corresponding cochain complex with coefficients in G is denoted by $C^*(Y;G), = \{C^q(Y;G), \delta_q\}$. When G = Z, the group of integers, we shall simply write $C_*(Y)$ and $C^*(Y)$. We let $f_\#$ and $f^\#$ denote the chain and cochain maps induced by a continuous map f from one topological space to another, and f^* the corresponding cohomology map.

Let $\mathcal{B} = (B, X, \pi)$ be an n-dimensional vector bundle.

Definition: A pseudo-group $J = \{u=(\check{u}, \hat{u})\}$ of local bundle maps on $\mathcal{B}$ is a set of bundle maps $u : \mathcal{B}|U \to \mathcal{B}$, where U is an open subset of X and $\mathcal{B}|U$ is the restriction of $\mathcal{B}$ to U, satisfying the following properties:

(i) the identity map $1 = (1_X, 1_B)$ is an element of J;

(ii) the inverse of each element of J is an element of J;

(iii) the restriction of an element of J to an open subset of its domain is an element of J;

(iv) the composition of elements of J, whenever defined, is an element of J.

In particular, condition ii) implies that if $(\check{u}, \hat{u}) \in J$, then $\check{u}$ and $\hat{u}$ are both homeomorphisms, and the inverse of $(\check{u}, \hat{u})$ is $(\check{u}^{-1}, \hat{u}^{-1})$. We shall use the notations $\check{u} \in J$ and $\hat{u} \in J$ to mean that there is a $u = (\check{u}, \hat{u})$ in J.

Now let $\mathcal{B}$ be a vector bundle and J a pseudo-group of local bundle maps on $\mathcal{B}$. We shall denote such a pair by $(\mathcal{B}, J)$; and shall call $(\mathcal{B}, J)$ a vector bundle with pseudo-group.

Let $(\mathcal{B}, J)$ and $(\mathcal{B}', J')$ be two n-dimensional vector bundles with pseudo-groups.

Definition: A bundle map $f = (\check{f}, \hat{f}) : \mathcal{B} \to \mathcal{B}'$ is called equivariant iff for each $u = (\check{u}, \hat{u}) \in J$, there is a $u' = (\check{u}', \hat{u}') \in J'$ such that $\check{f} \circ \check{u} = \check{u}' \circ \check{f}$ and $\hat{f} \circ \hat{u} = \hat{u}' \circ \hat{f}$ whenever these expressions are defined.

With this notion of map, we see that the set of vector bundles with pseudo-groups and equivariant bundle maps forms a category. This category will be called the category of vector bundles with pseudo-groups and will be denoted by $\mathcal{V}_{p.g.}$.

Let $I_{\mathcal{B}}$ be the pseudo-group of local bundle maps on $\mathcal{B}$ generated by the identity map of $\mathcal{B}$. Then we may identify the vector bundle $\mathcal{B}$ with the pair $(\mathcal{B}, I_{\mathcal{B}})$, and we note that under this identification, the category $\mathcal{V}$ of vector bundles is a full subcategory of $\mathcal{V}_{p.g.}$.

Now, let $(\mathcal{B}, J)$ be an object in $\mathcal{V}_{p.g.}$. The pseudo-group J determines a relation on the set $S_q(X)$ of singular q-simplexes of X as follows. For $\sigma, \sigma' \in S_q(X)$, we say σ is J-related to σ' iff there is a $\check{u} \in J$ such that $\sigma = \check{u} \circ \sigma'$. Because J is a pseudo-group, this relation is obviously an equivalence relation. Let $J(\sigma', \sigma)$ be the (possibly empty) subset of J consisting of all $(\check{u}, \hat{u}) \in J$ for which $\check{u} \circ \sigma' = \sigma$.

Definition: Two singular q-simplexes σ and σ' of X are J-equivalent iff $J(\sigma, \sigma')$ is non-empty.

The relation of J-equivalence partitions the set $S_q(X)$ into equivalence classes. Let $A(\sigma)$ be the J-equivalence class of $\sigma \in S_q(X)$, i.e., $A(\sigma) = \{\sigma' \in S_q(X) : J(\sigma, \sigma') \text{ is non-empty}\}$.

Because J-equivalence preserves incidence, the singular chain complex $C_*(X)$ induces a chain complex structure on the quotient of $C_*(X)$ by J-equivalence. This new complex, denoted by $C_*^e(X)$, is called the <u>equivariant chain complex of X induced by J.</u>

<u>Definition:</u> A q-cochain $c \in C^q(X;G)$ is called <u>equivariant</u> iff $c(\sigma) = c(\sigma')$ for all $\sigma, \sigma' \in S_q(X)$, with $\sigma' \in A(\sigma)$.

Thus, a q-cochain is equivariant iff it assigns the same element of the coefficient group to all J-equivalent singular q-simplexes. The equivariant cochains obviously constitute a subcomplex of the cochain complex $C^*(X;G)$ which we denote by $\underline{C_e^*(X;G)}$. The corresponding equivariant cohomology groups are denoted by $\underline{H_e^q(X;G)}$. We note that the cochain complex $C_e^*(X;G)$ is naturally isomorphic to the cochain complex corresponding to the equivariant chain complex $C_*^e(X)$ and coefficient group G.

<u>Definition:</u> A homomorphism $T_p : C_p(X;G) \to C_p(B;G)$ is called <u>equivariant</u> iff for each $(\check{u}, \hat{u}) \in J$, $T_p \circ \check{u}_\# = \hat{u}_\# \circ T_p$, whenever these expressions are defined. A set of equivariant homomorphisms $T = \{T_p : 0 \leq p \leq q,\ T_p : C_p(X;G) \to C_p(B;G)\}$ satisfying $\partial_p \circ T_p = T_{p-1} \circ \partial_p$

is called an <u>equivariant chain map</u> on dimensions less than $q+1$.

Now, let $(\mathcal{B}, J)$ be an object in $\mathcal{V}_{p.g.}$, with $\mathcal{B} = (B, X, \pi)$ and the dimension of $\mathcal{B}$ equal to n. For each q, $0 \leq q \leq n-1$, let $\mathcal{B}^q = (B^q, X, \pi^q, Y^q)$ be the associated bundle (5, p. 43) of $\mathcal{B}$ with fibre $Y^q = V_{n, n-q}$, the Stiefel manifold of $(n-q)$-frames in n-space. Thus, a point of $\mathcal{B}^q$ over $x \in X$ is a pair (x, v) where $v = (v_1, v_2, \ldots, v_{n-q})$ is a set of $(n-q)$ linearly independent vectors (not necessarily orthonormal) in the vector space $\pi^{-1}(x)$, and $\pi^q(x, v) = x$. The pseudo-group J induces a pseudo-group of local bundle maps on $\mathcal{B}^q$, for each $q = 0, 1, \ldots, n-1$, which we also denote by J. Thus, with each object $(\mathcal{B}, J)$ in $\mathcal{V}_{p.g.}$ of dimension n are associated n new pairs $(\mathcal{B}^0, J), (\mathcal{B}^1, J), \ldots, (\mathcal{B}^{n-1}, J)$. We note that the pair $(\mathcal{B}^q, J)$ is not an object of $\mathcal{V}_{p.g.}$. However, the notions of J-equivalence and equivariance introduced above have immediate analogues in the present context.

We shall define <u>the equivariant Stiefel-Whitney class $W_e^{q+1}(\mathcal{B}, J)$</u> of an object $(\mathcal{B}, J)$ in $\mathcal{V}_{p.g.}$ as the element of $H_e^{q+1}(X; Z_2)$ determined by a certain cocycle $c_\tau \in C_e^{q+1}(X; Z_2)$. The cocycle c_τ will be an obstruction to extending an equivariant chain map $\tau = \{\tau_p : C_p(X) \to C_p(B^q)\}$ on dimensions less than $q+1$ to an equivariant chain map on dimensions less than $q+2$. The equivariant chain map τ will also be required to satisfy $\pi_\# \circ \tau_p = 1_{C_p(X)}$ so that τ is a lifting of the singular chain complex of X to that of

B^q on dimensions less than $q+1$.

The condition that τ be an equivariant chain lifting will require that we make some assumption concerning the holonomy induced by J. The holonomy group of $(\mathcal{B}, J)$ at a point $x \in X$ can be defined as follows. Let $J_x = \{(\check{u}, \hat{u}) \in J : \check{u}(x) = x\}$. Then $(\check{u}, \hat{u}) \in J_x$ implies that $\hat{u}|\pi^{-1}(x)$ is a linear isomorphism of $\pi^{-1}(x)$ onto itself. Let $\Phi_x = \{\hat{u}|\pi^{-1}(x) : (\check{u}, \hat{u}) \in J_x\}$. Then Φ_x is a subset of the group of all automorphisms of the vector space $\pi^{-1}(x)$, and because J is a pseudo-group, the subset Φ_x is a subgroup. Φ_x is called the holonomy group of $(\mathcal{B}, J)$ at $x \in X$. The holonomy group Φ_x^q of $(\mathcal{B}^q, J)$ at $x \in X$ is defined similarly, and we note that Φ_x^q is induced by Φ_x.

Now, suppose the holonomy group Φ_{x_0} of $(\mathcal{B}, J)$ at $x_0 \in X$ is non-trivial, i. e., contains an element other than the identity. Then there is an element $(\check{u}, \hat{u}) \in J_{x_0}$ such that $\hat{u}|\pi^{-1}(x_0) \neq 1_{\pi^{-1}(x_0)}$. If $\sigma \in S_0(X)$ is defined by $\sigma(\Delta_0) = x_0$, then the equivariance of $\tau_0 : C_0(X) \to C_0(B^q)$ requires that $\tau_0(\sigma) = \tau_0(\check{u} \circ \sigma) = \hat{u}_\# \circ \tau_0(\sigma)$. But since $\hat{u}|\pi^{-1}(x_0)$ is not the identity, this last equation is not necessarily satisfied.

In Chapter III, we shall treat the case of trivial holonomy. In Chapter IV, we shall consider the case where the holonomy group Φ_x at each $x \in X$ is orientation preserving and leaves at least an $(n-1)$-dimensional subspace of $\pi^{-1}(x)$ fixed. We shall also comment on why our procedure cannot be carried out for more general holonomy.

III. THE CASE OF TRIVIAL HOLONOMY

Let $\mathcal{B} = (B, X, \pi)$ be an n-dimensional vector bundle and J a pseudo-group of local bundle maps on $\mathcal{B}$ such that Φ_x consists of only the identity automorphism for each $x \in X$. For each q, $0 \leq q \leq n - 1$, we have the associated bundle and pseudo-group $(\mathcal{B}^q, J)$ with fibre $V_{n, n-q}$. In the present case of no holonomy we note that if $(1, \hat{u}) \in J$, then $\hat{u} = 1$. Hence, if $(\check{u}, \hat{u})$ and $(\check{u}, \hat{v})$ are in J, we must have $\hat{u} = \hat{v}$.

Until stated otherwise, let q be a fixed integer, $0 \leq q \leq n - 1$. In the associated bundle $\mathcal{B}^q$ we shall drop the superscript q on the projection and fibre, so that $\mathcal{B}^q = (B^q, X, \pi, Y)$, with $Y = V_{n, n-q}$. The fibre over $x \in X$ will be designated as $Y_x = \pi^{-1}(x)$.

We wish to define an <u>equivariant lifting</u> of the singular chain complex of X to the singular chain complex of B^q. More precisely, we shall define homomorphisms $\tau_p : C_p(X) \to C_p(B^q)$ for $0 \leq p \leq q$ satisfying the following conditions:

(o) for each $\sigma : \Delta_p \to X$, $\tau_p(\sigma) : \Delta_p \to B^q$;

(i) $\pi_{\#} \circ \tau_p = 1_{C_p(X)}$;

(ii) $\partial_p \tau_p = \tau_{p-1} \partial_p$;

(iii) $\tau_p \circ \check{u}_{\#} = \hat{u}_{\#} \circ \tau_p$ whenever these expression are defined.

We shall refer to condition (i) as the <u>lifting condition</u> and (iii) as the <u>equivariance condition.</u> Condition (ii) simply states that the τ_p's form a chain map on dimensions less than $q+1$. A collection $\tau = \{\tau_p : 0 \leq p \leq q, \tau_p : C_p(X) \to C_p(B^q)\}$ satisfying conditions (o), (i), (ii) and (iii) will be called an equivariant lifting in $(\mathcal{B}^q, J)$ on dimensions less than $q+1$.

We shall define τ_p by finite induction on the dimension p. In order to facilitate this definition, we assume that the singular simplexes of X on each dimension have been well ordered, and let $<$ be the order relation on $S_p(X)$. We define τ_p on $S_p(X)$ by transfinite induction and extend τ_p to all of $C_p(X)$ by linearity. We further assume that the pseudo-group J, considered as a set, has been well ordered.

For each $\sigma \in S_p(X)$, the J-equivalence class $A(\sigma)$ of σ is a non-empty subset of the well ordered set $S_p(X)$, and consequently $A(\sigma)$ has a first element. We call this first element the base of $A(\sigma)$.

If the subset $J(\sigma, \sigma') = \{(\check{u}, \hat{u}) \in J : \check{u} \circ \sigma = \sigma'\}$ of the well ordered set J is non-empty, then it also has a first element which we call the base of $J(\sigma, \sigma')$.

Each $\sigma \in S_p(X)$, being a continuous mapping of Δ_p into X, induces a bundle over Δ_p (5, p. 47-48) with bundle space B^q_σ and a bundle map h_σ making the following diagram commutative:

(I)

$$\begin{array}{ccc} B^q_\sigma & \xrightarrow{h_\sigma} & B^q \\ \pi_\sigma \downarrow & & \downarrow \pi \\ \Delta_p & \xrightarrow[\sigma]{} & X \end{array} .$$

The bundle space B^q_σ of the induced bundle is the subspace of the product space $\Delta_p \times B^q$ given by $(x, b) \in B^q_\sigma$ iff $\sigma(s) = \pi(b)$. The maps π_σ and h_σ are the restrictions to B^q_σ of the natural projections to the first and second factors of $\Delta_p \times B^q$, respectively. The standard fibre of the induced bundle is again $V_{n, n-q}$.

The induced bundle in (I) has as its base space a finitely triangulable space. The obstruction theory for such bundles is treated in (5, p. 148-217), and we shall make use of several results proved there. In particular, the fibre $V_{n, n-q}$ is (q-1)-connected so that any cross-section over the boundary $\dot{\Delta}_p$ of Δ_p can be extended to a cross-section over Δ_p, $0 \leq p \leq q$ (5, p. 149).

The induced bundle in (I) is a product bundle and hence it admits cross-sections. We assume that the cross-sections of the induced bundle are well-ordered for each $\sigma \in S_p(X)$.

Note that if $\sigma \in A(\sigma')$, with $\sigma = \check{u} \circ \sigma'$, then the pair $(\check{u}, \hat{u}) \in J$ induces a bundle isomorphism $(1, \hat{u}) : B^q_{\sigma'} \to B^q_\sigma$ making the following diagram commutative:

(II)
$$\begin{array}{ccc} B^q_{\sigma'} & \xrightarrow{(1,\hat{u})} & B^q_{\sigma} \\ \pi_{\sigma'}\downarrow & & \downarrow \pi_{\sigma} \\ \Delta_p & \xrightarrow{1} & \Delta_p \end{array}$$

and furthermore $(1, \hat{u})^{-1} = (1, \hat{u}^{-1})$.

We are now ready to define τ_0. Let σ be the first element of $S_0(X)$, and let t_σ be the first cross-section of the induced bundle in (I). We define $\tau_0(\sigma) = h_\sigma \circ t_\sigma$. Then $\tau_0(\sigma) : \Delta_0 \to B^q$ and because (I) is commutative, $\pi \circ \tau_0(\sigma) = \pi \circ h_\sigma \circ t_\sigma = \sigma \circ \pi_\sigma \circ t_\sigma = \sigma$.

Now, let σ be an arbitrary element of $S_0(X)$ and suppose $\tau_0(\sigma')$ has been defined for all $\sigma' < \sigma$ in such a manner that conditions (o) and (i) are satisfied, i. e., $\tau_0(\sigma') : \Delta_0 \to B^q$ and $\pi_\# \circ \tau_0(\sigma') = \sigma'$. We distinguish two cases in defining $\tau_0(\sigma)$:

1) $\sigma \notin A(\sigma')$ for any $\sigma' < \sigma$;

2) $\sigma \in A(\sigma'')$ for some $\sigma'' < \sigma$.

In case 1) we define $\tau_0(\sigma)$ as above, i. e., let t_σ to be the first cross-section of the induced bundle in (I) and define $\tau_0(\sigma) = h_\sigma \circ t_\sigma$. Then $\tau_0(\sigma)$ is a lifting of σ to B^q.

In case 2), let σ' be the base of $A(\sigma'')$, i. e., the first element in $A(\sigma'')$, and let $(\check{u}, \hat{u})$ be the base of $J(\sigma', \sigma)$, so that $\sigma = \check{u} \circ \sigma'$. Since σ' is the base of $A(\sigma'')$, $\sigma' \leqq \sigma'' < \sigma$ so that $\tau_0(\sigma')$ is defined. Let $t_{\sigma'} : \Delta_0 \to B^q_{\sigma'}$ be the cross-section induced by the map $\tau_0(\sigma') : \Delta_0 \to B^q$ and the bundle map $h_{\sigma'} : B^q_{\sigma'} \to B^q$.

The bundle map $(1, \hat{u}^{-1}) : B^q_\sigma \to B^q_{\sigma'}$ of (II) and the cross-section $t_{\sigma'}$ induce a unique cross-section $t_\sigma : \Delta_0 \to B^q_\sigma$ making the following diagram commutative (5, p. 12):

(III$_0$)

$$\begin{array}{ccc} B^q_\sigma & \xrightarrow{(1,\hat{u}^{-1})} & B^q_{\sigma'} \\ t_\sigma \uparrow & & \uparrow t_{\sigma'} \\ \Delta_0 & \xrightarrow{1} & \Delta_0 \end{array} .$$

We define $\tau_0(\sigma) = h_\sigma \circ t_\sigma : \Delta_0 \to B^q$ and $\pi \circ \tau_0(\sigma) = \sigma$ again follows from (I).

By transfinite induction we have τ_0 defined on all of $S_0(X)$ and satisfying condition (o) $\tau_0(\sigma) : \Delta_0 \to B^q$, condition (i) $\pi \circ \tau_0(\sigma) = \sigma$, and condition (ii) (vacuously). It remains to show that condition (iii) is satisfied.

Before showing this, we wish to make two remarks.

Firstly, the map $\tau_0(\sigma) : \Delta_0 \to B^q$ together with the bundle map $h_\sigma : B^q_\sigma \to B^q$ uniquely determines a cross-section $t_\sigma : \Delta_0 \to B^q_\sigma$ such that $\tau_0(\sigma) = h_\sigma \circ t_\sigma$. If $d: \Delta_0 \to \Delta_0 \times \Delta_0$ is the diagonal map, then t_σ is defined by $t_\sigma = (1, \tau_0(\sigma)) \circ d$. This will also be true of the maps $\tau_p(\sigma) : \Delta_p \to B^q$ to be defined later. The cross-section determined by $\tau_p(\sigma)$ and h_σ will hereafter be designated by t_σ, and we shall use the representation $\tau_p(\sigma) = h_\sigma \circ t_\sigma$ without additional comment.

Secondly, if $\sigma, \sigma' \in A(\sigma'')$, with σ'' the base of $A(\sigma'')$, and if $\sigma = \check{u} \circ \sigma''$ and $\sigma' = \check{v} \circ \sigma''$, then $\sigma = \check{u} \circ \check{v}^{-1} \circ \sigma'$. By our definitions and diagram (III_0), $t_\sigma = (1, \hat{u}) \circ t_{\sigma''}$ and $t_{\sigma'} = (1, \hat{v}) \circ t_{\sigma''}$, so that $t_\sigma = (1, \hat{u}) \circ (1, \hat{v}^{-1}) \circ t_{\sigma'} = (1, \hat{u} \circ \hat{v}^{-1}) \circ t_{\sigma'}$. Consequently, whenever we have the relation $\sigma = \check{w} \circ \sigma'$, we have the companion relation $t_\sigma = (1, \hat{w}) \circ t_{\sigma'}$. This will also be true for dimensions other than zero, because our definitions will be similar, and we shall have a commutative diagram (III_p) corresponding to (III_0).

Now, let $\sigma = \check{u} \circ \sigma'$; $\sigma, \sigma' \in S_0(X)$ and let $\pi^{-1}(\sigma'(\Delta_0)) = B^q | \sigma'(\Delta_0)$. We have the commutative diagram:

$$(IV_0) \qquad \begin{array}{ccc} B^q | \sigma'(\Delta_0) & \xrightarrow{\hat{u}} & B^q \\ h_{\sigma'} \uparrow & & \uparrow h_\sigma \\ B^q_{\sigma'} & \xrightarrow[(1, \hat{u})]{} & B^q_\sigma \end{array} .$$

Consequently, $\tau_0(\check{u} \circ \sigma') = \tau_0(\sigma) = h_\sigma \circ t_\sigma = h_\sigma \circ (1, \hat{u}) \circ (1, \hat{u}^{-1}) \circ t_\sigma$ $= \hat{u} \circ h_{\sigma'} \circ t_{\sigma'} = \hat{u} \circ \tau_0(\sigma')$ and condition (iii), the equivariance of τ_0, is satisfied.

We now extend τ_0 to all of $C_0(X)$ by linearity and note that since conditions (o), (i), (ii) and (iii) are satisfied on the generators of $C_0(X)$, namely on $S_0(X)$, and $\pi_\#$, ∂ and τ_0 are homomorphisms, then conditions (o), (i), (ii) and (iii) are satisfied by the map $\tau_0 : C_0(X) \to C_0(B^q)$.

We now proceed to our inductive step. Thus, we assume that $T_m : C_m(X) \to C_m(B^q)$ has been defined for all m for which $0 \leq m < p \leq q$, satisfying the conditions (o) $T_m(\sigma) : \Delta_m \to B^q$ for each $\sigma \in S_m(X)$, (i) $\pi_\# \circ T_m = 1_{C_m(X)}$, (ii) $\partial_m T_m = T_{m-1}\partial_m$, and (iii) $\hat{u}_\# \circ T_m = T_m \circ \check{u}_\#$ whenever these expressions are defined.

Let σ be the first element of $S_p(X)$. For each (p-1)-face $\sigma^{(i)}$ of σ, $T_{p-1}(\sigma^{(i)}) : \Delta_{p-1} \to B^q$ is defined and induces a cross-section $t_{\sigma^{(i)}}$ in the bundle $B^q_{\sigma^{(i)}} \to \Delta_{p-1}$. Let $\Delta_p^{(i)}$ be the $i^{\underline{th}}$ (p-1)-face of Δ_p. We then have the commutative diagram:

(V)
$$\begin{array}{ccccc} B^q_{\sigma^{(i)}} & \xrightarrow{(e^i_p,\, 1)} & B^q_\sigma | \Delta^{(i)}_p & \subset & B^q_\sigma \\ \downarrow & & \downarrow & & \downarrow \\ \Delta_{p-1} & \xrightarrow[e^i_p]{} & \Delta^{(i)}_p & \subset & \Delta_p \end{array}.$$

Since e^i_p is a homeomorphism, we may define a cross-section $t^i_\sigma : \Delta^{(i)}_p \to B^q_\sigma | \Delta^{(i)}_p$ by $t^i_\sigma = (e^i_p, 1) \circ t_{\sigma^{(i)}} \circ (e^i_p)^{-1}$. Thus, t^i_σ is a cross-section of the part of the bundle $B^q_\sigma \to \Delta_p$ over the $i^{\underline{th}}$ (p-1)-face of Δ_p.

If the t^i_σ's fit together, i.e., if they are equal on their common domains, then we may define a cross-section $t_{\dot\sigma}$ over $\dot\Delta_p$ by $t_{\dot\sigma} | \Delta^{(i)}_p = t^i_\sigma$. This condition is superfluous, of course, for $p = 1$.

Before showing that the t^i_σ's do in fact fit together, we verify the following facts which are collected together as a lemma due to their frequent use.

Lemma III. 1: Let $\tau_{m-1} : C_{m-1}(X) \to C_{m-1}(B^q)$ *and* $\tau_m : C_m(X) \to C_m(B^q)$ *be homomorphisms satisfying* $\tau_{m-1} : S_{m-1}(X) \to S_{m-1}(B^q)$ *and* $\tau_m : S_m(X) \to S_m(B^q)$. *Let* $\rho \in S_m(X)$ *and let* t_ρ *and* h_ρ *be defined as before. Then*

a) *the following diagram is commutative:*

(VI)
$$\begin{array}{ccc} B^q_\rho & \xrightarrow{h_\rho} & B^q \\ \uparrow{\scriptstyle (e^i_m,\, 1)} & \nearrow{\scriptstyle h_{\rho^{(i)}}} & \\ B^q_{\rho^{(i)}} & & \end{array} .$$

Furthermore, consider the conditions (i) $\pi_{\#} \circ \tau_m(\rho) = \rho$ *and (ii)* $\partial_m \tau_m(\rho) = \tau_{m-1}\partial_m(\rho)$ *and the diagram:*

(VII)
$$\begin{array}{ccccc} B^q_{\rho^{(i)}} & \xrightarrow{(e^i_m,\, 1)} & B^q_\rho | \Delta^{(i)}_m & \subset & B^q_\rho \\ \uparrow{\scriptstyle t_{\rho^{(i)}}} & & \uparrow{\scriptstyle t^i_\rho} & & \uparrow{\scriptstyle t_\rho} \\ \Delta_{m-1} & \xrightarrow[e^i_m]{} & \Delta^{(i)}_m & \subset & \Delta_m \end{array} .$$

Then b) *if conditions (i) and (ii) are satisfied, then diagram (VII) is commutative;*

c) *if diagram (VII) is commutative, then conditions (i) and (ii)* are satisfied.

Proof: To prove a), let $(s, b) \in B^q_{\rho^{(i)}}$. Then $h_{\rho^{(i)}}((s, b)) = b$.

But $h_\rho \circ (e^i_m, 1)((s, b)) = h_\rho((e^i_m(s), b)) = b$. Thus, $h_{\rho^{(i)}} = h_\rho \circ (e^i_m, 1)$.

For b) we have from (ii) $\partial \tau_m(\rho) = \sum_{i=0}^{m} (-1)^i \tau_m(\rho) \circ e^i_m$

$= \sum_{i=0}^{m} (-1)^i \tau_{m-1}(\rho \circ e^i_m) = \tau_{m-1}\partial(\rho)$. Since τ_m and τ_{m-1} take simplexes into simplexes, this equation implies that

$\tau_m(\rho) \circ e^{i_k}_m = \pm \tau_{m-1}(\rho \circ e^{j_k}_m)$ for some pairings i_k and j_k. Suppose that $\tau_m(\rho) \circ e^{i_k}_m = \pm \tau_{m-1}(\rho \circ e^{j_k}_m)$ for some $i_k \neq j_k$. Then (i) implies that $\rho \circ e^{i_k}_m = \pm \rho \circ e^{j_k}_m$ so that $\tau_{m-1}(\rho \circ e^{i_k}_m) = \pm \tau_{m-1}(\rho \circ e^{j_k}_m)$ $= \tau_m(\rho) \circ e^{i_k}_m$. Thus $\tau_m(\rho) \circ e^i_m = \tau_{m-1}(\rho \circ e^i_m)$ for all i, and hence $h_\rho \circ t_\rho \circ e^i_m = h_{\rho^{(i)}} \circ t_{\rho^{(i)}}$. By part a) we have $h_\rho \circ t_\rho \circ e^i_m = h_\rho \circ (e^i_m, 1) \circ t_{\rho^{(i)}}$. Let $s \in \Delta_{m-1}$. Then $t_\rho \circ e^i_m(s) = (e^i_m(s), b)$ and $(e^i_m, 1) \circ t_{\rho^{(i)}}(s) = (e^i_m(s), \overline{b})$ for some b and $\overline{b}$ in B^q. Then $h_\rho \circ t_\rho \circ e^i_m(s) = b = \overline{b} = h_\rho \circ (e^i_m, 1) \circ t_{\rho^{(i)}}(s)$ so that $t_\rho \circ e^i_m = (e^i_m, 1) \circ t_{\rho^{(i)}}$, as was to be shown.

For part c) we have $\partial T_m(\rho) = \sum_{i=0}^{m} (-1)^i T_m(\rho) \circ e^i_m$

$= \sum_{i=0}^{m} (-1)^i h_\rho \circ t_\rho \circ e^i_m = \sum_{i=0}^{m} (-1)^i h_\rho \circ (e^i_m, 1) \circ t_{\rho^{(i)}} = \sum_{i=0}^{m} (-1)^i h_{\rho^{(i)}} \circ t_{\rho^{(i)}}$

$= T_{m-1}(\partial\rho).$

Q. E. D.

Now, to show that the t^i_σ's fit together, let $\Delta^{(i)}_p$ and $\Delta^{(j)}_p$ have a common face, say $\Delta^{(i)(k)}_p = \Delta^{(j)(\ell)}_p$. Then $e^i_p \circ e^k_{p-1} = e^j_p \circ e^\ell_{p-1}$ so that $\sigma^{(i)(k)} = \sigma^{(j)(\ell)}$ and $t_{\sigma^{(i)(k)}} = t_{\sigma^{(j)(\ell)}}$.

By definition, $t^i_\sigma | \Delta^{(i)(k)}_p = (e^i_p, 1) \circ t_{\sigma^{(i)}} \circ (e^i_p)^{-1} | \Delta^{(i)(k)}_p$

$= (e^i_p, 1) \circ t_{\sigma^{(i)}} | \Delta^{(k)}_{p-1} \circ (e^i_p)^{-1} | \Delta^{(i)(k)}_p$. Our induction hypotheses (o), (i) and (ii) imply by Lemma III. 1, b) that $t_{\sigma^{(i)}} | \Delta^{(k)}_{p-1} = t^k_{\sigma^{(i)}}$

$= (e^k_{p-1}, 1) \circ t_{\sigma^{(i)(k)}} \circ (e^k_{p-1})^{-1}$. Also, the two preceeding equations hold when the pair i, k is replaced by the pair, j, ℓ. Hence

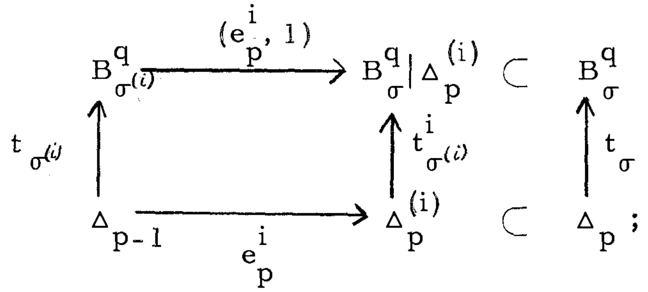
$t^i_\sigma | \Delta^{(i)(k)}_p = (e^i_p, 1) \circ (e^k_{p-1}, 1) \circ t_{\sigma^{(i)(k)}} \circ (e^k_{p-1})^{-1} \circ (e^i_p)^{-1} | \Delta^{(i)(k)}_p$

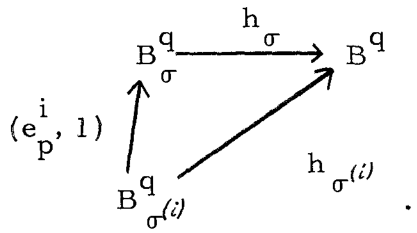
$= (e^i_p \circ e^k_{p-1}, 1) \circ t_{\sigma^{(i)(k)}} \circ (e^i_p \circ e^k_{p-1})^{-1} | \Delta^{(i)(k)}_p$

$= (e^j_p \circ e^\ell_{p-1}, 1) \circ t_{\sigma^{(j)(\ell)}} \circ (e^j_p \circ e^\ell_{p-1})^{-1} | \Delta^{(j)(\ell)}_p = t^j_\sigma | \Delta^{(j)(\ell)}_p$ and the t^i_σ's do in fact agree on their common domains. We define $t_\sigma : \dot{\Delta}_p \to B^q_\sigma$ by $t_\sigma | \Delta^{(i)}_p = t^i_\sigma$.

We recall that since the fibre $V_{n, n-q}$ of the induced bundle $B^q_\sigma \to \Delta_p$ is (q-1)-connected, the cross-section $t_{\dot{\sigma}}$ over $\dot{\Delta}_p$ can be extended to a cross-section over Δ_p. Let t_σ be the first such extension, and define $T_p(\sigma) = h_\sigma \circ t_\sigma : \Delta_p \to B^q$. Then condition (i) $\pi \circ T_p(\sigma) = \sigma$ follows again from the commutativity of (I).

By the definition of t_σ, the following diagrams are commutative:





Hence by Lemma III. 1, (c), condition (ii) $\partial_p T_p(\sigma) = T_{p-1}\partial_p(\sigma)$ is satisfied.

Now let σ be an arbitrary element of $S_p(X)$ and suppose $T_p(\sigma')$ has been defined for all $\sigma' < \sigma$ satisfying conditions (o), (i) and (ii). We again distinguish two cases in defining $T_p(\sigma)$:

1) $\sigma \notin A(\sigma')$ for any $\sigma' < \sigma$, i. e., σ is not J-equivalent to any $\sigma' < \sigma$;

2) $\sigma \in A(\sigma'')$ for some $\sigma'' < \sigma$.

In case 1) we define $T_p(\sigma)$ precisely as we did above for the first element of $S_p(X)$. Hence, for each (p-1)-face $\sigma^{(i)}$ of σ we get a cross-section $t^i_\sigma = (e^i_p, 1) \circ t_{\sigma^{(i)}} \circ (e^i_p)^{-1}$ over the (p-1)-face $\Delta^{(i)}_p$ of Δ_p and the t^i_σ's fit together to give a cross-section $t_{\dot\sigma}$ over the boundary $\dot\Delta_p$ of Δ_p. We let t_σ be the first cross-section over Δ_p which is an extension of $t_{\dot\sigma}$ and define $T_p(\sigma) = h_\sigma \circ t_\sigma : \Delta_p \to B^q$. Then conditions (i) and (ii) are satisfied exactly as before.

In case 2), let σ' be the base of $A(\sigma'')$ and $(\check{u}, \hat{u})$ the base of $J(\sigma', \sigma)$ so that $\sigma = \check{u} \circ \sigma'$. Since σ' is the base of $A(\sigma'')$, $\sigma' \leq \sigma'' < \sigma$, and $T_p(\sigma')$ is defined. Let $t_{\sigma'} : \Delta_p \to B^q_{\sigma'}$ be the cross-section induced by $T_p(\sigma')$ so that $T_p(\sigma') = h_{\sigma'} \circ t_{\sigma'}$. We let t_σ be the cross-section of the bundle $B^q_\sigma \to \Delta_p$ induced by the bundle map $(1, \hat{u}^{-1})$ and cross-section $t_{\sigma'}$, so that the following diagram is commutative:

$$(III_p) \qquad \begin{array}{ccc} B^q_\sigma & \xrightarrow{(1,\, \hat{u}^{-1})} & B^q_{\sigma'} \\ \uparrow t_\sigma & & \uparrow t_{\sigma'} \\ \Delta_p & \xrightarrow{\;1\;} & \Delta_p \end{array} .$$

We define $T_p(\sigma) = h_\sigma \circ t_\sigma : \Delta_p \to B^q$ and as before, condition (i) follows from diagram (I).

In order to show condition (ii), we shall show that

$t_\sigma \circ e^i_p = (e^i_p, 1) \circ t_{\sigma^{(i)}}$ and then apply Lemma III. 1, c).

We have the commutative diagram

$$(IV_p) \qquad \begin{array}{ccc} B^q \,|\, \sigma'(\Delta_p) & \xrightarrow{\ \hat{u}\ } & B^q \\ {\scriptstyle h_{\sigma'}} \uparrow & & \uparrow {\scriptstyle h_\sigma} \\ B^q_{\sigma'} & \xrightarrow[(1,\, \hat{u})]{} & B^q_\sigma \end{array} .$$

By our inductive hypothesis on σ' and Lemma III. 1, b), $t_{\sigma'} \circ e^i_p = (e^i_p, 1) \circ t_{\sigma'^{(i)}}$. Hence, by (III_{p-1}) and (III_p) we have $t_\sigma \circ e^i_p = (1, \hat{u}) \circ t_{\sigma'} \circ e^i_p = (1, \hat{u}) \circ (e^i_p, 1) \circ t_{\sigma'^{(i)}} = (e^i_p, 1) \circ (1, \hat{u}) \circ t_{\sigma'^{(i)}}$ $= (e^i_p, 1) \circ t_{\sigma^{(i)}}$, and thus $t_\sigma \circ e^i_p = (e^i_p, 1) \circ t_{\sigma^{(i)}}$, as was to be shown.

By transfinite induction, we have T_p defined on all of $S_p(X)$, satisfying conditions (o), (i) and (ii).

Now let $\sigma, \sigma' \in S_p(X)$, with $\sigma = \check{u} \circ \sigma'$. Then $t_\sigma = (1, \hat{u}) \circ t_{\sigma'}$ so that using (IV_p) we have $T_p(\check{u} \circ \sigma') = T_p(\sigma) = h_\sigma \circ t_\sigma$ $= h_\sigma \circ (1, \hat{u}) \circ (1, \hat{u}^{-1}) \circ t_\sigma = \hat{u} \circ h_{\sigma'} \circ t_{\sigma'} = \hat{u} \circ T_p(\sigma')$ and condition (iii) is satisfied.

We now extend T_p to all of $C_p(X)$ by linearity and note that since conditions (o), (i), (ii) and (iii) are satisfied on the generators of $C_p(X)$, namely $S_p(X)$, and $\pi_\#$, ∂_p and T_p are homomorphisms, conditions (o), (i), (ii) and (iii) are satisfied by the map $T_p : C_p(X) \to C_p(B^q)$.

The above construction of an equivariant lifting of $C_p(X)$ to $C_p(B^q)$ <u>fails on dimension q+1.</u> For let $\rho : \Delta_{q+1} \to X$. Then using conditions (o), (i) and (ii) for $\tau_q(\rho^{(i)})$ and applying Lemma III. 1, b), the cross-sections t^i_ρ over the q-faces $\Delta^{(i)}_{q+1}$ of Δ_{q+1} defined by $t^i_\rho = (e^i_{q+1}, 1) \circ t_{\rho^{(i)}} \circ (e^i_{q+1})^{-1}$ fit together as before to give a cross-section $t_{\dot\rho}$ over the boundary $\dot\Delta_{q+1}$ of Δ_{q+1}. Because $\pi_q(V_{n,n-q}) \neq 0$, the cross-section $t_{\dot\rho}$ may not be extendible to a cross-section over all of Δ_{q+1}. Hence we meet with an obstruction.

<u>We now define a singular cochain $C_\tau \in C^{q+1}(X;Z_2)$</u> which we show is <u>equivariant,</u> is a <u>cocycle,</u> and <u>obstructs equivariant extensions of τ_q</u>, in the sense that if there is an equivariant extension τ_{q+1}, then $C_\tau = 0$.

Let $\rho : \Delta_{q+1} \to X$ be a (q+1)-singular simplex of X and let $t_{\dot\rho}$ be the cross-section over $\dot\Delta_{q+1}$ in the bundle $B^q_\rho \to \Delta_{q+1}$, as described above. The obstruction to extending $t_{\dot\rho}$ to a cross-section over all of Δ_{q+1} is a cocycle $c(t_{\dot\rho}) \in C^{q+1}(\Delta_{q+1}, \pi_q(V_{n,n-q}))$ which is defined as follows (5, p. 148-151).

The oriented boundary $\dot\Delta_{q+1}$ of Δ_{q+1} is homeomorphic to the oriented q-sphere S^q, so that $t_{\dot\rho} : \dot\Delta_{q+1} \to B^q_\rho$ uniquely determines an element $\{t_{\dot\rho}\} \in \pi_q(B^q_\rho)$. Choosing a point $s \in \Delta_{q+1}$, say the leading vertex, and letting Y_s be the fibre over s, the inclusion map $Y_s \subset B^q_\rho$ induces an isomorphism $\pi_q(Y_s) \approx \pi_q(B^q_\rho)$. Since

Y_s is homeomorphic to $V_{n,n-q}$, we have an isomorphism $\pi_q(V_{n,n-q}) \approx \pi_q(Y_s)$. Letting $[t_\rho]$ be the element of $\pi_q(V_{n,n-q})$ corresponding to $\{t_\rho\}$ under the above isomorphisms, the cocycle $c(t_\rho)$ is defined by $c(t_\rho)(\Delta_{q+1}) = [t_\rho]$.

As noted in Chapter I, the Stiefel manifold $V_{n,n-q}$ had $q^{\underline{th}}$ homotopy group isomorphic to either Z or Z_2. Let $\underline{r : \pi_q(V_{n,n-q}) \to Z_2}$ be the unique non-zero homomorphism.

For each $\rho : \Delta_{q+1} \to X$, we define $c_T(\rho) = r(c(t_\rho)(\Delta_{q+1}) = r([t_\rho])$. Extending c_T to all of $C_{q+1}(X)$ by linearity we have a (q+1) singular cochain $c_T \in C^{q+1}(X; Z_2)$.

Theorem III. 2: The cochain $c_T \in C^{q+1}(X; Z_2)$ obstructs the equivariant extension of T_q.

Proof: Let T_{q+1} be an equivariant extension of T_q, i. e., T_{q+1} is a homomorphism from $C_{q+1}(X) \to C_{q+1}(B^q)$ such that (o) $T_{q+1}(\rho) : \Delta_{q+1} \to B^q$ for all $\rho \in S_{q+1}(X)$, (i) $\pi_\# \circ T_{q+1} = 1_{C_{q+1}(X)}$, (ii) $\partial_{q+1} T_{q+1} = T_q \partial_{q+1}$ and (iii) $\hat{u} \circ T_{q+1} = T_{q+1} \circ \check{u}$ whenever these expressions are defined. For each $\rho : \Delta_{q+1} \to X$, $T_{q+1}(\rho)$ induces a cross-section $t_\rho : \Delta_{q+1} \to B^q_\rho$. By Lemma III. 1, b), the following diagram is commutative:

$$\begin{array}{ccccc} B^q_{\rho^{(i)}} & \xrightarrow{(e^i_{q+1},\,1)} & B^q_\rho \mid \Delta^{(i)}_{q+1} & \subset & B^q_\rho \\ \uparrow t_{\rho^{(i)}} & & \uparrow t^i_\rho & & \uparrow t_\rho \\ \Delta_q & \xrightarrow[e^i_{q+1}]{} & \Delta^{(i)}_{q+1} & \subset & \Delta_{q+1} \end{array} .$$

Therefore, t_ρ is an extension of the cross-section $t_{\dot\rho}$ over $\dot\Delta_{q+1}$, so that $c(t_{\dot\rho}) = 0$. Consequently, $c_T(\rho) = r(c(t_{\dot\rho})(\Delta_{q+1})) = 0$.

Q. E. D.

Because of Theorem III. 2, we call c_T <u>the obstruction cochain of T</u>.

<u>Theorem III. 3:</u> <u>The obstruction cochain c_T is equivariant.</u>

<u>Proof:</u> We must show that if $\rho, \rho' \in S_{q+1}(X)$ with $\rho = \check{u} \circ \rho'$, then $c_T(\rho) = c_T(\rho')$. It therefore suffices to show that $r(\{t_{\dot\rho}\}) = r(\{t_{\dot\rho'}\})$. Now, $\rho = \check{u} \circ \rho'$ implies that $\rho^{(i)} = \check{u} \circ \rho'^{(i)}$. We have seen that the latter equation implies that $t_{\rho^{(i)}} = (1,\hat{u}) \circ t_{\rho'^{(i)}}$. Therefore $t_{\dot\rho} = (1, \hat{u}) \circ t_{\dot\rho'}$. Hence, $r(\{t_{\dot\rho}\}) = r(\{(1, \hat{u}) \circ t_{\dot\rho'}\}) = r \circ (1, \hat{u})_*(\{t_{\dot\rho'}\}) = r(\{t_{\dot\rho'}\})$, since the homeomorphism $(1, \hat{u})$ induces an isomorphism of the homotopy groups.

Q. E. D.

<u>Alternate Proof:</u> Suppose $\rho = \check{u} \circ \rho' : \Delta_{q+1} \to X$. Then as above $t_{\dot\rho} = (1, \hat{u}) \circ t_{\dot\rho'}$ so that $t_{\dot\rho}$ is the cross-section induced by

the bundle map $(1, \hat{u}^{-1})$ and cross-section $t_{\rho'}$. Then by the naturality of the obstruction cocycle $c(t_\rho)$ (5, p. 168), $c(t_\rho) = 1_\# c(t_{\rho'}) = c(t_{\rho'})$ and hence $c_\mathcal{T}(\rho) = c_\mathcal{T}(\rho')$.

Q. E. D.

Now, let $\rho : \Delta_m \to X$ be a singular m-simplex of X, and let $K(\rho) = \{\sigma : \sigma$ is a face of $\rho\}$. Let $K(\Delta_m)$ be the finite simplicial complex consisting of Δ_m and all of its proper faces. There is a natural identification of $K(\rho)$ with $K(\Delta_m)$ which we now describe. If Δ_p^i is a p-face of Δ_m, there are face maps $e_k^{i_k} : \Delta_{k-1} \to \Delta_k$ for $k=p+1, p+2, \ldots, m$ such that

$\Delta_p^i = e_m^{i_m} \circ e_{m-1}^{i_{m-1}} \circ \ldots \circ e_{p+1}^{i_{p+1}}(\Delta_p)$. Let σ^i be the p-face of ρ defined by $\sigma^i = \rho \circ e_m^{i_m} \circ \ldots \circ e_{p+1}^{i_{p+1}}$. Then the association $\Delta_p^i \to \sigma^i$ is a bijection between $K(\Delta_m)$ and $K(\rho)$ which respects face operations in the sense that if $(\Delta_p^i)^{(k)}$ is the $k^{\underline{th}}$ (p-1)-face of Δ_p^i and $(\Delta_p^i)^{(k)}$ is associated with σ^k, then σ^k is the $k^{\underline{th}}$ (p-1)-face of σ^i.

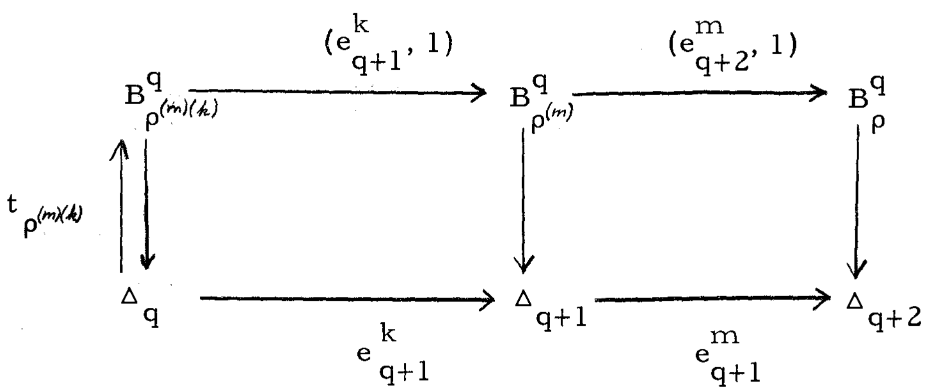
The above allows us to identify $K(\rho)$ with the finite simplicial complex $K(\Delta_m)$. This identification is assumed for $m=q+2$ in the proof of the following theorem, and we shall use $K(\rho)$ and $K(\Delta_{q+2})$ interchangeably.

<u>Theorem III. 4:</u> <u>The equivariant obstruction cochain is a cocycle.</u>

<u>Proof:</u> Let $\rho : \Delta_{q+2} \to X$ be given. Define a (q+1)-simplicial cochain $\not c$ of $K(\rho)$ with values in Z_2 by $\not c(\rho^{(i)}) = c_T(\rho^{(i)})$. Then $\delta c_T(\rho) = c_T(\partial\rho) = \not c(\partial\rho) = \delta\not c(\rho)$, so that $\delta c_T(\rho) = 0$ if and only if $\delta\not c(\rho) = 0$.

We prove that $\delta\not c(\rho) = 0$ by showing that $\not c$ is an obstruction cochain for a cross-section f over the q-skeleton of $K(\Delta_{q+2})$ in the bundle $B^q_\rho \to \Delta_{q+2}$ and is therefore a cocycle.

Let $\Delta^{(m)(k)}_{q+2} = e^m_{q+2} \circ e^k_{q+1}(\Delta_q)$ be a q-face of Δ_{q+1}. We then have the following commutative diagram:



where $t_{\rho^{(m)(k)}}$ is induced by $T_q(\rho^{(m)(k)})$. Therefore we have for each q-face $\Delta^{(m)(k)}_{q+2}$ of Δ_{q+2} a cross-section t^{mk} of the part of the bundle $B^q_\rho \to \Delta_{q+2}$ over $\Delta^{(m)(k)}_{q+2}$ defined by

$t^{mk} = (e^m_{q+2} \circ e^k_{q+1}, 1) \circ t_{\rho^{(m)(k)}} \circ (e^m_{q+2} \circ e^k_{q+1})^{-1}$.

Now, suppose two q-faces Δ^i_q and Δ^j_q of Δ_{q+2} have a common (q-1)-face. Then there is a (q+1)-face $\Delta^{(m)}_{q+2}$ of Δ_{q+2}

such that Δ_q^i and Δ_q^j are faces of $\Delta_{q+2}^{(m)}$. Thus let $\Delta_q^i = \Delta_{q+2}^{(m)(i)}$ and $\Delta_q^j = \Delta_{q+2}^{(m)(j)}$ have a common (q-1)-face, Δ_{q-1}^{ij}. We define a cross-section $t_{\rho^{(\dot{m})}}$ of the part of the bundle $B^q_{\rho^{(m)}} \to \Delta_{q+1}$ over $\dot{\Delta}_{q+1}$ by $t_{\rho^{(\dot{m})}} | \Delta_{q+1}^{(k)} = (e_{q+1}^k, 1) \circ t_{\rho^{(m)(k)}} \circ (e_{q+1}^k)^{-1}$. Then in the bundle $B_\rho^q \to \Delta_{q+2}$, we define a cross-section of the part of the bundle over the boundary $\Delta_{q+1}^{(\dot{m})}$ of $\Delta_{q+1}^{(m)}$ by $t^{\dot{m}} = (e_{q+2}^m, 1) \circ t_{\rho^{(\dot{m})}} \circ (e_{q+2}^m)^{-1} | \Delta_{q+2}^{(\dot{m})}$. We see that $t^{\dot{m}} | \Delta_{q+2}^{(m)(k)} = (e_{q+2}^m, 1) \circ (e_{q+1}^k, 1) \circ t_{\rho^{(m)(k)}} \circ (e_{q+1}^k)^{-1} \circ (e_{q+2}^m)^{-1} | \Delta_{q+2}^{(m)(k)}$ $= t^{mk}$. Consequently, $t^{mi} | \Delta_{q-1}^{ij} = t^{\dot{m}} | \Delta_{q-1}^{ij} = t^{mj} | \Delta_{q-1}^{ij}$. Thus, the cross-sections t^{mk} of the parts of the bundle $B_\rho^q \to \Delta_{q+2}$ over $\Delta_{q+2}^{(m)(k)}$ fit together to give a cross-section f of the part of the bundle over the q-skeleton of $K(\Delta_{q+2})$. Furthermore, for each $\Delta_{q+2}^{(m)}$, $f | \Delta_{q+2}^{(\dot{m})} = (e_{q+2}^m, 1) \circ t_{\rho^{(\dot{m})}} \circ (e_{q+2}^m)^{-1} | \Delta_{q+2}^{(\dot{m})}$.

Now, the obstruction to extending f to a cross-section over the (q+1)-skeleton of $K(\Delta_{q+2})$ is a (q+1)-cocycle $c(f)$. Let $\Delta_{q+2}^{(m)}$ be a (q+1)-simplex of $K(\Delta_{q+2})$. Then $f : \Delta_{q+2}^{(\dot{m})} \to B_\rho^q$ determines an element of $\pi_q(B_\rho^q)$, and this element is by definition $c(f)(\Delta_{q+2}^{(m)})$. But $f | \Delta_{q+2}^{(\dot{m})} = (e_{q+2}^m, 1) \circ t_{\rho^{(\dot{m})}} \circ (e_{q+2}^m)^{-1} | \Delta_{q+2}^{(\dot{m})}$ so that $c(f)(\Delta_{q+2}^{(m)}) = \{(e_{q+2}^m, 1) \circ t_{\rho^{(\dot{m})}}\} \in \pi_q(B_\rho^q)$. Hence the obstruction cocycle of f with values in Z_2 is $r(c(f))$ and $r(c(f))(\Delta_{q+2}^{(m)})$ $= r(\{(e_{q+2}^m, 1) \circ t_{\rho^{(\dot{m})}}\}) = r(\{t_{\rho^{(\dot{m})}}\})$ since the homeomorphism $(e_{q+2}^m, 1)$ induces an isomorphism of the homotopy groups.

Consequently, $\phi(\rho^{(m)}) = c_T(\rho^{(m)}) = r([t_{\rho^{(\dot{m})}}]) = r(\{t_{\rho^{(\dot{m})}}\}) =$

$= r(c(f)(\Delta^{(m)}_{q+2}))$, i.e., $\not{c} = r(c(f))$ and $\not{c}$ is an obstruction cocycle. Q. E. D.

We now wish to show that *the equivariant obstruction cocycle is natural with respect to equivariant bundle maps.* We recall that a vector bundle map $f = (\check{f}, \hat{f}) : (\mathcal{B}, J) \to (\mathcal{B}', J')$ is equivariant iff for each $u = (\check{u}, \hat{u}) \in J$ there is a $u' = (\check{u}, \hat{u}') \in J'$ such that $\check{f} \circ \check{u} = \check{u}' \circ \check{f}$ and $\hat{f} \circ \hat{u} = \hat{u}' \circ \hat{f}$ whenever these expressions are defined. Also, the associated bundle maps $(\mathcal{B}^q, J) \to (\mathcal{B}'^q, J')$ are equivariant in this same sense.

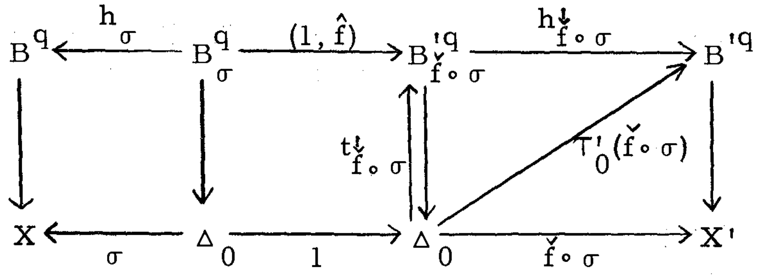
Before proving naturality, we establish the following two lemmas.

Lemma III. 5: If $f = (\check{f}, \hat{f}) : \mathcal{B}^q \to \mathcal{B}'^q$ *is a bundle map and* $T'_p : C_p(X') \to C_p(B'^q)$ *is defined for* $0 \leq p \leq q$ *satisfying* (o') $T'_p(\sigma) : \Delta_p \to B'^q$ *for* $\sigma \in S_p(X')$, (i') $\pi'_{\#} \circ T'_p = 1_{C_p(X')}$, *and* (ii') $\partial_p T'_p = T'_{p-1}\partial_p$, *then there is a* $T_p : C_p(X) \to C_p(B^q)$ *defined for* $0 \leq p \leq q$ *and satisfying* (o) $T_p(\sigma) : \Delta_p \to B^q$ *for* $\sigma \in S_p(X)$, (i) $\pi_{\#} \circ T_p = 1_{C_p(X)}$, (ii) $\partial_p T_p = T_{p-1}\partial_p$, *and* (iii) $\hat{f}_{\#} \circ T_p = T'_p \circ \check{f}_{\#}$.

Proof: Our proof is by induction on the dimension p.

First, for dimension 0, let $\sigma : \Delta_0 \to X$ be an arbitrary singular zero-simplex of X and consider the commutative diagram:

(VIII$_0$)



where the cross-section $t'_{\check{f}\circ\sigma}$ is induced by $\tau'_0(\check{f}\circ\sigma)$ so that $\tau'_0(\check{f}\circ\sigma) = h'_{\check{f}\circ\sigma}\circ t'_{\check{f}\circ\sigma}$. Let $t_\sigma : \Delta_0 \to B^q_\sigma$ be the cross-section induced by $t'_{\check{f}\circ\sigma}$ and the bundle map $(1, \hat{f})$, so that for each $s \in \Delta_0$, with $(1, \hat{f})_s = (1, \hat{f})|\pi_\sigma^{-1}(s)$, we have $t_\sigma(s) = [(1, \hat{f})_s]^{-1} \circ t'_{\check{f}\circ\sigma}$.

We define $\tau_0(\sigma) = h_\sigma \circ t_\sigma : \Delta_0 \to B^q$ and condition (i) follows immediately from diagram (VIII$_0$) and the fact that t_σ is a cross-section. Also, condition (ii) is vacuously satisfied.

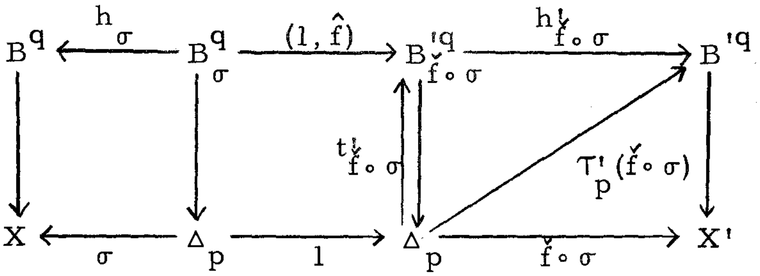
To check condition (iii), let $s \in \Delta_0$. Then $\hat{f}\circ\tau_0(\sigma)(s)$ $= \hat{f}\circ h_\sigma \circ t_\sigma(s) = \hat{f}\circ h_\sigma \circ [(1,\hat{f})_s]^{-1} \circ t'_{\check{f}\circ\sigma}(s)$. But
$\hat{f}\circ h_\sigma(s, b) = \hat{f}(b) = h'_{\check{f}\circ\sigma}(s, \hat{f}(b)) = h'_{\check{f}\circ\sigma} \circ (1, \hat{f})(s, b)$ for all $(s, b) \in B^q_\sigma$.
Hence, $\hat{f}\circ\tau_0(\sigma)(s) = h'_{\check{f}\circ\sigma}\circ(1,\hat{f})\circ[(1,\hat{f})_s]^{-1}\circ t'_{\check{f}\circ\sigma}(s) = h'_{\check{f}\circ\sigma}\circ t'_{\check{f}\circ\sigma}(s)$ $= \tau'_0(\check{f}\circ\sigma)(s)$, i. e., $\hat{f}\circ\tau_0(\sigma) = \tau'_0 \circ \check{f}(\sigma)$.

We define τ_0 on each singular zero-simplex on X as above and extend τ_0 to all of $C_0(X)$ by linearity. Then conditions (o), (i), (ii) and (iii) are satisfied since they are satisfied on the generators of $C_0(X)$, namely $S_0(X)$, and $\pi_\#$, τ_0, ∂, $\check{f}_\#$ and $\hat{f}_\#$ are homomorphisms.

Suppose now that $\tau_m : C_m(X) \to C_m(B^q)$ has been defined for

$0 \leq m < p \leq q$ satisfying (o), (i), (ii) and (iii), and let σ be an arbitrary singular p-simplex of X. We have the following commutative diagram:

$(VIII_p)$



where again the cross-section $t'_{\check{f}\circ\sigma}$ is induced by $T'_p(\check{f}\circ\sigma)$ so that $T'_p(\check{f}\circ\sigma) = h'_{\check{f}\circ\sigma} \circ t'_{\check{f}\circ\sigma}$. Let $t_\sigma : \Delta_p \to B^q_\sigma$ be the cross-section induced by $t'_{\check{f}\circ\sigma}$ and $(1, \hat{f})$, so that for each $s \in \Delta_p$,

$t_\sigma(s) = [(1, \hat{f})_s]^{-1} \circ t'_{\check{f}\circ\sigma}(s)$.

We define $T_p(\sigma) = h_\sigma \circ t_\sigma : \Delta_p \to B^q$ and condition (i) is again immediate. Condition (iii) follows exactly as in the case $p = 0$, since the computations used make no reference to the dimension.

In order to show that condition (ii) is satisfied, we first recall that conditions (o'), (i') and (ii'') on T'_p imply that

$t'_{\check{f}\circ\sigma} \circ e^i_p = (e^i_p, 1) \circ t'_{\check{f}\circ\sigma^{(i)}}$ by Lemma III. 1, b). We also have the relation $h'_{\check{f}\circ\sigma^{(i)}} = h'_{\check{f}\circ\sigma} \circ (e^i_p, 1)$ by Lemma III. 1, a) and, by the definitions, $\hat{f} \circ h_\sigma = h'_{\check{f}\circ\sigma} \circ (1, \hat{f})$.

Now let $s \in \Delta_{p-1}$. Then $h_\sigma \circ t_\sigma \circ e^i_p(s) =$

$h_\sigma \circ [(1, \hat{f})_{e^i_p(s)}]^{-1} \circ t'_{\check{f}\circ\sigma} \circ e^i_p(s) = h_\sigma \circ [(1, \hat{f})_{e^i_p(s)}]^{-1} \circ (e^i_p, 1) \circ t'_{\check{f}\circ\sigma^{(i)}}(s)$

$= [(\hat{f}_{\sigma} \circ e^i_p(s)]^{-1} \circ h'_{\check{f}\circ\sigma} \circ (e^i_p, 1) \circ t'_{\check{f}\circ\sigma^{(i)}}(s) = [\hat{f}_{\sigma^{(i)}}(s)]^{-1} \circ h'_{\check{f}\circ\sigma^{(i)}} \circ t'_{\check{f}\circ\sigma^{(i)}}(s) =$

$= h_{\sigma^{(i)}} \circ [(1, \hat{f})_{e^i_p(s)}]^{-1} \circ t_{\check{f} \circ \sigma^{(i)}}^{\downarrow}(s) = h_{\sigma^{(i)}} \circ t_{\sigma^{(i)}}(s)$, i. e., we have $h_\sigma \circ t_\sigma \circ e^i_p = h_{\sigma^{(i)}} \circ t_{\sigma^{(i)}}$ for each $i = 0, 1, \ldots, p$. Consequently,

$$\partial_p \tau_p(\sigma) = \sum_{i=0}^{p} (-1)^i \tau_p(\sigma) \circ e^i_p = \sum_{i=0}^{p} (-1)^i h_\sigma \circ t_\sigma \circ e^i_p = \sum_{i=0}^{p} (-1)^i h_{\sigma^{(i)}} \circ t_{\sigma^{(i)}}$$

$$= \sum_{i=0}^{p} (-1)^i \tau_{p-1}(\sigma^{(i)}) = \tau_{p-1}\partial_p(\sigma),$$ and condition (ii) is satisfied.

We define τ_p on each singular p-simplex of X as above and extend τ_p to all of $C_p(X)$ by linearity. Again, conditions (o), (i), (ii) and (iii) are satisfied on the generators of $C_p(X)$ and each of $\pi_\#$, τ_p, ∂, $\check{f}_\#$ and $\hat{f}_\#$ is a homomorphism so that the conditions are satisfied for $\tau_p : C_p(X) \to C_p(B^q)$.

Q. E. D.

The set $\tau = \{\tau_p : C_p(X) \to C_p(B^q) : 0 \leq p \leq q\}$ is called <u>the induced lifting.</u>

<u>Lemma III. 6: If in Lemma III. 5,</u> $\mathcal{B}^q$ <u>and</u> $\mathcal{B}'^q$ <u>have pseudo-groups</u> J <u>and</u> J', <u>respectively, and if</u> τ' <u>and</u> $f = (\check{f}, \hat{f})$ <u>are both equivariant, then the induced lifting</u> τ <u>is also equivariant.</u>

<u>Proof:</u> Let $\sigma : \Delta_p \to X$, $0 \leq p \leq q$, and suppose $\check{u} \circ \sigma$ is defined for some $(\check{u}, \hat{u}) \in J$. We must show that $\hat{u} \circ \tau_p(\sigma) = \tau_p(\check{u} \circ \sigma)$, i. e., $\hat{u} \circ h_\sigma \circ t_\sigma = h_{\check{u} \circ \sigma} \circ t_{\check{u} \circ \sigma}$. Now, by definition, for each $s \in \Delta_p$, $t_\sigma(s) = [(1, \hat{f})_s]^{-1} \circ t_{\check{f} \circ \sigma}^{\downarrow}$. Let $u' = (\check{u}', \hat{u}')$ be an element in J' associated with $u = (\check{u}, \hat{u})$ in J by the equivariance of f. Then for each $s \in \Delta_p$, $\hat{u} \circ h_\sigma \circ t_\sigma = \hat{u} \circ h_\sigma \circ [(1, \hat{f})_s]^{-1} \circ t_{\check{f} \circ \sigma}^{\downarrow}(s) =$

$= \hat{u} \circ [\hat{f}_{\sigma(s)}]^{-1} \circ h'_{\check{f}\circ\sigma} \circ t'_{\check{f}\circ\sigma}(s) = [\hat{f}_{\check{u}\circ\sigma(s)}]^{-1} \circ \hat{u}' \circ h'_{\check{f}\circ\sigma} \circ t'_{\check{f}\circ\sigma}(s)$

$= [\hat{f}_{\check{u}\circ\sigma(s)}]^{-1} \circ h'_{\check{u}'\circ\check{f}\circ\sigma} \circ t'_{\check{u}'\circ\check{f}\circ\sigma}(s) = [\hat{f}_{\check{u}\circ\sigma(s)}]^{-1} \circ h'_{\check{f}\circ\check{u}\circ\sigma} \circ t'_{\check{f}\circ\check{u}\circ\sigma}(s)$

$= h_{\check{u}\circ\sigma} \circ [(1, \hat{f})_s]^{-1} \circ t'_{\check{f}\circ\check{u}\circ\sigma}(s) = h_{\check{u}\circ\sigma} \circ t_{\check{u}\circ\sigma}(s).$ Thus,

$\hat{u} \circ h_\sigma \circ t_\sigma = h_{\check{u}\circ\sigma} \circ t_{\check{u}\circ\sigma}.$ Q. E. D.

Thoerem III. 7: If $f = (\check{f}, \hat{f}) : (\mathcal{B}, J) \to (\mathcal{B}', J')$ is an equivariant vector bundle map, and $\mathcal{T}'$ is an equivariant lifting in $(\mathcal{B}'^q, J')$ on dimensions less than $q + 1$, $0 \leq q \leq n - 1$, and if $\mathcal{T}$ is the induced equivariant lifting in $(\mathcal{B}^q, J)$, then $c_{\mathcal{T}} = \check{f}^{\#}(c_{\mathcal{T}'})$.

Proof: Let $\rho : \Delta_{q+1} \to X$. Then $\check{f}^{\#}(c_{\mathcal{T}'})(\rho) = c_{\mathcal{T}'}(\check{f}\circ\rho)$ $= r(c(t'_{\check{f}\circ\dot{\rho}})(\Delta_{q+1}))$ and $c_{\mathcal{T}}(\rho) = r(c(t_{\dot{\rho}})(\Delta_{q+1}))$. Therefore it suffices to show that $c(t'_{\check{f}\circ\dot{\rho}}) = c(t_{\dot{\rho}})$. By the definition of the induced lifting $\mathcal{T}$, the following diagram is commutative:

$$\begin{array}{ccc} B^q_\rho & \xrightarrow{(1,\hat{f})} & B'^q_{\check{f}\circ\rho} \\ t_{\dot{\rho}} \uparrow & & \uparrow t'_{\check{f}\circ\dot{\rho}} \\ \dot{\Delta}_{q+1} & \xrightarrow{1} & \dot{\Delta}_{q+1} \end{array},$$

i. e., the cross-section $t_{\dot{\rho}}$ is induced by $t'_{\check{f}\circ\dot{\rho}}$ and the bundle map $(1, \hat{f})$. Consequently, by the naturality of the obstruction cocycle (5, p. 168-169), $c(t_{\dot{\rho}}) = 1^{\#}c(t'_{\check{f}\circ\dot{\rho}}) = c(t'_{\check{f}\circ\dot{\rho}})$.

Q. E. D.

We have shown above that with each equivariant lifting $\mathcal{T}$ in $(\mathcal{B}^q, J)$ on dimensions less than or equal to q, there is associated an equivariant obstruction cocycle $c_T \in C^{q+1}(X;Z_2)$. We now wish to show that the equivariant cohomology class of c_T is independent of the particular choice of equivariant lifting, i.e., if T and $\overline{T}$ are equivariant liftings in $(\mathcal{B}^q, J)$ on dimensions less than $q+1$, then c_T and $c_{\overline{T}}$ are equivariantly cohomologous. We do this by defining an equivariant q-cochain $d(T,\overline{T}) \in C_e^q(X;Z_2)$ which satisfies $\delta d(T,\overline{T}) = c_T - c_{\overline{T}}$.

Before we define $d(T,\overline{T})$, we note the following. Let $\sigma : \Delta_m \to X$, $m=q$ or $q+1$. Then in the induced bundle $B_\sigma^q \to \Delta_m$ any two cross-sections t_σ and $\overline{t}_\sigma$ over the (q-1)-skeleton of Δ_m are homotopic (5, p. 181), and any such homotopy k_σ gives rise to a q-cochain $d_\sigma = d(t_\sigma, k_\sigma, \overline{t}_\sigma) \in C^q(\Delta_m, \pi_q(V_{n,n-q}))$, called the deformation cochain, which satisfies the following condition. If $c(t_\sigma)$ and $c(\overline{t}_\sigma)$ are the obstruction cocycles of t_σ and $\overline{t}_\sigma$, then $\delta d_\sigma = c(t_\sigma) - c(\overline{t}_\sigma)$ (5, p. 171-172). We suppose that for any given $\sigma : \Delta_m \to X$, $m=q$ or $q+1$, and any two cross-sections t_σ and $\overline{t}_\sigma$ over the (q-1)-skeleton of Δ_m in the induced bundle $B_\sigma^q \to \Delta_m$, the set of homotopies $k_\sigma : t_\sigma \simeq \overline{t}_\sigma$ is well-ordered.

Let $\mathfrak{A}$ be the set of all base elements of $S_q(X)$, $\mathfrak{A} = \{\sigma \in S_q(X) : \sigma$ is the base of some $A(\sigma')\}$. We first define $d(T,\overline{T})$ on $\mathfrak{A}$ by transfinite induction.

Thus, let σ be the first element of $\mathfrak{A}$. Then $T_q(\sigma)$ and $\overline{T}_q(\sigma)$ induce cross-sections t_σ and $\overline{t}_\sigma$ of the bundle $B_\sigma^q \to \Delta_q$.

Let k_σ be the first homotopy of the restrictions of t_σ and $\overline{t}_\sigma$ to the (q-1)-skeleton $\dot{\Delta}_q$ of Δ_q. Then $k_\sigma : t_\sigma | \dot{\Delta}_q \simeq \overline{t}_\sigma | \dot{\Delta}_q$ gives rise to $d_\sigma \in C^q(\Delta_q : \pi_q(V_{n,n-q}))$ and we define $d(T, \overline{T})(\sigma) = r(d_\sigma(\Delta_q))$, where r is as before the unique epimorphism $r : \pi_q(V_{n,n-q}) \to Z_2$.

Next, let σ' be the second element of $\mathfrak{A}$. We distinguish two cases in defining $d(T, \overline{T})(\sigma')$:

1) σ' has no face of any dimension in common with σ;

2) σ' has a face in common with σ.

Note that for $q=0$, case 1) is the only possibility.

In case 1), we define $d(T, \overline{T})(\sigma')$ in the same way that we defined $d(T, \overline{T})(\sigma)$, i.e., we let k_σ be the first homotopy of the restrictions of the induced cross-sections $t_{\sigma'}$ and $\overline{t}_{\sigma'}$ to the (q-1)-skeleton $\dot{\Delta}_q$ of Δ_q, $d_{\sigma'} = d(t_{\sigma'}, k_{\sigma'}, \overline{t}_{\sigma'})$ the corresponding deformation cochain, and define $d(T, \overline{T})(\sigma') = r(d_{\sigma'}(\Delta_q))$.

Before discussing case 2), we prove the following lemma.

<u>Lemma III. 8:</u> <u>For</u> $q \geq 1$, <u>let</u> $\dot{\Delta}_q^*$ <u>be a proper subset of the faces of</u> $\dot{\Delta}_q$. <u>Let</u> t_σ <u>and</u> $\overline{t}_\sigma$ <u>be cross-sections over</u> $\dot{\Delta}_q$ <u>in the bundle</u> $B_\sigma^q \to \Delta_q$. <u>Then any homotopy</u> $k_\sigma^* : t_\sigma | \dot{\Delta}_q^* \simeq \overline{t}_\sigma | \dot{\Delta}_q^*$ <u>can be extended to a homotopy</u> $k_\sigma : t_\sigma \simeq \overline{t}_\sigma$.

<u>Proof:</u> We can extend k_σ^* trivially to a map $k_\sigma^* : \dot{\Delta}_q^* \times I \cup \dot{\Delta}_q \times \{0\} \cup \dot{\Delta}_q \times \{1\}$ such that $k_\sigma^*(s, 0) = t_\sigma(s)$ and $k_\sigma^*(s, 1) = \overline{t}_\sigma(s)$ for all $s \in \dot{\Delta}_q$. Let $K = \dot{\Delta}_q \times I$ and

$L = \dot{\Delta}_q^* \times I \cup \dot{\Delta}_q \times \{0\} \cup \dot{\Delta}_q \times \{1\}$. Then L is a closed subset of K, and the pair (K, L) is a finitely triangulable pair. Therefore we wish to show that a cross-section over L into $B_\sigma^q \times I$ can be extended to a cross-section over K. By (5, p. 149), since $B_\sigma^q \times I$ is (q-1)-connected, the cross-section k_σ^* can be extended to a cross-section over $L \cup K^q$, where K^q is the q-skeleton of K. But the dimension of K is q, so that $K^q = K = L \cup K^q$ and k_σ^* can be extended to all of K.

Q. E. D.

Now, in case 2), we identify $K(\sigma)$ and $K(\sigma')$ with the finite simplicial complex $K(\Delta_q)$ as in the proof of Theorem III. 4. Let L be the subcomplex of $K(\Delta_q)$ consisting of all those faces of σ which are in common with faces of σ'. Let L' be the corresponding subcomplex for σ' and let $|L|$ and $|L'|$ be the corresponding subsets of Δ_q. Then $|L|$ and $|L'|$ are homeomorphic and we have the following commutative diagram:

(IX)
$$\begin{array}{ccccccc}
B_\sigma^q & \supset & B_\sigma^q \,\big|\, |L| & \xleftarrow{\hat{h}} & B_{\sigma'}^q \,\big|\, |L'| & \subset & B_{\sigma'}^q \\
t_\sigma \Big\uparrow\Big\uparrow \bar{t}_\sigma & & \Big\uparrow\Big\uparrow & & \Big\uparrow\Big\uparrow & & t_{\sigma'} \Big\uparrow\Big\uparrow \bar{t}_{\sigma'} \\
\Delta_q & \supset & |L| & \xleftarrow[\check{h}]{} & |L'| & \subset & \Delta_q
\end{array}$$

The map $\check{h}$ is a homeomorphism, and $\hat{h}$ is the corresponding bundle space homeomorphism. The inner vertical maps are those induced by the outer vertical maps in each outer rectangle.

The homotopy $k_\sigma : t_\sigma|\dot{\Delta}_q \simeq \overline{t}_\sigma|\dot{\Delta}_q$ together with diagram (IX) induces a homotopy $k^*_{\sigma'} : t_{\sigma'}\big|\,|L'| \simeq \overline{t}_{\sigma'}\big|\,|L'|$. By Lemma III. 8, this homotopy can be extended to a homotopy over all of $\dot{\Delta}_q$. Let $k_{\sigma'}$ be the first such extension. Then as before, we have the deformation cochain $d_{\sigma'} \in C^q(\Delta_q;\pi_q(V_{n,\,n-q}))$ and we define $d(T,\overline{T})(\sigma') = r(d_{\sigma'}(\Delta_q))$.

Now, let σ be an arbitrary element of $\mathfrak{O}$, and suppose that $d(T,\overline{T})(\sigma')$ has been defined as above for all $\sigma' < \sigma$ in such a manner that the following is true. If $\sigma'' < \sigma'$ and σ'' and σ' have a common face, then the homotopy $k_{\sigma'}$ in $d_{\sigma'} = d(t_{\sigma'}, k_{\sigma'}, \overline{t}_{\sigma'})$ used to define $d(T,\overline{T})(\sigma')$ is the first extension of the partial homotopy induced by $k_{\sigma''}$ in $d_{\sigma''}$ used to define $d(T,\overline{T})(\sigma'')$. We again distinguish two cases in defining $d(T,\overline{T})(\sigma)$:

1) σ has no face of any dimension in common with any $\sigma' < \sigma$;

2) σ has a face in common with some $\sigma' < \sigma$.

Note again that for $q=0$, case 1) is the only possibility.

In case 1) we define $d(T,\overline{T})(\sigma)$ exactly as in case 1) of the definition for the second element of $\mathfrak{O}$. Thus, we let $k_\sigma : t_\sigma|\dot{\Delta}_q \simeq \overline{t}_\sigma|\dot{\Delta}_q$ be the first homotopy and define $d(T,\overline{T})(\sigma) = r(d_\sigma(\Delta_q))$.

In case 2), suppose σ has a certain p-face, $0 \le p \le q-1$, in common with some preceding element of $\mathfrak{O}$ and let σ' be the first such element of $\mathfrak{O}$. Let $\dot{\Delta}^{p*}_q$ be the domain of this p-face of σ, and let k_{σ}^{p*} be the homotopy of t_{σ} and $\bar{t}_{\sigma}$ over $\dot{\Delta}_{q}^{p*}$ induced by the homotopy $k_{\sigma'}$ used in defining $d(T, \bar{T})(\sigma')$. By our inductive hypothesis, k_{σ}^{p*} then coincides with the homotopy induced by $k_{\sigma''}$, $k_{\sigma''}$ used in defining $d(T, \bar{T})(\sigma'')$, for any $\sigma'' < \sigma$ which has the given p-face of σ as a face. If we carry out the above procedure for each m-face of σ which is common with an m-face of some $\sigma' < \sigma$ in $\mathfrak{A}$, and let $\dot{\Delta}_{q}^{*}$ be the union of the domains of all such m-faces, we obtain a homotopy $k_{\sigma}^{*} : t_{\sigma}|\dot{\Delta}_{q}^{*} \simeq \bar{t}_{\sigma}|\dot{\Delta}_{q}^{*}$. By Lemma III. 8, this homotopy can be extended to all of $\dot{\Delta}_{q}$. Let k_{σ} be the first such extension and let $d_{\sigma} = d(t_{\sigma}, k_{\sigma}, \bar{t}_{\sigma})$ be the corresponding deformation cochain. We define $d(T, \bar{T})(\sigma) = r(d_{\sigma}(\Delta_{q}))$.

By transfinite induction, we have $d(T, \bar{T})$ defined on the subset $\mathfrak{A}$ of $S_{q}(X)$ and satisfying the condition that if $\sigma < \sigma'$ in $\mathfrak{A}$ and σ and σ' have a common p-face, $0 \leq p \leq q-1$, then the homotopy $k_{\sigma'}$, used to define $d(T, \bar{T})(\sigma')$, when restricted to the domain of this p-face is induced by the homotopy k_{σ} used to define $d(T, \bar{T})(\sigma)$.

Now let σ be an arbitrary element of $S_{q}(X)$, not in $\mathfrak{A}$. Let $\sigma' \in \mathfrak{A}$ be the base of $A(\sigma)$ and $(\check{u}, \hat{u})$ the base of $J(\sigma', \sigma)$. We then have the following commutative diagram:

$$\begin{array}{ccc} B^{q}_{\sigma} & \xrightarrow{(1,\, \hat{u}^{-1})} & B^{q}_{\sigma'} \\ t_{\sigma}\uparrow\uparrow\bar{t}_{\sigma} & & t_{\sigma'}\uparrow\uparrow\bar{t}_{\sigma'} \\ \Delta_{q} & \xrightarrow{1} & \Delta_{q} \end{array} \tag{IX}$$

i.e., $t_\sigma = (1, \hat{u}) \circ t_{\sigma'}$ and $\bar{t}_\sigma = (1, \hat{u}) \circ \bar{t}_{\sigma'}$. Let $k_{\sigma'}$ be the homotopy used to define $d(T, \bar{T})(\sigma')$ and let $k_\sigma : t_\sigma | \dot{\Delta}_q \simeq \bar{t}_\sigma | \dot{\Delta}_q$ be the induced homotopy. Then the deformation cochain

$d_\sigma = d(t_\sigma, k_\sigma, \bar{t}_\sigma) \in C^q(\Delta_q; \pi_q(V_{n,n-q}))$ is defined, and we define $d(T, \bar{T})(\sigma) = r(d_\sigma(\Delta_q))$.

We now have $d(T, \bar{T})$ defined on all of $S_q(X)$. Suppose that σ and σ' in $S_q(X)$ have a common p-face, $0 \leq p \leq q-1$, say $\sigma^p = \sigma'^p$. Let $\bar{\sigma}$ and $\bar{\sigma}'$ be the bases of $A(\sigma)$ and $A(\sigma')$, respectively, and $(\check{u}, \hat{u})$ and $(\check{v}, \hat{v})$ the bases of $J(\bar{\sigma}, \sigma)$ and $J(\bar{\sigma}', \sigma')$, respectively. Then $\bar{\sigma}$ and $\bar{\sigma}'$ in $\mathfrak{A}$ have a common p-face, namely $\check{u}^{-1} \circ \sigma^p = \check{v}^{-1} \circ \sigma'^p$ so that if $\bar{\sigma} < \bar{\sigma}'$, then $k_{\bar{\sigma}'}$ over the common p-face is induced by $k_{\bar{\sigma}}$. By our definition, k_σ and $k_{\sigma'}$ over the common p-face are induced by $k_{\bar{\sigma}}$ and $k_{\bar{\sigma}'}$, respectively, so that finally, since $(1, \hat{u}^{-1})$ is a homeomorphism, $\underline{k_{\sigma'}}$ <u>over the common p-face is induced by</u> $\underline{k_\sigma}$.

We extend $d(T, \bar{T})$ to all of $C_q(X)$ by linearity and we have $d(T, \bar{T}) \in C^q(X; Z_2)$.

<u>Theorem III. 9:</u> <u>The q-cochain</u> $d(T, \bar{T})$ <u>is equivariant.</u>

<u>Proof:</u> Let $\sigma = \check{u} \circ \sigma' \in S_q(X)$. We must show that $d(T, \bar{T})(\sigma) = d(T, \bar{T})(\sigma')$. Let $\sigma'' \in \mathfrak{A}$ be the base of $A(\sigma) = A(\sigma')$, and $(\check{v}, \hat{v})$, $(\check{w}, \hat{w})$ be the bases of $J(\sigma'', \sigma)$ and $J(\sigma'', \sigma')$, respectively. Then $\sigma = \check{v} \circ \sigma''$ and $\sigma' = \check{w} \circ \sigma''$ so that

$\sigma = \check{v} \circ \check{w}^{-1} \circ \sigma'$. Hence $\check{u} = \check{v} \circ \check{w}^{-1}$ and $(\check{u}, \hat{v} \circ \hat{w}^{-1}) \in J$. We then have the following commutative diagram:

$$(X) \qquad \begin{array}{ccccc} B^q_\sigma & \xrightarrow{(1, \hat{v}^{-1})} & B^q_{\sigma''} & \xleftarrow{(1, \hat{w}^{-1})} & B^q_{\sigma'} \\ t_\sigma \Big\uparrow\Big\uparrow \bar{t}_\sigma & & t_{\sigma''} \Big\uparrow\Big\uparrow \bar{t}_{\sigma''} & & t_{\sigma'} \Big\uparrow\Big\uparrow \bar{t}_{\sigma'} \\ \Delta_q & \xrightarrow{1} & \Delta_q & \xleftarrow{1} & \Delta_q \end{array} .$$

By definition, $d(T, \bar{T})(\sigma) = r(d(t_\sigma, k_\sigma, \bar{t}_\sigma)(\Delta_q))$ where $k_\sigma : t_\sigma | \dot{\Delta}_q \simeq \bar{t}_\sigma | \dot{\Delta}_q$ is induced by the left square in (X) from the homotopy $k_{\sigma''}$ used to define $d(T, \bar{T})(\sigma'')$, and similarly $d(T, \bar{T})(\sigma') = r(d(t_{\sigma'}, k_{\sigma'}, \bar{t}_{\sigma'})(\Delta_q))$. Because the bundle maps are homeomorphisms, k_σ is induced from $k_{\sigma'}$ by the bundle map $(1, \hat{w} \circ \hat{v}^{-1})$. Therefore, t_σ, k_σ, and $\bar{t}_\sigma$ are induced from $t_{\sigma'}$, $k_{\sigma'}$ and $\bar{t}_{\sigma'}$. By the naturality of the deformation cochain (5, p. 172), $d(t_\sigma, k_\sigma, \bar{t}_\sigma) = 1^{\#} d(t_{\sigma'}, k_{\sigma'}, \bar{t}_{\sigma'}) = d(t_{\sigma'}, k_{\sigma'}, \bar{t}_{\sigma'})$. Consequently, $d(T, \bar{T})(\sigma) = r(d_\sigma(\Delta_q)) = r(d_{\sigma'}(\Delta_q)) = d(T, \bar{T})(\sigma')$.

Q. E. D.

<u>Theorem III. 10:</u> $\delta d(T, \bar{T}) = c_T - c_{\bar{T}}$.

<u>Proof:</u> Let $\rho : \Delta_{q+1} \to X$. The cross-sections t_ρ and $\bar{t}_\rho$ over the q-skeleton of Δ_{q+1} in the bundle $B^q_\rho \to \Delta_{q+1}$ are homotopic as maps of the (q-1)-skeleton of Δ_{q+1}, and any such homotopy k_ρ gives rise to a deformation cochain $d_\rho = d(t_\rho, k_\rho, \bar{t}_\rho) \in C^q(\Delta_{q+1}; \pi_q(V_{n, n-q}))$ satisfying $\delta d_\rho = c(t_\rho) - c(\bar{t}_\rho)$.

Now, $\delta d(\mathsf{T},\overline{\mathsf{T}})(\rho) = d(\mathsf{T},\overline{\mathsf{T}})(\partial\rho) = \sum_{i=0}^{q+1}(-1)^i d(\mathsf{T},\overline{\mathsf{T}})(\rho^{(i)})$

$= r(\sum_{i=0}^{q+1}(-1)^i d_{\rho^{(i)}}(\Delta_q))$. Also, $(c_{\mathsf{T}} - c_{\overline{\mathsf{T}}})(\rho) = r([c(t_{\dot\rho}) - c(\overline{t}_{\dot\rho})](\Delta_{q+1}))$

$= r(\delta d_{\dot\rho}(\Delta_{q+1})) = r(\sum_{i=0}^{q+1}(-1)^i d_{\dot\rho}(\Delta_{q+1}^{(i)}))$ for any deformation cochain $d_{\dot\rho}$ defined as above. We shall construct a particular homotopy $k_{\dot\rho}$ below such that $d_{\dot\rho}$ satisfies the relation $e_{q+1}^{i\#} d_{\dot\rho} = d_{\rho^{(i)}}$ for each $i=0, 1, \ldots, q+1$. Then it will follow that $d_{\dot\rho}(\Delta_{q+1}^{(i)}) = e_{q+1}^{i\#} d_{\dot\rho}(\Delta_q)$

$= d_{\rho^{(i)}}(\Delta_q)$, so that $\delta d(\mathsf{T},\overline{\mathsf{T}})(\rho) = r(\sum_{i=0}^{q+1}(-1)^i d_{\rho^{(i)}}(\Delta_q)) = r(\sum_{i=0}^{q+1}(-1)^i d_{\dot\rho}(\Delta_{q+1}^{(i)}))$

$= (c_{\mathsf{T}} - c_{\overline{\mathsf{T}}})(\rho)$, as was to be shown.

Thus, it remains only to define the homotopy $k_{\dot\rho}$ making $d_{\dot\rho}$ satisfy the relations $e_{q+1}^{i\#} d_{\dot\rho} = d_{\rho^{(i)}}$, $0 \leq i \leq q+1$. For a fixed i, $0 \leq i \leq q+1$, consider the commutative diagram:

(XI)
$$\begin{array}{ccccc}
B^q_{\rho^{(i)}} & \xrightarrow{(e^i_{q+1},\,1)} & B^q_{\dot\rho} \,|\, \Delta^{(i)}_{q+1} & \subset & B^q_{\dot\rho} \,|\, \dot\Delta_{q+1} \\
t_{\rho^{(i)}} \Big\uparrow\Big\uparrow \overline{t}_{\rho^{(i)}} & & t^i_{\dot\rho} \Big\uparrow\Big\uparrow \overline{t}^i_{\dot\rho} & & t_{\dot\rho} \Big\uparrow\Big\uparrow \overline{t}_{\dot\rho} \\
\Delta_q & \xrightarrow[e^i_{q+1}]{} & \Delta^{(i)}_{q+1} & \subset & \dot\Delta_{q+1}
\end{array}$$

Because the bundle map $(e^i_{q+1}, 1)$ is a homeomorphism, the homotopy $k_{\rho^{(i)}}$ used to define $d(\mathsf{T},\overline{\mathsf{T}})(\rho^{(i)})$ induces a homotopy $k^i_{\dot\rho}$ of the maps $t^i_{\dot\rho}$ and $\overline{t}^i_{\dot\rho}$ restricted to the $(q-1)$-skeleton of $\Delta^{(i)}_{q+1}$. Thus, we have a homotopy of the cross-sections $t_{\dot\rho}$ and $\overline{t}_{\dot\rho}$ over the (q-1)-skeleton of each q-face of Δ_{q+1}. However, if $\Delta^{(i)}_{q+1}$ and $\Delta^{(j)}_{q+1}$ have a common (q-1)-face, then the homotopy $k_{\rho(i)}$ is induced by the homotopy $k_{\rho(j)}$ so that the homotopies $k^i_{\dot\rho}$ and $k^j_{\dot\rho}$ agree on this common (q-1)-face. Therefore, the homotopies $k^i_{\dot\rho}$ fit together to give a homotopy $k_{\dot\rho}$ of the maps $t_{\dot\rho}$ and $\bar{t}_{\dot\rho}$ over the (q-1)-skeleton of Δ_{q+1}. $k_{\dot\rho}$ is defined by $k_{\dot\rho}|\Delta^{(i)(j)}_{q+1} \times I = k^i_{\dot\rho}|\Delta^{(i)(j)}_{q+1} \times I$. Furthermore, by the naturality of the deformation cochain $d_{\dot\rho} = d(t_{\dot\rho}, k_{\dot\rho}, \bar{t}_{\dot\rho})$ and diagram (XI), $e^{i*}_{q+1} d_{\dot\rho} = d_{\rho(i)}$.

Q. E. D.

We have shown that for each q, $0 \leq q \leq n-1$, we can construct an equivariant lifting in $(\mathcal{B}^q, J)$ on dimensions less than q+1. Using this equivariant lifting, we can define a cocycle $c_\tau \in C^{q+1}_e(X;Z_2)$ and the equivariant cohomology class of c_τ is independent of the choice of equivariant lifting.

Definition: The cohomology class $\bar{c}_e(\mathcal{B}^q, J) \in H^{q+1}_e(X;Z_2)$ of c_τ is called the equivariant characteristic cohomology class of $(\mathcal{B}^q, J)$. It is the primary obstruction to an equivariant lifting in $\mathcal{B}^q$.

Theorem III. 11: $\bar{c}_e(\mathcal{B}^q, J)$ obstructs equivariant liftings in $(\mathcal{B}^q, J)$ on dimension q+1.

Proof: Suppose there is an equivariant lifting $\tau_p : C_p(X) \to C_p(B^q)$ for $0 \leq p \leq q+1$. Then by Theorem III. 2, the equivariant obstruction cocycle $c_{\mathcal{T}} \in C_e^{q+1}(X;Z_2)$ is zero. Therefore, its equivariant cohomology class $\bar{c}_e(\mathcal{B}^q, J)$ is also zero.

Q. E. D.

We now show that the equivariant characteristic cohomology class is natural with respect to equivariant bundle maps.

Theorem III. 12: If $f = (\check{f}, \hat{f}) : (\mathcal{B}, J) \to (\mathcal{B}', J')$ is an equivariant bundle map, then for each q, $0 \leq q \leq n - 1$, $\bar{c}_e(\mathcal{B}^q, J) = \check{f}^*(\bar{c}_e(\mathcal{B}'^q, J'))$.

Proof: Let q be a fixed integer $0 \leq q \leq n - 1$. Let $\mathcal{T}'_p : C_p(X') \to C_p(B'^q)$ be an equivariant lifting in $(\mathcal{B}'^q, J')$ for $0 \leq p \leq q$ and let $\mathcal{T}_p$ be the induced lifting in $(\mathcal{B}^q, J)$. Then $c_{\mathcal{T}'}$ and $c_{\mathcal{T}}$ are representative cocycles of $\bar{c}_e(\mathcal{B}'^q, J')$ and $\bar{c}_e(\mathcal{B}^q, J)$. By Theorem III. 7, $c_{\mathcal{T}} = \check{f}^{\#}(c_{\mathcal{T}'})$, so that $\bar{c}_e(\mathcal{B}^q, J) = f^*(\bar{c}_e(\mathcal{B}'^q, J'))$.

Q. E. D.

Let $\mathcal{V}^0_{p.g.}$ be the full subcategory of the category $\mathcal{V}_{p.g.}$ consisting of those objects $(\mathcal{B}, J)$ in $\mathcal{V}_{p.g.}$ which have trivial holonomy. For each $q \geq 0$, we define the function $\underline{W_e^{q+1}}$ on $\mathcal{V}^0_{p.g.}$ by

$$W_e^{q+1}(\mathcal{B}, J) = \begin{cases} \bar{c}_e(\mathcal{B}^q, J) & \text{if the dimension of } \mathcal{B} \text{ is greater than } q; \\ 0 \in H_e^{q+1}(X;Z_2) & \text{otherwise.} \end{cases}$$

Definition: The element $W_e^{q+1}(\mathcal{B}, J) \in H_e^{q+1}(X;Z_2)$ is called the $(q+1)^{\underline{st}}$ equivariant Stiefel-Whitney class of the object $(\mathcal{B}, J)$.

For $0 \leq q \leq n-1$, $W_e^{q+1}(\mathcal{B}, J)$ is by definition the primary obstruction to a lifting in $(\mathcal{B}^q, J)$.

Theorem III. 13: The equivariant Stiefel-Whitney classes are natural with respect to the morphisms in $\mathcal{V}^0_{p.g.}$.

Proof: Let $f = (\check{f}, \hat{f}) : (\mathcal{B}, J) \to (\mathcal{B}', J')$ be a morphism between n-dimensional objects of $\mathcal{V}^0_{p.g.}$. Then $W_e^{q+1}(\mathcal{B}', J') = 0 = W_e^{q+1}(\mathcal{B}, J)$ for $q > n-1$. For $0 \leq q \leq n-1$, $\check{f}^*(W_e^{q+1}(\mathcal{B}', J')) = \check{f}^*(\bar{c}_e(\mathcal{B}'^q, J')) = \bar{c}_e(\mathcal{B}^q, J) = W_e^{q+1}(\mathcal{B}, J)$ by Theorem III. 12.

Q. E. D.

If $(\mathcal{B}, J)$ is an object in the subcategory $\mathcal{V}$ of $\mathcal{V}^0_{p.g.}$, then $J = I_{\mathcal{B}}$ and singular cohomology and equivariant cohomology of the base space coincide. Furthermore, every lifting τ in $(\mathcal{B}^q, I_{\mathcal{B}})$ is equivariant and gives rise to an equivariant obstruction cocycle c_τ.

For any two such liftings τ and $\bar{\tau}$ in $(\mathcal{B}^q, I_{\mathcal{B}})$ the cochain $d(\tau, \bar{\tau})$ is equivariant. Therefore, our definition of $W_e^{q+1}(\mathcal{B}, I_{\mathcal{B}})$ coincides with the definition of $W^{q+1}(\mathcal{B})$ and we have the following.

Theorem III. 14: The function W_e^{q+1} on $\mathcal{V}^0_{p.g.}$ is an extension of the function W^{q+1} on $\mathcal{V}$.

IV. THE CASE OF ALLOWABLE HOLONOMY

Let $\mathcal{B} = (B, X, \pi)$ be an n-dimensional vector bundle and J a pseudo-group of local bundle maps on $\mathcal{B}$ such that for each $x \in X$, the holonomy group Φ_x of $(\mathcal{B}, J)$ at x is orientation preserving and leaves a subspace B_x^* of $\pi^{-1}(x)$ of dimension at least (n-1) fixed. With this restriction on the holonomy, we note that if $(\check{u}, \hat{u})$, $(\check{u}, \hat{v}) \in J$, then for each $x \in$ domain $\check{u}$, $\hat{u}|\pi^{-1}(x)$ and $\hat{v}|\pi^{-1}(x)$ differ only on a one-dimensional subspace of $\pi^{-1}(x)$, and $\hat{u}|\pi^{-1}(x)$ preserves orientation if and only if $\hat{v}|\pi^{-1}(x)$ preserves orientation.

For each q, $0 \leq q \leq n - 1$, the corresponding holonomy group of $(\mathcal{B}^q, J)$ at x will be denoted by Φ_x^q. An element of Φ_x^q acts only on one and the same component of each vector in the (n-q)-frames of the fibre $\pi^{-1}(x) = Y_x$. Now, Y_x is homeomorphic to $V_{n, n-q}$. For $q > 0$, the subset Y_x^* of Y_x which is left pointwise fixed by the action of Φ_x^q can therefore be identified with $V_{n-1, (n-1)-(q-1)}$. Therefore, the first non-zero homotopy group of Y_x^* is $\pi_{q-1}(Y_x^*)$ and $\pi_{q-1}(Y_x^*) \approx \pi_{q-1}(V_{n-1, (n-1)-(q-1)})$ is isomorphic to either Z or Z_2.

Now let q be a fixed integer, $0 \leq q \leq n - 1$, until stated otherwise, and let $B^{q*} = \cup_{x \in X} Y_x^*$. Then the pseudo-group J acts on the bundle $B^{q*} \to X$, denoted by $\mathcal{B}^{q*}$, with trivial holonomy.

Consequently, <u>the pair $(\mathcal{B}^{q*}, J)$ is of the type considered in Chapter III,</u> and we may apply the results of that chapter. In particular, we may define an equivariant lifting $\tau_p : C_p(X) \to C_p(B^{q*})$ for $0 \leq p \leq q - 1$ and we suppose we have done so. We identify $C_p(B^{q*})$ with its image in $C_p(B^q)$ under the monomorphism induced by the inclusion $B^{q*} \subset B^q$.

Thus, for each p, $0 \leq p \leq q - 1$, we have a homomorphism $\tau_p : C_p(X) \to C_p(B^q)$ satisfying:

(o) $\tau_p(\sigma) : \Delta_p \to B^q$ for each $\sigma \in S_p(X)$;

(i) $\pi_\# \circ \tau_p = 1_{C_p(X)}$;

(ii) $\partial_p \tau_p = \tau_{p-1} \partial_p$;

(iii) $\tau_p \circ \check{u}_\# = \hat{u}_\# \circ \tau_p$ whenever these expressions are defined.

We shall define an extension (not necessarily equivariant) $\tau_q : C_q(X) \to C_q(B_q)$ of τ_{q-1} in such a manner that the obstruction cochain c_τ, defined as before, is an equivariant cocycle. Thus, we shall define a homomorphism $\tau_q : C_q(X) \to C_q(B^q)$ satisfying:

(o) $\tau_q(\sigma) : \Delta_q \to B^q$ for $\sigma \in S_q(X)$;

(i) $\pi_\# \circ \tau_q = 1_{C_q(X)}$;

(ii) $\partial_q \tau_q = \tau_{q-1} \partial_q$;

(iii) $\delta c_\tau = 0$;

(iv) $c_\tau(\check{w} \circ \rho) = c_\tau(\rho)$ for all $\rho : \Delta_{q+1} \to X$ and $\check{w} \in J$ for which $\check{w} \circ \rho$ is defined.

In order to define τ_q, we shall again use transfinite induction.

Hence, as in Chapters II and III, suppose $S_q(X) = \{\sigma : \Delta_q \to X\}$ is well ordered, $A(\sigma) = \{\sigma' \in S_q(X) : \sigma'$ is J-equivalent to $\sigma\}$, and each set $A(\sigma)$ has as base the first element of $A(\sigma)$. Also, let J be well-ordered, $J(\sigma, \sigma') = \{(\check{u}, \hat{u}) \in J : \check{u} \circ \sigma = \sigma'\}$, and if $J(\sigma, \sigma')$ is non-empty, then its first element is the base of $J(\sigma, \sigma')$.

Let $\mathfrak{A}$ be the set of all base elements in $S_q(X)$, $\mathfrak{A} = \{\sigma \in S_q(X) : \sigma$ is the base of some $A(\sigma')\}$. We first define $\mathcal{T}_q$ on the subset $\mathfrak{A}$ of $S_q(X)$.

Let $\sigma \in \mathfrak{A}$ and consider the induced bundle $B^q_\sigma \to \Delta_q$ and bundle map $h_\sigma : B^q_\sigma \to B^q$ as before. Then exactly as in the induction step of the definition of $\mathcal{T}_p$ in Chapter III, for each (q-1)-face $\sigma^{(i)}$ of σ we get a cross-section $t^i_\sigma = (e^i_q, 1) \circ t_{\sigma^{(i)}} \circ (e^i_q)^{-1}$ over the (q-1)-face $\Delta^{(i)}_q$ in the induced bundle, and these fit together to give a cross-section $t_{\dot{\sigma}}$ over $\dot{\Delta}_q$, the boundary of Δ_q. Because the fibre $V_{n, n-q}$ is (q-1)-connected, $t_{\dot{\sigma}}$ can be extended to a cross-section over all of Δ_q. Let t_σ be the first such extension and define $\mathcal{T}_q(\sigma) = h_\sigma \circ t_\sigma : \Delta_q \to B^q$. Then also as before, $\pi \circ \mathcal{T}_q(\sigma) = \sigma$ and $\partial_q \mathcal{T}_q(\sigma) = \mathcal{T}_{q-1}(\partial_q \sigma)$.

This defines $\mathcal{T}_q$ on the subset $\mathfrak{A}$ of $S_q(X)$. Now let $\sigma : \Delta_q \to X$ be an arbitrary element of $S_q(X)$. If $\sigma \in \mathfrak{A}$, i.e., if σ is the base of some $A(\sigma')$, then $\mathcal{T}_q(\sigma)$ has been defined above. If $\sigma \notin \mathfrak{A}$, then there is a unique element $\sigma' \in \mathfrak{A}$ for which $\sigma \in A(\sigma')$. Let $(\check{u}, \hat{u})$ be the base of $J(\sigma', \sigma)$ and define

$\tau_q(\sigma) = \hat{u} \circ \tau_q(\sigma') : \Delta_q \to B^q$. Then $\pi \circ \tau_q(\sigma) = \pi \circ \hat{u} \circ \tau_q(\sigma')$ $= \check{u} \circ \pi \circ \tau_q(\sigma') = \check{u} \circ \sigma' = \sigma$ and condition (i) is satisfied.

Also, since τ_{q-1} is equivariant, we have $\hat{u}_\# \circ \tau_{q-1}(\partial\sigma')$ $= \tau_{q-1} \circ \check{u}_\#(\partial\sigma')$ so that $\partial\tau_q(\sigma) = \partial(\hat{u} \circ \tau_q(\sigma')) = \hat{u}_\# \circ \partial\tau_q(\sigma')$ $= \hat{u}_\# \circ \tau_{q-1}(\partial\sigma') = \tau_{q-1} \circ \check{u}_\#(\partial\sigma') = \tau_{q-1}\partial(\check{u} \circ \sigma') = \tau_{q-1}\partial(\sigma)$ and condition (ii) is satisfied.

The above defines τ_q on $S_q(X)$ satisfying conditions (o), (i) and (ii). We extend τ_q to all of $C_q(X)$ by linearity and note that conditions (o), (i) and (ii) are satisfied since $\pi_\#$, τ_q and ∂ are homomorphisms.

Before discussing conditions (iii) and (iv) concerning the cochain c_τ, we note the following. In the definition of τ_q, we relied on the equivariance of τ_{q-1} to show that condition (ii) was satisfied. The fact that τ_{q-1} could be defined equivariantly depended on the fact that the holonomy group Φ_x of $(\mathcal{B}, J)$ at $x \in X$ leaves an (n-1)-dimensional subspace of $\pi^{-1}(x)$ pointwise fixed. Therefore, we see that the construction of this section does not extend to vector bundles with pseudo-group $(\mathcal{B}, J)$ where the holonomy groups do not leave at least an (n-1)-dimensional subspace of the fibre pointwise fixed.

Now, the cochain $\underline{c_\tau \in C^{q+1}(X;Z_2)}$ is defined precisely as in Chapter III. Thus, for $\rho : \Delta_{q+1} \to X$, the cross-sections over the q-faces $\Delta_{q+1}^{(i)}$ of Δ_{q+1} in the bundle $B_\rho^q \to \Delta_{q+1}$, induced by the cross-sections $t_{\rho(i)} : \Delta_q \to B^q_{\rho(i)}$, fit together to give a cross-section $t_{\dot\rho} : \dot{\Delta}_{q+1} \to B^q_\rho$. The obstruction cocycle $c(t_{\dot\rho}) \in C^{q+1}(\Delta_{q+1}, \pi_q(V_{n,n-q}))$ is then defined by $c(t_{\dot\rho})(\Delta_{q+1}) = [t_{\dot\rho}] \in \pi_q(V_{n,n-q})$, and we define $c_\tau(\rho) = r(c(t_{\dot\rho})(\Delta_{q+1}) = r([t_{\dot\rho}])$.

Theorem IV. 1: The cochain c_τ obstructs the equivariant extension of τ_q.

Proof: The proof is identical to that of Theorem III.2, since the equivariance of τ_q played no part in the proof.

We state condition (iii) in the form of a theorem.

Theorem IV. 2: The cochain c_τ is a cocycle.

Proof: The proof is identical to that of Theorem III. 4, since the equivariance of τ_q played no part in the proof. Q. E. D.

We precede the proof of condition (iv) by the following lemma.

Lemma IV. 3: Let $\rho : \Delta_{q+1} \to X$ and $\rho' = \check{u} \circ \rho$, $(\check{u}, \hat{u}) \in J$. Then under the present holonomy assumptions, the following diagram is homotopy commutative:

$$\begin{array}{ccc} B^q_\rho & \xrightarrow{(1,\hat{u})} & B^q_{\rho'} \\ \uparrow{\scriptstyle t_{\dot\rho}} & & \uparrow{\scriptstyle t_{\dot\rho'}} \\ \dot{\Delta}_{q+1} & \xrightarrow{1} & \dot{\Delta}_{q+1} \end{array}.$$

<u>Proof</u>: In order to show $(1,\hat{u})\circ t_{\dot{\rho}} \simeq t_{\dot{\rho}'}$, it suffices to show that for each i, $0 \le i \le q$, $(1,\hat{u})\circ t_{\dot{\rho}}|\Delta^{(i)}_{q+1} \simeq t_{\check{u}\circ\dot{\rho}}|\Delta^{(i)}_{q+1}$ rel $(\Delta^{(i)}_{q+1})^{\bullet}$, where $(\Delta^{(i)}_{q+1})^{\bullet}$ is the boundary of $\Delta^{(i)}_{q+1}$, for since $\mathcal{T}_{q-1}$ is equivariant, we actually have $(1,\hat{u})\circ t_{\dot{\rho}}|(\Delta^{(i)}_{q+1})^{\bullet} = t_{\check{u}\circ\dot{\rho}}|(\Delta^{(i)}_{q+1})^{\bullet}$. Recall that $t_{\dot{\rho}}|\Delta^{(i)}_{q+1} = t^i_{\rho} = (e^i_{q+1}, 1)\circ t_{\rho^{(i)}}\circ(e^i_{q+1})^{-1}$ so that it suffices to show $(1,\hat{u})\circ(e^i_{q+1},1)\circ t_{\rho^{(i)}} \simeq (e^i_{q+1},1)\circ t_{\check{u}\circ\rho^{(i)}}$ rel $\dot{\Delta}_q$. Thus, let i be a fixed integer, $0 \le i \le q$.

For each $\sigma : \Delta_q \to X$, $t_\sigma = (1, \mathcal{T}_q(\sigma))\circ d$, where $d : \Delta_q \to \Delta_q x \Delta_q$ is the diagonal map. Hence it is sufficient to show that

$(1,\hat{u})\circ(e^i_{q+1},1)\circ(1,\mathcal{T}_q(\rho^{(i)}) = (e^i_{q+1}, \hat{u}\circ\mathcal{T}_q(\rho^{(i)}))$
$\simeq (e^i_{q+1}, \mathcal{T}_q(\check{u}\circ\rho^{(i)}))$ rel $d(\dot{\Delta}_q)$ and hence to show $\hat{u}\circ\mathcal{T}_q(\rho^{(i)})$
$\simeq \mathcal{T}_q(\check{u}\circ\rho^{(i)})$ rel $\dot{\Delta}_q$.

Now, let σ be the base of $A(\rho^{(i)})$, $(\check{w},\hat{w})$ the base of $J(\sigma,\rho^{(i)})$. Then by definition, $\mathcal{T}_q(\rho^{(i)}) = \hat{w}\circ\mathcal{T}_q(\sigma)$. Also, $\check{u}\circ\rho^{(i)} \in A(\rho^{(i)})$ so that $\check{u}\circ\rho^{(i)} = \check{v}\circ\sigma$, and if $(\check{v},\hat{v})$ is the base of $J(\sigma,\check{u}\circ\rho^{(i)})$, then $\mathcal{T}_q(\check{u}\circ\rho^{(i)}) = \hat{v}\circ\mathcal{T}_q(\sigma)$. Therefore $\mathcal{T}_q(\rho^{(i)}) = \hat{w}\circ\hat{v}^{-1}\circ\mathcal{T}_q(\check{u}\circ\rho^{(i)})$. But, $\rho^{(i)} = \check{w}\circ\sigma = \check{w}\circ\check{v}^{-1}\circ\check{u}\circ\rho^{(i)}$ so that $\check{w}\circ\check{v}^{-1} = \check{u}^{-1}$ and we have $(\check{u},\hat{u})$ and $(\check{u},\hat{v}\circ\hat{w}^{-1})$ in J and $\hat{v}\circ\hat{w}^{-1}\circ\mathcal{T}_q(\rho^{(i)}) = \mathcal{T}_q(\check{u}\circ\rho^{(i)})$. Let $\hat{v}\circ\hat{w}^{-1} = \hat{\hat{u}}$ so that $(\check{u},\hat{u})$, $(\check{u},\hat{\hat{u}}) \in J$ and $\hat{\hat{u}}\circ\mathcal{T}_q(\rho^{(i)}) = \mathcal{T}_q(\check{u}\circ\rho^{(i)})$

By the above, it is therefore sufficient to show that if $(\check{u},\hat{u})$, $(\check{u},\hat{\hat{u}}) \in J$, then $\hat{u}^{-1}\circ\hat{\hat{u}}\circ\mathcal{T}_q(\rho^{(i)}) \simeq \mathcal{T}_q(\rho^{(i)})$ rel $\dot{\Delta}_q$. We do this by exhibiting a homotopy.

Let F be defined on $\Delta_q \times I$ by $F(s,t) = t(\hat{u}^{-1} \circ \hat{\bar{u}} \circ T_q(\rho^{(i)})(s)) + (1-t)(T_q(\rho^{(i)})(s))$. Thus $F(s,0) = T_q(\rho^{(i)})(s)$, $F(s,1) = \hat{u}^{-1} \circ \hat{\bar{u}} \circ T_q(\rho^{(i)})(s)$ so that F is a homotopy between $T_q(\rho^{(i)})$ and $\hat{u}^{-1} \circ \hat{\bar{u}} \circ T_q(\rho^{(i)})$. Furthermore F is a homotopy relative to $\dot{\Delta}_q$ because for $s \in \dot{\Delta}_q$, $T_q(\rho^{(i)})(s)$ is in $Y^*_{\rho^{(i)}(s)}$, the subspace of $Y_{\rho^{(i)}(s)}$ fixed under the action of the holonomy group, and $\hat{u}^{-1} \circ \hat{\bar{u}} | Y_{\rho^{(i)}(s)} \in J_{\rho^{(i)}(s)}$. Hence for $s \in \dot{\Delta}_q$, $\hat{u}^{-1} \circ \hat{\bar{u}} \circ T_q(\rho^{(i)})(s) = T_q(\rho^{(i)})(s)$ and $F(s,t) = T_q(\rho^{(i)})(s)$ for all $t \in I$.

We wish to show that F is a homotopy in B^q, i. e., that $F : \Delta_q \times I \to B^q$ or $F(s,t)$ is an (n-q)-frame for each $s \in \Delta_q$, $t \in I$. Let $B_{\rho^{(i)}(s)}$ be the fibre in B over $\rho^{(i)}(s)$. $B_{\rho^{(i)}(s)}$ is homeomorphic to R^n. The holonomy group $\Phi_{\rho^{(i)}(s)}$ leaves an (n-1)-dimensional subspace $B^*_{\rho^{(i)}(s)}$ fixed. Let $B_{\rho^{(i)}(s)} = R^{n-1} \times R$ and $B^*_{\rho^{(i)}(s)} = R^{n-1} \times \{0\}$.

Now, the (n-q)-frame $T_q(\rho^{(i)})(s)$ can be written $T_q(\rho^{(i)})(s) = (v_1 + \bar{v}_1, v_2 + \bar{v}_2, \ldots, v_{n-q} + \bar{v}_{n-q})$ where $v_k = (v_{1k}, \ldots, v_{n-1,k}, 0)$ and $\bar{v}_k = (0, 0, \ldots, 0, \bar{v}_{nk})$. Let $\phi = \hat{u}^{-1} \circ \hat{\bar{u}} | Y_{\rho^{(i)}(s)} \in \Phi_{\rho^{(i)}(s)}$. By our assumption concerning holonomy, $\phi(T_q(\rho^{(i)}(s))) = (v_1 + a\bar{v}_1, \ldots, v_{n-q} + a\bar{v}_{n-q})$ for some positive real number a.

Thus, $F(s,t) = t(v_1 + a\bar{v}_1, \ldots, v_{n-q} + a\bar{v}_{n-q}) + (1-t)(v_1 + \bar{v}_1, \ldots, v_{n-q} + \bar{v}_{n-q}) = (v_1 + (at + 1 - t)\bar{v}_1, \ldots, v_{n-q} + (at + 1 - t)\bar{v}_{n-q})$. To show $F(s,t) \in B^q$, it suffices to show that the following matrix is of maximal rank (n-q):

$$(\alpha) \qquad \begin{bmatrix} v_{11} & \cdots & v_{1,n-1} & (ta+1-t)\bar{v}_{1n} \\ \vdots & & \vdots & \vdots \\ v_{n-q,1} & \cdots & v_{n-q,n-1} & (ta+1-t)\bar{v}_{n-q,n} \end{bmatrix} .$$

We know that the matrix

$$(\beta) \qquad \begin{bmatrix} v_{11} & \cdots & v_{1,n-1} & \bar{v}_{1n} \\ \vdots & & \vdots & \vdots \\ v_{n-q,1} & \cdots & v_{n-q,n-1} & \bar{v}_{n-q,n} \end{bmatrix}$$

is of rank (n-q), so that (β) has an (n-q) by (n-q) submatrix V with non-zero determinant. If this submatrix is of the form

$$V = \begin{bmatrix} v_{1,i_1} & \cdots & v_{1,i_{n-q}} \\ \vdots & & \vdots \\ v_{n-q,i_1} & \cdots & v_{n-q,i_{n-q}} \end{bmatrix}$$

then V is also a submatrix of (α) and (α) has rank (n-q).

If the submatrix V is of the form

$$V = \begin{bmatrix} v_{1,i_1} & \cdots & v_{1,i_{n-q-1}} & \bar{v}_{1n} \\ \vdots & & \vdots & \vdots \\ v_{n-q,i_1} & \cdots & v_{n-q,i_{n-q-1}} & \bar{v}_{n-q,n} \end{bmatrix}$$

then consider the following submatrix of (α):

$$V_t = \begin{bmatrix} v_{1,i_1} & \cdots & v_{1,i_{n-q-1}} & (ta+1-t)\bar{v}_{1,n} \\ \vdots & & \vdots & \vdots \\ v_{n-q,i_1} & \cdots & v_{n-q,i_{n-q-1}} & (ta+1-t)\bar{v}_{n-q,n} \end{bmatrix}.$$

Then, determinant of $V_t = \det V_t = (ta+1-t)\det V$ so that $\det V_t = 0$ if and only if $ta + 1 - t = t(a-1) + 1 = 0$. But, $a > 0$ and $t \in I$ implies that $t(a-1) > -1$. Therefore, $\det V_t \neq 0$ and (α) is of rank $(n-q)$. Q. E. D.

We now prove condition (iv) as a theorem.

Theorem IV. 4: The obstruction cocycle $c_{\mathcal{T}}$ is equivariant.

Proof: Let $\rho' = \check{u} \circ \rho : \Delta_{q+1} \to X$. Then the elements $\{(1, \hat{u}) \circ t_{\dot{\rho}}\}$ and $\{t_{\dot{\rho}'}\}$ in $\pi_q(B^q_{\rho'})$ are equal by Lemma IV. 3. Hence, the elements $[(1, \hat{u}) \circ t_{\dot{\rho}}]$ and $[t_{\dot{\rho}'}]$ are equal in $\pi_q(V_{n,n-q})$. Because $(1, \hat{u})$ is a homeomorphism, it induces an isomorphism $(1, \hat{u})_*$ on the homotopy groups. Therefore, $c_{\mathcal{T}}(\rho) = r(c(t_{\dot{\rho}})(\Delta_{q+1}))$ $= r([(1, \hat{u}) \circ t_{\dot{\rho}}]) = r \circ (1, \hat{u})_*([t_{\dot{\rho}}]) = r([t_{\dot{\rho}}])$. But also, $r([(1, \hat{u}) \circ t_{\dot{\rho}}])$ $= r([t_{\dot{\rho}'}]) = c_{\mathcal{T}}(\rho')$. Thus, $c_{\mathcal{T}}(\rho) = c_{\mathcal{T}}(\rho')$. Q. E. D.

We next show that the equivariant obstruction cocycle is natural with respect to equivariant bundle maps. This fact is contained in the following theorem.

Theorem IV. 5: Let $f = (\check{f}, \hat{f}) : (\mathcal{B}, J) \to (\mathcal{B}', J')$ be an equivariant bundle map. Let $\mathcal{T}'_p : C_p(X') \to C_p(B'^q)$ be an equivariant lifting for $0 \leq p \leq q-1$ and let $\mathcal{T}'_q : C_q(X') \to C_q(B'^q)$ be an extension of $\mathcal{T}'_{q-1}$ which yields an equivariant obstruction cocycle $c_{\mathcal{T}'} \in C_e^{q+1}(X;Z_2)$. Then the induced lifting $\mathcal{T}_p : C_p(X) \to C_p(B^q)$, $0 \leq p \leq q$, given by Lemma III. 5 is such that $\mathcal{T}_p$ is equivariant for $0 \leq p \leq q-1$, $\check{f}^{\#}(c_{\mathcal{T}'}) = c_{\mathcal{T}}$, and $c_{\mathcal{T}}$ is equivariant.

Proof: The fact that $\mathcal{T}_p$ is equivariant for $0 \leq p \leq q-1$ follows exactly as in Lemma III. 6.

Let $\rho : \Delta_{q+1} \to X$. By the definition of the induced lifting, the following diagram is commutative:

$$\begin{array}{ccc} B^q_\rho & \xrightarrow{(1,\hat{f})} & B'^q_{\check{f}\circ\rho} \\ \uparrow t_{\dot{\rho}} & & \uparrow t'_{\check{f}\circ\dot{\rho}} \\ \dot{\Delta}_{q+1} & \xrightarrow{1} & \dot{\Delta}_{q+1} \end{array}.$$

Therefore, as in the proof of Theorem III. 7, $c(t_{\dot{\rho}}) = c(t'_{\check{f}\circ\dot{\rho}})$ so that $c_{\mathcal{T}}(\rho) = r(c(t_{\dot{\rho}})(\Delta_{q+1})) = r(c(t'_{\check{f}\circ\dot{\rho}})(\Delta_{q+1})) = c_{\mathcal{T}'}(\check{f}\circ\rho)$, i. e., $c_{\mathcal{T}} = \check{f}^{\#}(c_{\mathcal{T}'})$.

Finally, let $\rho' = \check{u}\circ\rho : \Delta_{q+1} \to X$. Since f is equivariant, there is a $\check{u}' \in J'$ such that $\check{f}\circ\check{u} = \check{u}'\circ\check{f}$. Therefore $c_{\mathcal{T}}(\check{u}\circ\rho) = c_{\mathcal{T}'}(\check{f}\circ\check{u}\circ\rho) = c_{\mathcal{T}'}(\check{u}'\circ\check{f}\circ\rho) = c_{\mathcal{T}'}(\check{f}\circ\rho)$ since $c_{\mathcal{T}'}$ is equivariant. Consequently, $c_{\mathcal{T}}(\check{u}\circ\rho) = c_{\mathcal{T}}(\rho)$. Q. E. D.

We have now shown that with every equivariant lifting $\mathcal{T}$ in

$(\mathcal{B}^q, J)$ on dimensions less than or equal to $q - 1$, we may define an extension τ_q which yields an equivariant obstruction cocycle $c_\tau \in C^{q+1}(X; Z_2)$. We now wish to show that the equivariant cohomology class of c_τ is independent of the particular equivariant lifting on dimension less than q and the extension τ_q, as long as τ_q yields an equivariant obstruction cocycle.

Thus, suppose $\tau_p, \overline{\tau}_p : C_p(X) \to C_p(B^q)$ are equivariant liftings for $0 \leq p \leq q - 1$, and $\tau_q, \overline{\tau}_q : C_q(X) \to C_q(B^q)$ are extensions which yield equivariant obstruction cocycles c_τ, $c_{\overline{\tau}}$. We shall define an equivariant q-cochain $d(\tau, \overline{\tau}) \in C^q(X; Z_2)$ such that $\delta d(\tau, \overline{\tau}) = c_\tau - c_{\overline{\tau}}$.

The cochain <u>$d(\tau, \overline{\tau})$ is defined precisely as in Chapter III.</u> Thus, for $\sigma \in \mathfrak{A}$, $\sigma : \Delta_q \to X$, we associate the cross-sections t_σ and $\overline{t}_\sigma$ induced by $\tau_q(\sigma)$ and $\overline{\tau}_q(\sigma)$ and let $k_\sigma : t_\sigma | \dot{\Delta}_q \simeq \overline{t}_\sigma | \dot{\Delta}_q$ be the first homotopy satisfying the condition that if $\sigma' < \sigma$ and σ' and σ have a common face, then the homotopy k_σ restricted to the domain of this common face is induced by the homotopy $k_{\sigma'}$ used to define $d(\tau, \overline{\tau})(\sigma')$. Associated with t_σ, k_σ and $\overline{t}_\sigma$ is the deformation cochain $d_\sigma \in C^q(\Delta_q, \pi_q(V_{n, n-q}))$ and we define $d(\tau, \overline{\tau})(\sigma) = r(d_\sigma(\Delta_q))$.

If $\sigma \in S_q(X) - \mathfrak{A}$, i. e., σ is not a base element in $S_q(X)$, then there is a unique $\sigma' \in \mathfrak{A}$ such that $\sigma' \in A(\sigma)$. Let $(\check{u}, \hat{u})$ be the base element of $J(\sigma', \sigma)$. Then $k_{\sigma'}$, the homotopy used to define $d(T, \overline{T})(\sigma')$, and the bundle map $(1, \hat{u}^{-1}) : B^q_\sigma \to B^q_{\sigma'}$, induces a homotopy $k_\sigma : t_\sigma | \dot{\Delta}_q \simeq \overline{t}_\sigma | \dot{\Delta}_q$. This yields the deformation cochain d_σ, and we define $d(T, \overline{T})(\sigma) = r(d_\sigma(\Delta_q))$.

This defines $d(T, \overline{T})$ on $S_q(X)$, satisfying the condition that if σ and σ' have a common face and k_σ and $k_{\sigma'}$ are the homotopies used to define $d(T, \overline{T})(\sigma)$ and $d(T, \overline{T})(\sigma')$, then the homotopy k_σ restricted to the domain of the common face, is induced by $k_{\sigma'}$.

We extend $d(T, \overline{T})$ to all of $C_q(X)$ by linearity, and we have $d(T, \overline{T}) \in C^q(X; Z_2)$.

Because the definition of $d(T, \overline{T})$ depends upon what occurs on dimension $(q-1)$, and T_{q-1} and $\overline{T}_{q-1}$ are equivariant, the proof of the following theorem is identical to the proof of Theorem III. 9.

Theorem IV. 6: The q-cochain $d(T, \overline{T})$ is equivariant.

Since c_T, $c_{\overline{T}}$ and $d(T, \overline{T})$ are defined precisely as in Chapter III, the following theorem is proved exactly as is Theorem III. 10.

Theorem IV. 7: $\delta d(T, \overline{T}) = c_T - c_{\overline{T}}$.

We have now shown that for each q, $0 \leq q \leq n-1$, we can construct an equivariant lifting in $(\mathcal{B}^q, J)$ on dimensions less than q, and an extension to dimension q which yields a cocycle

$c_{\tau} \in C_e^{q+1}(X;Z_2)$. Furthermore, the equivariant cohomology class of c_{τ} is independent of the particular equivariant lifting and extension to dimension q.

Definition: The cohomology class $\bar{c}_e(\mathcal{B}^q, J) \in H_e^{q+1}(X;Z_2)$ of c_{τ} is called the equivariant characteristic cohomology class of $(\mathcal{B}^q, J)$. It is the primary obstruction to an equivariant lifting in $\mathcal{B}^q$.

Theorem IV. 8: If $f = (\check{f}, \hat{f}) : (\mathcal{B}, J) \to (\mathcal{B}', J')$ is an equivariant bundle map, then for each q, $0 \leq q \leq n-1$,
$\bar{c}_e(\mathcal{B}^q, J) = \check{f}^*(\bar{c}_e(\mathcal{B}'^q, J'))$.

Proof: The proof is identical to that of Theorem III. 12, except that we refer to Theorem IV. 5 rather than Theorem III. 7.

Q. E. D.

Let $\mathcal{V}^1_{p.g.}$ be the full subcategory of the category $\mathcal{V}_{p.g.}$ consisting of those objects $(\mathcal{B}, J)$ in $\mathcal{V}_{p.g.}$ for which the holonomy group Φ_x of $(\mathcal{B}, J)$ at each $x \in X$ is orientation preserving and leaves a subspace of $\pi^{-1}(x)$ of dimension at least $(n-1)$ pointwise fixed, when the dimension of $(\mathcal{B}, J)$ is n. For each $q \geq 0$, we define the function w_e^{q+1} on $\mathcal{V}^1_{p.g.}$ by

$$w_e^{q+1}(\mathcal{B}, J) = \begin{cases} \bar{c}_e(\mathcal{B}^q, J) & \text{if the dimension of } (\mathcal{B}, J) \text{ is greater than } q; \\ 0 \in H_e^{q+1}(X;Z_2) & \text{otherwise.} \end{cases}$$

Definition: The element $W_e^{q+1}(\mathcal{B}, J) \in H_e^{q+1}(X; Z_2)$ is called the $(q+1)^{\underline{st}}$ equivariant Stiefel-Whitney class of the object $(\mathcal{B}, J)$.

Theorem IV. 9: The equivariant Stiefel-Whitney classes are natural with respect to the morphisms in $\mathcal{V}^1_{p.g.}$.

Proof: The proof is identical to the proof of Theorem III. 13, except that we refer to Theorem IV. 8 rather than Theorem III. 12.

Q. E. D.

As in Chapter III, the definition of $W_e^{q+1}(\mathcal{B}, I_{\mathcal{B}})$ for an object $(\mathcal{B}, I_{\mathcal{B}}) \in \mathcal{V}$ coincides with the definition of $W^{q+1}(\mathcal{B})$. Therefore we have the following.

Theorem IV. 10: The function W_e^{q+1} on $\mathcal{V}^1_{p.g}$ is an extension of the function W^{q+1} on $\mathcal{V}$.

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