Discrete Comput Geom 26:429–461 (2001)
DOI: 10.1007/s00454-001-0008-0

Discrete & Computational
Geometry

Equivelar Polyhedra with Few Vertices

B. Datta and N. Nilakantan

Department of Mathematics, Indian Institute of Science,
Bangalore 560 012, India
{dattab,nandini}@math.iisc.ernet.in

Abstract. We know that the polyhedra corresponding to the Platonic solids are equivelar. In this article we have classified completely all the simplicial equivelar polyhedra on ≤ 11 vertices. There are exactly 27 such polyhedra. For each $n \geq -4$, we have classified all the (p, q) such that there exists an equivelar polyhedron of type $\{p, q\}$ and of Euler characteristic n. We have also constructed five types of equivelar polyhedra of Euler characteristic $-2m$, for each $m \geq 2$.

1. Introduction

A finite collection K of cycles, edges and vertices of a complete graph is called a *complex* (of dimension 2) if (i) each edge of a cycle in K is in K, (ii) each vertex of each edge in K is in K and (iii) any two cycles have at most one common edge. The cycles, edges and vertices in a complex are called the *faces*, *edges* and *vertices* in that complex, respectively. We denote a face $u_1 \cdots u_m u_1$ by $u_1 \cdots u_m$ also.

For a complex K, the *edge graph* $\mathrm{EG}(K)$ of K is the graph whose vertices and edges are the vertices and edges of K, respectively. $\mathrm{EG}(K)$, is also called the 1-*skeleton* of K. The graph theoretic complement of $\mathrm{EG}(K)$ is called the *non-edge graph* of K and is denoted by $\mathrm{NEG}(K)$. So, e is an edge in $\mathrm{NEG}(K)$ if and only if e is not an edge in K. See [2] for the graph-theoretic terms used in this paper.

If K is a complex, then we associate another graph $\Lambda(K)$ with K as follows. The vertices of $\Lambda(K)$ are the faces in K and for faces $F_1, F_2 \in K$, F_1F_2 is an edge in $\Lambda(K)$ whenever F_1 and F_2 have a common edge. For a vertex u in K let $\mathcal{F}_u$ be the set of faces containing u. A complex K is called an *abstract polyhedron* (or simply a *polyhedron*) (of dimension 2) if (iv) for each vertex v there is a face F containing v, (v) each edge is in exactly two faces, (vi) the induced subgraph $L(u) = \Lambda(K)[\mathcal{F}_u]$ is a cycle for each vertex

u in K and (vii) the graph $\Lambda(K)$ is connected. Since all the polyhedra considered in this paper are two-dimensional, we drop the qualification "two-dimensional". Clearly, the faces of a polyhedron determine the polyhedron. Because of this we identify a polyhedron with the set of faces in it.

A complex may be thought of as a prescription for the construction of a topological space by pasting together plane polygons. The topological space thus obtained from a complex K is called the *geometric carrier* of K and is denoted by $|K|$. It is easy to see that the geometric carrier of a polyhedron is a connected two-dimensional manifold.

Two complexes K and L are called *isomorphic* (denoted by $K \cong L$) if there exists a bijective map φ from the vertex-set of K to the vertex-set of L such that $v_1 \cdots v_k$ is a face in K if and only if $\varphi(v_1) \cdots \varphi(v_k)$ is a face in L. We identify two complexes if they are isomorphic.

If uv is an edge in a complex K, then we say u and v are adjacent in K. For a vertex v in a complex K, the number of edges through v is called the *degree* of v in K. If $f_0(K)$, $f_1(K)$ and $f_2(K)$ are the number of vertices, edges and faces, respectively, of a polyhedron K, then the number $\chi(K) := f_0(K) - f_1(K) + f_2(K)$ is called the *Euler characteristic* of K.

A polyhedron K is called *equivelar of type* $\{p, q\}$ (or $\{p, q\}$-*equivelar*) if each face is a p-gon (i.e., $\Lambda(K)$ is a p-regular graph) and the degree of each vertex is q (see [4]). A polyhedron is called *equivelar* if it is equivelar of type $\{p, q\}$ for some p and q.

A complex is called *simplicial* if each face consists of three vertices. If u is a vertex of a simplicial complex K, then the *link* of u in K (denoted by $\mathrm{Lk}_K(u)$) is the graph whose vertices are those vertices of K which are adjacent to u and whose edges are those edges vw in K such that uvw is a face in K. A simplicial complex with properties (iv)–(vi) is called a *combinatorial* 2-*manifold*. Observe that in this case the link of any vertex is a cycle. So, a connected combinatorial 2-manifold in which the degree of each vertex is the same is a simplicial equivelar polyhedron and hence is called an *equivelar combinatorial* 2-*manifold*.

In [18]–[20] McMullen et al. considered equivelar polyhedra with geometric carriers in $\mathbb{R}^3$ (and hence orientable). We consider both the orientable and non-orientable cases.

Example 1. Some equivelar polyhedra:

$$
\begin{aligned}
S^2_4 &= \{abc, abd, acd, bcd\},\\
O &= \{a_ib_jc_k \colon 1 \le i, j, k \le 2\},\\
C &= \{a_1b_2c_1d_2, a_1b_2d_1c_2, a_1c_2b_1d_2, a_2b_1c_2d_1, a_2b_1d_2c_1, a_2c_1b_2d_1\},\\
I &= \{uu_iu_{i+1}, u_iu_{i+1}v_{i+3}, v_iv_{i+1}u_{i+3}, vv_iv_{i+1} \colon 1 \le i \le 5\},\\
D &= \{v_1v_2v_3v_4v_5, v_iv_{i+1}u_{i+1}v_{i,i+1}u_i, v_{i,i+1}u_{i+1}v_{i+1,i+2}u_{i+1,i+2}u_{i,i+1},\\
&\qquad u_{12}u_{23}u_{34}u_{45}u_{51} \colon 1 \le i \le 5\},\\
\mathbb{R}P^2_6 &= \{uu_iu_{i+1}, u_iu_{i+1}u_{i+3} \colon 1 \le i \le 5\},\\
R &= \{u_{1,2}u_{2,3}u_{3,4}u_{4,5}u_{5,1}, u_{i,i+1}u_{i,i+1,i+3}u_{i+2,i+3,i}u_{i+5,i,i+2}u_{i+5,i} \colon 1 \le i \le 5\},\\
M_1 &= \{u_{1+p}u_{4+p}u_{7+p}, u_{i+3p}u_{j+3p}u_{k+3p} \colon (i, j, k)\\
&\qquad \in \{(1,2,5), (1,3,5), (1,3,4), (1,8,9), (1,6,8), (1,2,6), (2,3,6)\},\\
&\qquad 0 \le p \le 2\},
\end{aligned}
$$

$N_1 = \{uu_iu_{i+1}, u_iu_{i+1}u_{i+4}, u_iu_{i+2}u_{i+4}, u_iu_{i+3}u_{i+6}\colon 1 \leq i \leq 9\}$.
(Additions in the subscripts are modulo 5 in I, D, $\mathbb{R}P_6^2$, R and are modulo 9 in M_1, N_1.)

Here S_4^2 is equivelar of type $\{3,3\}$, O is equivelar of type $\{3,4\}$, C is equivelar of type $\{4,3\}$, I and $\mathbb{R}P_6^2$ are equivelar of type $\{3,5\}$, D and R are equivelar of type $\{5,3\}$, M_1 is equivelar of type $\{3,8\}$ and N_1 is equivelar of type $\{3,9\}$.

The geometric carrier of each of S_4^2, O, C, I and D is the 2-sphere and they correspond to the Platonic solids [5], [6], [12], [8], [24], namely, tetrahedron, octahedron, cube, icosahedron and dodecahedron, respectively. The polyhedron $\mathbb{R}P_6^2$ [1] is called the hemi-icosahedron and the polyhedron R is called the hemi-dodecahedron. The geometric carrier of each of $\mathbb{R}P_6^2$ and R is the real projective plane. The geometric carrier of M_1 is the non-orientable surface of Euler characteristic -3. The geometric carrier of N_1 is the non-orientable surface of Euler characteristic -5.

Let K be a polyhedron with faces $F_1, \ldots, F_m$. Consider a complex $\widetilde{K}$ with vertex-set $\{w_1, \ldots, w_m\}$ as: $w_{i_1} \cdots w_{i_k}$ is a face in $\widetilde{K}$ if and only if there exists a vertex u in K such that $F_{i_1} \cdots F_{i_k} F_{i_1}$ is the cycle $L(u)$ defined above. Then $\widetilde{K}$ is a polyhedron. $\widetilde{K}$ is called the *dual* of K. It is easy to show that the dual of $\widetilde{K}$ is isomorphic to K and $\chi(\widetilde{K}) = \chi(K)$. It is also not difficult to see that $\widetilde{S_4^2} \cong S_4^2$, $\widetilde{C} \cong O$, $\widetilde{I} \cong D$ and $\widetilde{\mathbb{R}P_6^2} \cong R$. Observe that the graph $\Lambda(K)$ is isomorphic to $\mathrm{EG}(\widetilde{K})$. Because of this, for a polyhedron K, $\Lambda(K)$ is called the *dual* 1-*skeleton* of K.

A *pattern* on a connected surface M is a non-empty, connected locally finite graph Γ contained in M, such that each component of $M\backslash\Gamma$ is simply connected and has compact closure. The closure of a component of $M\backslash\Gamma$ is called a face of Γ. A pattern decomposes the surface into faces. Such a decomposition is called a *map* (see [7], [15] and [16]). A pattern Γ is called *non-singular* if each edge of Γ lies in two faces. A pattern Γ is called *equivelar* of type $\{p,q\}$ (or a $\{p,q\}$-*pattern*) if each face contains p (counted with multiplicity) edges and each vertex has degree q.

Let $\Phi(n) = \{(p,q)$: there exists a $\{p,q\}$-equivelar polyhedron of Euler characteristic $n\}$ and $\Sigma(n)$ denote the set of all equivelar polyhedra of Euler characteristics n. Clearly, $\Phi(m) = \emptyset$ for $m \geq 3$. It is known (e.g., see [17]) that if $(p,q) \in \Phi(n)$ for some $n < 0$, then $(p,q) \in \Phi(n)$ for infinitely many negative n. Here we prove:

Theorem 1. *If* $\Phi(n)$ *and* $\Sigma(n)$ *are as above, then*

(i) $\Phi(n) \cap \Phi(-m) = \emptyset$ *for all* $n \geq 0$ *and* $m \geq 1$,
(ii) $\Phi(2) = \{(3,3), (3,4), (4,3), (3,5), (5,3)\}$,
(iii) $\Phi(1) = \{(3,5), (5,3)\}$,
(iv) $\Phi(0) = \{(3,6), (6,3), (4,4)\}$,
(v) $\Phi(-1) = \emptyset$,
(vi) $\Phi(-2) = \{(3,7), (7,3), (4,5), (5,4)\}$,
(vii) $\Phi(-3) = \{(3,7), (7,3), (3,8), (8,3), (4,5), (5,4), (5,5)\}$,
(viii) $\Phi(-4) = \{(3,7), (7,3), (3,8), (8,3), (4,5), (5,4), (4,6), (6,4), (5,5)\} \subseteq \Phi(-2m)$, *for all* $m \geq 3$,
(ix) $(3, 3k-1) \in \Phi(-k(3k-7)/2)$, $(3,3k) \in \Phi(1 - k(3k-5)/2)$, *for all* $k \geq 3$,

(x) $\Phi(n)$ *is a finite set for each integer n and*
(xi) *for each* $n \geq 7$*, there exists an n-vertex* $\{3, 6\}$*-equivelar polyhedron in* $\Sigma(0)$.

Corollary 2. *For each* $n \neq 0$*, there exist only finitely many equivelar polyhedra of Euler characteristic* n.

In [11] (also see in [7]) Edmonds et al. proved the existence and uniqueness of a $\{p, q\}$-pattern on surfaces. Clearly, a $\{p, q\}$-equivelar polyhedron K gives a non-singular $\{p, q\}$-pattern on $|K|$ with the property that any two faces have at most one common edge. So, the existence of a $\{p, q\}$-equivelar polyhedron implies the existence of a $\{p, q\}$-pattern but not conversely. For example, by Theorem 2.4(i) of [11], a $\{3, 7\}$-pattern and a $\{4, 6\}$-pattern exist on a non-orientable surface of Euler characteristic -1 but, from Theorem 1(v) above, an equivelar polyhedron of Euler characteristic -1 does not exist. Also, by the same theorem in [11], a $\{5, 6\}$-pattern exists on an orientable surface of Euler characteristic -4 but, from Theorem 1(viii) above a $\{5, 6\}$-equivelar polyhedron of Euler characteristic -4 does not exist. There are five choices of (p, q) for $\{p, q\}$-equivelar polyhedra of Euler characteristic 2, where as there are infinitely many choices of (p, q) for $\{p, q\}$-patterns (e.g., $\{p, 2\}$-patterns exist for all $p \geq 3$) on the 2-sphere (see [11] and [7]). However, in each case unique pattern exists (see Classification in [11]). Similarly for Euler characteristic 1. From these (Classification in [11]) and Theorem 1(iii) and (iv) above we get:

Corollary 3. *If the Euler characteristic of an equivelar polyhedron is positive, then the polyhedron is* S^2_4, C, O, I, D, $\mathbb{R}P^2_6$ *or* R *defined in Example* 1.

Corollary 2 says that $\Sigma(n)$ is a finite set for $n \neq 0$ and Theorem 1(xi) shows that $\Sigma(0)$ is an infinite set. If the Euler characteristic is ≤ 0, then (unlike when the Euler characteristic is > 0) it is in general difficult to classify all the non-singular $\{p, q\}$-patterns. In particular, it is very difficult to classify all the $\{p, q\}$-equivelar polyhedra of a given non-positive Euler characteristic. Even for a negative Euler characteristic, there can exist more than one $\{p, q\}$-equivelar polyhedra of the same Euler characteristic (e.g., $N_1, \dots, N_{14}$ in Examples 1 and 8). For simplicial polyhedra on few vertices we have:

Theorem 4. *Let K be an n-vertex simplicial equivelar polyhedron of Euler characteristic* 0*. If* $n \leq 11$*, then K is isomorphic to* $T_7, \dots, T_{11}$, $A_{3,3}$, $B_{3,3}$ *or Q defined below.*

Theorem 5. *If* M_1 *and* M_2 *are as in Examples* 1 *and* 7*, then* $M_1 \ncong M_2$ *and any* 9*-vertex neighbourly simplicial equivelar polyhedron is isomorphic to* M_1 *or* M_2.

Theorem 6. *Let* $N_1, \dots, N_{14}$ *be as in Examples* 1 *and* 8*. We have the following*:

(i) N_i *is not isomorphic to* N_j *for* $1 \leq i \neq j \leq 14$.
(ii) *If M is a* 10*-vertex neighbourly simplicial equivelar polyhedron, then M is isomorphic to* N_i *for some* $i \in \{1, \dots, 14\}$.

Theorem 7. *There are exactly* 27 (*up to isomorphism*) *simplicial equivelar polyhedra on* ≤ 11 *vertices, namely,* S^2_4, O, $\mathbb{R}P^2_6$, $A_{3,3}$, $B_{3,3}$, $T_7, \ldots, T_{11}$, M_1, M_2, $N_1, \ldots, N_{14}$ *and* Q, *defined in Examples* 1–8.

Remark 1. Observe that M_1 and M_2 are non-isomorphic but they have the same 1-skeleton. Similarly, $N_1, \ldots, N_{14}$ have the same 1-skeleton but they are pairwise non-isomorphic combinatorial 2-manifolds.

Remark 2. Corollary 3 is classically known. We have added it here as an immediate consequence of Theorem 1. Corollary 2 is also known (e.g., see [25]). We have added it for the sake of completeness.

Remark 3. In this article we consider polyhedra from a combinatorial point of view. For some polyhedra (e.g., L_2, L_3, Q, G, D_n's, H_n's, ...) we have given their geometric realizations in Section 2. A polyhedron K is also called a *polyhedral* 2*-manifold* (see [4]).

Remark 4. Property (vii), in the definition of a polyhedron, implies that the geometric carrier of a polyhedron is connected. A complex with properties (iv)–(vi) is said to be a *weak polyhedron*. Similarly, we can define $\{p, q\}$-equivelar and equivelar weak polyhedra. Clearly, an equivelar weak polyhedron is the disjoint union of equivelar polyhedra whose Euler characteristic is the sum of the Euler characteristics of the components. So, it is sufficient to consider only equivelar polyhedra.

2. Examples

In this section we construct infinitely many equivelar polyhedra. Some of them have already been mentioned in the previous section. We use others in the next section. Recall that we identify a polyhedron with the set of faces in it. At the end of this section we give the geometric realizations of some of the polyhedra.

Example 2. Some equivelar polyhedra of Euler characteristic 0:

$$\begin{aligned}
A_{m,n} &= \{u_{i,j}u_{i+1,j}u_{i+1,j+1},\ u_{i,j}u_{i,j+1}u_{i+1,j+1}\colon 1 \leq i \leq m, 1 \leq j \leq n\}, \quad m, n \geq 3.\\
B_{m,n} &= \{u_{i,j}u_{i+1,j}u_{i+1,j+1},\ u_{i,j}u_{i,j+1}u_{i+1,j+1}\colon 1 \leq i \leq m-1, 1 \leq j \leq n\}\\
&\quad \cup \{u_{m,j}u_{1,n+2-j}u_{1,n+1-j},\ u_{m,j}u_{m,j+1}u_{1,n+1-j}\colon 1 \leq j \leq n\}, \quad m, n \geq 3.\\
C_{m,n} &= \{u_{i,j}u_{i+1,j}u_{i+1,j+1}u_{i,j+1}\colon 1 \leq i \leq m, 1 \leq j \leq n\}, \quad m, n \geq 3.
\end{aligned}$$

(Additions in the first and second subscripts are modulo m and n, respectively.)

$$T_n = \{u_i u_{i+1} u_{i+3}, u_i u_{i+2} u_{i+3}\colon 1 \leq i \leq n\}, \quad n \geq 7.$$

(Additions in the subscripts are modulo n.)

$$Q = \{012, 023, 034, 045, 056, 016, 127, 136, 138, 178, 236, 269, 279, 348, 457, 479, 489, 569, 578, 589\}.$$

$A_{m,n}$, $B_{m,n}$, T_n and Q are equivelar of type $\{3, 6\}$ and $C_{m,n}$ is equivelar of type $\{4, 4\}$. The geometric carriers of $A_{m,n}$, $C_{m,n}$ and T_n are the torus and the geometric carriers of $B_{m,n}$ and Q are the Klein bottle.

Example 3. Some equivelar polyhedra of type $\{4, 5\}$:

$$\begin{aligned}
D_n = {} & [\{a_{i,j,k}a_{i,j+1,k}a_{i,j+1,k+1}a_{i,j,k+1}\colon 1 \le j,k \le 4,\ 1 \le i \le n\}\setminus \\
& \{a_{i,j,k}a_{i,j+1,k}a_{i,j+1,k+1}a_{i,j,k+1}\colon (j,k) \\
& \in \{(1,1),(1,3),(3,1),(3,3)\}, 1 \le i \le n\}] \\
& \cup \{a_{i,3,k}a_{i,3,k+1}a_{i+1,2,k+1}a_{i+1,2,k},\ a_{i,3,k+1}a_{i,4,k+1}a_{i+1,1,k+1}a_{i+1,2,k+1}, \\
& a_{i,4,k+1}a_{i,4,k}a_{i+1,1,k}a_{i+1,1,k+1},\ a_{i,4,k}a_{i,3,k}a_{i+1,2,k}a_{i+1,1,k}\colon \\
& k \in \{1,3\}, 1 \le i \le n\}, \qquad \text{for} \quad n \ge 2.
\end{aligned}$$

$$\begin{aligned}
E_n = {} & [\{a_{i,j,k}a_{i,j+1,k}a_{i,j+1,k+1}a_{i,j,k+1}\colon 1 \le j,k \le 4, \quad 1 \le i \le n\}\setminus \\
& \{a_{i,j,k}a_{i,j+1,k}a_{i,j+1,k+1}a_{i,j,k+1}\colon (j,k) \\
& \in \{(1,1),(1,3),(3,2),(3,4)\}, 1 \le i \le n\}] \\
& \cup \{a_{i,3,k}a_{i,3,k+1}a_{i+1,2,k}a_{i+1,2,k+3},\ a_{i,3,k+1}a_{i,4,k+1}a_{i+1,1,k}a_{i+1,2,k}, \\
& a_{i,4,k+1}a_{i,4,k}a_{i+1,1,k+3}a_{i+1,1,k},\ a_{i,4,k}a_{i,3,k}a_{i+1,2,k+3}a_{i+1,1,k+3}\colon \\
& k \in \{2,4\}, 1 \le i \le n\}. \qquad \text{for} \quad n \ge 1.
\end{aligned}$$

(Additions in the first subscripts are modulo n and in the second and third subscripts are modulo 4.)

$$\begin{aligned}
F_1 = {} & \{a_ic_id_{i+1}d_i,\ a_{i+1}a_id_ib_{i+1},\ a_ib_ib_{i+1}c_{i+1},\ b_id_ic_{i+1}c_i,\ a_1c_1a_2c_2,\ b_1d_1b_2d_2\colon \\
& 1 \le i \le 2\}.
\end{aligned}$$

(Additions in the subscripts are modulo 2.)

$$\begin{aligned}
F_n = {} & [\{a_{i,j}a_{i,j+1}a_{i+1,j+1}a_{i+1,j}\colon 1 \le j \le 4, 1 \le i \le 2n\}\setminus \\
& \{a_{2k-1,j}a_{2k-1,j+1}a_{2k,j+1}a_{2k,j}\colon j \in \{1,3\}, 1 \le k \le n\}] \\
& \cup \{a_{2k-1,1}a_{2k-1,2}a_{2k+2,4}a_{2k+2,3},\ a_{2k-1,2}a_{2k,2}a_{2k+1,4}a_{2k+2,4}, \\
& a_{2k,2}a_{2k,1}a_{2k+1,3}a_{2k+1,4},\ a_{2k,1}a_{2k-1,1}a_{2k+2,3}a_{2k+1,3}\colon \\
& 1 \le k \le n\}, \qquad \text{for} \quad n \ge 2.
\end{aligned}$$

$$\begin{aligned}
G_n = {} & [\{a_{i,j}a_{i,j+1}a_{i+1,j+1}a_{i+1,j}\colon 1 \le j \le 4, 1 \le i \le 2n\}\setminus \\
& \{a_{2k-1,j}a_{2k-1,j+1}a_{2k,j+1}a_{2k,j}\colon j \in \{1,3\}, 1 \le k \le n\}] \\
& \cup \{a_{2k-1,1}a_{2k-1,2}a_{2k+1,4}a_{2k+1,3},\ a_{2k-1,2}a_{2k,2}a_{2k+2,4}a_{2k+1,4}, \\
& a_{2k,2}a_{2k,1}a_{2k+2,3}a_{2k+2,4},\ a_{2k,1}a_{2k-1,1}a_{2k+1,3}a_{2k+2,3}\colon \\
& 1 \le k \le n\}, \qquad \text{for} \quad n \ge 2.
\end{aligned}$$

(Additions in the first and second subscripts are modulo $2n$ and 4, respectively.)

$$\begin{aligned}
G = {} & \{abcd, adef, bagh, cbij, dckl, feij, hgkl, aljg, alhf, bkeh, bkgi, ciek, cigj, dfhe, \\
& dfjl\}.
\end{aligned}$$

$\chi(D_n) = 16n - 40n + 20n = -4n$ and the geometric carrier of D_n is the orientable surface of genus $2n + 1$ for all $n \geq 2$. $\chi(E_n) = -4n$ and the geometric carrier of E_n is the orientable surface of genus $2n + 1$ for all $n \geq 1$. $\chi(F_n) = 8n - 20n + 10n = -2n$ and the geometric carrier of F_n is the orientable surface of genus $n + 1$ for all $n \geq 1$. $\chi(G_n) = -2n$ and the geometric carrier of G_n is a non-orientable surface for all $n \geq 2$. The geometric carrier of G is the non-orientable surface of Euler characteristic -3.

Example 4. Some equivelar polyhedra of type {3, 7}:

$$\begin{aligned} H_n = \{&u_{i,1}v_{i,1}u_{i,2},\ u_{i,2}v_{i,1}v_{i,2},\ u_{i,2}v_{i,2}v_{i,3},\ u_{i,2}v_{i,3}u_{i,3},\ u_{i,4}v_{i,4}u_{i,1},\ u_{i,1}v_{i,4}v_{i,1},\\ &v_{i,2}w_{i,2}w_{i,3},\ v_{i,2}w_{i,3}v_{i,3},\ v_{i,3}w_{i,3}v_{i,4},\ v_{i,4}w_{i,3}w_{i,4},\ v_{i,4}w_{i,4}v_{i,1},\ v_{i,1}w_{i,4}w_{i,1},\\ &w_{i,1}u_{i,1}w_{i,2},\ w_{i,2}u_{i,1}u_{i,2},\ w_{i,2}u_{i,2}w_{i,3},\ w_{i,3}u_{i,2}u_{i,3},\ w_{i,3}u_{i,3}w_{i,4},\ w_{i,4}u_{i,3}u_{i,4},\\ &w_{i,4}u_{i,4}u_{i,1},\ w_{i,4}u_{i,1}w_{i,1}\}\\ &\cup\{u_{i+1,3}v_{i+1,3}w_{i,2},\ u_{i+1,3}w_{i,2}v_{i,2},\ u_{i+1,3}u_{i+1,4}v_{i,2}, u_{i+1,4}v_{i,2}v_{i,1},\\ &\quad u_{i+1,4}v_{i+1,4}w_{i,1},\ u_{i+1,4}v_{i,1}w_{i,1},\ v_{i+1,4}v_{i+1,3}w_{i,1},\ v_{i+1,3}w_{i,1}w_{i,2}\colon\\ &\quad 1 \leq i \leq n\}, \qquad \text{for} \quad n \geq 1. \end{aligned}$$

(Additions in the subscripts are modulo n.)

$$\begin{aligned} H = \{&a_1b_4b_1, a_1b_1b_3, a_1b_3a_2, a_1a_2c_2, a_1c_2c_1, a_2b_3b_2, a_2b_2c_5, a_2c_5c_4, a_2c_4a_3, a_2a_3c_2,\\ &a_3c_4b_5, a_3b_5b_6, a_3b_6a_4, a_3a_4c_3, a_3c_3c_2, a_4b_6b_4, a_4b_4b_1, a_4b_1a_5, a_4a_5c_1, a_4c_1c_3,\\ &a_5b_1b_2, a_5b_2c_5, a_5c_5c_6, a_5c_6a_6, a_5a_6c_1, a_6c_6c_4, a_6c_4b_5, a_6b_5b_4, a_6b_4a_1, a_6a_1c_1,\\ &b_1b_2b_6, b_1b_3b_6, b_2b_3b_5, b_2b_5b_6, b_3b_4b_5, b_3b_4b_6, c_1c_2c_6, c_1c_3c_6, c_2c_3c_5, c_2c_5c_6,\\ &c_3c_4c_5, c_3c_4c_6\}. \end{aligned}$$

The geometric carrier of H is the non-orientable surface of Euler characteristic -3. The geometric carrier of H_n is an orientable surface and $\chi(H_n) = 12n - 42n + 28n = -2n$ for all $n \geq 1$.

Example 5. Some sequences of equivelar polyhedra of type {3, 8}:

$$\begin{aligned} J_n = \{&u_{i,j}u_{i+1,j}u_{i+1,j+1},\ u_{i,j}u_{i,j+1}u_{i+1,j+1}\colon 1 \leq i \leq 2n, 1 \leq j \leq 3\}\setminus\\ &\{u_{2k-1,j}u_{2k-1,j+1}u_{2k,j+1}\colon 1 \leq k \leq n,\ j \in \{1,3\}\}\\ &\cup\{u_{2k-1,1}u_{2k+1,3}u_{2k-1,2}, u_{2k-1,2}u_{2k+1,3}u_{2k+2,3},\ u_{2k-1,2}u_{2k+2,3}u_{2k,2},\\ &\quad u_{2k,2}u_{2k+2,3}u_{2k+2,1}, u_{2k,2}u_{2k+2,1}u_{2k-1,1},\ u_{2k-1,1}u_{2k+2,1}u_{2k+1,3}\colon\\ &\quad 1 \leq k \leq n\}, \qquad \text{for} \quad n \geq 3. \end{aligned}$$

(Additions in the first and second subscripts are modulo $2n$ and 3, respectively.)

$$\begin{aligned} K_n = [\{&u_{i,j,k}u_{i,j+1,k}u_{i,j+1,k+1},\ u_{i,j,k}u_{i,j,k+1}u_{i,j+1,k+1},\ u_{i,l,k}v_{i,l+1,k}u_{i,l,k+1},\\ &u_{i,l,k+1}u_{i,l+1,k}u_{i,l+1,k+1}\colon 1 \leq j \leq 2, 3 \leq l \leq 4, 1 \leq k \leq 3, 1 \leq i \leq n\}\setminus\\ &\{u_{i,1,1}u_{i,2,2}u_{i,1,2},\ u_{i,1,3}u_{i,2,3}u_{i,2,1},\ u_{i,3,2}u_{i,4,1}u_{i,4,2},\ u_{i,3,3}u_{i,4,3}u_{i,3,1}\}]\\ &\cup\{u_{i,1,1}u_{i,4,3}u_{i,3,3},\ u_{i,1,1}u_{i,3,3}u_{i,1,2},\ u_{i,1,2}u_{i,3,3}u_{i,3,1},\ u_{i,1,2}u_{i,3,1}u_{i,2,2},\\ &\quad u_{i,2,2}u_{i,3,1}u_{i,4,3},\ u_{i,2,2}u_{i,4,3}u_{i,1,1},\ u_{i,4,1}u_{i+1,2,1}u_{i+1,2,3}, \end{aligned}$$

$$u_{i,4,1}u_{i+1,2,3}u_{i,3,2},\ u_{i,3,2}u_{i+1,2,3}u_{i+1,1,3},\ u_{i,3,2}u_{i+1,1,3}u_{i,4,2},$$
$$u_{i,4,2}u_{i+1,1,3}u_{i+1,2,1},\ u_{i,4,2}u_{i+1,2,1}u_{i,4,1}\colon 1 \le i \le n\}, \qquad \text{for} \quad n \ge 1.$$

(Additions in the first and second subscripts are modulo n and 3, respectively.)

$\chi(J_n) = 6n - 24n + 16n = -2n$ and the geometric carrier of J_n is the orientable surface of genus $n+1$ for all $n \ge 3$. $\chi(K_n) = 12n - 48n + 32n = -4n$ and the geometric carrier of K_n is a non-orientable surface for all $n \ge 1$.

Example 6. Some sequences of equivelar polyhedra of type {4, 6}:

$$L_2 = \{1263, 1374, 1425, 1536, 1647, 1752, 8273, 8346, 8657, 8724, 8435, 8562\}.$$

$$L_n = \{a_{i,1}a_{i+1,1}a_{i+1,2}a_{i,2},\ a_{i,2}a_{i+1,2}b_{i+1,3}b_{i,3},\ b_{i,3}b_{i+1,3}b_{i+1,4}b_{i,4},\ b_{i,4}a_{i+1,4}a_{i+1,1}a_{i,1}\colon$$
$$1 \le i \le 2n\}\setminus\{a_{2k-1,1}a_{2k,1}a_{2k,2}a_{2k-1,2},\ b_{2k-1,3}b_{2k,3}b_{2k,4}b_{2k-1,4}\colon 1 \le k \le n\},$$
$$\text{where} \quad a_{2k-1,1} = b_{2k+1,4}, \quad a_{2k-1,2} = b_{2k+2,4}, \quad a_{2k,2} = b_{2k+2,3},$$
$$a_{2k,1} = b_{2k+1,3},$$
$$\text{for} \quad n \ge 3.$$

(Additions in the subscripts are modulo $2n$.)

$$P_n = \{a_{i,j,k}a_{i,j+1,k}a_{i,j+1,k+1}a_{i,j,k+1}\colon 1 \le j, k \le 4, 1 \le i \le n\}\setminus$$
$$\{a_{i,j,k}a_{i,j+1,k}a_{i,j+1,k+1}a_{i,j,k+1}\colon (j,k)$$
$$\in \{(1,1),(1,3),(3,1),(3,3)\},\ 1 \le i \le n\},$$
$$\text{where} \quad a_{i,j,k} = a_{i+1,j+2,k},$$
$$\text{for} \quad (j,k) \in \{(1,1),(1,2),(2,1),(2,2),(3,3),(3,4),(4,3),(4,4)\}$$
$$\text{and} \quad 1 \le i \le n \quad \text{for} \quad n \ge 3.$$

(Additions in the first subscripts are modulo n and in the second and third subscripts are modulo 4.)

$\chi(L_n) = 4n - 12n + 6n = -2n$ for all $n \ge 2$. The geometric carrier of L_2 is the orientable surface of genus 3 and the geometric carrier of L_n is a non-orientable surface for all $n \ge 3$. $\chi(P_n) = 8n - 24n + 12n = -4n$ and the geometric carrier of P_n is the orientable surface of genus $2n+1$ for all $n \ge 3$. (Note that L_n, for $n \ge 3$, is obtained from an $8n$-vertex ($\{a_{i,j}, b_{i,k}\colon 1 \le i \le 2n, 1 \le j \le 2, 3 \le k \le 4\}$) complex (torus with $2n$ holes) by identifying $b_{l,k}$'s with $a_{i,j}$'s. P_n is also obtained by some identifications.)

Example 7. Another 9-vertex neighbourly simplicial equivelar polyhedra:

$$M_2 = \{129, 239, \ldots, 789, 189, 124, 136, 138, 147, 156, 157,$$
$$237, 245, 258, 267, 268, 346, 357, 358, 468, 478\},$$

where the vertex set of M_2 is $\{0, 1, \ldots, 8\}$. Clearly, $\chi(M_2) = -3$.

Example 8. Thirteen more 10-vertex neighbourly simplicial equivelar polyhedra:

$$N_2 = A \cup \{134, 136, 156, 158, 178, 179, 237, 248, 257, 259, 268,$$
$$269, 358, 359, 368, 379, 457, 467, 469, 489\},$$

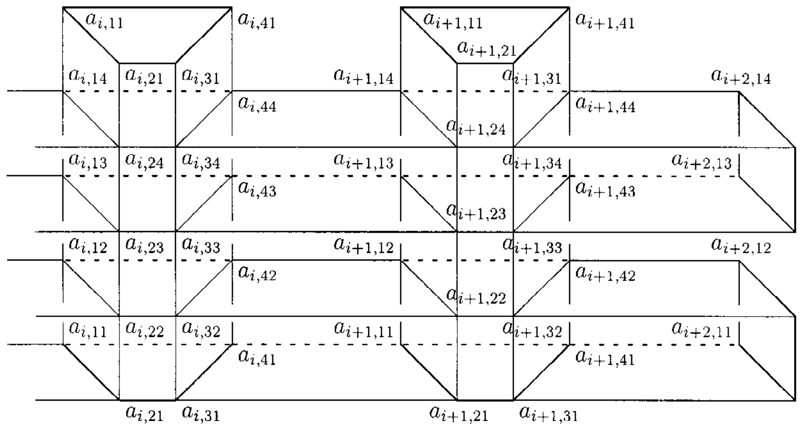
$$
\begin{aligned}
N_3 &= A \cup \{137, 139, 146, 156, 158, 178, 237, 248, 257, 259, 268,\\
&\qquad 269, 346, 358, 359, 368, 457, 479, 489, 679\},\\
N_4 &= A \cup \{136, 137, 145, 158, 168, 179, 238, 249, 256, 257, 269,\\
&\qquad 278, 347, 358, 359, 369, 467, 468, 489, 579\},\\
N_5 &= A \cup \{134, 136, 157, 158, 168, 179, 237, 248, 256, 259, 268,\\
&\qquad 279, 358, 359, 369, 378, 457, 467, 469, 489\},\\
N_6 &= A \cup \{134, 136, 157, 158, 168, 179, 238, 247, 256, 259, 268,\\
&\qquad 279, 357, 359, 369, 378, 458, 467, 469, 489\},\\
N_7 &= A \cup \{134, 138, 156, 157, 168, 179, 236, 247, 258, 259, 269,\\
&\qquad 278, 357, 359, 367, 389, 458, 468, 469, 479\},\\
N_8 &= A \cup \{134, 138, 156, 157, 168, 179, 236, 247, 257, 258, 269,\\
&\qquad 289, 358, 359, 367, 379, 459, 468, 469, 478\},\\
N_9 &= A \cup \{138, 139, 146, 157, 158, 167, 236, 245, 258, 269, 278,\\
&\qquad 279, 347, 357, 359, 368, 468, 479, 489, 569\},\\
N_{10} &= A \cup \{136, 138, 145, 158, 167, 179, 238, 249, 256, 257, 268,\\
&\qquad 279, 347, 357, 359, 369, 468, 469, 478, 589\},\\
N_{11} &= A \cup \{134, 138, 156, 157, 168, 179, 237, 246, 257, 259, 268,\\
&\qquad 289, 358, 359, 367, 369, 458, 469, 478, 479\},\\
N_{12} &= A \cup \{134, 138, 156, 157, 168, 179, 236, 247, 258, 259, 268,\\
&\qquad 279, 357, 359, 369, 378, 458, 467, 469, 489\},\\
N_{13} &= A \cup \{136, 137, 145, 158, 168, 179, 238, 246, 257, 259, 267,\\
&\qquad 289, 349, 357, 358, 369, 468, 478, 479, 569\},\\
N_{14} &= A \cup \{134, 138, 156, 157, 168, 179, 237, 246, 257, 258, 269,\\
&\qquad 289, 358, 359, 367, 369, 459, 468, 478, 479\},
\end{aligned}
$$

where the vertex set of N_i $(2 \le i \le 14)$ is $\{0, 1, \ldots, 9\}$ and $A = \{012, \ldots, 089, 019, 124\}$. Clearly, $\chi(N_i) = -5$ for $1 \le i \le 14$. Thus, all of them triangulate the same non-orientable surface of Euler characteristic -5.

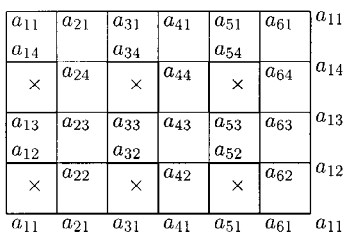
Ringel and Jungerman [13], [21]–[23], [14] have shown that there exist neighbourly simplicial polyhedra on $3k$ and $3k + 1$ vertices, for each $k \ge 3$, i.e.,

Proposition 1. *For $k \ge 2$, if $n = 3k$ or $3k + 1$, then there exists an n-vertex equivelar polyhedron of type $\{3, n - 1\}$.*

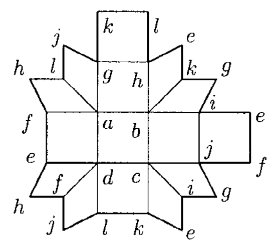
Thus, if $m = -k(3k - 7)/2$ or $1 - k(3k - 5)/2$, for $k \ge 3$, then there exists an equivelar polyhedron of type $\{3, f_0 - 1\}$ of Euler characteristic m. In particular, there exist neighbourly equivelar polyhedra on nine and ten vertices (M_1 and N_1, respectively, in Example 1).



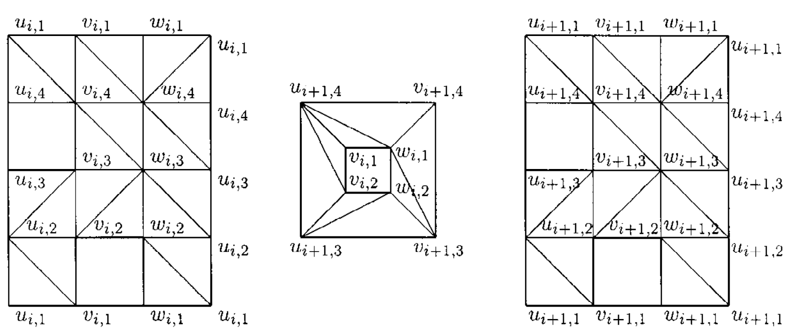
D_n



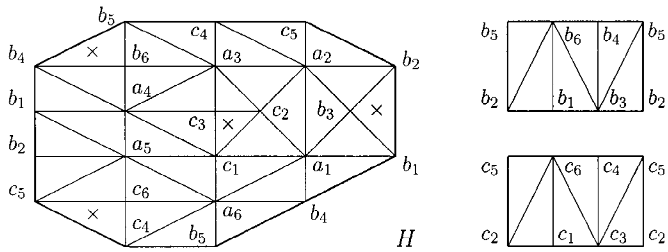
$\subseteq F_3,\ G_3$



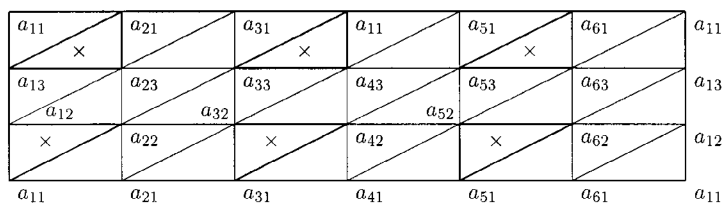
G



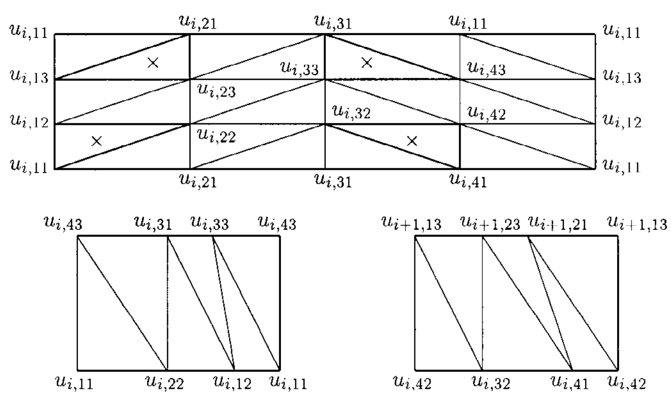
H_n



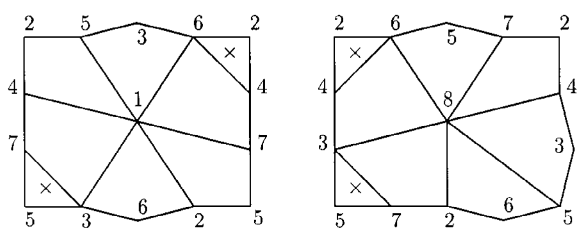
H



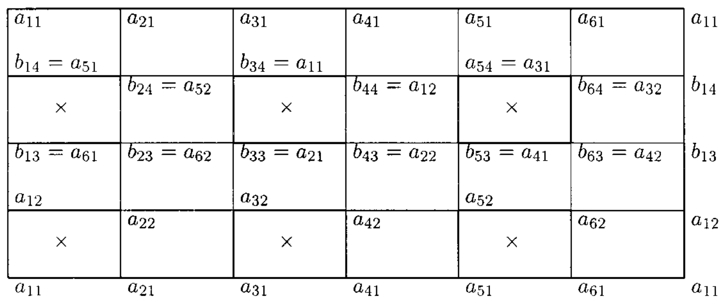
$\subseteq J_3$



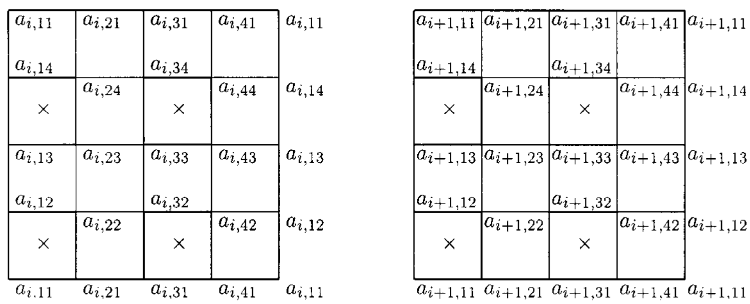
K_n



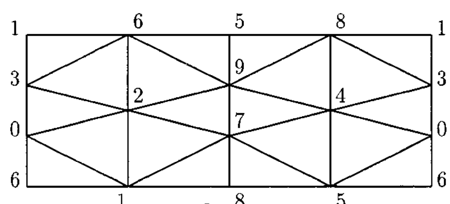
L_2


a11 a21 a31 a41 a51 a61 a11
b14 = a51 b34 = a11 a54 = a31
b24 = a52 b44 = a12 b64 = a32 b14
× × ×
b13 = a61 b23 = a62 b33 = a21 b43 = a22 b53 = a41 b63 = a42 b13
a12 a32 a52
a22 a42 a62 a12
× × ×
a11 a21 a31 a41 a51 a61 a11

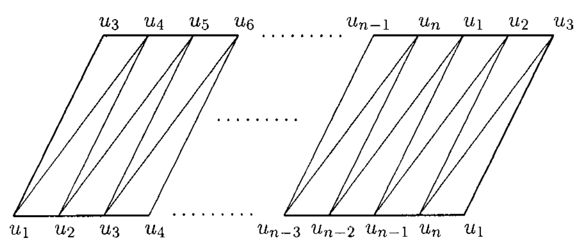
L_3


a_{i,11} a_{i,21} a_{i,31} a_{i,41} a_{i,11}
a_{i,14} a_{i,34}
a_{i,24} a_{i,44} a_{i,14}
a_{i,13} a_{i,23} a_{i,33} a_{i,43} a_{i,13}
a_{i,12} a_{i,32}
a_{i,22} a_{i,42} a_{i,12}
a_{i,11} a_{i,21} a_{i,31} a_{i,41} a_{i,11}
a_{i+1,11} a_{i+1,21} a_{i+1,31} a_{i+1,41} a_{i+1,11}
a_{i+1,14} a_{i+1,34}
a_{i+1,24} a_{i+1,44} a_{i+1,14}
a_{i+1,13} a_{i+1,23} a_{i+1,33} a_{i+1,43} a_{i+1,13}
a_{i+1,12} a_{i+1,32}
a_{i+1,22} a_{i+1,42} a_{i+1,12}
a_{i+1,11} a_{i+1,21} a_{i+1,31} a_{i+1,41} a_{i+1,11}

P_n


1 6 5 8 1
3 9 3
2 4
0 7 0
6 6
1 8 5

Q


u3 u4 u5 u6 .... u_{n-1} u_n u1 u2 u3
....
u1 u2 u3 u4 .... u_{n-3} u_{n-2} u_{n-1} u_n u1

T_n

3. Proofs

Proof of Theorem 1. First observe that if K is an equivelar polyhedron of type $\{p, q\}$, then $\widetilde{K}$ is an equivelar polyhedron of type $\{q, p\}$. Thus, $(p, q) \in \Phi(m)$ implies $(q, p) \in \Phi(m)$.

If K is a $\{p, q\}$-equivelar polyhedron with f_0 vertices, f_1 edges and f_2 faces, then

$$q f_0 = 2 f_1 = p f_2. \tag{1}$$

This gives

$$\frac{1}{p} + \frac{1}{q} - \frac{1}{2} = \frac{\chi(K)}{2 f_1} = \frac{\chi(K)}{q f_0}. \tag{2}$$

Thus, $(p, q) \in \Phi(n)$, for $n \geq 0$, implies $1/p + 1/q - \frac{1}{2} \geq 0$ and $(p, q) \in \Phi(-m)$, for $m > 0$, implies $1/p + 1/q - \frac{1}{2} < 0$. This implies (i).

If there esists a $\{p, q\}$-equivelar polyhedron K with $\chi(K) > 0$, then $1/p + 1/q > \frac{1}{2}$ or $(p-2)(q-2) < 4$. This implies $(p, q) \in \{(3, 3), (3, 4), (4, 3), (3, 5), (5, 3)\}$. Thus

$$\Phi(1), \Phi(2) \subseteq \{(3, 3), (3, 4), (4, 3), (3, 5), (5, 3)\}. \tag{3}$$

From Example 1 we have $(3, 3), (3, 4), (4, 3), (3, 5), (5, 3) \in \Phi(2)$. This proves (ii).

Let K be an equivelar polyhedron of type $\{p, q\}$ and $\chi(K) = 1$. If $(p, q) = (3, 3)$, then, from (2), we get $f_1 = 3$, which is not possible. If $(p, q) = (3, 4)$, then, from (2), $f_0 = 3$, which is again not possible. Similarly, $(p, q) \neq (4, 3)$. Therefore, from (3), $(p, q) = (3, 5)$ or $(5, 3)$. This and Example 1 ($\mathbb{R}P^2_6$ and R) imply (iii).

If there exists a $\{p, q\}$-equivelar polyhedron K with $\chi(K) = 0$, then, by (2), $1/p + 1/q = \frac{1}{2}$. Since $p, q \geq 3$, $(p, q) = (3, 6)$, $(6, 3)$ or $(4, 4)$. This together with Example 2 proves (iv).

Let K be an equivelar polyhedron of type $\{p, q\}$ and $\chi(K) = -1$. Since, the existence of an equivelar polyhedron of type $\{p, q\}$ implies the existence of an equivelar polyhedron of type $\{q, p\}$, therefore we may assume $q \geq p$. Also $p, q \geq 3$ and $f_0 \geq p + 1$.

If $p = 3$, then from (1) we get $-1 = f_0 - f_1/3 = f_0 - q f_0/6$ or $(q-6) f_0 = 6$. This implies $f_0 = 6$ and $q = 7$, a contradiction. If $p = 4$, then $-1 = f_0 - f_1/2 = f_0 - q f_0/4$ or $(q-4) f_0 = 4$, which is not possible. If $p \geq 5$, then $f_0 \geq 6$ and hence $f_1 \geq (6 \times 3)/2 = 9$. Then, from (2), $1/q = \frac{1}{2} - 1/p - 1/(2 f_1) > \frac{1}{2} - \frac{1}{5} - \frac{1}{18} > \frac{1}{5}$. This implies $q < 5$, a contradiction to the assumption that $p \leq q$. This proves (v).

Observe that $F_1, H_1, \widetilde{F}_1, \widetilde{H}_1 \in \Sigma(-2)$ and F_1 and H_1 are equivelar of type $\{4, 5\}$ and $\{3, 7\}$, respectively. Therefore, $\Phi(-2) \supseteq \{(5, 4), (4, 5), (7, 3), (3, 7)\}$.

Let K be an equivelar polyhedron of type $\{p, q\}$ and $\chi(K) = -2$. In this case also we may assume that $3 \leq p \leq q \leq f_0 - 1$.

If $p \geq 5$, then $f_0 \geq q + 1 \geq 6$. Then, from (2), $\frac{1}{2} = 1/p + 1/q + 2/(q f_0) \leq \frac{1}{5} + \frac{1}{5} + \frac{2}{5 \times 6} = \frac{14}{30}$, a contradiction.

If $p = 4$, then $-2 = f_0 - f_1/2 = f_0 - q f_0/4$ or $(q-4) f_0 = 8$, which implies $(f_0, q) = (8, 5)$.

If $p = 3$, then from (1) we get $-2 = f_0 - f_1/3 = f_0 - q f_0/6$ or $(q-6) f_0 = 12$. This implies $(f_0, q) = (12, 7)$. These imply $\Phi(-2) \subseteq \{(5, 4), (4, 5), (7, 3), (3, 7)\}$. This proves (vi).

Observe that H, M_1, $G \in \Sigma(-3)$ and H, M_1, G are equivelar of type $(3, 7)$, $(3, 8)$ and $(4, 5)$, respectively. Thus, $\{(3, 7), (7, 3), (3, 8), (8, 3), (4, 5), (5, 4)\} \subseteq \Phi(-3)$.

Let K be an equivelar polyhedron of type $\{p, q\}$ and $\chi(K) = -3$. Assume $p \leq q \leq f_0 - 1$.

If $p = 3$, then, by (1), $(q - 6) f_0 = 18$. This implies $(f_0, q) = (9, 8)$ or $(18, 7)$.

If $p = 4$, then, by (1), $(q - 4) f_0 = 12$, which implies $(f_0, q) = (12, 5)$.

If $p = q = 5$, then $f_0 = 6$. Now, $S_1 = \{12345, 15264, 14536, 16423, 13652, 24356\}$ is a $\{5, 5\}$-equivelar polyhedron of Euler characteristic -3. So, $(5, 5) \in \Phi(-3)$.

Finally, assume that $p \geq 5$ and $q \geq 6$. Then, by (2), $\frac{1}{2} \leq \frac{1}{5} + \frac{1}{6} + \frac{3}{6\times 7} = \frac{92}{210}$, a contradiction. This proves (vii).

H_m (in Example 4) is an equivelar polyhedron of type $\{3, 7\}$ and of Euler characteristic $-2m$ for all $m \geq 1$, therefore $(3, 7)$ (and hence $(7, 3)$) $\in \Phi(-2m)$ for all $m \geq 1$.

K_1 (in Example 5) is an equivelar polyhedron of type $\{3, 8\}$ of Euler characteristic -4 and J_m (in Example 5) is an equivelar polyhedron of type $\{3, 8\}$ and of Euler characteristic $-2m$ for all $m \geq 3$. Therefore $(3, 8)$ (and hence $(8, 3)$) $\in \Phi(-2m)$ for all $m \geq 2$.

F_m (in Example 3) is an equivelar polyhedron of type $\{4, 5\}$ of Euler characteristic $-2m$ for all $m \geq 2$. Therefore $(4, 5)$ (and hence $(5, 4)$) $\in \Phi(-2m)$ for all $m \geq 2$.

L_m (in Example 6) is an equivelar polyhedron of type $\{4, 6\}$ of Euler characteristic $-2m$ for all $m \geq 2$. Therefore $(4, 6)$ (and hence $(6, 4)$) $\in \Phi(-2m)$ for all $m \geq 2$. Thus,

$$\{(3, 7), (7, 3), (3, 8), (8, 3), (4, 5), (5, 4), (4, 6), (6, 4)\} \subseteq \Phi(-2m), \qquad \text{for all} \quad m \geq 2. \tag{4}$$

Let K be an f_0-vertex equivelar polyhedron of type $\{p, q\}$ and $\chi(K) = -4$. Assume $p \leq q$, i.e., $3 \leq p \leq q \leq f_0 - 1$.

If $p \geq 5$ and $q \geq 6$, then $f_0 \geq q + 1 \geq 7$ and hence $\frac{1}{2} = 1/p + 1/q + 4/(q f_0) \leq \frac{1}{5} + \frac{1}{6} + \frac{4}{6\times 7} = \frac{97}{210}$, a contradiction.

If $p = q = 5$, then, by (2), $(f_0, f_1) = (8, 20)$ and hence $f_2 = 8$. For each $m \geq 2$, $S_{2m} = \{a_i a_{i+1} b_{i+m+1} b_{i+m} b_{i+m-1}, a_i a_{i+1} b_{i+1} a_{i+m+1} b_i : 1 \leq i \leq 2m$, additions in the subscripts are modulo $2m\}$ is $\{5, 5\}$-equivelar and belongs to $\Sigma(-2m)$. So, $(5, 5) \in \Phi(-2m)$ for all $m \geq 2$.

If $p = 3$, then, from (1), $-4 = f_0 - f_1/3 = f_0 - q f_0/6$ or $(q - 6) f_0 = 24$. This implies $(f_0, q) = (12, 8)$ or $(24, 7)$ and hence $(p, q) = (3, 8)$ or $(3, 7)$.

If $p = 4$, then, from (1), $-4 = f_0 - q f_0/4$ or $(q - 4) f_0 = 16$. This implies $(f_0, q) = (16, 5)$ or $(8, 6)$ and hence $(p, q) = (4, 5)$ or $(4, 6)$. Thus, $(p, q) \in \Phi(-4)$ and $p \leq q$ imply $(p, q) = (3, 8), (3, 7), (4, 5), (5, 5)$ or $(4, 6)$. So, $\Phi(-4) \subseteq \{(3, 7), (7, 3), (3, 8), (8, 3), (4, 5), (5, 4), (4, 6), (6, 4), (5, 5)\}$. This, the examples S_{2m}'s and (4) imply (viii).

(ix) follows from the proposition stated in the previous section.

Let K be an equivelar polyhedron of type $\{p, q\}$ and $\chi(K) = -m$, where $m > 0$. In this case also, we may first assume that $p \leq q$, i.e., $3 \leq p \leq q \leq f_0 - 1$. Then, by (2), $\frac{1}{2} - 1/p = 1/q + m/(q f_0) \leq 1/q + m/(q(q + 1))$. This gives

$$2p(q + 1 + m) \ \geq \ q(q + 1)(p - 2) \geq p(q + 1)(p - 2) \quad \text{or} \tag{5}$$

$$2m \ \geq \ (q + 1)(p - 4). \tag{6}$$

Clearly, if $p > 4$, then there are only finitely many q such that (p, q) satisfies (6). So, there are only finitely many $(p, q) \in \Phi(-m)$ such that $p, q > 4$.

If $p = 3$, then from (5) we get $6(q + 1 + m) \geq 3q(q + 1)$ or

$$6m \geq (q - 6)(q + 1). \tag{7}$$

Clearly, given $m > 0$, only finitely many q (≥ 3) satisfy (7). This shows that there exists only finitely many q such that $(3, q) \in \Phi(-m)$.

If $q \geq p = 4$, then from (5) we get $4(q + 1 + m) \geq q(q + 1)$ or

$$4m \geq (q - 4)(q + 1). \tag{8}$$

Clearly, given $m > 0$, only finitely many q (≥ 4) satisfy (8). This shows that there exists only finitely many $q \geq 4$ such that $(4, q) \in \Phi(-m)$. Therefore there are only finitely many $(p, q) \in \Phi(-m)$ such that p or q is 4. So, $\Phi(-m)$ is finite for all $m > 0$. This together with (ii), (iii) and (iv) imply (x).

For each $n \geq 7$, T_n is an n-vertex $\{3, 6\}$-equivelar polyhedron. This proves (xi). □

Proof of Corollary 2. Let $(p, q) \in \Phi(n)$. Let K be an equivelar polyhedron of type $\{p, q\}$ and $\chi(K) = n$. If $n \neq 0$, then, by (2) and (1), $(f_0(K), f_1(K), f_2(K))$ is uniquely determined by (p, q). Since, for each f_0, there exist finitely many polyhedra on f_0 vertices, therefore for a given $(p, q) \in \Phi(n)$ there are only finitely many equivelar polyhedra of type $\{p, q\}$. The corollary now follows from Theorem 1(x). □

Lemma 1. *If K is a* 9*-vertex* $\{3, 6\}$*-equivelar polyhedron, then K is isomorphic to T_9, $A_{3,3}$ or $B_{3,3}$ defined in Example* 2.

Proof. Let K be a 9-vertex $\{3, 6\}$-equivelar polyhedron. Then $f_2(K) = 18$ and $\mathrm{NEG}(K)$ is a 2-regular graph and hence is either a cycle or disjoint union of cycles on nine vertices. So, $\mathrm{NEG}(K)$ is isomorphic to C_9, $C_6 \sqcup C_3$, $C_5 \sqcup C_4$ or $3C_3 := C_3 \sqcup C_3 \sqcup C_3$. (Here C_n denotes the cycle with n vertices. A cycle with edges $v_1v_2, \ldots, v_{n-1}v_n, v_nv_1$ is also denoted by $C_n(v_1, \ldots, v_n)$.)

If $\mathrm{NEG}(K) = C_5 \sqcup C_4$, then there exist four vertices, a, b, c, d say, such that ac and bd are edges in K but ab, bc, cd and da are not edges in K. Consider the following six sets of faces, $S_{ac} = \{\sigma\colon a, c \in \sigma\}$, $S_{bd} = \{\sigma\colon b, d \in \sigma\}$, $S_a = \{\sigma\colon a \in \sigma, c \notin \sigma\}$, $S_b = \{\sigma\colon b \in \sigma, d \notin \sigma\}$, $S_c = \{\sigma\colon c \in \sigma, a \notin \sigma\}$ and $S_d = \{\sigma\colon d \in \sigma, b \notin \sigma\}$. Clearly, these six sets are pairwise disjoint and $\#(S_{ac}) = \#(S_{bd}) = 2$ and $\#(S_a) = \#(S_b) = \#(S_c) = \#(S_d) = 4$. This implies that $f_2(K) \geq 20$, a contradiction. So, $\mathrm{NEG}(K)$ is C_9, $C_6 \sqcup C_3$ or $3C_3$.

First consider the case when the non-edge graph consists of three 3-cycles, i.e., $\mathrm{NEG}(K) = C_3(1, 2, 3) \sqcup C_3(4, 5, 6) \sqcup C_3(7, 8, 9) \cong \mathrm{NEG}(A_{3,3})$. Then, up to an isomorphism, $\mathrm{Lk}_K(1) = C_6(4, 7, 5, 9, 6, 8)$ and hence we may assume, without loss, $\mathrm{Lk}_K(4) = C_6(1, 7, 2, 9, 3, 8)$. These imply $\mathrm{Lk}_K(7) = C_6(1, 4, 2, 6, 3, 5)$, $\mathrm{Lk}_K(8) = C_6(1, 6, 2, 5, 3, 4)$. These imply $\mathrm{Lk}_K(2) = C_6(4, 9, 5, 8, 6, 7)$, $\mathrm{Lk}_K(3) = C_6(4, 8, 5, 7, 6, 9)$, $\mathrm{Lk}_K(5) = C_6(1, 9, 2, 8, 3, 7)$, $\mathrm{Lk}_K(6) = C_6(1, 8, 2, 7, 3, 9)$ and hence $\mathrm{Lk}_K(9) = C_6(1, 5, 2, 4, 3, 6)$. Clearly, these imply that K is unique, up to an isomorphism, in this case and hence is isomorphic to $A_{3,3}$.

Now assume $\mathrm{NEG}(K) = C_6(1, \ldots, 6) \sqcup C_3(7, 8, 9) \cong \mathrm{NEG}(B_{3,3})$. Then, we may assume, $\mathrm{Lk}_K(1) = C_6(3, 8, 4, 9, 5, 7)$. This implies $\mathrm{Lk}_K(7) = C_6(1, 5, 2, 4, 6, 3)$. Then $\mathrm{Lk}_K(3) = C_6(1, 8, 5, 9, 6, 7)$ and $\mathrm{Lk}_K(5) = C_6(1, 7, 2, 8, 3, 9)$. These imply $\mathrm{Lk}_K(8) =$

$C_6(1, 3, 5, 2, 6, 4), \mathrm{Lk}_K(9) = C_6(1, 4, 2, 6, 3, 5), \mathrm{Lk}_K(2) = C_6(4, 7, 5, 8, 6, 9), \mathrm{Lk}_K(6) = C_6(2, 9, 3, 7, 4, 8)$. So, $\mathrm{Lk}_K(4) = C_6(1, 9, 2, 7, 6, 8)$. These determine the polyhedron uniquely and therefore K is isomorphic to $B_{3,3}$.

Finally assume $\mathrm{NEG}(K) = C_9(1, \ldots, 9)$. Then $\mathrm{Lk}_K(1) = C_6(4, 6, 8, 3, 5, 7)$, $C_6(4, 6, 8, 5, 3, 7)$, $C_6(4, 6, 3, 8, 5, 7)$, $C_6(4, 7, 5, 3, 6, 8)$ or $C_6(4, 6, 3, 7, 5, 8)$.

If $\mathrm{Lk}_K(1) = C_6(4, 6, 8, 3, 5, 7)$, then $\mathrm{Lk}_K(3) = C_6(1, 5, 7, 9, 6, 8)$. These imply $\mathrm{Lk}_K(5) = C_3(1, 3, 7)$, a contradiction. If $\mathrm{Lk}_K(1) = C_6(4, 6, 3, 8, 5, 7)$, then $\mathrm{Lk}_K(3) = C_6(1, 6, 9, 7, 5, 8)$. These give $\mathrm{Lk}_K(5) = C_4(1, 7, 3, 8)$, a contradiction. If $\mathrm{Lk}_K(1) = C_6(4, 7, 5, 3, 6, 8)$, then 15, 16 are edges and 67, 78, 89 are non-edges in $\mathrm{Lk}_K(3)$. It is then not possible to construct $\mathrm{Lk}_K(3)$, which is a 6-cycle with vertex-set $\{1, 5, 6, 7, 8, 9\}$. Similarly, it is not possible to construct $\mathrm{Lk}_K(8)$ when $\mathrm{Lk}_K(1) = C_6(4, 6, 8, 5, 3, 7)$. Thus, $\mathrm{Lk}_K(1) = C_6(4, 6, 3, 7, 5, 8)$.

Similarly, replacing 1 by i, $\mathrm{Lk}_K(i) = C_6(i + 3, i + 5, i + 2, i + 6, i + 4, i + 7)$, for $1 \leq i \leq 9$ (additions in the subscripts are modulo 9). This shows that the simplicial polyhedron is unique, up to an isomorphism, with the non-edge graph a 9-cycle. Therefore, K is isomorphic to T_9. This completes the proof. □

Lemma 2. *If K is a* 10*-vertex* $\{3, 6\}$*-equivelar polyhedron, then K is isomorphic to T_{10} or Q defined in Example* 2.

Proof. Let K be a 10-vertex $\{3, 6\}$-equivelar polyhedron. Choose a vertex, say u_0. Let the link of u_0 be $C_6(u_1, \ldots, u_6)$. Since the link of each vertex is a 6-cycle, $u_iu_{i+1}u_{i+2}$ is not a face in K for $i = 1, \ldots, 6$ (additions in the subscripts are modulo 6). If either $u_iu_{i+1}u_{i+3}$ or $u_iu_{i+1}u_{i+4}$ is a face for each i, then we get 12 faces and hence the number of faces through the remaining three vertices is ≤ 8, a contradiction. So, assume $u_1u_2u_7$ is a face, where u_7 is one of the remaining three vertices. Then $u_1u_6u_7$ and $u_2u_3u_7$ cannot be faces.

Case 1: *$u_1u_3u_6$ or $u_2u_3u_6$ is a face*. Assume, without loss, that $u_1u_3u_6$ is a face.

If u_5 is in the link of u_1, then $u_1u_3u_5$, $u_1u_5u_7$ are faces and hence $V(\mathrm{Lk}(u_3)) = \{u_0, u_1, u_2, u_4, u_5, u_6\}$. This gives 12 faces through u_0, u_1 and u_3. Hence, the number of faces through the remaining two vertices is ≤ 8, a contradiction. If u_4 is in the link of u_1, then $u_1u_3u_4$ and $u_1u_4u_7$ are faces. Here, five vertices of each of $\mathrm{Lk}(u_3)$ and $\mathrm{Lk}(u_4)$ are known. This implies $V(\mathrm{Lk}(u_8)) = \{u_2, u_3, u_5, u_6, u_7, u_9\}$ and $V(\mathrm{Lk}(u_9)) = \{u_2, u_4, u_5, u_6, u_7, u_8\}$, where u_8 and u_9 are the remaining two vertices. Therefore, $\mathrm{Lk}(u_4) = C_6(u_7, u_1, u_3, u_0, u_5, u_9)$ and $\mathrm{Lk}(u_3) = C_6(u_2, u_0, u_4, u_1, u_6, u_8)$. Then $\mathrm{Lk}(u_2) = C_6(u_7, u_1, u_0, u_3, u_8, u_9)$. However, now $\mathrm{Lk}(u_7)$ contains $C_4(u_1, u_4, u_9, u_2)$, which is impossible. Therefore, the sixth vertex in the link of u_1 is one of the remaining two vertices, say u_8.

Clearly, $u_1u_3u_8$ and $u_1u_7u_8$ are faces. Thus, $V(\mathrm{Lk}(u_3)) = \{u_0, u_1, u_2, u_4, u_6, u_8\}$. Let u_9 be the remaining vertex. Then $V(\mathrm{Lk}(u_9)) = \{u_2, u_4, u_5, u_6, u_7, u_8\}$. The link of u_2 shows that $u_2u_7u_9$ is a face. Now, from the link of u_3, either $u_2u_3u_8$ and $u_3u_4u_6$ are faces or $u_2u_3u_6$ and $u_3u_4u_8$ are faces. In either case, u_2u_5 is not an edge and hence $V(\mathrm{Lk}(u_5)) = \{u_0, u_4, u_6, u_7, u_8, u_9\}$.

Subcase 1.1: *$u_2u_3u_6$ and $u_3u_4u_8$ are faces*. Then, by considering $\mathrm{Lk}(u_2)$, $\mathrm{Lk}(u_6)$ and $\mathrm{Lk}(u_5)$ successively, $u_2u_6u_9$, $u_5u_6u_9$ are faces and $u_4u_5u_9$ is not a face. Finally, by completing $\mathrm{Lk}(u_9)$ and $\mathrm{Lk}(u_5)$ successively, the other faces are $u_4u_7u_9$, $u_4u_8u_9$, $u_5u_8u_9$, $u_4u_5u_7$ and $u_5u_7u_8$. In this case K is isomorphic, via the map $f\colon u_i \mapsto i$, to Q.

Subcase 1.2: *$u_2u_3u_8$ and $u_3u_4u_6$ are faces*. Then, by considering $\mathrm{Lk}(u_2)$, $\mathrm{Lk}(u_8)$, $\mathrm{Lk}(u_5)$ and $\mathrm{Lk}(u_4)$ successively, the other faces are $u_2u_8u_9$, $u_5u_7u_8$, $u_5u_8u_9$, $u_4u_5u_7$, $u_5u_6u_9$, $u_4u_6u_9$, $u_4u_7u_9$. So, K is isomorphic, via the composition of f and $(1, 2, 3)(4, 8)(5, 7)(6, 9)$, to T_{10}.

Case 2: *$u_1u_4u_6$ or $u_2u_3u_5$ is a face*. Assume, without loss, that $u_1u_4u_6$ is a face. Let the remaining two vertices be u_8 and u_9.

If either u_1u_8 or u_1u_9 is an edge, say u_1u_8 is an edge, then, by considering $\mathrm{Lk}(u_1)$ and $\mathrm{Lk}(u_4)$, successively, $u_1u_4u_8$, $u_1u_7u_8$, $u_3u_4u_6$ and $u_4u_5u_8$ are faces. Clearly, $V(\mathrm{Lk}(u_9)) = \{u_2, u_3, u_5, u_6, u_7, u_8\}$. By considering $\mathrm{Lk}(u_6)$ and $\mathrm{Lk}(u_5)$, $u_3u_6u_9$ and $u_5u_6u_9$ are faces and $u_2u_3u_5$ is not a face. Then, from $\mathrm{Lk}(u_3)$, $u_2u_3u_8$ and $u_3u_8u_9$ are faces. This gives $\deg(u_8) > 6$, a contradiction. So, the sixth vertex in the link of u_1 is either u_3 or u_5.

If neither u_8 nor u_9 is in the link of u_4, then from the links of u_0, u_1 and u_4 we get 12 faces. Therefore, the number of faces containing u_8 or u_9 is ≤ 8, a contradiction. So, $\mathrm{Lk}(u_4)$ contains one of u_8 or u_9, say u_8. Then $V(\mathrm{Lk}(u_9)) = \{u_2, u_3, u_5, u_6, u_7, u_8\}$.

Subcase 2.1: *u_3 is in the link of u_1*. The links of u_1 and u_4 show that $u_1u_3u_4$, $u_1u_3u_7$, $u_4u_5u_8$ and $u_4u_6u_8$ are faces. It is easy to see, by considering the link of u_3, that $u_2u_3u_9$ and $u_3u_7u_9$ are faces. However, this implies $V(\mathrm{Lk}(u_8)) = \{u_2, u_4, u_5, u_6, u_7, u_9\}$. Then, by completing $\mathrm{Lk}(u_2)$, $\mathrm{Lk}(u_6)$ and $\mathrm{Lk}(u_5)$ successively, the other faces are $u_2u_7u_8$, $u_2u_8u_9$, $u_5u_6u_9$, $u_6u_8u_9$, $u_5u_7u_8$ and $u_5u_7u_9$. In this case K is isomorphic, via the composition of f and $(1, 9, 5, 3, 8, 4)(2, 7, 6)$, to T_{10}.

Subcase 2.2: *u_5 is in the link of u_1*. Clearly, $u_1u_4u_5$ and $u_1u_5u_7$ are faces. In this case, by considering $\mathrm{Lk}(u_4)$, $\mathrm{Lk}(u_5)$, $\mathrm{Lk}(u_6)$, $\mathrm{Lk}(u_7)$ and $\mathrm{Lk}(u_2)$, $u_3u_4u_8$, $u_4u_6u_8$, $u_5u_6u_9$, $u_5u_7u_9$, $u_6u_8u_9$, $u_2u_7u_8$, $u_3u_7u_8$, $u_3u_7u_9$, $u_2u_3u_9$ and $u_2u_8u_9$ are faces. Then K is isomorphic, via the map $(u_1, u_6)(u_2, u_5)(u_3, u_4)(u_7, u_9)$, to Q.

Case 3: *None of $u_1u_3u_6$, $u_1u_4u_6$, $u_2u_3u_5$ or $u_2u_3u_6$ is a face*. In this case we can assume that $u_1u_6u_8$ is a face, where u_8 is one of the remaining two vertices. Let u_9 be the remaining vertex.

If $u_3 \in \mathrm{Lk}(u_1)$, then, by considering the links of u_1 and u_3, $u_1u_3u_7$, $u_1u_3u_8$, $u_2u_3u_8$ and $u_3u_4u_7$ are faces in K. Clearly, $V(\mathrm{Lk}(u_9)) = \{u_2, u_4, u_5, u_6, u_7, u_8\}$. The links of u_2 and u_8 show that $\deg(u_7) > 6$, a contradiction. Similarly, we get a contradiction if $u_5 \in \mathrm{Lk}(u_1)$.

If $u_4 \in \mathrm{Lk}(u_1)$, then $u_1u_4u_7$ and $u_1u_4u_8$ are faces in K. Here, $V(\mathrm{Lk}(u_9)) = \{u_2, u_3, u_5, u_6, u_7, u_8\}$. To complete $\mathrm{Lk}(u_4)$, either $u_3u_4u_7$ or $u_3u_4u_8$ has to be a face. In both cases we see that $u_6u_8u_9 \in K$. In the first case, the links of u_4 and u_5 show that $u_4u_5u_8$ and $u_5u_8u_9$ are faces, which imply that $\deg(u_8) < 6$, a contradiction. In the second case, $\mathrm{Lk}(u_4)$, $\mathrm{Lk}(u_5)$ and $\mathrm{Lk}(u_7)$ show that $u_4u_5u_7$, $u_5u_7u_9$, $u_2u_7u_8$ and $u_7u_8u_9$ are faces. This implies $\deg(u_8) > 6$, a contradiction. Hence, the sixth vertex in $\mathrm{Lk}(u_1)$ is u_9.

Clearly, the edge u_2u_3 belongs to either $u_2u_3u_8$ or $u_2u_3u_9$.

If $u_2u_3u_8$ is a face, the sixth vertex, y, in $\mathrm{Lk}(u_2)$, is one of u_4, u_5 or u_6. If $y = u_4$ or u_5, the links of u_2 and y show that $\deg(u_8) > 6$, a contradiction. If $y = u_6$, by considering the links of u_2 and u_6, we see that $u_2u_6u_7$, $u_2u_6u_8$ and $u_5u_6u_7$ are faces.

Clearly, $u_3u_4u_9 \in K$. The sixth vertex in $\mathrm{Lk}(u_7)$ has to be u_3. However, this implies that $\deg(u_3) > 6$, a contradiction. Hence, $u_2u_3u_9 \in K$.

The sixth vertex, say x, in $\mathrm{Lk}(u_2)$, is one of u_4, u_5 or u_8.

If $x = u_4$ or u_5, by considering the links of x and u_9, we observe that $\deg(u_9) > 6$, a contradiction. Hence, $x = u_8$.

The links of u_2, u_8, u_4, u_7 and u_3 show that $u_2u_7u_8$, $u_2u_8u_9$, $u_4u_6u_8$, $u_4u_7u_8$, $u_3u_4u_6$, $u_4u_5u_7$, $u_5u_7u_9$, $u_3u_5u_6$ and $u_3u_5u_9$ are faces. Here, K is isomorphic, via the map $(1,4)(2,5)\ (3,6)$, to Q. This completes the proof. □

Lemma 3. *If K is an* 11*-vertex* $\{3,6\}$*-equivelar polyhedron, then K is isomorphic to T_{11} defined in Example* 2.

Proof. Let K be an 11-vertex $\{3,6\}$-equivelar polyhedron. Choose a vertex, say u_0, and let the link of u_0 be $C_6(u_1,\dots,u_6)$. By an argument similar to that in the previous lemma, we assume that $u_1u_2u_7$ is a face.

Claim. *One of $u_1u_3u_6$, $u_1u_4u_6$, $u_2u_3u_5$ or $u_2u_3u_6$ has to be a face.*

If not assume $u_1u_6u_8$ is a face, where u_8 is one of the remaining three vertices. Let u_9 and u_{10} be the remaining two vertices.

If $u_3 \in \mathrm{Lk}(u_1)$, then the links of u_1 and u_3 show that $u_1u_3u_7$, $u_1u_3u_8$, $u_2u_3u_8$ and $u_3u_4u_7$ are faces in K. It is clear that $\mathrm{Lk}(u_2)$ contains either u_9 or u_{10}, say u_9. Then $V(\mathrm{Lk}(u_{10})) = \{u_4, u_5, u_6, u_7, u_8, u_9\}$. The links of u_8 and u_7 now show that $\deg(u_9) < 6$, a contradiction.

If $u_4 \in \mathrm{Lk}(u_1)$, then it is clear that the edge u_5u_6 belongs to $u_5u_6u_7$ or $u_5u_6u_9$ (or $u_5u_6u_{10}$). If $u_5u_6u_7 \in K$, then $\deg(u_9), \deg(u_{10}) < 6$. If $u_5u_6u_9$ (or $u_5u_6u_{10}$) is a face, then, by considering $\mathrm{Lk}(u_4)$, either $u_3u_4u_7$ or $u_3u_4u_8 \in K$. In both cases we see that the link of u_5 has only five vertices, a contradiction.

Finally, assume (without loss) that $u_9 \in \mathrm{Lk}(u_1)$. The edge u_2u_3 can belong to one of $u_2u_3u_8$, $u_2u_3u_9$ or $u_2u_3u_{10}$.

If $u_2u_3u_8 \in K$, then the sixth vertex, say y, in $\mathrm{Lk}(u_2)$ is u_4, u_5, u_6 or u_{10}. If $y = u_4$ or u_5, then $V(\mathrm{Lk}(u_{10})) = \{u_3, u_4, u_5, u_6, u_7, u_9\} \backslash \{y\}$ and hence $\deg(u_{10}) < 6$. If $y = u_6$, we can easily see that $V(\mathrm{Lk}(u_{10})) = \{u_3, u_4, u_5, u_7, u_8, u_9\}$. Then, by completing the links of u_6, u_7 and u_8, we see that $\deg(u_9) = 4$, a contradiction. If $y = u_{10}$, then we see that either $u_3u_6u_8$ or $u_6u_8u_{10}$ has to be a face to complete $\mathrm{Lk}(u_8)$. In both the cases we see that there exists no vertex x ($\neq u_0$) such that u_3u_4x is a face in K, a contradiction.

If $u_2u_3u_9$ is a face in K, then, by an argument similar to the one above, we see that $u_4, u_5 \notin \mathrm{Lk}(u_2)$. Hence either u_8 or $u_{10} \in \mathrm{Lk}(u_2)$. If $u_8 \in \mathrm{Lk}(u_2)$, it is easy to see that $u_{10} \in \mathrm{Lk}(u_8)$. The last vertex in $\mathrm{Lk}(u_9)$ has to be u_5. Completing the link of u_7, we observe that $\deg(u_5) > 6$, a contradiction. If $u_{10} \in \mathrm{Lk}(u_2)$, then $u_2u_7u_{10}$ and $u_2u_9u_{10}$ are simplices. To complete $\mathrm{Lk}(u_9)$, $u_3u_7u_9$ and $u_8u_9u_{10}$ have to be faces. It is clear that $u_4u_6u_8 \in K$. The last vertex in $\mathrm{Lk}(u_7)$ is one of u_4, u_5 or u_6, all of which are impossible.

If $u_2u_3u_{10} \in K$, then the sixth vertex, z, in $\mathrm{Lk}(u_2)$ is u_4, u_5 or u_8. If $z = u_4$, the links of u_2 and u_4 show that $u_2u_4u_7$, $u_2u_4u_{10}$, $u_4u_5u_{10}$ and $u_3u_4u_7$ have to be faces. It is easy to see that $\mathrm{Lk}(u_3)$ has only five vertices, a contradiction. If $z = u_8$, considering the links of u_2, u_8, u_9, u_7, u_3 and u_6, we see that $u_4u_6u_7$ and $u_4u_5u_6$ are faces, a contradiction. If

$z = u_5$, to complete $\mathrm{Lk}(u_5)$, we see that either $u_4u_5u_7$ or $u_4u_5u_{10}$ is a face. In the first case the links of u_7 and u_4 show that $\deg(u_{10}) > 6$. In the second case we see that there exists no vertex x $(\neq u_1)$ such that u_7u_9x is a face. This proves the claim.

By the claim we can assume without loss that $u_1u_3u_6$ or $u_1u_4u_6$ is a face.

Case 1: $u_1u_3u_6$ *is a face in* K. Let u be the sixth vertex in $\mathrm{Lk}(u_1)$.

If $u = u_5$, we see from the links of u_1 and u_3 that $\deg(u_5) > 6$, a contradiction. If $u = u_4$, the links of u_1 and u_3 show that $u_1u_3u_4$, $u_1u_4u_7$, $u_2u_3u_8$ and $u_3u_6u_8$ are faces, where u_8 is one of the remaining three vertices. Let u_9 and u_{10} be the other two vertices. Now, one of u_9 or u_{10}, say u_9, has to be in the link of u_2. Hence, $V(\mathrm{Lk}(u_{10})) = \{u_4, u_5, u_6, u_7, u_8, u_9\}$. Then it is easy to see that $V(\mathrm{Lk}(u_9)) = \{u_2, u_5, u_7, u_8, u_{10}\}$, a contradiction. So, u is one of the remaining three vertices, say u_8.

Clearly, $u_1u_3u_8$ and $u_1u_7u_8$ are faces. Let u_9 and u_{10} be the other two vertices. To complete $\mathrm{Lk}(u_3)$ either $u_2u_3u_6$ or $u_2u_3u_8 \in K$. If $u_2u_3u_6 \in K$, we see that the sixth vertex in both $\mathrm{Lk}(u_2)$ and $\mathrm{Lk}(u_6)$ is the same, say u_9. Then $\deg(u_{10}) < 6$, a contradiction. So, $u_2u_3u_8$ and $u_3u_4u_6 \in K$.

Clearly, $V(\mathrm{Lk}(u_9)) = \{u_2, u_4, u_5, u_7, u_8, u_{10}\}$ and $V(\mathrm{Lk}(u_{10})) = \{u_4, u_5, u_6, u_7, u_8, u_9\}$. Then, by considering the links of u_2, u_6, u_4, u_8 and u_5, $u_2u_7u_9$, $u_2u_8u_9$, $u_4u_6u_{10}$, $u_5u_6u_{10}$, $u_4u_5u_9$, $u_4u_9u_{10}$, $u_7u_8u_{10}$, $u_8u_9u_{10}$, $u_5u_7u_9$ and $u_5u_7u_{10}$ are faces in K. Here, K is isomorphic, via the map $(1, 2, 3)(4, 9, 6, 10, 7, 5, 8)$, to T_{11}.

Case 2: $u_1u_4u_6$ *is a face in* K. Let u be the sixth vertex in $\mathrm{Lk}(u_1)$.

If $u = u_5$, then $V(\mathrm{Lk}(u_4)) = \{u_0, u_1, u_3, u_5, u_6, u_8\}$ and $V(\mathrm{Lk}(u_6)) = \{u_0, u_1, u_4, u_5, u_8, u_9\}$ (otherwise $f_2 < 22$), where u_8 and u_9 are two of the remaining three vertices. Then the vertex set of the link of the remaining vertex is a subset of $\{u_2, u_3, u_7, u_8, u_9\}$, a contradiction. If u is one of the remaining three vertices, say u_8, then the links of u_1 and u_4 show that $u_1u_4u_8$, $u_1u_7u_8$, $u_3u_4u_6$ and $u_4u_5u_8$ are faces in K. The face other that $u_0u_5u_6$ having u_5u_6 as an edge has to be $u_5u_6u_9$ (in all other cases $f_2 < 22$), where u_9 is one of the remaining two vertices. If u_{10} is the remaining vertex, then $V(\mathrm{Lk}(u_{10})) = \{u_2, u_3, u_5, u_7, u_8, u_9\}$. The links of u_3, u_5 and u_{10} imply that $\deg(u_2) = 5$, a contradiction. So, $u = u_3$.

Clearly, $u_1u_3u_4$ and $u_1u_3u_7$ are faces. The sixth vertex in $\mathrm{Lk}(u_4)$ has to be one of the three remaining vertices, say u_8. Then $u_4u_5u_8$ and $u_4u_6u_8 \in K$. If u_9 and u_{10} are the remaining two vertices, then $V(\mathrm{Lk}(u_9)) = \{u_2, u_3, u_5, u_7, u_8, u_{10}\}$ and $V(\mathrm{Lk}(u_{10})) = \{u_2, u_5, u_6, u_7, u_8, u_9\}$. The links of u_3, u_2, u_6, u_5 and u_7 show that $u_2u_3u_9$, $u_3u_7u_9$, $u_2u_7u_{10}$, $u_2u_9u_{10}$, $u_5u_6u_{10}$, $u_6u_8u_{10}$, $u_5u_8u_9$, $u_5u_9u_{10}$, $u_7u_8u_9$ and $u_7u_8u_{10}$ are faces in K. In this case, K is isomorphic, via the map $(0, 1)(2, 9, 7, 8, 5, 4)(3, 10, 6)$, to T_{11}. This completes the proof of the lemma. □

Proof of Theorem 4. Let K be a $\{3, q\}$-equivelar simplicial polyhedron on n (≤ 11) vertices.

Since $\chi(K) = 0$ we have $q = 6$ and hence $n \geq 7$.

It is not difficult to show (also see in [9]) that T_7 is the only (up to isomorphism) 7-vertex neighbourly combinatorial 2-manifold. Hence, if $n = 7$, then K is isomorphic to T_7.

The case $n = 8$ follows from the classification of combinatorial 2-manifolds on eight vertices in [10]. (It is also not difficult, by a similar argument as in the proof of Lemma 1, to show that there exists a unique combinatorial 2-manifold with non-edge graph $4K_2$ (the disjoint union of four edges). This implies that K is isomorphic to T_8.)

If $n = 9$, then, by Lemma 1, K is isomorphic to $A_{3,3}$, $B_{3,3}$ or T_9.

If $n = 10$, then, by Lemma 2, K is isomorphic to or T_{10} or Q.

If $n = 11$, then, by Lemma 3, K is isomorphic to T_{11}. This completes the proof. □

Lemma 4. *If M_1 and M_2 are as in Examples* 1 *and* 7, *then* $M_1 \not\cong M_2$.

Proof. For $1 \le i \le 2$, let $A(\Lambda(M_i))$ denote the adjacency matrix of the graph $\Lambda(M_i)$. Let $\mathcal{P}_i(x)$ denote the characteristic polynomial of $A(\Lambda(M_i))$. Then

$$\begin{aligned} \mathcal{P}_1(x) &= (x-3)(x-2)^6(x-1)^3x^4(x+1)^3(x+2)^6(x+3), \\ \mathcal{P}_2(x) &= (x-3)(x-2)^4(x-1)x^4(x+1)^2(x+2)^2(x^2-3)(x^2+2x-1) \\ &\quad \cdot (x^3+2x^2-4x-6)^2. \end{aligned}$$

If M_1 and M_2 are isomorphic, then $\Lambda(M_1)$ and $\Lambda(M_2)$ are isomorphic as graphs and hence $\mathcal{P}_1(x) = \mathcal{P}_2(x)$. Clearly, $\mathcal{P}_1(x) \neq \mathcal{P}_2(x)$. Hence, $M_1 \not\cong M_2$. □

Lemma 5. *If $N_1, \dots, N_{14}$ are as in Examples* 1 *and* 8, *then $N_i \not\cong N_j$ for* $1 \le i \neq j \le 14$.

Proof. For $1 \le i \le 14$, let $A(\Lambda(N_i))$ denote the adjacency matrix of the graph $\Lambda(N_i)$. Let $\mathcal{P}_i(x)$ denote the characteristic polynomial of $A(\Lambda(N_i))$. Then

$$\begin{aligned} \mathcal{P}_1(x) &= (x-3)(x-2)(x-1)(x+1)^2(x+2)(x^3+x^2-5x-3)^2 \\ &\quad \cdot (x^9-12x^7+2x^6+45x^5-12x^4-52x^3+9x^2+15x+1)^2, \\ \mathcal{P}_2(x) &= (x-3)(x-1)^9(x+2)^4(x^2+3x+1)^3(x^2-x-3)^5, \\ \mathcal{P}_3(x) &= (x-3)(x-1)^3(x+2)(x^2-x-3)^2(x^2+x-1)^3 \\ &\quad \cdot (x^5+x^4-8x^3-5x^2+13x+6)^3, \\ \mathcal{P}_4(x) &= (x-3)(x-1)^2x^2(x^{13}-20x^{11}+154x^9+8x^8-576x^7-82x^6+1073x^5 \\ &\qquad +272x^4-893x^3-316x^2+235x+96) \\ &\quad \cdot (x^{12}+5x^{11}-7x^{10}-67x^9-25x^8+315x^7+315x^6-599x^5-850x^4 \\ &\qquad +358x^3+781x^2+57x-144), \\ \mathcal{P}_5(x) &= (x-3)(x-1)x^3(x^{12}-20x^{10}+154x^8+4x^7-570x^6-38x^5+1015x^4 \\ &\qquad +110x^3-723x^2-90x+93) \\ &\quad \cdot (x^{13}+4x^{12}-12x^{11}-60x^{10}+42x^9+340x^8 \\ &\qquad +2x^7-910x^6-269x^5+1170x^4+453x^3-650x^2-207x+112), \\ \mathcal{P}_6(x) &= (x-3)(x-1)x(x^2-5)(x^2+3x+1)(x^2-x-1) \\ &\quad \cdot (x^{11}+2x^{10}-14x^9-25x^8+72x^7+111x^6-162x^5-210x^4+145x^3 \\ &\qquad +156x^2-28x-24) \\ &\quad \cdot (x^{10}-14x^8-x^7+68x^6+11x^5-132x^4-30x^3+81x^2+12x-8), \end{aligned}$$

$$\begin{aligned}
\mathcal{P}_7(x) &= (x-3)(x-1)(x+1)\\
&\quad \cdot (x^{27}+3x^{26}-35x^{25}-105x^{24}+541x^{23}+1621x^{22}-4851x^{21}\\
&\quad -14513x^{20}+27857x^{19}+83335x^{18}-106646x^{17}-320442x^{16}\\
&\quad +274411x^{15}+836409x^{14}-466133x^{13}-1470839x^{12}+497081x^{11}\\
&\quad +1695261x^{10}-297277x^{9}-1217051x^{8}+70531x^{7}+501773x^{6}\\
&\quad +9836x^{5}-105408x^{4}-6268x^{3}+9416x^{2}+568x-228),\\
\mathcal{P}_8(x) &= (x-3)x^2(x+1)^5(x^2-5)^3(x^2-x-1)^2\\
&\quad \cdot (x^6-10x^4+5x^3+25x^2-25x+5)^2,\\
\mathcal{P}_9(x) &= (x-3)x^2(x^3+x^2-5x-1)(x^3+x^2-5x-4)^2\\
&\quad \cdot (x^9-12x^7+x^6+48x^5-6x^4-74x^3+12x^2+36x-10)^2,\\
\mathcal{P}_{10}(x) &= (x-3)x^2(x^{27}+3x^{26}-36x^{25}-108x^{24}+576x^{23}+1728x^{22}-5384x^{21}\\
&\quad -16184x^{20}+32482x^{19}+98344x^{18}-131768x^{17}-406048x^{16}\\
&\quad +362570x^{15}+1159262x^{14}-664274x^{13}-2283392x^{12}\\
&\quad +766277x^{11}+3040125x^{10}-481011x^{9}-2623483x^{8}\\
&\quad +78215x^{7}+1363257x^{6}+75039x^{5}-373481x^{4}\\
&\quad -37464x^{3}+40010x^{2}+4277x-493),\\
\mathcal{P}_{11}(x) &= (x-3)x(x+1)(x^{27}+2x^{26}-38x^{25}-70x^{24}+646x^{23}+1080x^{22}\\
&\quad -6472x^{21}-9660x^{20}+42370x^{19}+55418x^{18}\\
&\quad -189975x^{17}-213030x^{16}+594709x^{15}+556564x^{14}\\
&\quad -1301093x^{13}-981238x^{12}+1959990x^{11}+1134608x^{10}\\
&\quad -1967022x^{9}-812454x^{8}+1239001x^{7}+322970x^{6}\\
&\quad -439776x^{5}-56142x^{4}+71163x^{3}+1616x^{2}\\
&\quad -3360x+160),\\
\mathcal{P}_{12}(x) &= (x-3)x(x^2-5)(x^2-x-1)(x^4+2x^3-4x^2-5x+2)\\
&\quad \cdot (x^{10}-14x^8-x^7+68x^6+11x^5-132x^4-30x^3+81x^2+12x-8)\\
&\quad \cdot (x^{10}+2x^9-12x^8-21x^7+50x^6+71x^5-86x^4-84x^3+57x^2\\
&\quad +22x-8),\\
\mathcal{P}_{13}(x) &= (x-3)(x+1)(x^2-3)(x^3-x^2-3x+1)(x^3+3x^2-x-5)\\
&\quad \cdot (x^{10}-14x^8+2x^7+65x^6-14x^5-114x^4+25x^3+66x^2-12x-9)^2,\\
\mathcal{P}_{14}(x) &= (x-3)(x^{29}+3x^{28}-36x^{27}-108x^{26}+576x^{25}+1726x^{24}-5388x^{23}\\
&\quad -16128x^{22}+32588x^{21}+97682x^{20}-132955x^{19}-401795x^{18}\\
&\quad +369861x^{17}+1143465x^{16}-690891x^{15}-2250991x^{14}\\
&\quad +824419x^{13}+3013939x^{12}-552352x^{11}-2648342x^{10}\\
&\quad +117160x^{9}+1435760x^{8}+77028x^{7}-433570x^{6}-45664x^{5}\\
&\quad +61804x^{4}+5564x^{3}-3748x^{2}-119x+63).
\end{aligned}$$

Now, $N_i \cong N_j$ implies $\Lambda(N_i)$ and $\Lambda(N_j)$ are isomorphic as graphs and hence $\mathcal{P}_i(x) =$

$\mathcal{P}_j(x)$. Since $\mathcal{P}_i(x)$ and $\mathcal{P}_j(x)$ have different prime factorizations $\mathcal{P}_i(x) \neq \mathcal{P}_j(x)$, for $1 \leq i \neq j \leq 14$. Hence, for $1 \leq i \neq j \leq 14$, $N_i \ncong N_j$. □

Proof of Theorem 5. Let K be a 9-vertex neighbourly simplicial equivelar polyhedron. Choose a vertex, say u_9, and let the link of u_9 be $C_8(u_1, \dots, u_8)$.

Claim. *There exists a face of the form* $u_iu_{i+1}u_{i+3}$ *or* $u_iu_{i+1}u_{i+6}$ *for some* $i \in \{1, \dots, 8\}$ (*additions in the subscripts are modulo* 8).

If possible, assume that either $u_iu_{i+1}u_{i+4}$ or $u_iu_{i+1}u_{i+5}$ is a face for each i. Then we can assume without loss that $u_1u_2u_5$ is a face in K. Then, by repeated use of the assumption, it is clear that $u_1u_4u_8$, $u_4u_5u_8$, $u_3u_7u_8$ and $u_3u_4u_7$ are faces in K. To complete $\mathrm{Lk}(u_4)$ and $\mathrm{Lk}(u_7)$, $u_2u_4u_6$, $u_2u_4u_7$, $u_1u_4u_6$ and $u_1u_6u_7$ have to be faces, a contradiction. This proves the claim.

By the claim, we can assume without loss that $u_1u_2u_4 \in K$. We observe that the edge u_1u_8 belongs to one of $u_1u_3u_8$, $u_1u_5u_8$ or $u_1u_6u_8$.

Case 1: $u_1u_3u_8$ *is a face*. By considering $\mathrm{Lk}(u_1)$, one of $u_1u_3u_5$, $u_1u_3u_6$ or $u_1u_3u_7$ is a face.

If $u_1u_3u_7 \in K$, then, by considering the links of u_1 and u_5, $u_1u_5u_6$, $u_1u_4u_6$ and $u_1u_5u_7$ are faces and hence, by considering $\mathrm{Lk}(u_7)$, $u_3u_5u_7 \notin K$. Then, by considering the links of u_5, u_8 and u_6, $u_2u_3u_5$, $u_3u_5u_8 \in K$, $u_5u_7u_8 \notin K$ and hence $u_2u_5u_7$, $u_4u_5u_8$, $u_2u_7u_8$, $u_2u_6u_8$, $u_4u_6u_8$, $u_2u_3u_6$ and $u_3u_6u_7$ are faces. Then $u_2u_3u_5$, $u_2u_3u_6$ and $u_2u_3u_9$ are the faces through u_2u_3, a contradiction.

If $u_1u_3u_5 \in K$, then, by considering $\mathrm{Lk}(u_1)$ and $\mathrm{Lk}(u_6)$, $u_1u_6u_7$, $u_1u_4u_6$, $u_1u_5u_7$ are faces. The edge u_4u_6, belongs to either $u_3u_4u_6$ or $u_4u_6u_8$. In the first case, $\mathrm{Lk}(u_3)$ and $\mathrm{Lk}(u_6)$ show that $u_2u_3u_6 \notin K$ and $u_2u_5u_6$, $u_2u_6u_8$, $u_3u_6u_8$, $u_2u_3u_7$, $u_3u_5u_7 \in K$. However, this implies that $\mathrm{Lk}(u_5)$ contains $C_3(u_3, u_1, u_7)$, a contradiction. In the second case $u_2u_3 \in \mathrm{Lk}(u_6)$. Considering $\mathrm{Lk}(u_7)$ and $\mathrm{Lk}(u_5)$, $u_2u_4u_7$, $u_3u_4u_7$, $u_2u_5u_7$ and $u_3u_7u_8 \in K$. Then $\mathrm{Lk}(u_8)$ contains $C_4(u_7, u_9, u_1, u_3)$, a contradiction. So, $u_1u_3u_6 \in K$.

From $\mathrm{Lk}(u_1)$, $u_1u_5u_7$ is a face and either $u_1u_4u_5$ or $u_1u_4u_7$ is a face.

If $u_1u_4u_5 \in K$, then so is $u_1u_6u_7$. To complete $\mathrm{Lk}(u_7)$, either $u_2u_4u_7$ and $u_3u_4u_7$ or $u_2u_4u_7$ and $u_2u_3u_7$ have to be faces. This shows that $\deg(u_4) < 8$ in the first case and $\deg(u_2) < 8$ in the second case, a contradiction. Thus, $u_1u_4u_7 \in K$.

Clearly, $u_1u_5u_6$ is a face. By considering the links of u_3 and u_6, $u_3u_6u_8 \notin K$ and $u_4u_6u_8$, $u_2u_6u_8 \in K$. Then, by considering the links of u_6, u_4 and u_8, $u_2u_4u_6$, $u_2u_4u_7$ and $u_2u_4u_8$ are not faces and hence $u_2u_4u_5$ is a face. Now, by considering the links of u_2, u_5 and u_4, $u_2u_5u_7$ is not a face and $u_2u_5u_8$, $u_2u_6u_7$, $u_2u_3u_7$, $u_3u_5u_7$, $u_3u_5u_8$ $u_3u_4u_6$, $u_4u_7u_8$ are faces. Here, K is M_2.

Case 2: $u_1u_5u_8$ *is a face*. Clearly, one of $u_1u_3u_5$, $u_1u_5u_6$ or $u_1u_5u_7$ is a face.

Subcase 2.1: $u_1u_3u_5$ *is a face*. If $u_1u_3u_7$ is a face, then the link of u_1 shows that $u_1u_4u_6$ and $u_1u_6u_7$ are faces. By considering the links of u_3, u_8, u_7, u_4 and u_2, $u_2u_5u_7$, $u_4u_5u_7$, $u_2u_3u_7$, $u_4u_7u_8$, $u_3u_4u_6$, $u_2u_4u_8$ and $u_2u_5u_6 \in K$. Then $\mathrm{Lk}(u_5)$ is $C_5(u_7, u_4, u_9, u_6, u_2)$, a contradiction. So, by considering $\mathrm{Lk}(u_1)$, $u_1u_3u_6$, $u_1u_6u_7$ and $u_1u_4u_7 \in K$.

Clearly, either u_3u_4 or $u_4u_5 \in \mathrm{Lk}(u_7)$. If $u_4u_5u_7 \in K$, then $u_2u_3u_7$, $u_2u_7u_8$ and $u_3u_5u_7$ are faces (if $u_3u_7u_8 \in K$, then, to complete $\mathrm{Lk}(u_3)$, $u_3u_5u_8 \in K$, which implies

$\deg(u_5) < 8$). $\mathrm{Lk}(u_5)$ shows that $u_2u_5u_6$ and $u_2u_5u_8$ are faces. The link of u_8 now contains $C_5(u_2, u_7, u_9, u_1, u_5)$, a contradiction. So, $u_3u_4 \in \mathrm{Lk}(u_7)$.

From the links of u_7, u_3, u_8 and u_2, $u_3u_5u_7$, $u_2u_5u_7$, $u_2u_7u_8$, $u_2u_3u_8$, $u_3u_6u_8$, $u_4u_5u_8$, $u_4u_6u_8$, $u_2u_4u_6$ and $u_2u_5u_6$ are faces. Here, K is isomorphic, via the map $(1, 3, 5, 7)(2, 4, 6, 8)$, to M_2.

Subcase 2.2: $u_1u_5u_6$ *is a face*. To complete $\mathrm{Lk}(u_1)$, $u_1u_3u_6$, $u_1u_3u_7$ and $u_1u_4u_7$ have to be faces. $\mathrm{Lk}(u_5)$ shows that the edge u_4u_5 belongs to either $u_2u_4u_5$ or $u_4u_5u_7$.

If $u_4u_5u_7 \in K$, then, by considering $\mathrm{Lk}(u_5)$ and $\mathrm{Lk}(u_7)$, $u_2u_3u_5 \in K$, $u_3u_5u_7 \notin K$ and hence $u_2u_5u_7$ and $u_3u_5u_8 \in K$. To complete $\mathrm{Lk}(u_3)$, $u_3u_7u_8$ and $u_3u_4u_6$ have to be faces (since $\mathrm{Lk}(u_4)$ shows that $u_3u_4u_7 \notin K$). This implies that the link of u_8 contains $C_5(u_3, u_7, u_9, u_1, u_5)$, a contradiction. So, $u_2u_4u_5 \in K$.

The links of u_5, u_2, u_8 and u_4 show that $u_3u_5u_7$, $u_2u_5u_7$, $u_3u_5u_8$, $u_2u_6u_8$ and $u_4u_6u_8$ are faces. Then, from $\mathrm{Lk}(u_2)$, either $u_2u_3u_6 \in K$ or $u_2u_3u_8 \in K$.

In the first case the links of u_2, u_6 and u_8 show that $u_2u_7u_8$, $u_4u_6u_7$ and $u_3u_4u_8$ are faces. Here, K is isomorphic, via the map $(1, 7, 2, 5, 8, 4, 3, 6, 9)$, to M_2.

In the second case the links of u_2, u_6 and u_8 show that $u_2u_6u_7$, $u_3u_4u_6$ and $u_4u_7u_8$ are faces. Here, K is isomorphic, via the map $(1, 2, 7, 5, 3)(4, 8)$, to M_1.

Subcase 2.3: $u_1u_5u_7$ *is a face*. From $\mathrm{Lk}(u_1)$, $u_1u_3u_6$ and one of $u_1u_3u_4$ or $u_1u_4u_6$ are faces.

If $u_1u_3u_4 \in K$, then so is $u_1u_6u_7$. The links of u_3, u_6 and u_4 show that $u_3u_5u_6$, $u_2u_3u_6 \notin K$, $u_2u_4u_6$, $u_4u_6u_8$, $u_3u_6u_8$, $u_4u_5u_7$ and $u_4u_7u_8$ are faces. This shows that the link of u_7 contains $C_6(u_4, u_5, u_1, u_6, u_9, u_8)$, a contradiction. So, $u_1u_4u_6$ is a face.

Clearly, $u_1u_3u_7 \in K$. We now observe that either $u_3u_4u_7$ or $u_3u_4u_8$ is a face (since, $\mathrm{Lk}(u_6)$ shows that $u_3u_4u_6 \notin K$).

In the first case the links of u_3, u_4, u_5 and u_6 show that $u_3u_5u_8 \in K$, $u_4u_5u_7 \notin K$ and hence $u_2u_5u_7$, $u_2u_4u_5$, $u_3u_5u_6$, $u_2u_3u_8$, $u_4u_6u_8$, $u_4u_7u_8$, $u_2u_6u_8$ and $u_2u_6u_7$ are faces. Here, K is isomorphic, via the map $(1, 4, 6, 3, 9, 7, 5, 2, 8)$, to M_2.

In the second case $u_4u_5u_7$ has to be a face (if $u_2u_4u_5 \in K$, then, from $\mathrm{Lk}(u_4)$ and $\mathrm{Lk}(u_7)$, $\deg(u_7) < 8$). Considering the links of u_5, u_3, u_7 and u_6, $u_2u_3u_5 \in K$, $u_3u_5u_8 \notin K$ and $u_2u_5u_8$, $u_3u_5u_6$, $u_3u_7u_8$, $u_2u_4u_7$, $u_2u_6u_7$, $u_2u_6u_8$ and $u_4u_6u_8$ are faces. Here, K is isomorphic, via the map $(1, 3, 6, 4, 9, 7)(5, 8)$, to M_2.

Case 3: $u_1u_6u_8$ *is a face*. Clearly, u_3u_6, u_5u_6 or $u_6u_7 \in \mathrm{Lk}(u_1)$.

If $u_1u_6u_7 \in K$, then $u_1u_3u_5 \in K$. The links of u_6, u_2 and u_1 show that $u_2u_4u_6$, $u_3u_4u_6$, $u_1u_3u_7$ and $u_1u_4u_5$ are faces. Here, $\mathrm{Lk}(u_4)$ contains $C_6(u_6, u_3, u_9, u_5, u_1, u_2)$, a contradiction.

If $u_1u_5u_6 \in K$, then $u_1u_3u_7 \in K$. It is clear from $\mathrm{Lk}(u_6)$ that $u_2u_4u_6$ and $u_3u_4u_6$ are faces. Considering the links of u_1, u_4, u_8 and u_5, $u_1u_4u_7$, $u_1u_3u_5$, $u_4u_5u_8$, $u_4u_7u_8$, $u_3u_5u_7 \in K$, a contradiction. So, $u_3u_6 \in \mathrm{Lk}(u_1)$.

Clearly, $u_5u_7 \in \mathrm{Lk}(u_1)$. Now, either $u_1u_4u_5$ or $u_1u_4u_7 \in K$.

If $u_1u_4u_5 \in K$, then $u_1u_3u_7 \in K$. The links of u_4, u_8, u_6 and u_3 show that $u_4u_6u_8$, $u_4u_6u_7$, $u_2u_5u_6$, $u_2u_3u_6$ and $u_3u_5u_7$ which imply $\deg(u_7) < 8$. So, $u_1u_4u_7$ is a face.

The link of u_1 shows that $u_1u_3u_5$ is a face. Now, $u_4u_5u_8$ ($\neq u_4u_5u_9$) is the face having u_4u_5 as an edge. (If $u_2u_4u_5 \in K$, considering the links of u_4, u_7 and u_2, we observe that $u_3u_5 \in \mathrm{Lk}(u_7)$, a contradiction. $\mathrm{Lk}(u_7)$ shows that $u_4u_5u_7 \notin K$.) The links of u_3 and u_5

show that $u_2u_5u_6$ is a face. To complete $\mathrm{Lk}(u_5)$, $u_2u_5u_7$ and $u_3u_5u_8$ have to be faces (if $u_2u_3u_5 \in K$, then $u_2u_3 \in \mathrm{Lk}(u_8)$, a contradiction). Now, the second face through u_6u_8 is $u_2u_6u_8$ or $u_4u_6u_8$.

In the first case the links of u_6, u_4 and u_7 show that $u_3u_4u_6$, $u_4u_6u_7$, $u_2u_4u_8$, $u_2u_3u_7$ and $u_3u_7u_8$ are faces. Here, K is isomorphic, via the map $(2, 5)(3, 9, 6, 8)(4, 7)$, to M_2.

In the second case the links of u_8, u_7 and u_6 show that $u_2u_3u_8$, $u_2u_7u_8$, $u_3u_4u_7$, $u_3u_6u_7$ and $u_2u_4u_6$ are faces. Here, K is isomorphic, via the map $(2, 7, 3)$, to M_1.

The theorem now follows from Lemma 4. □

Proof of Theorem 6. Let K be a 10-vertex neighbourly simplicial equivelar polyhedron. Choose a vertex, say u, and let the link of u be $C_9(u_1, \ldots, u_9)$.

Case 1: *There exists no face of the form* $u_iu_{i+1}u_{i+3}$ *or* $u_iu_{i+1}u_{i+7}$. If $u_iu_{i+1}u_{i+5}$ is a face for each $i \in \{1, \ldots, 9\}$ (additions in the subscript are modulo 9), then $u_1u_2u_6$, $u_1u_5u_6$ and $u_1u_5u_9 \in K$. This implies that $C_5(u_2, u_6, u_5, u_9, u)$ is in $\mathrm{Lk}(u_1)$, a contradiction. So assume, without loss, that $u_1u_2u_5 \in K$.

Claim 1. $u_1u_4u_9$ *is a face.*

Since $u_1u_8u_9$ is not a face, the second face through u_1u_9 is $u_1u_4u_9$, $u_1u_6u_9$ or $u_1u_7u_9$. However, by the assumption, $u_1u_7u_9$ is not a face.

If $u_1u_6u_9 \in K$, then u_1u_5 is in one of $u_1u_3u_5$, $u_1u_4u_5$, $u_1u_5u_7$ or $u_1u_5u_8$.

If $u_1u_3u_5 \in K$, then $u_1u_4u_7$ and $u_1u_4u_8 \in K$ (since u_3u_4 and $u_7u_8 \notin \mathrm{Lk}(u_1)$). The edge u_5u_6 belongs to either $u_2u_5u_6$ or $u_5u_6u_9$. If $u_2u_5u_6 \in K$, the links of u_5, u_1, u_3 and u_6 show that $u_4u_5u_8$, $u_5u_8u_9$, $u_3u_5u_7$, $u_1u_3u_8$, $u_1u_6u_7$, $u_3u_4u_9$, $u_4u_6u_9$, $u_2u_4u_6$ and $u_2u_4u_7 \in K$. The link of u_6 contains $C_7(u_2, u_5, u, u_7, u_1, u_9, u_4)$, a contradiction. If $u_5u_6u_9 \in K$, then the links of u_6, u_1 and u_8 imply $u_1u_7 \notin \mathrm{Lk}(u_6)$, $u_1u_3u_7$, $u_1u_6u_8 \in K$, $u_3u_4u_6$, $u_4u_6u_8 \notin K$ and hence $u_4u_6u_7 \in K$, a contradiction to the assumption.

If $u_1u_4u_5 \in K$, then, by considering the links of u_5, u_1, u_6, u_4 and u_3, $u_1u_3u_7$, $u_1u_3u_8$, $u_5u_6u_9$, $u_1u_4u_7$, $u_1u_6u_8$, $u_4u_6u_8$, $u_2u_4u_6$, $u_2u_3u_6$, $u_3u_6u_7$ (since u_3u_4, $u_3u_8 \notin \mathrm{Lk}(u_6)$), $u_3u_4u_9$, $u_4u_7u_9$, $u_4u_7u_8$, $u_3u_5u_8$ and $u_5u_7u_8$ are faces. This gives a contradiction to the assumption.

If $u_1u_5u_7 \in K$, then $u_1u_3u_8$ and $u_1u_4u_8 \in K$. If $u_2u_5u_6 \in K$, we see that $u_8u_9 \in \mathrm{Lk}(u_5)$ (if not, $\mathrm{Lk}(u_5)$ shows that $u_3u_5u_8$ and $u_4u_5u_8 \in K$ which implies that $C_4(u_3, u_1, u_4, u_5)$ is in $\mathrm{Lk}(u_8)$). Hence, the links of u_5, u_1, u_6, u_7 and u_9 show that $u_3u_5u_7$, $u_1u_4u_7$, $u_1u_3u_6$, $u_2u_6u_8$, $u_4u_6u_8$, $u_4u_6u_9$, $u_4u_7u_9$, $u_2u_7u_9$, $u_2u_7u_8$, $u_2u_3u_9$, $u_3u_5u_9$ and $u_4u_5u_8$ are faces. Here, u_4u_8 is an edge in $u_1u_4u_8$, $u_4u_5u_8$ and $u_4u_6u_8$, a contradiction. If $u_5u_6u_9 \in K$, then $u_4u_8 \in \mathrm{Lk}(u_5)$, $u_3u_5 \notin \mathrm{Lk}(u_8)$. The links of u_5, u_1 and u_9 show that $u_3u_5u_7$, $u_3u_5u_9$, $u_1u_4u_7$, $u_1u_3u_6$, $u_3u_4u_9$ and $u_4u_6u_9 \in K$. Here u_6u_9 is an edge in $u_1u_6u_9$, $u_5u_6u_9$ and $u_4u_6u_9$, a contradiction.

Finally, let $u_1u_5u_8 \in K$. Then $u_1u_3u_7$, $u_1u_4u_7 \in K$. If $u_2u_5u_6 \in K$, then the links of u_5, u_1, u_6, u_9 and u_2 show that $u_4u_5u_9$, $u_5u_7u_9$, $u_3u_5u_7$, $u_3u_5u_8$, $u_1u_4u_8$, $u_1u_3u_6$, $u_3u_6u_7$, $u_4u_6u_9$, $u_4u_6u_8$, $u_2u_6u_8$, $u_3u_8u_9$, $u_2u_3u_9$, $u_2u_7u_9$ and $u_2u_4u_8 \in K$. Hence $\mathrm{Lk}(u_8)$ contains $C_3(u_2, u_6, u_4)$, a contradiction. If $u_5u_6u_9 \in K$, then $\mathrm{Lk}(u_5)$ shows that $u_2u_4u_5$, $u_3u_5u_7$, $u_3u_5u_8$ and $u_5u_8u_9 \in K$. Hence $\mathrm{Lk}(u_9)$ contains $C_5(u_5, u_8, u, u_1, u_6)$, a contradiction. This proves Claim 1.

Clearly, the second face through u_1u_4 is $u_1u_4u_6$, $u_1u_4u_7$ or $u_1u_4u_8$.

Claim 2. *$u_1u_4u_7$ is a face.*

If $u_1u_4u_6 \in K$, then $u_1u_3u_7$, $u_1u_3u_8$ are faces. We see that u_4u_5 is an edge in either $u_4u_5u_8$ or $u_4u_5u_9$. If $u_4u_5u_9 \in K$, then the links of u_5, u_1 and u_4 imply that $u_2u_5u_6$, $u_3u_5u_7$, $u_3u_5u_8$, $u_1u_5u_8$, $u_5u_7u_9$, $u_1u_6u_7$, $u_3u_4u_7$ and $u_2u_4u_7 \in K$. This implies u_3u_7 is in $u_1u_3u_7$, $u_3u_5u_7$ and $u_3u_4u_7$, a contradiction. Hence, $u_4u_5u_8 \in K$ and either $u_1u_5u_8$ or $u_1u_5u_7 \in K$. In the former case the links of u_1, u_4 and u_6 show that $u_1u_6u_8 \in K$, $u_4u_6u_8$ and $u_4u_6u_9 \notin K$ and hence $u_2u_4u_6$ is a face. It is clear that $u_1u_6u_8$ is the only face having u_6u_8 as an edge, a contradiction. In the latter case the links of u_1, u_5, u_7 and u_4 show that $u_1u_6u_7$, $u_2u_5u_7$, $u_3u_5u_7$, $u_3u_5u_9$, $u_2u_7u_9$, $u_4u_7u_9$, $u_4u_8u_9$ and $u_3u_4u_5 \in K$, a contradiction.

If $u_1u_4u_8 \in K$, either $u_4u_5u_9$ or $u_4u_5u_8 \in K$. In either case $\mathrm{Lk}(u_4)$ implies that $u_3u_4u_7$, $u_2u_4u_7$ and $u_2u_4u_6$ are faces. Hence $u_1u_6u_7 \in K$. To complete $\mathrm{Lk}(u_1)$, either $u_1u_3u_8$ or $u_1u_6u_8 \in K$. If $u_1u_3u_8 \in K$, then to complete $\mathrm{Lk}(u_6)$, either $u_6u_8u_9$ or $u_5u_6u_8 \in K$, a contradiction. Hence $u_1u_6u_8$ and therefore $u_1u_3u_5$ and $u_1u_3u_7 \in K$. Clearly, $u_2u_7u_8 \in K$. This implies $\deg(u_7) < 9$, a contradiction. This proves Claim 2.

Again, either $u_4u_5u_9$ or $u_4u_5u_8 \in K$. If $u_4u_5u_9 \in K$, then it is clear from $\mathrm{Lk}(u_4)$ that $u_3u_4u_8$, $u_4u_6u_8$, $u_2u_4u_6$ and $u_2u_4u_7 \in K$. It is easy to see that $u_3u_6u_7$, $u_2u_5u_6$, $u_2u_7u_8$, $u_2u_3u_9$ and $u_5u_8u_9 \in K$. Hence $\deg(u_9) < 9$, a contradiction. Thus, $u_4u_5u_8 \in K$.

Clearly, $u_3u_4u_7$ or $u_3u_4u_9$ is a face through u_3u_4. If $u_3u_4u_9 \in K$, the links of u_4, u_1 and u_6 show that $u_2u_4u_7$, $u_4u_6u_8$, $u_1u_3u_7$, $u_1u_6u_8$ and $u_3u_6u_7 \in K$. To complete $\mathrm{Lk}(u_7)$, either $u_5u_7u_8$ or $u_7u_8u_9 \in K$, a contradiction. So, $u_3u_4u_7 \in K$.

The links of u_4, u_1 and u_7 show that $u_1u_6u_7$, $u_1u_3u_8$, $u_2u_4u_6$, $u_2u_7u_8$ and $u_5u_7u_9$ are faces.

Now, either $u_2u_5u_8$ or $u_5u_8u_9 \in K$. If $u_5u_8u_9 \in K$, the links of u_5, u_1, u_4, u_6 show that $C_3(u_3, u_1, u_6)$ is in $\mathrm{Lk}(u_8)$, a contradiction. So, $u_2u_5u_8 \in K$.

Now, the links of u_1, u_5, u_6, u_8 and u_2 show that $u_1u_3u_5$, $u_1u_6u_8$, $u_3u_5u_7$, $u_5u_6u_9$, $u_4u_6u_8$, $u_2u_3u_6$, $u_3u_6u_9$, $u_3u_8u_9$, $u_2u_7u_9$ and $u_2u_4u_9$ are faces. Here, K is N_1.

Case 2: *There exists a face of the form $u_iu_{i+1}u_{i+3}$ or $u_iu_{i+1}u_{i+7}$*. We can assume without loss that $u_1u_2u_4 \in K$. From $\mathrm{Lk}(u_1)$ we see that u_1u_9 is an edge in one of $u_1u_3u_9$, $u_1u_5u_9$, $u_1u_6u_9$ or $u_1u_7u_9$.

Subcase 2.1: $u_1u_7u_9 \in K$. The edge u_1u_4 belongs to one of $u_1u_3u_4$, $u_1u_4u_5$, $u_1u_4u_6$ or $u_1u_4u_8$.

2.1.1: $u_1u_3u_4 \in K$. From $\mathrm{Lk}(u_1)$ we see that one of $u_1u_5u_7$, $u_1u_6u_7$ or $u_1u_7u_8 \in K$.

If $u_1u_6u_7 \in K$, we see from $\mathrm{Lk}(u_1)$ and $\mathrm{Lk}(u_6)$ that $u_1u_6u_8$, $u_1u_5u_8$ and $u_1u_3u_5 \in K$. The link of u_4 shows that one of $u_4u_5u_7$, $u_4u_5u_8$ or $u_4u_5u_9 \in K$.

If $u_4u_5u_7$ is a face, the links of u_4 and u_7 show that either $u_4u_7u_8$ or $u_4u_7u_9 \in K$. In both cases, after completing $\mathrm{Lk}(u_4)$, we see from $\mathrm{Lk}(u_6)$ and $\mathrm{Lk}(u_7)$ that $u_2u_3u_6$ and $u_2u_3u_7$ are faces, a contradiction (since uu_2u_3 is already a face).

If $u_4u_5u_8$ is a face, then it is clear that $u_4u_6 \notin \mathrm{Lk}(u_8)$ and $u_4u_8u_9 \notin K$ (since, if $u_4u_8u_9 \in K$, then $\mathrm{Lk}(u_4)$ implies that $u_4u_6u_7 \in K$, a contradiction). The links of u_4 and u_7 show that $u_4u_7u_8$ and $u_4u_7u_9 \in K$, which imply that $C_6(u_9, u_1, u_6, u, u_8, u_4)$ is in $\mathrm{Lk}(u_7)$. So, $u_4u_5u_9 \in K$.

Since $u_6u_7 \notin \mathrm{Lk}(u_4)$, we see on completing $\mathrm{Lk}(u_4)$ that $u_4u_6u_8$ and $u_4u_7u_8 \in K$. The links of u_7, u_4, u_8 and u_6 show that $u_4u_6u_9$, $u_2u_4u_7$, $u_2u_3u_8$ and $u_2u_3u_6 \in K$, a contradiction. Therefore, either $u_1u_5u_7$ or $u_1u_7u_8 \in K$.

2.1.1.1: $u_1u_5u_7 \in K$. The link of u_1 shows that either $u_1u_5u_6$ or $u_1u_5u_8$ is a face.

2.1.1.1.1: $u_1u_5u_6$ *is a face*. From $\mathrm{Lk}(u_1)$ we see that $u_1u_6u_8, u_1u_3u_8 \in K$. Considering the links of u_4 and u_5, either $u_4u_5u_8$ or $u_4u_5u_9 \in K$.

Subcase A: $u_4u_5u_8$ *is a face*. The links of u_5 and u_7 show that either $u_2u_5u_7$ or $u_3u_5u_7 \in K$.

A.1: $u_2u_5u_7$ *is a face*. If $u_2u_3u_5 \in K$, $\mathrm{Lk}(u_5)$, $\mathrm{Lk}(u_3)$ and $\mathrm{Lk}(u_8)$ show that $u_3u_5u_9$, $u_3u_6u_7$ and $u_3u_6u_8$ are faces. This implies that $C_3(u_3, u_1, u_6)$ is in $\mathrm{Lk}(u_8)$.

To complete $\mathrm{Lk}(u_5)$, $u_2u_5u_9$, $u_3u_5u_8$ and $u_3u_5u_9 \in K$. The links of u_3 and u_6 show that $u_3u_6u_7$, $u_4u_6u_9$ and $u_2u_4u_6 \in K$ (since, $u_4u_6 \notin \mathrm{L}(u_8)$ and $\mathrm{Lk}(u_3)$). The links of u_4, u_8, u_7 and u_6 show that $u_4u_7u_9$, $u_4u_7u_8$, $u_2u_6u_8$, $u_2u_8u_9$, $u_2u_3u_7$ and $u_3u_6u_9$ are faces. In this case K is N_{11} (more precisely, K is isomorphic to N_{11} by the map φ, where $\varphi(u) = 0$ and $\varphi(u_i) = i$, for $1 \le i \le 9$).

A.2: $u_3u_5u_7 \in K$. If $u_2u_3u_5 \in K$, the links of u_5 and u_3 show that $u_5u_8u_9$ and $u_3u_8u_9 \in K$ (since $u_3u_6 \notin \mathrm{L}(u_8)$), a contradiction. Hence, $u_2u_5u_9$, $u_2u_5u_8$ and $u_3u_5u_9$ are the faces required to complete $\mathrm{Lk}(u_5)$. If $u_2u_3u_7 \in K$, $\mathrm{Lk}(u_3)$ shows that $u_3u_5u_6 \in K$, a contradiction.

We now observe from the links of u_3, u_5 and u_9 that the edge u_2u_3 belongs to $u_2u_3u_6$. The edge u_3u_6 is in either $u_3u_6u_7$ or $u_3u_6u_9$.

In the first case the links of u_3, u_7, u_4 and u_2 show that $u_3u_8u_9$, $u_2u_4u_7$, $u_4u_7u_9$, $u_4u_6u_9$, $u_4u_6u_8$, $u_2u_7u_8$ and $u_2u_6u_9$ are faces. Here, K is N_7.

In the second case the links of u_3, u_7, u_2 and u_4 show that $u_3u_7u_8$, $u_2u_4u_7$, $u_4u_6u_7$, $u_2u_7u_9$, $u_2u_6u_8$, $u_4u_6u_9$ and $u_4u_8u_9$ are faces. Now, K is N_{12}.

Subcase B: $u_4u_5u_9 \in K$. If $u_2u_4u_8 \in K$, the links of u_4 and u_7 show that $u_4u_6u_7$, $u_4u_6u_8$ and $u_4u_7u_9 \in K$. Here $C_6(u_8, u_4, u_7, u, u_5, u_1)$ is in $\mathrm{Lk}(u_6)$. We now observe from $\mathrm{Lk}(u_4)$ that either $u_2u_4u_6$ or $u_2u_4u_7 \in K$.

In the first case the links of u_4, u_8, u_7, u_6 and u_9 show that $u_4u_6u_8$, $u_4u_7u_8$, $u_4u_7u_9$, $u_2u_5u_8$, $u_2u_3u_7$, $u_3u_5u_8$ $u_2u_8u_9$, $u_3u_6u_7$, $u_3u_6u_9$, $u_2u_6u_9$, $u_2u_5u_7$ and $u_3u_5u_9$ are faces. Now, K is N_{14}.

In the second case the links of u_4, u_6, u_7, u_2 and u_3 show that $u_4u_6u_8$, $u_4u_7u_8$, $u_4u_6u_9$, $u_2u_3u_6$, $u_3u_6u_7$, $u_2u_6u_9$, $u_2u_5u_7$, $u_3u_7u_9$, $u_2u_5u_8$, $u_2u_8u_9$, $u_3u_5u_9$ and $u_3u_5u_8$ are faces. Here, K is N_8.

2.1.1.1.2: $u_1u_5u_8 \in K$. The links of u_1 and u_4 show that $u_1u_6u_8$, $u_1u_3u_6 \in K$ and one of $u_4u_5u_7$, $u_4u_5u_8$ or $u_4u_5u_9 \in K$.

Subcase A: $u_4u_5u_7 \in K$. If $u_2u_4u_6 \in K$, then $\mathrm{Lk}(u_4)$ and $\mathrm{Lk}(u_8)$ show that $u_4u_7u_9$ and $u_4u_8u_9 \in K$, thereby showing that $\deg(u_9) < 9$. In the case when $u_2u_4u_9 \in K$, $\mathrm{Lk}(u_4)$ and $\mathrm{Lk}(u_7)$ show that $u_4u_6u_8$ and $u_2u_3u_7$ are faces (since, from $\mathrm{Lk}(u_4)$, either $u_4u_7u_8$ or $u_4u_6u_7 \in K$). Now, the links of u_5 and u_2 show that $u_2u_5u_9$, $u_3u_5u_9$ and $u_2u_6u_8$ are faces. This implies that u_6u_8 is an edge in $u_1u_6u_8$, $u_2u_6u_8$ and $u_4u_6u_8$, a contradiction. Thus, from $\mathrm{Lk}(u_4)$, $u_2u_4u_8 \in K$.

The links of u_4, u_7, u_8, u_2 and u_9 show that $u_4u_6u_9$, $u_4u_6u_7$, $u_4u_8u_9$, $u_2u_3u_7$, $u_2u_7u_9$, $u_3u_7u_8$, $u_2u_5u_6$, $u_3u_5u_9$, $u_2u_5u_9$, $u_3u_6u_9$, $u_2u_6u_8$ and $u_3u_5u_8$ are faces. Here, K is N_5.

Subcase B: $u_4u_5u_8 \in K$. It is clear that $C_4(u_4, u_5, u_1, u_6)$ is in $\mathrm{Lk}(u_8)$ if $u_4u_6u_8 \in K$. If $u_4u_7u_8 \in K$, considering the links of u_4, u_7 and u_8 we see that $u_2u_3u_7$ and $u_2u_3u_8$ are faces, a contradiction. Hence, from $\mathrm{Lk}(u_4)$, $u_4u_8u_9 \in K$. Now, on considering $\mathrm{Lk}(u_9)$ also, $u_4u_6u_9$, $u_4u_6u_7$ and $u_2u_4u_7 \in K$. The links of u_8, u_2 and u_9 show that $u_2u_3u_8$, $u_3u_7u_8$, $u_2u_6u_8$, $u_2u_5u_9$ and $u_3u_5u_9$ are faces. To complete $\mathrm{Lk}(u_5)$, either $u_2u_5u_6$ or $u_3u_5u_6$ is a face.

In the first case the links of u_2, u_5 and u_3 show that $u_2u_7u_9$, $u_3u_5u_7$ and $u_3u_6u_9$ are faces. Clearly, K is N_6.

In the second case the links of u_3, u_9 and u_5 show that $u_3u_7u_9$, $u_2u_6u_9$ and $u_2u_5u_7$ are faces and therefore K is isomorphic, via the map $(0, 4, 2, 3, 1)(5, 7, 6, 9)$, to N_7.

Subcase C: $u_4u_5u_9 \in K$. Considering the link of u_4, we see that one of $u_4u_6u_9$, $u_4u_7u_9$ or $u_4u_8u_9$ is a face.

C.1: $u_4u_6u_9 \in K$. The links of u_4 and u_8 show that $u_4u_7u_8$, $u_2u_4u_7$, $u_4u_6u_8$ and $u_2u_3u_8 \in K$. If $u_6u_7u_9 \in K$, $\mathrm{Lk}(u_7)$ shows that $u_2u_3u_7 \in K$, a contradiction. Hence, we can conclude from $\mathrm{Lk}(u_7)$ that $u_3u_6u_7 \in K$. From $\mathrm{Lk}(u_6)$ and $\mathrm{Lk}(u_3)$, we see that $u_2u_6u_9$, $u_2u_5u_6$ and $u_3u_5u_9 \in K$. To complete $\mathrm{Lk}(u_8)$, either $u_2u_8u_9$ or $u_3u_8u_9$ is a face.

In the first case the links of u_8, u_2 and u_7 show that $u_3u_5u_8$, $u_2u_5u_7$ and $u_3u_7u_9$ are faces. Here, K is isomorphic, via the map $(0, 1, 3, 2, 4)(5, 9, 8, 6, 7)$, to N_{14}.

In the second case the links of u_8, u_3 and u_2 show that $u_2u_5u_8$, $u_3u_5u_7$ and $u_2u_7u_9$ are faces. Here, K is isomorphic, via the map, $(0, 1, 3, 2, 4)(5, 9, 8, 6, 7)$, to N_{11}.

C.2: $u_4u_7u_9 \in K$. The link of u_4 shows that either $u_4u_6u_7$ or $u_4u_7u_8 \in K$. In both cases $u_2u_3 \in \mathrm{Lk}(u_7)$. If $u_3u_5u_6 \in K$, the link of u_3 and u_8 show that $u_3u_7u_9$ and $u_3u_8u_9 \in K$, a contradiction. Hence, $u_2u_5u_6 \in K$ (the links of u_8 and u_5 show that $u_5u_6u_8$ and $u_5u_6u_9 \notin K$). Considering the links of u_5 and u_9, we see that $u_3u_5u_9$, $u_2u_6u_9$, $u_2u_8u_9$ and $u_3u_6u_9 \in K$. The links of u_6, u_4, u_2 and u_8 show that $u_4u_6u_8$, $u_4u_6u_7$, $u_2u_4u_8$, $u_2u_5u_7$, $u_3u_7u_8$ and $u_3u_5u_8$ are faces. In this case K is isomorphic, via the map $(0, 2, 1, 4, 3)(5, 7, 9, 6)$, to N_{11}.

C.3: $u_4u_8u_9 \in K$. The links of u_4, u_6, u_3, u_8, u_2 and u_5 imply that $u_4u_6u_8$, $u_4u_6u_7$, $u_2u_4u_7$, $u_2u_5u_6$, $u_2u_6u_9$, $u_3u_6u_9$, $u_2u_3u_8$, $u_2u_5u_8$, $u_2u_7u_9$, $u_3u_5u_7$, $u_3u_7u_8$ and $u_3u_5u_9$ are faces. Here, K is isomorphic, via the map $(0, 2, 1, 4, 3)(5, 7, 8, 9, 6)$, to N_{14}.

2.1.1.2: $u_1u_7u_8 \in K$. The link of u_1 shows that either $u_1u_5u_8$ or $u_1u_6u_8$ is a face.

Claim 3. $u_1u_5u_8$ *is a face.*

If $u_1u_6u_8 \in K$, the links of u_1 and u_4 show that $u_1u_5u_6$, $u_1u_3u_5$ and one of $u_4u_5u_7$, $u_4u_5u_8$, $u_4u_5u_9 \in K$. If $u_4u_5u_7 \in K$, from $\mathrm{Lk}(u_4)$, either $u_4u_6u_7$ or $u_4u_6u_9 \in K$ (since $u_4u_7u_8 \notin K$). In both the cases the links of u_2, u_4, u_6 and u_7 show that uu_2u_3, $u_2u_3u_6$, $u_2u_3u_7 \in K$, a contradiction. If $u_4u_5u_8 \in K$, $\mathrm{Lk}(u_4)$ and $\mathrm{Lk}(u_8)$ show that either $u_4u_6u_8$ or $u_4u_8u_9 \in K$. Again, in both cases, considering $\mathrm{Lk}(u_4)$, $\mathrm{Lk}(u_9)$ and $\mathrm{Lk}(u_8)$ we see that $u_2u_3u_6$, $u_2u_3u_8 \in K$. So, $u_4u_5u_9 \in K$. Since u_7u_8 already belongs to two faces, $u_4u_7u_8 \notin K$. Hence, $u_4u_6u_7$ and $u_4u_6u_8 \in K$, thereby showing that $C_6(u_8, u_4, u_7, u, u_5, u_1)$ is in $\mathrm{Lk}(u_6)$. This proves Claim 3.

The links of u_1, u_4 and u_5 show that $u_1u_3u_6$, $u_1u_5u_6$ and either $u_4u_5u_7$ or $u_4u_5u_9 \in K$.

Subcase A: $u_4u_5u_7 \in K$. Considering $\mathrm{Lk}(u_4)$, either $u_4u_6u_7$ or $u_4u_7u_9 \in K$. In both cases $u_2u_3 \in \mathrm{Lk}(u_7)$ and therefore, from $\mathrm{Lk}(u_5)$, $u_2u_5u_9$ and $u_3u_5u_9 \in K$.

If $u_4u_6u_7 \in K$, the links of u_4, u_6, u_8, u_3 and u_5 show that $u_4u_8u_9$, $u_2u_6u_8$, $u_3u_6u_8$, $u_2u_6u_9$, $u_2u_4u_8$, $u_3u_5u_8$, $u_4u_6u_9$, $u_3u_7u_9$ and $u_2u_5u_7$ are faces and, thus, K is N_2.

If $u_4u_7u_9 \in K$, the links of u_4, u_9, u_6, u_7, u_5 and u_8 show that $u_2u_4u_8, u_4u_6u_8, u_4u_6u_9$, $u_3u_5u_7$, $u_2u_6u_7$, $u_2u_5u_8$, $u_3u_6u_8$, $u_3u_8u_9$ and $u_2u_6u_9$ are faces. Here, K is isomorphic, via the map $(0, 4, 2, 3, 1)(5, 8, 7, 6, 9)$, to N_5.

Subcase B: $u_4u_5u_9 \in K$. The links of u_4, u_7, u_6, u_8, u_5 and u_9 show that $u_4u_6u_8$, $u_4u_6u_7$, $u_2u_4u_7$, $u_4u_8u_9$, $u_3u_5u_7$, $u_3u_6u_9$, $u_2u_6u_9$, $u_2u_6u_8$, $u_2u_3u_8$, $u_3u_5u_8$, $u_2u_5u_7$, $u_2u_5u_9$ and $u_3u_7u_9 \in K$. In this case K is isomorphic, via the map $(0, 4, 6, 8, 1)(2, 5, 9, 3, 7)$, to N_5.

2.1.2: $u_1u_4u_5 \in K$.

Claim 4. *For a vertex $x \neq u, u_2, u_3$ and u_4 cannot occur together in any order in the link of any of the vertices.*

Let x be any vertex in K. If u_2u_3x and u_3u_4x are faces, then $C_4(u_2, x, u_4, u)$ is in $\mathrm{Lk}(u_3)$. If u_2u_4x and u_3u_4x are faces, then $C_6(u_3, u, u_5, u_1, u_2, x)$ is in $\mathrm{Lk}(u_4)$ and finally if u_2u_3x and u_2u_4x are faces, then $C_5(u_4, u_1, u, u_3, x)$ is in $\mathrm{Lk}(u_2)$. Hence, Claim 4 is proved.

The links of u_1 and u_5 show that $u_1u_3u_5$ or $u_1u_5u_8 \in K$.

Claim 5. *$u_1u_5u_8$ is a face.*

If $u_1u_3u_5 \in K$, then $\mathrm{Lk}(u_1)$ and $\mathrm{Lk}(u_5)$ show that $u_1u_6u_8$ and one of $u_2u_5u_6$, $u_5u_6u_8$ or $u_5u_6u_9 \in K$.

If $u_2u_5u_6 \in K$, $\mathrm{Lk}(u_5)$ shows that $u_5u_7u_8$ and $u_5u_7u_9 \in K$ (if $u_5u_8u_9$ and $u_5u_7u_9$ are faces, then $\deg(u_9) < 9$). Hence $u_7u_8 \notin \mathrm{Lk}(u_1)$ and therefore $u_1u_6u_7$ and $u_1u_3u_8 \in K$. This shows that $C_6(u_1, u_9, u_5, u_8, u, u_6)$ is in $\mathrm{Lk}(u_7)$.

If $u_5u_6u_8 \in K$, considering the links of u_6 and u_1, we see that $u_1u_3u_6$ and $u_1u_7u_8$ are faces, which imply that u_2, u_3 and u_4 are together in $\mathrm{Lk}(u_8)$, which is impossible from Claim 4. Thus, $u_5u_6u_9 \in K$.

It is clear that $u_1u_7u_8$ and $u_1u_3u_6$ are faces (since $u_1u_6u_7 \notin K$, from $\mathrm{Lk}(u_6)$ and Claim 4). Since $u_7u_8 \notin \mathrm{Lk}(u_5)$, $u_2u_5u_7$ and $u_2u_5u_8$ are faces. To complete $\mathrm{Lk}(u_5)$, either $u_5u_7u_9$ or $u_5u_8u_9 \in K$, both of which are impossible by looking at $\mathrm{Lk}(u_7)$, $\mathrm{Lk}(u_8)$ and Claim 4. This proves Claim 5.

From Claim 5 and $\mathrm{Lk}(u_1)$, $u_1u_5u_8$ and therefore $u_1u_3u_6 \in K$.

The link of u_5 shows that u_5u_6 is an edge in one of $u_2u_5u_6$, $u_3u_5u_6$ or $u_5u_6u_9$.

If $u_3u_5u_6 \in K$, considering the links of u_6 and u_1, $u_1u_6u_8$ and $u_1u_3u_7$ are faces. The links of u_5, u_7 and Claim 4 show that $u_2u_3u_5 \notin K$. Hence, $\mathrm{Lk}(u_5)$ and $\mathrm{Lk}(u_3)$ show that $u_3u_5u_9$ and $u_2u_5u_7 \in K$. The links of u_8, u_5 and Claim 4 show that $u_5u_8u_9 \notin K$, $u_5u_7u_9$ and $u_2u_5u_8 \in K$. Since $u_6u_9 \notin \mathrm{Lk}(u_3)$, it is clear from $\mathrm{Lk}(u_9)$ and Claim 4 that $u_2u_6u_9$ and $u_4u_6u_9$ are faces. Since either $u_2u_3u_9$ or $u_3u_4u_9$ is a face, $\mathrm{Lk}(u_3)$ implies that $u_3u_7u_8 \in K$. The links of u_7, u_6 and u_4 show that $u_2u_4u_7$, $u_4u_6u_7$, $u_2u_6u_8$, $u_4u_8u_9$ and $u_3u_4u_8$ are faces, thereby showing that $C_5(u_3, u_4, u_9, u, u_7)$ is in $\mathrm{Lk}(u_8)$. Therefore, either $u_2u_5u_6$ or $u_5u_6u_9$ is a face.

Subcase A: $u_2u_5u_6 \in K$. The links of u_5 and u_7 and Claim 4 show that $u_2u_3u_5 \notin K$.

If $u_2u_5u_9 \in K$, so is $u_3u_5u_7$. If $u_5u_7u_8 \in K$, $\mathrm{Lk}(u_7)$, $\mathrm{Lk}(u_8)$ and Claim 4 show that neither $u_1u_6u_7$ nor $u_1u_6u_8$ is a face, a contradiction. Hence, from $\mathrm{Lk}(u_5)$, $u_3u_5u_8$ and $u_5u_7u_9 \in K$. Considering the links of u_1 and u_7 we see that $u_1u_6u_7$ and $u_1u_3u_8$ are faces. This is seen to be impossible by applying Claim 4 to $\mathrm{Lk}(u_7)$. Hence, $\mathrm{Lk}(u_5)$ shows that $u_2u_5u_7 \in K$.

We now observe, from $\mathrm{Lk}(u_5)$, that either $u_3u_5u_7$ or $u_5u_7u_9$ is a face.

In the first case the links of u_5, u_8 and u_1 show that $u_5u_8u_9$, $u_3u_5u_9$, $u_1u_3u_8$ and $u_1u_6u_7$ are faces (if $u_1u_6u_8 \in K$, the remaining three vertices in $\mathrm{Lk}(u_8)$ are u_2, u_3 and u_4). Since $u_6u_8 \notin \mathrm{Lk}(u_3)$, $\mathrm{Lk}(u_8)$ shows that $u_2u_6u_8$ and $u_4u_6u_8$ are faces. The links of u_6, u_9, u_7 and u_8 show that $u_3u_6u_9, u_4u_6u_9, u_2u_4u_9, u_2u_7u_9, u_3u_4u_7, u_4u_7u_8$ and $u_2u_3u_8$ are faces and hence K is N_{10}.

In the second case the links of u_5, u_8 and u_1 show that $u_3u_5u_8$, $u_3u_5u_9$, $u_1u_6u_8$ and $u_1u_3u_7$ are faces. It is easy to see from $\mathrm{Lk}(u_3)$, $\mathrm{Lk}(u_6)$ and $\mathrm{Lk}(u_9)$ that $u_4u_6u_7$ and $u_3u_6u_9$ are faces (if either $u_2u_3u_9$ or $u_3u_4u_9$ is a face, then from Claim 4 and on completing $\mathrm{Lk}(u_9)$, we get $\deg(u_6) < 9$). The links of u_9, u_4, u_6, u_7 and u_8 show that $u_2u_4u_9$, $u_2u_6u_9$, $u_4u_8u_9$, $u_4u_6u_8$, $u_3u_4u_7$, $u_2u_7u_8$, and $u_2u_3u_8$ are faces. Here, K is N_4.

Subcase B: $u_5u_6u_9$ *be a face*. The link of u_9 and Claim 4 show that $u_5u_7u_9 \notin K$ and hence $\mathrm{Lk}(u_5)$ shows that either $u_2u_5u_9$ or $u_3u_5u_9$ is a face.

B.1: $u_2u_5u_9 \in K$. Here, $u_3u_7 \in \mathrm{Lk}(u_5)$ and the links of u_5, u_7, u_8 and Claim 4 show that $u_5u_7u_8 \notin K$. Now, $\mathrm{Lk}(u_5)$, $\mathrm{Lk}(u_8)$ and $\mathrm{Lk}(u_1)$ show that $u_3u_5u_8$, $u_2u_5u_7$, $u_1u_6u_8$ and $u_1u_3u_7 \in K$. Now, the links of u_7 and u_6 show that the edge u_7u_9 belongs to $u_4u_7u_9$ and therefore $u_2u_6u_7$ and $u_4u_7u_8$ are faces (if $u_4u_6u_7 \in K$, then considering $\mathrm{Lk}(u_4)$, $\mathrm{Lk}(u_9)$ and $\mathrm{Lk}(u_6)$, $u_4u_6u_8$ and $u_2u_6u_9$ are faces which show that $C_3(u_2, u_5, u_6)$ is in $\mathrm{Lk}(u_9)$). To complete $\mathrm{Lk}(u_6)$, either $u_2u_4u_6$ or $u_4u_6u_9 \in K$.

In the first case the links of u_2, u_3 and u_8 show that $u_2u_3u_8$, $u_2u_8u_9$, $u_3u_6u_9$, $u_3u_4u_9$ and $u_4u_6u_8$ are faces. In this case, K is N_{13}.

In the second case the links of u_9, u_8 and u_6 show that $u_3u_8u_9$, $u_2u_3u_9$, $u_2u_4u_8$, $u_2u_6u_8$ and $u_3u_4u_6 \in K$. Here, K is isomorphic, via the map $(0, 1, 5, 7)(2, 8)(3, 6, 9, 4)$, to N_{13}.

B.2: $u_3u_5u_9 \in K$. It is clear that if $u_2u_3u_5$ and $u_5u_7u_8 \in K$, then the links of u_2, u_7, u_1 and u_3 show that $C_7(u_3, u_6, u_1, u_5, u_7, u, u_9)$ is in $\mathrm{Lk}(u_8)$. Therefore, from $\mathrm{Lk}(u_5)$, $u_2u_5u_8$, $u_2u_5u_7$ and $u_3u_5u_7 \in K$. Now, if $u_3u_7u_8 \in K$, the links of u_7, u_1, u_4 and u_8 show that $C_5(u_4, u_8, u, u_1, u_7)$ is in $\mathrm{Lk}(u_9)$. Hence, $u_4u_7u_8$ is a face.

In case $u_4u_7u_9 \in K$, the links of u_7, u_1, u_6 and u_2 show that $u_1u_3u_7$, $u_2u_6u_7$, $u_1u_6u_8 \in K$ and $u_2u_6u_8, u_2u_3u_6 \notin K$ (if $u_2u_3u_6 \in K$, to complete $\mathrm{Lk}(u_6)$, $u_4u_6u_8$ and $u_4u_6u_9 \in K$ which implies that $C_4(u_6, u_9, u_7, u_8)$ is in $\mathrm{Lk}(u_4)$). Since, $u_3u_6u_9 \notin K$ (from $\mathrm{Lk}(u_9)$), to complete $\mathrm{Lk}(u_6)$, $u_2u_4u_6$ and $u_3u_4u_6 \in K$, a contradiction. Further, $u_2u_4u_7 \notin K$ (if $u_2u_4u_7 \in K$, completing $\mathrm{Lk}(u_7)$, we get $u_2u_4u_9 \in K$, a contradiction). Hence, from $\mathrm{Lk}(u_7)$, $u_3u_4u_7 \in K$.

The links of u_7, u_1, u_4, u_6 and u_9 show that $u_1u_6u_7, u_2u_7u_9, u_1u_3u_8, u_4u_6u_8, u_4u_6u_9$, $u_2u_4u_9, u_2u_3u_6, u_2u_6u_8$ and $u_3u_8u_9$ are faces. In this case, K is isomorphic, via the map $(2, 9)(3, 8)(4, 7)(5, 6)$, to N_{10}.

2.1.3: $u_1u_4u_6 \in K$. Using the same method as the one above, we find that K is isomorphic to one of $N_2, N_4, N_5, N_7, \dots, N_{12}$.

2.1.4: $u_1u_4u_8 \in K$. In this case we find that K is isomorphic to one of $N_4, \dots, N_7$, $N_9, \dots, N_{14}$.

Subcase 2.2: $u_1u_3u_9 \in K$. The link of u_1 shows that one of $u_1u_4u_6$, $u_1u_4u_5$, $u_1u_4u_7$ or $u_1u_4u_8 \in K$.

2.2.1: $u_1u_4u_6 \in K$. We observe that either $u_7u_8 \in \mathrm{Lk}(u_1)$ or $u_7u_8 \notin \mathrm{Lk}(u_1)$

2.2.1.1: $u_7u_8 \in \mathrm{Lk}(u_1)$. The link of u_1 shows that either $u_1u_5u_7$ or $u_1u_5u_8 \in K$.

2.2.1.1.1: $u_1u_5u_7 \in K$. In this case, K is isomorphic to one of N_1, N_7, N_{11} or N_{14}.

2.2.1.1.2: $u_1u_5u_8 \in K$. The links of u_1 and u_7 show that $u_1u_5u_6$ and $u_1u_3u_7 \in K$. Considering $\mathrm{Lk}(u_6)$ and $\mathrm{Lk}(u_7)$, we see that either $u_2u_6u_7$ or $u_6u_7u_9 \in K$.

Claim 6. *$u_6u_7u_9$ is a face.*

If $u_2u_6u_7 \in K$, then $\mathrm{Lk}(u_7)$ and $\mathrm{Lk}(u_3)$ shows that either $u_3u_4u_7$ or $u_3u_5u_7 \in K$.

If $u_3u_4u_7 \in K$, the links of u_7, u_4, u_5 and u_3 show that $u_2u_5u_7$, $u_5u_7u_9$, $u_4u_7u_9$, $u_2u_4u_5$, $u_4u_6u_8$, $u_4u_8u_9$, $u_3u_5u_8$, $u_3u_5u_9$, $u_2u_3u_6$ and $u_3u_6u_8 \in K$. Here, $\deg(u_6) < 9$. Hence, $u_3u_5u_7 \in K$.

To complete $\mathrm{Lk}(u_7)$, $u_2u_4u_7$, $u_4u_7u_9$ and $u_5u_7u_9$ have to be faces (if $u_4u_5u_7 \in K$, $\mathrm{Lk}(u_5)$ implies that either $u_3u_5u_9$ or $u_5u_8u_9$ is a face, which is seen to be impossible from the links of u_3 and u_8). This shows that the face ($\neq uu_4u_5$) having u_4u_5 as an edge is $u_2u_4u_5$, a contradiction. This proves Claim 6.

From $\mathrm{Lk}(u_7)$ and Claim 6, we see that one of $u_2u_3u_7$, $u_3u_4u_7$ or $u_3u_5u_7$ is a face.

If $u_3u_4u_7$ is a face, the links of u_7, u_4, u_8 and u_5 show that $u_2u_4u_7$, $u_2u_5u_7$, $u_5u_7u_9$, $u_4u_5u_9$ and $u_2u_3u_5$ are faces which implies that $\deg(u_2) < 9$.

If $u_2u_3u_7$ is a face, the links of u_7, u_2, u_5, u_3 and u_6 show that $u_2u_5u_7$, $u_4u_5u_7$, $u_4u_7u_9$, $u_2u_5u_9$, $u_3u_5u_9$, $u_3u_5u_8$, $u_3u_4u_6$, $u_3u_6u_8$, $u_2u_6u_8$, $u_2u_6u_9$, $u_2u_4u_8$ and $u_4u_8u_9$ are faces. Here, K is N_3.

In the last case the link of u_7 shows that $u_2u_4u_7$, $u_2u_5u_7$ and $u_4u_7u_9 \in K$ (if $u_4u_5u_7 \in K$, to complete $\mathrm{Lk}(u_5)$, either $u_3u_5u_9$ or $u_5u_8u_9 \in K$, a contradiction). It is clear that $u_4u_5u_9 \in K$, from $\mathrm{Lk}(u_4)$ and $\mathrm{Lk}(u_5)$. The links of u_4, u_6, u_2 and u_3 show that $u_3u_4u_8$, $u_4u_6u_8$, $u_2u_3u_6$, $u_2u_6u_8$, $u_3u_6u_9$, $u_2u_8u_9$, $u_2u_5u_9$ and $u_3u_5u_8$ are faces. Here, K is isomorphic, via the map $(0, 9, 7)(1, 2, 5, 4, 6, 8)$, to N_5.

2.2.1.2: $u_7u_8 \notin \mathrm{Lk}(u_1)$. Hence, $u_1u_5u_7$ and $u_1u_5u_8$ are faces. To complete $\mathrm{Lk}(u_1)$, either $u_1u_6u_7$ and $u_1u_3u_8$ or $u_1u_6u_8$ and $u_1u_3u_7$ are faces

2.2.1.2.1: $u_1u_6u_7$ and $u_1u_3u_8$ are faces. The edge u_7u_8 belongs to one of $u_2u_7u_8$, $u_3u_7u_8$ or $u_4u_7u_8$.

Subcase A: $u_2u_7u_8 \in K$. If $u_2u_3u_7 \in K$, the links of u_7 and u_3 show that $u_3u_7u_9$, $u_4u_7u_9$, $u_4u_5u_7$ and $u_3u_5u_8$ are faces. This implies that $C_3(u_5, u_1, u_3)$ is in $\mathrm{Lk}(u_8)$. Hence, either $u_2u_4u_7$ or $u_2u_7u_9 \in K$.

A.1: $u_2u_7u_9 \in K$. To complete $\mathrm{Lk}(u_7)$, $u_3u_5u_7$, $u_3u_4u_7$ and $u_4u_7u_9$ have to be faces. Considering $\mathrm{Lk}(u_4)$, we see that either $u_4u_6u_9$ or $u_4u_8u_9 \in K$ (since, $u_2u_4 \notin \mathrm{Lk}(u_9)$).

In the first case the links of u_4, u_8, u_9 and u_6 show that $u_2u_4u_8$, $u_4u_5u_8$, $u_3u_6u_8$, $u_6u_8u_9$, $u_2u_5u_9$, $u_3u_5u_9$, $u_2u_3u_6$ and $u_2u_5u_6$ are faces and, hence, K is isomorphic, via the map $(0, 3, 1, 2, 8, 4, 6, 5, 9)$, to N_{10}.

In the second case the links of u_4, u_6, u_8, u_2 and u_9 show that $u_4u_6u_8$, $u_2u_4u_5$, $u_2u_5u_8$, $u_3u_6u_8$, $u_2u_3u_6$, $u_2u_6u_9$, $u_5u_6u_9$ and $u_3u_5u_9$ are faces. Here, K is N_9.

A.2: $u_2u_4u_7 \in K$. The link of (u_7) shows that $u_3u_7u_9 \in K$. Considering the links of u_8 and u_9, $u_2u_8u_9$ and $u_2u_3u_9 \notin K$ and hence the link of u_2 shows that $u_2u_5u_9$ and $u_2u_6u_9$ are faces. To complete $\mathrm{Lk}(u_7)$, either $u_3u_4u_7$ or $u_4u_7u_9 \in K$.

In the first case the links of u_7, u_9, u_4, u_8 and u_5 show that $u_5u_7u_9$, $u_4u_6u_9$, $u_4u_8u_9$, $u_4u_5u_8$, $u_2u_6u_8$, $u_3u_6u_8$, $u_2u_3u_5$ and $u_3u_5u_6$ are faces. Now, K is isomorphic, via the map $(0, 2)(1, 3)(5, 9, 8, 6, 7)$, to N_{10}.

In the second case $\mathrm{Lk}(u_4)$ shows that $u_4u_6u_9 \notin K$. The links of u_7, u_9, u_4, u_8 and u_6 now show that $u_3u_5u_7$, $u_4u_5u_9$, $u_6u_8u_9$, $u_2u_5u_8$, $u_2u_3u_6$, $u_3u_4u_8$, $u_4u_6u_8$ and $u_3u_5u_6$ are faces. Here, K is isomorphic, via the map $(0, 4, 5, 1, 7, 9, 8, 6, 2, 3)$, to N_{10}.

Subcase B: $u_3u_7u_8 \in K$. If $u_2u_3u_7 \in K$, the links of u_7, u_2 and u_3 show that $u_2u_7u_9$, $u_4u_7u_9$, $u_4u_5u_7$, $u_3u_5u_6$, $u_3u_5u_9$ and $u_3u_4u_6$ are faces. This implies that $C_6(u_3, u_4, u_1, u_7, u, u_5)$ is in $\mathrm{Lk}(u_6)$. It is easy to see that $C_4(u_7, u_8, u_1, u_9)$ is in $\mathrm{Lk}(u_3)$ if $u_3u_7u_9 \in K$. Hence, from $\mathrm{Lk}(u_7)$, $u_3u_4u_7$ and $u_2u_7u_9$ are faces.

The link of u_3 shows that $u_3u_5u_6$, $u_2u_3u_5$ and $u_3u_6u_9 \in K$. (If $u_2u_3u_6$ and $u_3u_5u_9 \in K$, the links of u_6, u_8, u_7 and u_2 show that $u_6u_8u_9$, $u_2u_4u_8$, $u_2u_5u_7$, $u_4u_7u_9$, $u_2u_5u_8$ and $u_4u_6u_8 \in K$. Here, $C_4(u_8, u_2, u_1, u_6)$ is in $\mathrm{Lk}(u_4)$.) The links of u_6, u_9, u_7, u_4 and u_8 show that $u_4u_6u_8$, $u_2u_6u_8$, $u_2u_6u_9$, $u_4u_5u_9$, $u_4u_8u_9$, $u_5u_7u_9$, $u_2u_4u_7$ and $u_2u_5u_8$ are faces. Here, K is isomorphic, via the map $(0, 9, 5, 2)(1, 8, 3)(4, 7, 6)$, to N_{10}.

Subcase C: $u_4u_7u_8 \in K$. $\mathrm{Lk}(u_7)$ shows that one of $u_2u_4u_7$, $u_3u_4u_7$ or $u_4u_7u_9$ is a face.

C.1: $u_2u_4u_7 \in K$. The links of u_2 and u_7 show that $u_2u_7u_9$, $u_3u_7u_9$ and $u_3u_5u_9 \in K$. Since $u_4u_9 \notin \mathrm{Lk}(u_2)$ and $\mathrm{Lk}(u_9)$, the links of u_9, u_4, u_3, u_6 and u_5 show that $u_4u_5u_9$, $u_4u_6u_9$, $u_3u_4u_8$, $u_2u_3u_6$, $u_3u_5u_6$, $u_2u_6u_8$, $u_6u_8u_9$, $u_2u_5u_8$ and $u_2u_5u_9$ are faces. In this case, K is isomorphic, via the map $(0, 7, 2, 8, 5, 1, 4, 6, 9, 3)$, to N_{10}.

C.2: $u_3u_4u_7 \in K$. It is easy to see from $\mathrm{Lk}(u_4)$ that $u_4u_5u_9 \in K$ (since $u_4u_5u_8 \notin K$ from $\mathrm{Lk}(u_8)$). The links of u_7, u_3, u_8, u_5 and u_6 show that $u_2u_5u_7$, $u_2u_7u_9$, $u_3u_7u_9$, $u_2u_3u_5$, $u_3u_5u_6$, $u_3u_6u_8$, $u_5u_8u_9$, $u_2u_6u_8$, $u_2u_4u_8$, $u_2u_6u_9$ and $u_4u_6u_9$ are faces. In this case, K is isomorphic, via the map $(0, 5, 3, 1, 2, 8)(4, 7, 6, 9)$, to N_9.

C.3: $u_4u_7u_9 \in K$. The link of (u_7) shows that $u_3u_5u_7$, $u_2u_3u_7$ and $u_2u_7u_9 \in K$ (if $u_3u_7u_9 \in K$, the links of u_7, u_3 and u_8 show that $u_3u_4u_5 \in K$). Since $u_3u_5 \notin \mathrm{Lk}(u_8)$, either $u_3u_5u_6$ or $u_3u_5u_9 \in K$. If $u_3u_5u_6 \in K$, the links of u_5 and u_2 show that $u_2u_4u_6 \in K$, which implies that $C_3(u_2, u_1, u_6)$ is in $\mathrm{Lk}(u_4)$. Hence, $u_3u_5u_9 \in K$.

The links of u_3, u_2, u_4, u_5 and u_6 show that $u_3u_4u_6$, $u_3u_6u_8$, $u_2u_4u_8$, $u_4u_5u_9$, $u_2u_5u_6$, $u_2u_5u_8$, $u_2u_6u_9$ and $u_6u_8u_9 \in K$. Now, K is isomorphic, via the map $(0, 8, 7, 4, 3, 1, 9)(2, 5)$, to N_{10}.

2.2.1.2.2: $u_1u_6u_8$ and $u_1u_3u_7$ are faces. In this case, using the above method, K is isomorphic to one of N_5, N_6, N_7, N_9, N_{10} or N_{12}.

2.2.2: $u_1u_4u_5 \in K$. In this case, K is isomorphic to N_{11}.

2.2.3: $u_1u_4u_7 \in K$. In this case, K is isomorphic to N_4, N_5, N_7, $N_{10}, \dots, N_{12}$ or N_{14}.

2.2.4: $u_1u_4u_8 \in K$. In this case, K is isomorphic to one of N_4, N_7, N_{10}, N_{11} or N_{14}.

Subcase 2.3: $u_1u_5u_9 \in K$. In this case, K is isomorphic to one of N_1, $N_4, \dots, N_{14}$.

Subcase 2.4: $u_1u_6u_9 \in K$. In this case, K is isomorphic to one of N_1, $N_3, \dots, N_{14}$.

The theorem now follows from Lemma 5. □

Proof of Theorem 7. Let K be an n-vertex ($n \leq 11$) $\{3, q\}$-equivelar polyhedron. If $\chi(K) > 0$, then, by Corollary 3, K is isomorphic to S^2_4, O or $\mathbb{R}P^2_6$.

If $\chi(K) \leq 0$, then, by (2), $q \geq 6$ and hence $n \geq 7$. From (1), nq is divisible by 6. So, $(n, q) = (7, 6)$, $(8, 6)$, $(9, 6)$, $(9, 8)$, $(10, 6)$, $(10, 9)$ or $(11, 6)$.

If $q = 6$, then $\chi(K) = 0$ and hence, by Theorem 4, K is isomorphic to $T_7, \ldots, T_{11}$, $A_{3,3}$, $B_{3,3}$ or Q. Since, the non-edge graphs of T_9, $A_{3,3}$ and $B_{3,3}$ are pairwise non-isomorphic, T_9, $A_{3,3}$ and $B_{3,3}$ are pairwise non-isomorphic. Observe that $\mathrm{NEG}(Q)$ is a bipartite graph. As $\mathrm{NEG}(T_{10})$ contains an induced pentagon, it is therefore not isomorphic to $\mathrm{NEG}(Q)$. Hence $Q \ncong T_{10}$. Thus, all these eight polyhedra are distinct (non-isomorphic).

If $(n, q) = (9, 8)$, then, by Theorem 5, K is isomorphic to M_1 or M_2. Moreover, $M_1 \ncong M_2$.

If $(n, q) = (10, 9)$, then, by Theorem 6, K is isomorphic to $N_1, \ldots, N_{14}$. Moreover, $N_i \ncong N_j$ for $1 \leq i \neq j \leq 14$. This completes the proof. □

Acknowledgments

The authors thank Bhaskar Bagchi for the idea to calculate the characteristic polynomials of adjacency matrices to prove Lemmas 4 and 5 and for making them aware of several important papers in the literature related to this work. The authors are thankful to C. R. Pranesachar for useful conversations. Finally, the authors thank an anonymous referee for many useful comments which led to substantial improvements in the presentation of this paper and for making them aware of several important papers in the literature related to this work. Remark 4 is also due to the referee.

References

[1] B. Bagchi and B. Datta, A structure theorem for pseudomanifolds, *Discrete Math*. **168** (1998), 41–60.
[2] J. A. Bondy and U. S. R. Murty, *Graph Theory with Applications*, Elsevier North-Holland, Amsterdam, 1976.
[3] U. Brehm, Polyhedral maps with few edges, in *Topics in Combinatorics and Graph Theory* (*Ringel-Fetstschrift*) (eds. R. Bodendiek and R. Henn), Physica-Verlag, Heidelberg, 1990, pp. 153–162.
[4] U. Brehm and E. Schulte, Polyhedral maps, in *Handbooks of Discrete and Computational Geometry* (eds. J. E. Goodman and J. O'Rourke), CRC Press, Boca Raton, FL, 1997, pp. 345–358.
[5] U. Brehm and J. M. Wills, Polyhedral manifolds, in *Handbook of Convex Geometry* (eds. P. M. Gruber and J. M. Wills), Elsevier, Dordrecht, 1993, pp. 535–554.
[6] H. S. M. Coxeter, *Regular Polytopes*, Dover, New York, 1973.
[7] H. S. M. Coxeter and W. O. J. Moser, *Generators and Relations for Discrete Groups*, Springer-Verlag, New York, 1972.
[8] L. Danzer and E. Schulte, Reguläre Inzidenzkomplexe I, *Geom. Dedicata* **13** (1982), 295–308.
[9] B. Datta, Two dimensional weak pseudomanifolds on seven vertices, *Bol. Soc. Mat. Mexicana* **5** (1999), 419–426.
[10] B. Datta and N. Nilakantan, Two dimensional weak pseudomanifolds on 8 vertices (preprint).
[11] A. L. Edmonds, J. H. Ewing and R. S. Kulkarni, Regular tessellations of surfaces and $(p, q, 2)$-triangle groups, *Ann. of Math*. **116** (1982), 113–132.
[12] B. Grünbaum, *Convex Polytopes*, Interscience, New York, 1967.
[13] M. Jungerman and G. Ringel, Minimal triangulations on orientable surfaces, *Acta Math*. **145** (1980), 121–154.
[14] W. Kühnel, Triangulations of manifolds with few vertices, in *Advances in Differential Geometry and Topology* (ed. F. Tricerri), World Scientific, Singapore, 1990, pp. 59–114.

[15] W. Magnas, *Noneuclidean Tesselations and Their Groups*, Academic Press, New York, 1974.
[16] P. McMullen, B. Monson and A. Ivic Weiss, Regular maps constructed from linear groups, *European J. Combin*. **14** (1993), 541–552.
[17] P. McMullen and E. Schulz, Finite quotients of infinite universal polytopes, in *Discrete and Computational Geometry* (eds. J. Goodman, R. Pollack and W. Steiger), DIMACS Series in Discrete Mathematics and Theoretical Computer Science, vol. 6, American Mathematical Society, Providence, RI, 1991, pp. 231–236.
[18] P. McMullen, Ch. Schulz and J. M. Wills, Equivelar polyhedral manifolds in E^3, *Israel J. Math*. **41** (1982), 331–346.
[19] P. McMullen, Ch. Schulz and J. M. Wills, Polyhedral 2-manifolds in E^3 with unusually large genus, *Israel J. Math*. **46** (1983), 127–144.
[20] P. McMullen, Ch. Schulz and J. M. Wills, Two remarks on equivelar manifolds, *Israel J. Math*. **52** (1985), 28–32.
[21] G. Ringel, Bestimmung der Maximalzahl der Nachbargebiete auf nichtorientierbaren Flächen, *Math. Ann*. **127** (1954), 181–214.
[22] G. Ringel, Wie man die geschlossenen nichtorientierbaren Flächen in möglichst wenig Dreiecke zerlegen kann, *Math. Ann*. **130** (1955), 317–326.
[23] G. Ringel, *Map Color Theorem*, Springer-Verlag, Berlin, 1974.
[24] E. Schulte, Reguläre Inzidenzkomplexe II & III, *Geom. Dedicata* **14** (1983), 33–56 & 57–79.
[25] A. Vince, Regular combinatorial maps, *J. Combin. Theory Ser. B* **35** (1983), 256–277.

Received February 14, 2000, *and in revised form August* 15, 2000. *Online publication March* 26, 2001.