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Equivariant Whitney immersion theorem

A. Wasserman has proved in 1969 a generalization of the classical Whitney immersion theorem to the case of G-manifolds (G being a compact Lie group). In the present note another equivariant Whitney theorem is proved for a compact G-manifold and G finite. In that case our theorem implies Wasserman's result.

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Let G be a finite group. Let us begin with some remarks on G-maps. Suppose that H is a subgroup of G, X is a topological space and $G/H\times X$ is a G-space with obvious G-action (trivial on X). Let Y be a G-space and $f\colon G/H\times X\to Y$ be a G-map. Since f is equivariant, we get $f(H/H\times X)\subset Y^H$ ($Y^H = \{y\in Y\colon G_y \geqslant H\}$). Therefore we may define a map $\bar{f}\colon X\to Y^H$ by the formula $\bar{f}(x) = f(H/H, x)$. On the other hand, let $\bar{f}\colon X\to Y^H$ be any map. Define $f\colon G/H\to Y$ by the formula $f(gH/H, x) = g\bar{f}(x)$ for $x\in X$, $g\in G$.

LEMMA 1. *In the above situation the correspondence $f\to\bar{f}$ yields an isomorphism of sets*: G-maps$(G/H\times X, Y)$ *and* Maps(X, Y^H).

For detailed proof see [4].

Suppose that M is the n-dimensional C^∞ manifold with a smooth action of G. If $m\in M^H$, then T_mM is a representation of H. Let $T_mM = \oplus\, l_m(V)\,V$, where V is an irreducible representation of H, $l_m(V)$ its multiple in T_mM. Then by $l(M, V)$ we denote $\max_{m\in M} l_m(V)$. Note that $l(M, V)\leqslant n$.

Let M be compact. We are already prepared to formulate the first theorem.

THEOREM 1. *If the representation W satisfies the following condition: for every subgroup H of G such that there exists the point $m\in M$ with $G_m = H$ and every irreducible representation V of H with $l(M, V)\neq 0$, the inequality*

$$(*)\qquad l(W, V)\geqslant \dim M^H + l(M, V)\dim V$$

holds, then there exists a G-immersion $f\colon M\to W$.

Proof. The set M is compact, therefore it has the structure of the finite G-CW complex (see [5] for the proof). We construct the immersion by an induction on G-cells attached. Let (H) be the maximal orbit type on M, $G/H\times d$ 0-dimensional cell, $\bar{f}^0\colon d\to W^H$ any map. We define an H-equivariant monomorphism $\bar{\varphi}_0\colon T_dM\to W$, afterwards $\varphi_0\colon TM|_{Gd}\to W$ by the formula $\varphi_0(v_{gd}) = g\bar{\varphi}_0(g^{-1}v_{gd})$, where $v_{gd}\in T_{gd}M$. A small neighbourhood of the orbit Gd in M is G-diffeomorphic to a neighbourhood of the base in the normal bundle $\nu(\mathrm{Gd}, M)$ (see [2]). By means of the map exp we obtain the immersion from a G-neighbourhood of the orbit Gd in W. Let us denote it by f_1^0. Suppose that M^{i-1} is $(i-1)$-dimensional invariant skeleton of M and σ_k^i an i-dimensional G-cell. Assume that we have already defined an immersion f_p^{i-1} from a G-neighbourhood $U_{i-1,p}$ of $M^{i-1}\cup\bigcup_{k=1}^{p}\sigma_k^i$. We extend this map to the immersion f_{p+1}^{i-1} which will be defined on a G-neighbourhood of $M^{i-1}\cup\bigcup_{i=1}^{p+1}\sigma_k^i$.

Let $\bar{\sigma}_{p+1}^i = \sigma_{p+1}^i\setminus\partial_\varepsilon\sigma_{p+1}^i$, where $\partial_\varepsilon\sigma_{p+1}^i$ is the open invariant neighbourhood of $\partial\sigma_{p+1}^i$ on which we have defined the immersion f_p^{i-1}. Observe that $\bar{\sigma}_{p+1}^i\cong G/K\times D^i$; moreover, on $\partial\bar{\sigma}_{p+1}^i\cong G/K\times S^{i-1}$ we have the map f which is a restriction of f_p^{i-1}. The map f is a G-immersion, thus $\bar{f}\colon S^{i-1}\to W^K$ is an immersion. We also have the map $\bar{\varphi}$ defined to be equal to Tf_p^{i-1} restricted to $K/K\times S^{i-1}$, which has rank $n = \dim M$ in every point of S^{i-1}. We want to extend that map to the monomorphism $\psi\colon TM|_{K/K\times D^i}\to W^K\times W$. But:

$$TM|_{K/K\times D^i}\cong D^i\times T_0M\cong D^i\times T_0M^K\times\nu_0(M^K, M)\qquad (0\in D^i).$$

Let $\bar{\varphi}_1 = \bar{\varphi}$ be restricted to $S^{i-1}\times T_0M^K$, $\bar{\varphi}_2 = \bar{\varphi}$ restricted to $S^{i-1}\times\times\nu_0(M^K, M)$. Observe that the image of $\bar{\varphi}_1$ is contained in $W^K\times W^K$. We look for an extension of $\bar{\varphi}_1$ to $\psi_1\colon D^i\times T_0M^K\to W^K\times W^K$ such that $\psi_1|_{TD^i}$ is a map tangent to the immersion $\bar{g}$ being the extension of $\bar{f}$ over the disk D^i. The obstruction to extending lies in $\pi_{i-1}(V_\varkappa^k)$, where $\varkappa = \dim M^K$, $k = \dim W^K$, $V_\varkappa^k$ is the Stiefel manifold of $\varkappa$-frames in an Euclidean k-dimensional space (see [3]). This obstruction vanishes if $i\leqslant k-\varkappa$. That inequality follows from condition $(*)$ which for trivial irreducible representation R of the group K has the form $2\dim M^K\leqslant\dim W^K$ (in that case $l(M, R) = \dim M^K$).

Note that the image of $\bar{\varphi}_2$ is contained in $W^K\times W^{K\perp} = W$. We look for an extension of $\bar{\varphi}_2$ to $\psi_2\colon D^i\times\nu_0(M^K, M)\to W$. Such an extension exists by assumption $(*)$ and Lemma 2 (bellow) applied to each irreducible representation separately.

Thus we have the K-equivariant monomorphism $\psi\colon D^i\times T_0M\to W^K\times W$ covering the immersion $\bar{g}\colon D^i\to W^K$. We choose a neighbourhood U of D^i so small that the map exp is a diffeomorphism of a neighbourhood of the base in the normal bundle $\nu(D^i, M)$ with U. By means of the map exp we obtain an immersion $\bar{g}'$ of U in W (and G-immersion g': $GU \to W$ by equivariantness). Note that $\bar{g}'|_{S^{i-1}} = \bar{f} = f_p^{i-1}|_{S^{i-1}}$, $T\bar{g}'|_{S^{i-1}} = Tf|_{S^{i-1}}$. We can choose an open neighbourhood V of S^{i-1} in $U \cap U_{i-1,p}$ such that $\bar{V} \subset U \cap U_{i-1,p}$ and homotopy G_t: $U \cap U_{i-1,p} \to W$ such that

(a) for every t, G_t is the immersion,

(b) $(G_t, TG_t) = (\bar{g}', T\bar{g}')$ on any small neighbourhood of $\partial(U \cap U_{i-1,p})$,

(c) $(G_1, TG_1)|_V = (f, Tf)|_V$ $(f = f_p^{i-1}|_{U \cap U_{i-1,p}})$,

(d) $(G_t, TG_t)(x) = (\bar{g}', T\bar{g}')(x)$ if $(\bar{g}', T\bar{g}')(x) = (f, Tf)(x)$ (see [3], Lemma 2.5).

Let $Z \subset U$ be a neighbourhood of D^i such that $Z \cap U_{i-1,p} = V$. We define

$$U_{i-1,p+1} = U_{i-1,p} \cup GZ \supset M^{i-1} \cup \bigcup_{k=1}^{p+1} \sigma_k^p$$

$$\text{and} \quad f_{p+1}^{i-1}(x) = \begin{cases} f_p^{i-1}(x) & \text{if} \quad x \in U_{i-1,p}, \\ g'(x) & \text{if} \quad x \in GZ. \end{cases}$$

Let π: $E \to N$ be a vector bundle over N with the fibre lV, where V is an irreducible representation of G, N is a trivial G-space and l is a natural number. Then $\mathrm{Hom}(E, tV)$ is a G-bundle over N with the fibre $\mathrm{Hom}(\pi^{-1}(n), tV)$. Suppose that C is a closed subspace of N, s: $C \to \mathrm{Hom}(E, tV)$ is an equivariant non-singular section (it means that $s(c)$ is an equivariant monomorphism for every $c \in C$).

THEOREM. *If $t \geqslant \dim N + l \dim V$, then s can be extended to an equivariant non-singular section $\bar{s}$: $N \to \mathrm{Hom}(E, tV)$.*

This theorem is a special case of Theorem 2.1 in [7].

Let $E = D^i \times lV$ be a trivial G-bundle, W a trivial contractible G-space, and F a representation of G containing exactly t copies of V. Let φ be a G monomorphism

$$\begin{array}{ccc} E|_{S^{i-1}} & \xrightarrow{\varphi} & W \times F \\ {\scriptstyle\pi}\downarrow & & \downarrow{\scriptstyle \mathrm{pr}_1} \\ S^{i-1} & \longrightarrow & W \end{array}$$

LEMMA 2. *If $t \geqslant i + l \dim V$, then the monomorphism φ can be extended to the monomorphism of G-bundles ψ: $E \to W \times F$.*

Proof. Obviously, $\varphi(S^{i-1} \times lV) \subset W \times tV$; thus it is sufficient to show that $\varphi' = h \cdot \varphi$ can be extended over the disk (h is a homotopy contracting W to $*$). But we have the same situation in the theorem above:

$$\begin{array}{ccc} E|_{S^{i-1}} & \to & tV \\ {\scriptstyle\pi}\downarrow & & \downarrow \\ S^{i-1} & \longrightarrow & * \end{array}$$

CORROLARY 1. *If M can be immersed in tW, then M can be immersed in $2nW$* (Proposition 1.1 in [7]).

Proof. The representation $2nW$ satisfies condition $(*)$: if H is a subgroup of G such that there exists the point $m \in M$ with $G_m = H$ and V is an irreducible representation of H and $l(M, V) \neq 0$, *then* $l(M, V)V$ has an H-monomorphism in tW. Thus V has an H-monomorphism in W and $l(2nW, V) \geqslant 2n \geqslant \dim M^H + l(M, V)\dim V$.

THEOREM 2. *If M and W satisfy the assumptions of Theorem* 1, *then every smooth G-map $g\colon M \to W$ can be C^k-approximated by a G-immersion. The approximation is also uniform.*

Proof. From Theorem 1 we know that there exists an immersion $f\colon M \to W$. The approximation $\bar{g}$ will be of the form $\bar{g} = g + \delta f$, where $\delta = \varepsilon / \sup_{x \in M} N_k f(x)$, $N_k f(x) = \sum_{j=0}^{k} \|D^j f\varphi(\varphi^{-1}x)\|$ and $\varphi\colon R^n \to M$ is a local coordinate chart. Let $m \in M$ and φ be the local coordinate chart such that $\varphi(0) = m$; the vectors $\partial/\partial x_i$ $(i = 1, \ldots, n)$ span the space $T_m M$. Note that the vectors

$$T_m f\left(\frac{\partial}{\partial x_i}\right) = \left[\frac{\partial f_1}{\partial x_i}(0), \ldots, \frac{\partial f_s}{\partial x_i}(0)\right] \quad (\text{where } s = \dim W)$$

are lineary independent in W. We define an isomorphism $A\colon W \to W$ and show that the vectors $A \cdot T_m \bar{g}(\partial/\partial x_i)$ are independent; thus also $T_m \bar{g}(\partial/\partial x_i)$ are lineary independent.

Let

$$A^{-1} = \begin{bmatrix} \frac{\partial f_1}{\partial x_1}(0) & \ldots & \frac{\partial f_1}{\partial x_n}(0) & 0 & \ldots & & 0 \\ \cdot & \cdot & \cdot & \cdot & \cdot & & \cdot \\ \frac{\partial f_n}{\partial x_1}(0) & \ldots & \frac{\partial f_n}{\partial x_n}(0) & 0 & \ldots & & 0 \\ \frac{\partial f_{n+1}}{\partial x_1}(0) & \ldots & \frac{\partial f_{n+1}}{\partial x_n}(0) & 1 & \ldots & & 0 \\ \cdot & \cdot & \cdot & \cdot & \cdot & & \cdot \\ \frac{\partial f_s}{\partial x_1}(0) & \ldots & \frac{\partial f_s}{\partial x_n}(0) & 0 & \ldots & 0 & 1 \end{bmatrix}.$$

We may assume that the determinant of the matrix $\left(\frac{\partial f_j}{\partial x_i}(0)\right)$ $(1 \leqslant i, j \leqslant n)$ does not vanish, thus $|A| \neq 0$

$$A \circ T_m f\left(\frac{\partial}{\partial x_i}\right) = A \circ \left[\frac{\partial f_1}{\partial x_i}(0) \ldots \frac{\partial f_s}{\partial x_i}(0)\right] = [0 \ldots \underset{i}{1} \ldots 0],$$

$$A\circ T_m\bar{g}\left(\frac{\partial}{\partial x_i}\right) = A\circ\left(\frac{\partial \bar{g}_j}{\partial x_i}(0)\right) = A\circ\left(\frac{\partial g_j}{\partial x_i}(0)\right) + \delta A\circ\left(\frac{\partial f_j}{\partial x_i}(0)\right)$$

$$= A\circ\left(\frac{\partial g_j}{\partial x_i}(0)\right) + \delta\begin{bmatrix} 1 & 0 & \dots & 0 \\ 0 & 1 & \dots & 0 \\ \cdot & \cdot & \cdot & \cdot \\ 0 & 0 & \dots & 1 \\ \cdot & \cdot & \cdot & \cdot \\ 0 & 0 & \dots & 0 \end{bmatrix}.$$

Let B denote the matrix build up from initial n rows of $A\circ\left(\frac{\partial \bar{g}_j}{\partial x_i}(0)\right)$ and C be the matrix build up from initial n rows of $A\circ\left(\frac{\partial g_j}{\partial x_i}(0)\right)$. Note that $|B| = 0$ iff some eigenvalue Λ_i of C is equal to $-\delta$. If δ satisfies the inequality $0<\delta<\min_i\{|\Lambda_i|:\ \Lambda_i\neq 0\}$, then the vectors $\frac{\partial \bar{g}_j}{\partial x_i}(0)$ $(1\leqslant i, j \leqslant n)$ are lineary independent in m therefore also in a neighbourhood of m. The manifold M is compact thus we can choose a finite covering $\{U_k\}$ and numbers δ_k such that the map $\bar{g}$ is an immersion on U_k for $\delta<\delta_k$. Let $\delta_0 = \min_k \delta_k$; the map $\bar{g}$ is an immersion if $\delta<\delta_0$.

Let G be a finite group, W its representation, and M a compact n-dimensional G-manifold with the following property:

(**) *every point $m\in M$ has a G-neighbourhood U_m and an equivariant embedding of U_m in $tW\setminus 0$ for some t.*

COROLLARY 2. *If M has property* (**) *and $s\geqslant 2n$, then every smooth map $f\colon M\to sW$ can be C^k uniformly approximated by an equivariant immersion.* (Cf. [7], Corollary 1.10.)

Proof. It follows from (**) that $l(M, V)V$ has an H-monomorphism in tW (H is a subgroup of G occurring on M and V is an irreducible representation of H). Thus it is easy to see that the representation sW satisfies condition (*).

EXAMPLE 1. Let Z_2 acts on $R^{n+1} = R^{p+1}\times R^{n-p}$ by reflection in R^{p+1} and on $R^{k+q} = R^q\times R^k$ by reflection in R^q. We consider equivariant immersions of the unit sphere $S^n\subset R^{n+1}$ with the induced Z_2 action into R^{k+q}. Since $(S^n)^{Z_2} = S^p$, inequality (*) holds if

(a) $2p\leqslant q$.

Since $(S^n)^e = S^n$, inequality (*) holds if

(b) $2n\leqslant k+q$ (for trivial group e).

If $\tilde{R}$ denote the non-trivial representation of Z_2, then $l(S^n, \tilde{R}) = n-p$ and $l(R^{k+q}, \tilde{R}) = k$. Inequality (*) holds if

(c) $n\leqslant k$.

Therefore, if inequalities (a), (b), (c) hold, then every smooth $\boldsymbol{Z}_2$-map $f\colon S^n \to R^{q+k}$ (with the actions described above) can be approximated by a $\boldsymbol{Z}_2$-immersion.

In particular, if we consider S^n with the antipodal action of $\boldsymbol{Z}_2$ (it means that $p = -1$), we only need the condition: $2n \leqslant k+q$.

Let W be a representation of G.

DEFINITION. G-manifold M is said to be a *W-manifold* if for every $m \in M$ its tangent space $T_m M$ is G_m-isomorphic to W with the action resrticted to G_m. (For properties and applications of W-manifolds see [6].)

Observe that if M is a W-manifold, then $l(M, V) = l(W, V)$ for every subgroup occurring on M and every V. In particular, every component of M^H has the dimension equal to $\dim W^H$. Thus the representation $2W$ satisfies condition $(*)$ for an irreducible trivial representation of every subgroup H occurring on M.

EXAMPLE 2. Let $V = kR \oplus l\tilde{R}$ be the representation of $\boldsymbol{Z}_2$ and let M be a compact V-manifold. Any smooth $\boldsymbol{Z}_2$-map $f\colon M \to 2V \oplus t\tilde{R}$ can be C^k-approximated by a $\boldsymbol{Z}_2$-immersion for $t \geqslant \max\{k-l, 0\}$.

Proof. It is sufficient to check condition $(*)$ for a non-trivial representation of $\boldsymbol{Z}_2$: $l(2V \oplus t\tilde{R}, \tilde{R}) = 2l + t \geqslant k + l = \dim M^{\boldsymbol{Z}_2} + l$.

EXAMPLE 3. Let W be a representation of $\boldsymbol{Z}_p$, $p \neq 2$ prime. Then $W = kR \oplus k_1 V_1 \oplus \ldots \oplus k_{p-1} V_{p-1}$, where V_j is a 2-dimensional representation of $\boldsymbol{Z}_p$ with the action:

$$\begin{bmatrix} \cos j\theta, & \sin j\theta \\ -\sin j\theta, & \cos j\theta \end{bmatrix}, \qquad \theta = \frac{2\pi}{p}.$$

Suppose that M is a compact W-manifold,

$$\delta_i = \begin{cases} 0, & \text{if} \quad k_i = 0, \\ 1, & \text{if} \quad k_i \neq 0, \end{cases}$$

then:

Any $\boldsymbol{Z}_p$-map (smooth) $f\colon M \to 2W \oplus k \circ \bigoplus_{i=1}^{p} \delta_i V_i$ can be approximated by $\boldsymbol{Z}_p$-immersion.

Proof. Denote the representation $2W \oplus k \bigoplus_{i=1}^{p-1} \delta_i V_i$ by V.

If $M^{\boldsymbol{Z}_p} \neq \emptyset$ and $\delta_i \neq 0$, then $l(V, V_i) = 2k_i + k = \dim M^p + k_i \dim V_i$.

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