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ERGODIC CONDITIONS
AND SPECTRAL PROPERTIES
FOR A-CONTRACTIONS

Abstract. In this paper the canonical representation of an A-contraction T on a Hilbert space $\mathcal{H}$ is used to obtain some conditions concerning the concept of A-ergodicity studied in [14–17]. The regular case and the case of $\mathcal{R}(A)$ closed are considered, and specifically, the TT^*-contractions are studied. Some spectral properties are also given for certain particular class of A-isometries.

1. PRELIMINARIES

Let $\mathcal{H}$ be a complex Hilbert space and $\mathcal{B}(\mathcal{H})$ be the C^*-algebra of all bounded linear operators on $\mathcal{H}$. For $T \in \mathcal{B}(\mathcal{H})$ we denote the range and the kernel of T by $\mathcal{R}(T)$ and $\mathcal{N}(T)$, respectively. We also write $\sigma(T), \sigma_a(T)$ and $\sigma_p(T)$ to designate the spectrum, the approximate point spectrum and the point spectrum of T, respectively.

An operator T is *quasinilpotent* if $\sigma(T) = \{0\}$, and T is *nilpotent* if $T^n = 0$ for some integer $n \geq 2$.

The operator T is *strongly stable* if $\{T^n\}$ converges to zero in the strong operator topology of $\mathcal{B}(\mathcal{H})$. In this case, T is a *power bounded operator*, that is it satisfies the condition

$$\sup_{n \geq 1} \|T^n\| < \infty.$$

For a (closed) subspace $\mathcal{M} \subset \mathcal{H}$, by $P_{\mathcal{M}} \in \mathcal{B}(\mathcal{M})$, we denote the orthogonal projection associated to $\mathcal{M}$. We also use $P_{\mathcal{H},\mathcal{M}} \in \mathcal{B}(\mathcal{H}, \mathcal{M})$ to denote the corresponding projection from $\mathcal{H}$ onto $\mathcal{M}$.

If $T \in \mathcal{B}(\mathcal{H})$ and $S \in \mathcal{B}(\mathcal{K})$ with $\mathcal{H} \subset \mathcal{K}$, then S is called a *lifting* of T if there is $P_{\mathcal{K},\mathcal{H}}S = TP_{\mathcal{K},\mathcal{H}}$, that is $P_{\mathcal{K},\mathcal{H}}$ intertwines S with T. Further, S is an *extension* of T

if $P_{\mathcal{K},\mathcal{H}}^*$ (the canonical embedding of $\mathcal{H}$ into $\mathcal{K}$) intertwines T with S. Clearly, S is a lifting of T if and only if S^* is an extension of T^*.

An operator T is *quasinormal* if T and T^*T commute, and T is *hyponormal* if $TT^* \leq T^*T$. T is *n-hyponormal* for some (integer) $n \geq 2$ if $(TT^*)^n \leq (T^*T)^n$, and T is *∞-hyponormal* if T is n-hyponormal for all $n \geq 1$. A quasinormal operator is ∞-hyponormal (see [10]).

All classes of operators mentioned above are closely related to that of contractions, that is the family of such $T \in \mathcal{B}(\mathcal{H})$ that $T^*T \leq I$, where T^* is the adjoint operator of T and $I = I_{\mathcal{H}}$ is the identity operator on $\mathcal{H}$. The contractions and their different generalizations have been intensively studied recently (see, for instance, [2–9]). In [14–18] there was considered a larger class of operators which generalize the contractions; their structure and ergodic properties were studied there. We refer to such operators below.

Let $A, T \in \mathcal{B}(\mathcal{H})$, $A \neq 0$ being a positive operator. The operator T is called an *A-contraction* if it satisfies the inequality

$$T^*AT \leq A. \tag{1.1}$$

T is an *A-isometry* if the equality holds in (1.1).

The case of $A = I$ in (1.1) leads to the ordinary contractions and isometries, respectively.

From (1.1), it is clear that one can define a contraction $\widehat{T}$ on $\overline{\mathcal{R}(A)} = \overline{\mathcal{R}(A^{1/2})}$ ($A^{1/2}$ being the square root of A) by

$$\widehat{T}A^{1/2}h = A^{1/2}Th \quad (h \in \mathcal{H}). \tag{1.2}$$

The contraction $\widehat{T}$ plays an important role in the study of A-contractions. Obviously, $\widehat{T}$ is an isometry if and only if T is an A-isometry.

When $A = T^{*m}T^m$ in (1.1) for some integer $m \geq 1$, that is T is a $T^{*m}T^m$-contraction, then T is called an *m-quasicontraction*. A $T^{*m}T^m$-isometry is called an *m-quasi-isometry*. If this happens for $m = 1$, then we briefly say that T is a *quasicontraction* (respectively, a *quasi-isometry*) (see [2, 9, 11, 12]). In particular, an m-nilpotent operator is an m-quasi-isometry.

An operator T on $\mathcal{H}$ is *orthogonally mean ergodic* if

$$\lim_{n \to \infty} \frac{1}{n} \sum_{j=0}^{n-1} T^j h = P_T h \quad (h \in \mathcal{H}) \tag{1.3}$$

where $P_T = P_{\mathcal{N}(I-T)}$. Clearly, in this case,

$$\mathcal{H} = \overline{\mathcal{R}(I-T)} \oplus \mathcal{N}(I-T), \tag{1.4}$$

and if T is power bounded, then (1.4) also implies (1.3).

It is well known that any contraction is orthogonally mean ergodic, but not any power bounded operator is.

In the case that there exists limit (1.3) and P_T is only a projection (non necessary orthogonal) onto $\mathcal{N}(I - T)$ with $\mathcal{N}(P_T) = \overline{\mathcal{R}(I - T)}$, the operator T is called *Cesàro ergodic*. For instance, the power bounded operators are Cesàro ergodic ([1, 19]).

In [15–17] a concept of ergodicity was introduced in the context of A-contractions and different results were obtained.

In this paper we give some ergodic conditions for A-contractions, involving certain related null-spaces. Two important cases are considered, when either $\mathcal{R}(A)$ is closed or T is A-regular which means that $AT = A^{1/2}TA^{1/2}$. As an application, we study the TT^*-contractions for $T \in \mathcal{B}(\mathcal{H})$, and we find their unitary (isometric) parts, thus generalizing a result from [8]. We also show that TT^*-isometries and the contractive quasi-isometries are ∞-hyponormal, improving in the last case a result from [11, 12], and we prove that such operators are similar to partial isometries.

Finally, we obtain some spectral properties of m-isometries, TT^*-isometries and some more general A-contractions T, concerning the subsets $\sigma_p(T)$ and $\sigma_a(T)$. Essentially, we see that these subsets are invariant with respect to complex conjugation, under the involution map $T \longmapsto T^*$.

2. NULL-SPACES AND ERGODIC CONDITIONS

Let $A, T \in \mathcal{B}(\mathcal{H})$ with $A \geq 0$ and such that $\mathcal{N}(A)$ is an invariant subspace for T. Then T has the following matrix representation with respect to the decomposition $\mathcal{H} = \overline{\mathcal{R}(A)} \oplus \mathcal{N}(A)$:

$$T = \begin{pmatrix} T_0 & 0 \\ T_1 & T_2 \end{pmatrix} \tag{2.1}$$

where $T_0^* = T^*|_{\overline{\mathcal{R}(A)}}$, $T_2 = T|_{\mathcal{N}(A)}$ and $T_1 = P_{\mathcal{N}(A)}T|_{\overline{\mathcal{R}(A)}}$.

We use representation (2.1) when T is an A-contraction, to obtain some relations between certain null-subspaces which appear in the A-ergodicity context (see [15–17]).

Lemma 2.1. *Let A, T be as above and $A_0 = A|_{\overline{\mathcal{R}(A)}}$. Then T is an A-contraction (A-isometry) if and only if T_0 is an A_0-contraction (A_0-isometry), T_0 being as in (2.1).*

When T is an A-contraction, then $A_0^{1/2}T_0 = \widehat{T}A_0^{1/2}$, and T_0 is similar to $\widehat{T}$ by $A_0^{1/2}$ if $\mathcal{R}(A)$ is closed, $\widehat{T}$ being as in (1.2).

Proof. Since $A = A_0 \oplus 0$ on $\mathcal{H} = \overline{\mathcal{R}(A)} \oplus \mathcal{N}(A)$, it follows that $T^*AT = T_0^*A_0T_0 \oplus 0$ with respect to the same decomposition of $\mathcal{H}$. So $T^*AT \leq A$ (respectively, $T^*AT = A$) if and only if $T_0^*A_0T_0 \leq A_0$ (respectively, $T_0^*A_0T_0 = A_0$). This yields the first assertion of the lemma.

Suppose now that $T^*AT \leq A$ and let $\widehat{T}$ be the contraction on $\overline{\mathcal{R}(A)}$ defined by $\widehat{T}A^{1/2}h = A^{1/2}Th$, $h \in \mathcal{H}$. Then

$$\widehat{T}A_0^{1/2} = (A^{1/2}T)|_{\overline{\mathcal{R}(A)}} = A_0^{1/2}T_0,$$

because it is easy to see that $A^{1/2}T = A_0^{1/2}T_0 \oplus 0$ on $\mathcal{H} = \overline{\mathcal{R}(A)} \oplus \mathcal{N}(A)$.

When $\mathcal{R}(A)$ is closed, then $\mathcal{R}(A) = \mathcal{R}(A^{1/2})$ and A_0 is invertible in $\mathcal{B}(\mathcal{R}(A))$, hence the above relation between T_0 and $\widehat{T}$ shows that T_0 is similar to $\widehat{T}$ by the invertible operator $A_0^{1/2}$. $\hspace{8cm}\square$

Denote $\mathcal{N} = \mathcal{N}(A^{1/2} - A^{1/2}T)$ and $\mathcal{N}_* = \mathcal{N}(A^{1/2} - T^*A^{1/2})$, and $A_0 = A|_{\overline{\mathcal{R}(A)}}$.

Proposition 2.2. *Let T be an A-contraction on $\mathcal{H}$ with representation (2.1). Then the following relations hold:*

$$\mathcal{N} = \mathcal{N}(A - AT) = \mathcal{N}(A - T^*A) = \mathcal{N}(I - T_0) \oplus \mathcal{N}(A), \qquad (2.2)$$

$$\mathcal{N}_* = \mathcal{N}(I - \widehat{T}) \oplus \mathcal{N}(A), \qquad (2.3)$$

$$\mathcal{N}(I - T_0) = \mathcal{N}((I - \widehat{T})A_0^{1/2}) = (A_0^{1/2})^{-1}\mathcal{N}(I - \widehat{T}), \qquad (2.4)$$

$$\mathcal{N}(I - \widehat{T}) = \mathcal{N}((I - T_0^*)A_0^{1/2}) = (A_0^{1/2})^{-1}\mathcal{N}(I - T_0^*), \qquad (2.5)$$

$$A^{1/2}\mathcal{N} = A^{1/2}\mathcal{N}(I - T_0) = \mathcal{R}(A^{1/2}) \cap \mathcal{N}(I - \widehat{T}), \qquad (2.6)$$

$$A^{1/2}\mathcal{N}_* = A^{1/2}\mathcal{N}(I - \widehat{T}) = \mathcal{R}(A^{1/2}) \cap \mathcal{N}(I - T_0^*) =$$
$$= \mathcal{R}(A^{1/2}) \cap \mathcal{N}(I - T^*). \qquad (2.7)$$

Proof. From representation (2.1) we infer that $A - AT = (A_0 - A_0T_0) \oplus 0$ and so $A^{1/2} - A^{1/2}T = (A_0^{1/2} - A_0^{1/2}T_0) \oplus 0$ on $\mathcal{H} = \overline{\mathcal{R}(A)} \oplus \mathcal{N}(A)$. Therefore,

$$\mathcal{N} = \mathcal{N}(A^{1/2} - A^{1/2}T) = \mathcal{N}(A_0^{1/2} - A_0^{1/2}T_0) \oplus \mathcal{N}(A) =$$
$$= \mathcal{N}(I - T_0) \oplus \mathcal{N}(A) = \mathcal{N}(A_0 - A_0T_0) \oplus \mathcal{N}(A) = \mathcal{N}(A - AT),$$

because A_0 (and also $A_0^{1/2}$) is injective on $\overline{\mathcal{R}(A)}$. This gives the two equalities in (2.2), and it remains to prove that $\mathcal{N} = \mathcal{N}(A - T^*A)$. But $A - T^*A = (A_0 - T_0^*A_0) \oplus 0$ and by Lemma 2.1 there is $T_0^*A_0^{1/2} = A_0^{1/2}\widehat{T}^*$ and so $T_0^*A_0 = A_0^{1/2}\widehat{T}^*A_0^{1/2}$. Hence we obtain

$$\mathcal{N}(A - T^*A) = \mathcal{N}(A_0 - T_0^*A_0) \oplus \mathcal{N}(A) =$$
$$= \mathcal{N}(A_0^{1/2} - \widehat{T}^*A_0^{1/2}) \oplus \mathcal{N}(A) =$$
$$= \mathcal{N}(A_0^{1/2} - \widehat{T}A_0^{1/2}) \oplus \mathcal{N}(A) =$$
$$= \mathcal{N}(A_0^{1/2} - A_0^{1/2}T_0) \oplus \mathcal{N}(A) = \mathcal{N},$$

because $\mathcal{N}(I - \widehat{T}^*) = \mathcal{N}(I - \widehat{T})$, $\widehat{T}$ being a contraction. Thus all equalities in (2.2) are proved.

Now from the above remark there follows

$$A^{1/2} - T^*A^{1/2} = (A_0^{1/2} - T_0^*A_0^{1/2}) \oplus 0,$$

whence we infer

$$\mathcal{N}_* = \mathcal{N}(A_0^{1/2} - T_0^*A_0^{1/2}) \oplus \mathcal{N}(A) = \mathcal{N}(A_0^{1/2} - A_0^{1/2}\widehat{T}^*) \oplus \mathcal{N}(A) =$$
$$= \mathcal{N}(I - \widehat{T}^*) \oplus \mathcal{N}(A) = \mathcal{N}(I - \widehat{T}) \oplus \mathcal{N}(A),$$

that is relation (2.3).

In passing we also proved

$$\mathcal{N}(I - T_0) = \mathcal{N}(A_0^{1/2} - A_0^{1/2}T_0) = \mathcal{N}(A_0^{1/2} - \widehat{T}A_0^{1/2}) = (A_0^{1/2})^{-1}\mathcal{N}(I - \widehat{T})$$

and

$$\mathcal{N}(I - \widehat{T}) = \mathcal{N}(A_0^{1/2} - A_0^{1/2}\widehat{T}^*) = \mathcal{N}(A_0^{1/2} - T_0^*A_0^{1/2}) = (A_0^{1/2})^{-1}\mathcal{N}(I - T_0^*),$$

that is relations (2.4) and (2.5).

Next, from (2.4) and (2.5), we infer, respectively

$$A^{1/2}\mathcal{N} = A^{1/2}\mathcal{N}(I - T_0) = \mathcal{R}(A^{1/2}) \cap \mathcal{N}(I - \widehat{T}),$$

and

$$A^{1/2}\mathcal{N}_* = A^{1/2}\mathcal{N}(I - \widehat{T}) = \mathcal{R}(A^{1/2}) \cap \mathcal{N}(I - T_0^*) = \mathcal{R}(A^{1/2}) \cap \mathcal{N}(I - T^*)$$

because $T_0^* = T^*|_{\overline{\mathcal{R}(A)}}$. Thus relations (2.6) and (2.7) also hold. $\square$

Theorem 2.3. *For an A-contraction T on $\mathcal{H}$ having the matrix representation* (2.1), *the following statements are equivalent:*

(i) $\mathcal{N}(I - T_0) = \mathcal{N}(I - \widehat{T})$ *(equivalently, $\mathcal{N} = \mathcal{N}_*$);*
(ii) $\mathcal{N}(I - \widehat{T}) \subset \mathcal{N}(I - T_0)$ *($\mathcal{N}_* \subset \mathcal{N}$);*
(iii) $A^{1/2}\mathcal{N}(I - \widehat{T}) \subset \mathcal{N}(I - \widehat{T})$ *($A^{1/2}\mathcal{N}_* \subset \mathcal{N}_*$);*
(iv) $A^{1/2}\mathcal{N}(I - \widehat{T}) \subset \mathcal{N}(I - T_0)$ *($A^{1/2}\mathcal{N}_* \subset \mathcal{N}$);*

If these conditions are satisfied then also:

(v) $A^{1/2}\mathcal{N}(I - T_0) \subset \mathcal{N}(I - T_0)$ *($A^{1/2}\mathcal{N} \subset \mathcal{N}$),*

or equivalently

(vi) $A\mathcal{N}(I - T_0) \subset \mathcal{N}(I - \widehat{T})$ *($A\mathcal{N} \subset \mathcal{N}_*$).*

Proof. Clearly, (i) implies (ii), and (ii) implies (iii) by relation (2.4).

Suppose now that inclusion (iii) holds. So $A\mathcal{N}(I - \widehat{T}) \subset \mathcal{N}(I - \widehat{T})$ and using also (2.4) we infer that $A^{1/2}\mathcal{N}(I - \widehat{T}) \subset \mathcal{N}(I - T_0)$ that is (iv). Thus we proved that (iii) implies (iv).

Next, inclusion from (iv) and relation (2.6) lead to the inclusion $A\mathcal{N}(I - T_0) \subset \mathcal{N}(I - T_0)$. Hence (iv) implies (v), and obviously, (v) implies (vi) by relation (2.4).

Conversely, by (2.4), inclusion (vi) means that

$$A^{1/2}\mathcal{N}(I - T_0) \subset (A_0^{1/2})^{-1}\mathcal{N}(I - \widehat{T}) = \mathcal{N}(I - T_0),$$

that is inclusion (v), and so relations (v) and (vi) are equivalent.

It is clear that any of inclusions (i)–(iv) is equivalent to the corresponding one from the bracket.

Let us now assume inclusion (iv). Then by (2.4) we obtain

$$A\mathcal{N}(I - \widehat{T}) \subset A^{1/2}\mathcal{N}(I - T_0) \subset \mathcal{N}(I - \widehat{T}),$$

whence
$$A^{1/2}\mathcal{N}(I - \widehat{T}) \subset \mathcal{R}(A^{1/2}) \cap \mathcal{N}(I - \widehat{T}) = A^{1/2}\mathcal{N}(I - T_0).$$

By the injectivity of $A^{1/2}$ on $\overline{\mathcal{R}(A)}$, this means that $\mathcal{N}(I - \widehat{T}) \subset \mathcal{N}(I - T_0)$, which gives $\mathcal{N}_* \subset \mathcal{N}$.

Let P, P_* be the orthogonal projections in $\mathcal{B}(\mathcal{H})$ onto $\mathcal{N}, \mathcal{N}_*$ respectively. Since assumption (iv) implies (vi) (by the above remark), then $A\mathcal{N} \subset \mathcal{N}_*$ and so $P_*AP = AP$. On the other hand, because $\mathcal{N}_* \subset \mathcal{N}$, there is $A(\mathcal{H} \ominus \mathcal{N}) \subset \mathcal{H} \ominus \mathcal{N}_*$; therefore

$$P_*A = P_*AP + P_*A(I - P) = P_*AP = AP.$$

But (iv) also implies (v), that is $A\mathcal{N} \subset \mathcal{N}$, which means that $\mathcal{N}$ reduces A, hence $AP = PA$. Thus $P_*A = PA$ that is $P_* = P$ on $\overline{\mathcal{R}(A)}$. As $\mathcal{N}(A) \subset \mathcal{N} \cap \mathcal{N}_*$ there also is $P_* = I = P$ on $\mathcal{N}(A)$. Consequently, $P_* = P$ or, equivalently, $\mathcal{N}_* = \mathcal{N}$. We conclude that (iv) implies (i), which ends the proof. $\square$

Remark 2.4. According to [15, 17], equivalent conditions (i)–(iv) mean that T is an *ergodic A-contraction*, or briefly that T is *A-ergodic*. In fact, this means that the following limits

$$\lim_{n \to \infty} \frac{1}{n} \sum_{j=0}^{n-1} A^{1/2}T^j h = P_*A^{1/2}h \quad (h \in \mathcal{H})$$

and

$$\lim_{n \to \infty} \frac{1}{n} \sum_{j=0}^{n-1} T^{*j}A^{1/2}h = A^{1/2}P_*h \quad (h \in \mathcal{H})$$

are equal, that is $P_*A^{1/2} = A^{1/2}P_* = A^{1/2}P$, P_* and P being as in the above proof. These limits always exist, but in general they are different (see [15, 17]).

By this remark, condition (i) from Theorem 2.3 says that T is A-ergodic if and only if T_0 is A_0-ergodic (A_0 being injective). So, the case of A injective deserves a special attention. Also, in certain cases conditions (i)–(iv) from Theorem 2.3 are also equivalent to (v) and (vi), respectively, as we will see this in the sequel.

Corollary 2.5. *Let T be an A-contraction on $\mathcal{H}$ with the range $\mathcal{R}(A)$ closed. Then T is A-ergodic if and only if $\mathcal{N}$ reduces A.*

Proof. Since (iv) implies (v) in Theorem 2.3, it remains to prove the converse implication. If $A\mathcal{N} \subset \mathcal{N}$ or, equivalently, $A^{1/2}\mathcal{N} \subset \mathcal{N}$, then by (2.6) and the assumed closedness of $\mathcal{R}(A)$ (hence $\mathcal{R}(A) = \mathcal{R}(A^{1/2})$) we obtain $\mathcal{N}(I - \widehat{T}) \subset \mathcal{N}(I - T_0)$, that is $\mathcal{N}_* \subset \mathcal{N}$. By Theorem 2.3, the last inclusion just means that $\mathcal{N}_* = \mathcal{N}$; consequently, T is A-ergodic (by Remark 2.4). $\square$

Corollary 2.6. *Let T be an A-contraction on $\mathcal{H}$ with an injective operator A. If $A\mathcal{N} \subset \mathcal{N}$, then*

$$\mathcal{N}(I - T) \subset \mathcal{N}(I - T^*). \tag{2.8}$$

Proof. Since $\mathcal{N}(A) = \{0\}$, then $\mathcal{N} = \mathcal{N}(I - T)$. So the inclusion $A\mathcal{N} \subset \mathcal{N}$ becomes $A\mathcal{N}(I - T) \subset \mathcal{N}(I - T)$, which implies $\overline{A\mathcal{N}(I - T)} = \mathcal{N}(I - T)$, because A is positive and injective. On the other hand, by (2.6) and (2.7), we get $A\mathcal{N}(I - T) \subset A^{1/2}\mathcal{N}_* \subset \mathcal{N}(I - T^*)$, and by the above remark we obtain inclusion (2.8). $\qquad\square$

Even if T is A-ergodic (which implies $A\mathcal{N} \subset \mathcal{N}$), we cannot obtain the equality in (2.8); hence in general, T may be A-ergodic without being orthogonally mean ergodic. But, in the case of Cesàro ergodicity, the following holds true.

Corollary 2.7. *Let T be an A-contraction on $\mathcal{H}$ with A injective and such that T is Cesàro ergodic. Then the following statements are equivalent:*

(i) *T is A-ergodic;*
(ii) *$A\mathcal{N} \subset \mathcal{N}$;*
(iii) *$\mathcal{N}(I - T) = \mathcal{N}(I - T^*)$;*
(iv) *T is orthogonally mean ergodic.*

Proof. Clearly, (i) implies (ii) by Theorem 2.3. Now, by Corollary 2.6 and the assumption that T is Cesàro ergodic (which assures that $\mathcal{H} = \overline{\mathcal{R}(I - T)} + \mathcal{N}(I - T)$ as a direct sum), one infers that (ii) implies (iii). Also, (iii) and (iv) are equivalent, because T is Cesàro ergodic. Finally, (iii) implies (i), since by (iii) and (2.7) there is

$$A^{1/2}\mathcal{N}_* \subset \mathcal{N}(I - T^*) = \mathcal{N},$$

which by Theorem 2.3 means that T is A-ergodic. $\qquad\square$

In particular, if T is a power bounded A-contraction (A non injective), then from the previous corollary and the above remark there follows that T is A-ergodic if and only if T_0 is orthogonally mean ergodic.

A more special case is mentioned in the following

Corollary 2.8. *Let T be an A-contraction on $\mathcal{H}$ with A an invertible operator. Then the following assertions are equivalent:*

(i) *T is A-ergodic;*
(ii) *$A\mathcal{N}(I - T) = \mathcal{N}(I - T)$;*
(iii) *$\mathcal{N}(I - T) = \mathcal{N}(I - T^*)$;*
(iv) *T is orthogonally mean ergodic;*
(v) *T^* is orthogonally mean ergodic;*
(vi) *T^* is A^{-1}-ergodic.*

Proof. The equivalences (i)–(iv) follow from Corollary 2.7 and the assumption that A is invertible, which assures that T is a power bounded operator and so T and T^* are Cesàro ergodic. Then statements (iii), (v) and (vi) are also equivalent since T^* is an A^{-1}-contraction on $\mathcal{H}$. To prove the last assertion, we use the fact that $T^*AT \leq A$, whence $A^{-1/2}T^*ATA^{-1/2} \leq I$, which means that $A^{1/2}TA^{-1/2}$ is a contraction, or equivalently $A^{1/2}TA^{-1}T^*A^{1/2} \leq I$, that is $TA^{-1}T^* \leq A^{-1}$. This ends the proof. $\qquad\square$

Next, we use representation (2.1) to obtain further properties of the subspaces $\mathcal{N}(I - T)$ and $\mathcal{N}(I - T^*)$.

Proposition 2.9. *Let T be an A-contraction on $\mathcal{H}$ having the representation* (2.1). *Then:*

(i) $\mathcal{N}(I - T_0) \cap \mathcal{N}(T_1) \oplus \mathcal{N}(I - T_2) \subset \mathcal{N}(I - T)$, *and the equality occurs if and only if*

$$\mathcal{N}(I - T) \subset \mathcal{N}(T_1) \oplus \mathcal{N}(I - T_2).$$

In particular, the equality in (i) *holds if*

$$\mathcal{R}(T_1) \cap \mathcal{R}(I - T_2) = \{0\}.$$

(ii) $\mathcal{N}(I - T_0^*) \oplus \mathcal{N}(T_1^*) \cap \mathcal{N}(I - T_2^*) \subset \mathcal{N}(I - T^*)$, *and the equality occurs if and only if*

$$\mathcal{N}(I - T^*) \subset \mathcal{N}(I - T_0^*) \oplus \mathcal{N}(T_1^*).$$

In particular, the equality in (ii) *holds if*

$$\mathcal{R}(I - T_0^*) \cap \mathcal{R}(T_1^*) = \{0\}.$$

Proof. Using (2.1), we get

$$I - T = \begin{pmatrix} I - T_0 & 0 \\ -T_1 & I - T_2 \end{pmatrix},$$

hence $(h, k) \in \mathcal{N}(I - T)$ if and only if $h \in \mathcal{N}(I - T_0)$ and $T_1 h = (I - T_2)k$, where $h \in \overline{\mathcal{R}(A)}$ and $k \in \mathcal{N}(A)$. This gives immediately the inclusion from (i). Furthermore, since there always is $\mathcal{N}(I - T) \cap \overline{\mathcal{R}(A)} \subset \mathcal{N}(I - T_0)$, it follows that

$$\mathcal{N}(I - T) = \mathcal{N}(I - T_0) \cap \mathcal{N}(T_1) \oplus \mathcal{N}(I - T_2)$$

if and only if $\mathcal{N}(I - T) \subset \mathcal{N}(T_1) \oplus \mathcal{N}(I - T_2)$. Clearly, if $\mathcal{R}(T_1) \cap \mathcal{R}(I - T_2) = \{0\}$ and $(h, k) \in \mathcal{N}(I - T)$, then $T_1 h = (I - T_2)k = 0$, so $h \in \mathcal{N}(T_1)$, $k \in \mathcal{N}(I - T_2)$ and the above equality for $\mathcal{N}(I - T)$ holds.

Now, by (2.1), there holds

$$I - T^* = \begin{pmatrix} I - T_0^* & -T_1^* \\ 0 & I - T_2^* \end{pmatrix},$$

whence we infer that $(h, k) \in \mathcal{N}(I - T^*)$ if and only if $(I - T_0^*)h = T_1^* k$ and $k \in \mathcal{N}(I - T_2^*)$. This obviously yields the inclusion from (ii). Since there always is $\mathcal{N}(I - T^*) \cap \mathcal{N}(A) \subset \mathcal{N}(I - T_2^*)$, we infer that

$$\mathcal{N}(I - T^*) = \mathcal{N}(I - T_0^*) \oplus \mathcal{N}(T_1^*) \cap \mathcal{N}(I - T_2^*)$$

if and only if $\mathcal{N}(I - T^*) \subset \mathcal{N}(I - T_0^*) \oplus \mathcal{N}(T_1^*)$. If $\mathcal{R}(I - T_0^*) \cap \mathcal{R}(T_1^*) = \{0\}$ and $(h, k) \in \mathcal{N}(I - T^*)$ then $(I - T_0^*)h = T_1^* k = 0$, hence the above equality for $\mathcal{N}(I - T^*)$ holds. The proof is finished. $\qquad\square$

Notice that, in general, $\mathcal{N}(I - T) \subset \mathcal{N}$, but $\mathcal{N}(I - T^*)$ does not contain in $\mathcal{N}_*$. We see the relation between $\mathcal{N}(I - T^*)$ and $\mathcal{N}_*$ in the following

Proposition 2.10. *For an A-contraction T on $\mathcal{H}$, the following statements are equivalent:*

(i) $\mathcal{N}(I - T^*) \subset \mathcal{N}$;
(ii) $\mathcal{N}(I - T^*) \subset \mathcal{N}_*$;
(iii) $A\mathcal{N}(I - T^*) \subset \mathcal{N}(I - T^*)$.

Moreover, if these conditions are satisfied, then T is A-ergodic.

Proof. Since by (2.2), there is $\mathcal{N} = \mathcal{N}(A - T^*A)$, condition (i) is equivalent to $\overline{\mathcal{R}(A - AT)} \subset \overline{\mathcal{R}(I - T)}$ and further to $A\mathcal{R}(I - T) \subset \mathcal{R}(I - T)$, which also means $A\mathcal{N}(I - T^*) \subset \mathcal{N}(I - T^*)$. Thus conditions (i) and (iii) are equivalent.

Now from (ii), by (2.7), we infer

$$A^{1/2}\mathcal{N}(I - T^*) \subset A^{1/2}\mathcal{N}_* \subset \mathcal{N}(I - T^*),$$

that is $A\mathcal{N}(I - T^*) \subset \mathcal{N}(I - T^*)$. Therefore (ii) implies (iii). Conversely, let us assume (iii) and let $(h, k) \in \mathcal{N}(I - T^*)$ with $h \in \overline{\mathcal{R}(A)}$, $k \in \mathcal{N}(A)$. Then by (2.7),

$$A^{1/2}h = A^{1/2}(h, k) \in \mathcal{R}(A^{1/2}) \cap \mathcal{N}(I - T^*) = A^{1/2}\mathcal{N}(I - \widehat{T}),$$

and since $A^{1/2}$ is injective on $\overline{\mathcal{R}(A)}$ it follows that $h \in \mathcal{N}(I - \widehat{T})$. Hence $(h, k) \in \mathcal{N}(I - \widehat{T}) \oplus \mathcal{N}(A) = \mathcal{N}_*$, and we have proved the inclusion $\mathcal{N}(I - T^*) \subset \mathcal{N}_*$. Consequently (iii) implies (ii) and, in fact, the two conditions are equivalent.

If conditions (i)-(iii) hold, by (2.7), we obtain

$$A^{1/2}\mathcal{N}_* \subset \mathcal{N}(I - T^*) \subset \mathcal{N}_*,$$

and by Theorem 2.3 we infer that T is A-ergodic. $\qquad\square$

Next we refer to a class of A-ergodic operators, namely to the A-regular ones. Recall that an A-contraction T on $\mathcal{H}$ is *regular* (briefly, T is *A-regular*) if $AT = A^{1/2}TA^{1/2}$.

Proposition 2.11. *For an A-contraction T on $\mathcal{H}$, with representation (2.1), the following statements are equivalent:*

(i) T *is A-regular;*
(ii) $\widehat{T}$ *is A_0-regular, where $A_0 = A|_{\overline{\mathcal{R}(A)}}$;*
(iii) T *is a lifting of $\widehat{T}$;*
(iv) $\widehat{T} = T_0$;
(v) $A_0^{1/2}\widehat{T} = \widehat{T}A_0^{1/2}$;
(vi) $AT = T_A A$, *where $T_A = \widehat{T} \oplus \{0\}$ on $\mathcal{H} = \overline{\mathcal{R}(A)} \oplus \mathcal{N}(A)$.*

Proof. From (i), for $h \in \mathcal{H}$, we obtain

$$ATh = A^{1/2}TA^{1/2}h = \widehat{T}Ah = A_0^{1/2}T_0A^{1/2}h$$

whence $\widehat{T}A^{1/2}h = A^{1/2}Th = T_0A^{1/2}h$. This gives $\widehat{T} = T_0$ and so (i) implies (iv). Also (iv) implies (iii), having in view the matrix (2.1).

Now assertion (iii) says that T^* is an extension of $\widehat{T}^*$, that is $T^*|_{\overline{\mathcal{R}(A)}} = \widehat{T}^*$. But from the definition of T_A in (vi), there follows $ATh = A^{1/2}\widehat{T}A^{1/2}h = A^{1/2}T_A A^{1/2}h$ for $h \in \mathcal{H}$, hence

$$AT = A^{1/2}T_A A^{1/2}. \tag{2.9}$$

Equivalently, $T^*A = A^{1/2}T_A^* A^{1/2}$ and since $T^*|_{\overline{\mathcal{R}(A)}} = \widehat{T}^* = T_A^*|_{\overline{\mathcal{R}(A)}}$, we infer that $\widehat{T}^*A_0 = A_0^{1/2}\widehat{T}^* A_0^{1/2}$. This means that $A_0\widehat{T} = A_0^{1/2}\widehat{T}A_0^{1/2}$, whence $A_0^{1/2}\widehat{T} = \widehat{T}A_0^{1/2}$, because $A_0^{1/2}$ is an injective operator. Thus (iii) implies (v), and since $\widehat{T}$ is a contraction, it follows that (v) implies (ii).

Next, assertion (ii) means $A_0\widehat{T} = A_0^{1/2}\widehat{T}A_0^{1/2}$ and this condition is equivalent to $A_0^{1/2}\widehat{T} = \widehat{T}A_0^{1/2}$. Then from (2.9), for $h \in \mathcal{H}$ there follows,

$$ATh = A^{1/2}T_A A^{1/2}h = A^{1/2}\widehat{T}A^{1/2}h = \widehat{T}Ah = T_A Ah,$$

hence $AT = T_A A$. Thus (ii) implies (vi). Also, (vi) implies (i) because by (vi) one has for $h \in \mathcal{H}$,

$$ATh = T_A Ah = \widehat{T}Ah = A^{1/2}TA^{1/2}h.$$

Finally, if T is A-regular then $(I - T^*)A = A^{1/2}(I - T^*)A^{1/2}$, which shows that $\mathcal{N}_* \subset \mathcal{N}$, that is T is A-ergodic. The proof is finished. $\qquad\square$

Remark 2.12. The operator T_A defined by condition (vi) above is the *Douglas contraction* ([3]) associated to an A-contraction T on $\mathcal{H}$. Our condition (vi) shows that T is A-regular iff T_A also satisfies the condition $AT = T_A A$.

Remark 2.13. From representation (2.1) of an operator T on $\mathcal{H}$ satisfying $T\mathcal{N}(A) \subset \mathcal{N}(A)$ it follows that T is a $P|_{\overline{\mathcal{R}(A)}}$-contraction iff T_0 is a contraction on $\overline{\mathcal{R}(A)}$, and in this case T is a lifting of a contraction. In particular, from Proposition 2.11 we infer that if T is A-regular then T is a $P|_{\overline{\mathcal{R}(A)}}$-contraction. Conversely, any operator T on $\mathcal{H}$ which is a lifting of a contraction C on a subspace $\mathcal{M} \subset \mathcal{H}$ is a regular $P_\mathcal{M}$-contraction. A particular case is mentioned in the following

Corollary 2.14. *Let T be an operator on $\mathcal{H}$, with representation (2.1) and $\|T_0\| \leq 1$. If $\sigma_p(T_0^*) \neq \emptyset$, then T is a P_λ-contraction for any $\lambda \in \sigma_p(T_0^*)$, where P_λ is the orthogonal projection onto $\mathcal{N}(T_0^* - \lambda I)$.*

Remark 2.15. If T is an A-contraction such that $\sigma_p(\widehat{T}^*) \neq \emptyset$, then $\sigma_p(\widehat{T}^*) \subset \sigma_p(T_0^*)$. Indeed, if $\lambda \in \sigma_p(\widehat{T}^*)$ and $0 \neq h \in \mathcal{H}$ such that $\widehat{T}^*h = \lambda h$, then $\lambda A_0^{1/2}h = A_0^{1/2}\widehat{T}^*h = T_0^* A_0^{1/2}h$ and $A_0^{1/2}h \neq 0$, hence $\lambda \in \sigma_p(T_0^*)$. Thus, from previous corollary, we infer the following

Corollary 2.16. *Let T be an A-isometry on $\mathcal{H}$ such that $\widehat{T}$ is a non-unitary operator. Then T is a P_λ-contraction for all λ with $|\lambda| < 1$.*

Proof. The hypothesis that T is an A-isometry, that is $T^*AT = A$, ensures that $\widehat{T}$ is an isometry on $\overline{\mathcal{R}(A)}$. Since $\widehat{T}$ is non-unitary, by the Wold decomposition, $\widehat{T}^*$ has a

non-zero coshift part, hence $\sigma_p(\widehat{T}^*) = \mathbb{D}$ (the open unit disc). By the above remark, $\mathbb{D} \subset \sigma_p(T_0^*)$, and the first statement of the corollary follows from Corollary 2.14. $\quad\square$

We also mention the following facts concerning the regularity condition.

Proposition 2.17. *Let T be a regular A-contraction on $\mathcal{H}$ such that $\mathcal{N}(A) \subset \mathcal{N}(T)$. Then T^* is a quasicontraction on $\mathcal{H}$.*

Proof. Since $\mathcal{N}(A) \subset \mathcal{N}(T)$, representation (2.1) of T becomes

$$T = \begin{pmatrix} T_0 & 0 \\ T_1 & 0 \end{pmatrix}$$

on $\mathcal{H} = \overline{\mathcal{R}(A)} \oplus \mathcal{N}(A)$, and $\|T_0\| \leq 1$ by Proposition 2.11. But $\overline{\mathcal{R}(T^*)} \subset \overline{\mathcal{R}(A)}$ and is invariant to T^*, hence also to T_0^*. Thus $T^*|_{\overline{\mathcal{R}(T^*)}} = T_0^*|_{\overline{\mathcal{R}(T^*)}}$ is a contraction, which means that T^* is a quasicontraction on $\mathcal{H}$. $\quad\square$

Corollary 2.18. *If $T \in \mathcal{B}(\mathcal{H})$ is a regular m-quasicontraction such that $\mathcal{N}(T) = \mathcal{N}(T^m)$ then T^* is a quasicontraction.*

Proof. The hypothesis on T means that T is a regular $T^{*m}T^m$-contraction and $\mathcal{N}(T) = \mathcal{N}(T^{*m}T^m)$. So the conclusion follows from Proposition 2.17. $\quad\square$

We remark that any m-quasicontraction is similar to a contraction ([2]), hence the spectral radius of such an operator is less or equal to 1. In addition, an injective regular quasicontraction is necessarily a quasinormal operator, and so a contraction. But a regular m-quasicontraction for $m > 2$ is not necessarily quasinormal, for example a m-nilpotent operator.

Remark also that Proposition 2.17 generalizes the fact that if T is a regular A-contraction with A an injective operator, then T^* is a contraction ($T = \widehat{T}$).

3. THE CASE OF A TT^*-CONTRACTION

A major default of an A-contraction T when $A \neq I$ is that T^* is not an A-contraction, in general. But this obviously happens if $\|T\| \leq 1$ and $AT = TA$.

In fact, when T and T^* are A-contractions, $\mathcal{N}(A)$ is a reducing subspace for T, but the converse assertions are not true in general. The converse holds true in the case considered above.

In particular, the study of quasicontractions T with T^* also a T^*T-contraction, leads to considering the class of TT^*-contractions, for $T \in \mathcal{B}(\mathcal{H})$.

Proposition 3.1.

(i) *If $T \neq 0$ is a TT^*-contraction, then $\|T\| \leq 1$ and $\mathcal{N}(T^*) \subset \mathcal{N}(T)$.*
(ii) *If $T \neq 0$ is a TT^*-isometry, then $\|T\| = 1$ and $\mathcal{N}(T) = \mathcal{N}(T^*)$.*
(iii) *T is TT^*-regular if and only if T^* is quasinormal.*

Proof. Let $0 \neq T \in \mathcal{B}(\mathcal{H})$ satisfy $T^*TT^*T \leq TT^*$. This means that $\|T^*Th\| \leq \|T^*h\|$ for $h \in \mathcal{H}$, whence $\|T\|^2 \leq \|T\|$, and thus $\|T\| \leq 1$. It also follows that $\mathcal{N}(T^*) \subset \mathcal{N}(T)$. When T is a TT^*-isometry, that is $(T^*T)^2 = TT^*$, then all above inequalities become equalities; thus assertions (i) and (ii) are proved.

Now, since $\mathcal{N}(T^*) \subset \mathcal{N}(T)$, it follows that $\mathcal{N}(TT^*) = \mathcal{N}(T^*)$ reduces T. Having in view representation (2.1) of T in this case, one can easily see that the condition on T to be TT^*-regular means that $TT^*T = T^2T^*$, or equivalently that T^* is quasinormal. This gives assertion (iii). $\qquad\square$

As we remarked above, $\mathcal{N}(T^*) = \mathcal{N}(TT^*)$ reduces T, when T is a TT^*-contraction, and T^* is also a TT^*-contraction, being a contraction.

Corollary 3.2. *Any TT^*-contraction T is TT^*-ergodic and the corresponding subspace $\mathcal{N} = \mathcal{N}(I - T) \oplus \mathcal{N}(T^*)$ reduces T.*

Proof. We can assume $T \neq 0$, therefore $\|T\| \leq 1$. Considering now the null-subspaces $\mathcal{N}$ and $\mathcal{N}_*$ corresponding to the TT^*-contraction T, by (2.7), we get

$$(TT^*)^{1/2}\mathcal{N}_* \subset \mathcal{N}(I - T^*) = \mathcal{N}(I - T) \subset \mathcal{N},$$

and by Theorem 2.3, T is TT^*-ergodic. In fact, since matrix (2.1) of T in this case gives $T = T_0 \oplus 0$ on $\mathcal{H} = \overline{\mathcal{R}(T)} \oplus \mathcal{N}(T^*)$, it follows that $\mathcal{N}(I - T) = \mathcal{N}(I - T_0)$. Hence $\mathcal{N} = \mathcal{N}(I - T) \oplus \mathcal{N}(T^*)$, and so $\mathcal{N}$ reduces T because both subspaces of $\mathcal{N}$ reduce T. $\qquad\square$

Recall that in [11, 12] it was proved that a non-zero contractive quasi-isometry is hyponormal. We can now improve this result, and we obtain a similar result for TT^*-isometries.

Theorem 3.3. *Let $T \in \mathcal{B}(\mathcal{H})$, $T \neq 0$. If either T is a contractive quasi-isometry, or it is a TT^*-isometry, then T is an ∞-hyponormal operator.*

Proof. First suppose that T is a quasi-isometry with $\|T\| = 1$. Then $\{(T^*T)^n\}_{n \geq 0}$ is a decreasing sequence of positive operators which strongly converges to $P_0 = P_{\mathcal{N}(I-T^*T)}$, the orthogonal projection onto $\mathcal{N}(I - T^*T)$. But for $n \geq 1$, there holds

$$\mathcal{R}((TT^*)^n) \subset \mathcal{R}(T) \subset \mathcal{N}(I - T^*T),$$

the second inclusion being based on the fact that T is a quasi-isometry, that is $T|_{\mathcal{R}(T)}$ is an isometry. Since $(TT^*)^n$ is a positive contraction, for $h \in \mathcal{H}$ of the form $h = h_0 + h_1$ with $h \in \mathcal{N}(I - T^*T)$ and $h_1 \perp \mathcal{N}(I - T^*T)$, there is

$$\langle (TT^*)^n h, h \rangle = \langle (TT^*)^n h_0, h_0 \rangle \leq \|h_0\|^2 = \langle P_0 h, h \rangle.$$

Therefore, $(TT^*)^n \leq P_0 \leq (T^*T)^n$ for any $n \geq 1$, which means that T is an ∞-hyponormal operator.

Now assume that T is a TT^*-isometry, that is $(T^*T)^2 = TT^*$. By Proposition 3.1, $\|T\| = 1$ and for $n \geq 1$, we infer

$$(TT^*)^{2n} \leq (TT^*)^n = (T^*T)^{2n}.$$

This, by Lowner-Heinz inequality [5], implies that $(TT^*)^n \leq (T^*T)^n$ for $n \geq 1$; consequently, T is an ∞-hyponormal operator. The proof is finished. □

We can complete M. Patel's result [11, 12] concerning the quasi-isometries as follows.

Corollary 3.4. *For a quasi-isometry $T \neq 0$, the following statements are equivalent:*

(i) $\|T\| = 1$;
(ii) T *is hyponormal;*
(iii) T *is ∞-hyponormal.*

Remark 3.5. This corollary clearly shows that the class of quasinormal quasi-isometries is strictly included in the one of ∞-hyponormal quasi-isometries. In fact, it is easy to see that a quasi-isometry is quasinormal iff it is a partial isometry, that is it has the form $V \oplus 0$ with V an isometry.

Next, for a contraction T on $\mathcal{H}$, we denote by S_T the strong limit of the (bounded decreasing) sequence $\{T^{*n}T^n\}_{n \geq 0}$. So, S_T is the asymptotic limit associated to T ([4,6]), S_T being a positive operator, but it is not an orthogonal projection (that is $S_T \neq S_T^2$), in general.

However, if T is a cohyponormal contraction, that is it satisfies $T^*T \leq TT^*$, then T is a TT^*-contraction, because

$$T^*TT^*T = (T^*T)^2 \leq T^*T \leq TT^*.$$

In this case, it is known (see [6]) that $S_T = S_T^2$.

We now formulate the following

Theorem 3.6. *Let T be a TT^*-contraction on $\mathcal{H}$. The following statements hold:*

(i) $\mathcal{N}(I - S_T)$ *is the maximum subspace reducing T to an isometry, equivalently, to a unitary operator, and there is*

$$\mathcal{N}(I - S_T) = \bigcap_{n \geq 0} T^{*n}\mathcal{N}(I - T^*T) \subset \mathcal{N}(I - S_{T^*}). \tag{3.1}$$

(ii) *If $\mathcal{R}(T^* - \overline{\lambda}I) \subset \mathcal{R}(T - \lambda I)$ for every $\overline{\lambda} \in \sigma_a(T^*)$, then $S_T = S_T^2$ and $\mathcal{N}(I - S_T) \oplus \mathcal{N}(T^*)$ is the maximum subspace reducing T to a quasi-isometry, or equivalently, to a normal partial isometry.*

Proof. Since T is a TT^*-contraction, then for $h \in \mathcal{N}(I - T^*T)$,

$$\|h\| = \|T^*Th\| \leq \|T^*h\|,$$

hence $\|T^*h\| = \|h\|$, T being a contraction. This means that $h = TT^*h$ and also $T^*h = T^*TT^*h$. Therefore, $h \in \mathcal{N}(I - TT^*)$ and $T^*h \in \mathcal{N}(I - T^*T)$. We infer that $\mathcal{N}(I - T^*T)$ is an invariant subspace for T^* and T^* is an isometry on this subspace. Since $\mathcal{N}(I - S_{T^*})$ is the maximum invariant subspace for T^* on which T^* is an isometry, we get the following inclusions:

$$\mathcal{N}(I - S_T) \subset \mathcal{N}(I - T^*T) \subset \mathcal{N}(I - S_{T^*}) \subset \mathcal{N}(I - TT^*).$$

Hence $\mathcal{N}(I - S_T) = \mathcal{N}(I - S_T) \cap \mathcal{N}(I - S_{T^*})$ is the maximum subspace reducing T to a unitary operator, or equivalently to an isometry, because $\mathcal{N}(I - S_T)$ has the same maximal property as an invariant subspace for T on which T is an isometry.

From the above remarks there follows

$$\mathcal{N}(I - S_T) \subset \bigcap_{n \geq 0} T^{*n} \mathcal{N}(I - T^*T) =: \mathcal{M},$$

and it is clear that $\mathcal{M}$ is an invariant subspace for T^*. Now, if $h \in \mathcal{M}$, then for any $n \geq 1$ there exists $h_n \in \mathcal{N}(I - T^*T)$ such that $h = T^{*n}h_n$. As $T^{*(n-1)}h_n \in \mathcal{N}(I - T^*T) \subset \mathcal{N}(I - TT^*)$, we infer that

$$Th = TT^*T^{*(n-1)}h_n = T^{*(n-1)}h_n \in T^{*(n-1)}\mathcal{N}(I - T^*T),$$

for all $n \geq 1$. So $Th \in \mathcal{M}$, and it follows that $\mathcal{M}$ is an invariant subspace for T. Hence $\mathcal{M}$ reduces T and T is a unitary operator on $\mathcal{M}$ because $\mathcal{M} \subset \mathcal{N}(I - T^*T) \subset \mathcal{N}(I - TT^*)$. By the maximality of $\mathcal{N}(I - S_T)$, one obtains $\mathcal{M} \subset \mathcal{N}(I - S_T)$; in fact $\mathcal{N}(I - S_T) = \mathcal{M}$. Assertion (i) is proved.

Next suppose that $\mathcal{R}(T^* - \overline{\lambda}I) \subset \mathcal{R}(T - \lambda I)$ for $\overline{\lambda} \in \sigma_a(T^*)$. Since for $\overline{\lambda} \notin \sigma_a(T^*)$ we have that $T^* - \overline{\lambda}I$ is injective with closed range, it results that our assumed inclusion is true for every scalar λ. This means ([7]) that T^* is a dominant contraction, T being also a TT^*-contraction. Finally by [7] it follows that S_T is an orthogonal projection.

The subspace $\mathcal{M}_q := \mathcal{N}(I - S_T) \oplus \mathcal{N}(T^*)$ clearly reduces T to an operator of the form $T|_{\mathcal{M}_q} = U \oplus 0$ with U a unitary operator. So $T|_{\mathcal{M}_q}$ is a normal partial isometry, hence a quasi-isometry.

Now let $\mathcal{M} \subset \mathcal{H}$ be a (closed) subspace reducing T to a quasi-isometry. Then it is easy to see that $\mathcal{M} \subset \mathcal{N}(T^*T - S_T)$. But as $S_T = S_T^2$, there is $\mathcal{H} = \mathcal{N}(I - S_T) \oplus \mathcal{N}(S_T)$, and from the inclusions

$$\mathcal{N}(I - S_T) \oplus \mathcal{N}(T) \subset \mathcal{N}(T^*T - S_T) \subset \mathcal{N}(I - S_T) \oplus \mathcal{N}(S_T)$$

we infer that

$$\mathcal{N}(T^*T - S_T) = \mathcal{N}(I - S_T) \oplus \mathcal{N}(T).$$

Hence every $h \in \mathcal{M}$ can be written $h = h_1 \oplus h_0$ with $h_1 \in \mathcal{N}(I - S_T)$, $h_0 \in \mathcal{N}(T)$. But $T^*h = T^*h_1 + T^*h_0$ and $T^*h_0 \in T^*\mathcal{N}(T) \subset T^*\mathcal{N}(S_T) \subset \mathcal{N}(S_T)$, and so $T^*h_0 \in \mathcal{N}(S_T) \ominus \mathcal{N}(T)$. On the other hand, $T^*h \in \mathcal{M} \subset \mathcal{N}(I - S_T) \oplus \mathcal{N}(T)$, therefore $T^*h_0 = 0$.

Thus $\mathcal{M} \subset \mathcal{N}(I - S_T) \oplus \mathcal{N}(T^*) = \mathcal{M}_q$; consequently, the subspace $\mathcal{M}_q$ has the required maximal property from (ii). The proof is finished. $\qquad\square$

Corollary 3.7. *If T is a TT^*-contraction on $\mathcal{H}$ with $\mathcal{R}(T^* - \overline{\lambda}I) \subset \mathcal{R}(T - \lambda I)$ for $\overline{\lambda} \in \sigma_a(T^*)$, then the completely non-unitary part of T is strongly stable.*

Proof. The assumption ensures that $S_T = S_T^2$ by above assertion (ii). Therefore $\mathcal{H} = \mathcal{N}(I - S_T) \oplus \mathcal{N}(S_T)$, $\mathcal{N}(S_T)$ being the completely non-unitary part in $\mathcal{H}$ for T in this case. Clearly, $h \in \mathcal{N}(S_T)$ if and only if $T^n h \to 0$ $(n \to \infty)$, hence T is a strongly stable contraction on $\mathcal{N}(S_T)$. $\qquad\square$

Recall ([6]) that if T is a cohyponormal contraction on $\mathcal{H}$, the conclusion from Corollary 3.7 holds true.

Corollary 3.8. *Let T be a TT^*-contraction on $\mathcal{H}$. Then the completely non-unitary part of T is a proper contraction if and only if $\mathcal{N}(I - S_T) = \mathcal{N}(I - T^*T)$.*

Proof. If $\mathcal{N}(I - S_T) = \mathcal{N}(I - T^*T)$, then for $0 \neq h \in \mathcal{H} \ominus \mathcal{N}(I - S_T)$ there is $h \neq T^*Th$, or equivalently $\|Th\| < \|h\|$, that is T is a proper contraction on $\mathcal{H} \ominus \mathcal{N}(I - S_T)$ in this case. Clearly, by Theorem 3.6, $\mathcal{H} \ominus \mathcal{N}(I - S_T)$ is the completely non-unitary part in $\mathcal{H}$ for T.

Conversely, if T is a proper contraction on $\mathcal{H} \ominus \mathcal{N}(I - S_T)$, then $[\mathcal{H} \ominus \mathcal{N}(I - S_T)] \cap \mathcal{N}(I - T^*T) = \{0\}$ which leads to $\mathcal{N}(I - S_T) = \mathcal{N}(I - T^*T)$. $\qquad\square$

Proposition 3.9. *Let T be a TT^*-isometry on $\mathcal{H}$. Then $\mathcal{N}(T) = \mathcal{N}(S_T)$ and, furthermore, $S_T = S_T^2$ if and only if T is a unitary operator on $\overline{\mathcal{R}(T)}$. In this last case, T is a normal partial isometry.*

Proof. Since $(T^*T)^2 = TT^*$, there is $T^{*n}TT^*T^n = TT^*$, for any $n \geq 1$. But T is hyponormal by Theorem 3.3, and so we infer for $n \geq 1$,

$$TT^* = T^{*n}TT^*T^n \leq T^{*(n+1)}T^{n+1} \leq T^*T.$$

From $\mathcal{N}(T^*) = \mathcal{N}(T)$ and the definition of S_T, we get $\mathcal{N}(T^*) = \mathcal{N}(S_T) = \mathcal{N}(T)$, hence S_T is an injective operator on $\overline{\mathcal{R}(T)} = \mathcal{H} \ominus \mathcal{N}(S_T)$. Clearly, $\overline{\mathcal{R}(T)}$ reduces S_T, and this operator is positive.

Suppose $S_T = S_T^2$, that is S_T is an orthogonal projection. Then we have $\mathcal{R}(S_T) = \overline{\mathcal{R}(T)}$, hence S_T is the identity operator on $\overline{\mathcal{R}(T)}$. Since $S_T \leq T^*T$ (by an above inequality), for $h \in \overline{\mathcal{R}(T)}$ we obtain

$$\|h\|^2 = \|S_T h\|^2 = \langle S_T h, h \rangle \leq \|Th\|^2 \leq \|h\|^2,$$

hence $\|Th\| = \|h\|$. So T is an isometry on $\overline{\mathcal{R}(T)}$, and it follows that $\overline{\mathcal{R}(T)} \subset \mathcal{N}(I - S_T)$. But $\overline{\mathcal{R}(T)}$ reduces T, and $\mathcal{N}(I - S_T)$ is the unitary part in $\mathcal{H}$ for T by Theorem 3.6. In conclusion, T is unitary on $\overline{\mathcal{R}(T)}$.

Conversely, if $T|_{\overline{\mathcal{R}(T)}}$ is unitary, from the definition of S_T we infer that S_T is the identity on $\overline{\mathcal{R}(T)} = \mathcal{H} \ominus \mathcal{N}(S_T)$; consequently, S_T is an orthogonal projection on $\mathcal{H}$. This ends the proof. $\qquad\square$

As a consequence of Theorem 3.6, we give W. Mlak's following result [8], which was obtained using the unitary dilation of a contraction.

Corollary 3.10 (Mlak)**.** *If T is a hyponormal contraction on $\mathcal{H}$, then the maximum subspace which reduces T to a unitary operator is*

$$\mathcal{N}(I - S_{T^*}) = \bigcap_{n \geq 0} T^n \mathcal{N}(I - TT^*). \tag{3.2}$$

Proof. We apply (3.1) to T^*, because T^* is a T^*T-contraction. $\qquad\square$

In the sequel, we propose the construction of some TT^*-isometries without assuming S_T to be an orthogonal projection.

Example 3.11. *Let $\{e_n\}_{n\geq 0}$ be the canonical basis of $l^2(\mathbb{N})$ and let $\{a_n\}_{n\geq 0}$ be a bounded increasing and strictly positive sequence of scalars. Let T be the unilateral weighted shift with weight sequence $\{a_n\}$, defined by $Te_n = a_n e_{n+1}$, $n \geq 0$. It is known ([7]) that T is hyponormal, in fact even ∞-hyponormal.*

Now the condition of TT^*-isometry for T implies that $a_n^2 = a_{n-1}$ and $0 < a_0 < 1$. On the other hand, T is quasinormal if and only if there exists $n_0 \in \mathbb{N}$ such that $a_n = 0$ for $n < n_0$ and $a_n = a_{n_0}$ for $n \geq n_0$. So, we can get such unilateral weighted shifts T which are TT^*-isometries, but are not quasinormal, and in this case we have $S_T \neq S_T^2$ by Proposition 3.9.

In [11] Patel showed that left invertible quasi-isometries are similar to isometries. Concerning the operators considered in Theorem 3.3, we may now prove their similarity to partial isometries.

Theorem 3.12. *Let $T \in \mathcal{B}(\mathcal{H})$ such that the range $\mathcal{R}(T)$ is closed. If either T is a TT^*-isometry, or T is a quasi-isometry with $\mathcal{N}(T) \subset \mathcal{N}(T^*)$, then T is similar to a partial isometry.*

Proof. Suppose that T is a TT^*-isometry. Then $\mathcal{N}(T) = \mathcal{N}(T^*)$ is a reducing subspace for T, hence in the case representation (2.1) for T reduces to $T = T_0 \oplus 0$ on $\mathcal{H} = \mathcal{R}(T) \oplus \mathcal{N}(T^*)$. If V is the isometry on $\mathcal{R}(T)$ associated to T and satisfying $V(TT^*)^{1/2}h = (TT^*)^{1/2}Vh$ for $h \in \mathcal{H}$, then since $\mathcal{R}(T)$ is closed, by Lemma 2.1 it follows that T_0 is similar to V by the invertible operator $S = (TT^*)^{1/2}|_{\mathcal{R}(T)}$ on $\mathcal{R}(T)$. But this clearly implies that $T = T_0 \oplus 0$ is similar to $V \oplus 0$ by $S \oplus I_{\mathcal{N}(T^*)}$; consequently, T is similar to a partial isometry.

When T is a quasi-isometry (T^*T-isometry) with $\mathcal{N}(T) \subset \mathcal{N}(T^*)$ one can use the same argument. $\square$

It is clear that a contractive quasi-isometry T, being hyponormal, satisfies the condition $\mathcal{N}(T) \subset \mathcal{N}(T^*)$. In fact, this condition may be dropped for a quasi-isometry in the above theorem, as can be seen in [9]. Here we only remark that in the presence of this condition, the similarity of a quasi-isometry with closed range to a partial isometry is immediate.

Corollary 3.13. *Let T be an injective operator on $\mathcal{H}$ with $\mathcal{R}(T)$ closed. If either T is a quasi-isometry, or it is a TT^*-isometry, then T is similar to an isometry.*

4. SPECTRAL PROPERTIES

In the sequel we give some spectral properties for certain A-contractions, particularly for m-isometries.

Proposition 4.1. *Let T be an A-contraction on $\mathcal{H}$ with representation* (2.1).*Then:*

(i) *If $\mathcal{N}(A) \neq \{0\}$ and $T|_{\mathcal{N}(A)}$ is nilpotent, then $0 \in \sigma_p(T)$.*
(ii) *Suppose that $T|_{\mathcal{N}(A)}$ is quasinilpotent and that either T_0 is hyponormal, or it is a quasi-isometry. If $\lambda \neq 0$ is an isolated point in $\sigma(T)$ then $\lambda \in \sigma_p(T_0)$. If furthermore $\mathcal{N}(T_0 - \lambda I) \cap \mathcal{N}(T_1) \neq \{0\}$ then $\lambda \in \sigma_p(T)$.*
(iii) *If $\lambda \in \sigma_p(T)$ then either $|\lambda| \leq 1$, or $\mathcal{N}(T - \lambda I) \subset \mathcal{N}(A)$.*

Proof. (i) Since $T_2 = T|_{\mathcal{N}(A)}$ is nilpotent one has $0 \in \sigma(T_2)$. When $T_2 \neq 0$, there exists $n \geq 2$ such that $T_2^n = 0$ and $T_2^{n-1} \neq 0$, hence $\mathcal{N}(T_2) \neq \{0\}$. If $T_2 = 0$ then $\mathcal{N}(T_2) = \mathcal{N}(A) \neq \{0\}$. Now if $0 \neq h \in \mathcal{N}(T_2)$ then $T(0, h) = (0, T_2 h) = 0$ and so $\mathcal{N}(T) \neq \{0\}$, that is $0 \in \sigma_p(T)$.

(ii) Let $0 \neq \lambda \in \sigma(T)$ be an isolated point in $\sigma(T)$. Since T_2 is quasinilpotent, then $\sigma(T_2) = \{0\}$; therefore, $\lambda \notin \sigma(T_2)$. Hence $\lambda \in \sigma(T_0)$ and both hypotheses on T_0 in (ii) ensure that $\lambda \in \sigma_p(T_0)$ (see [11–13]). Now, if there exists $h \in \mathcal{N}(T_0 - \lambda I) \cap \mathcal{N}(T_1)$, $h \neq 0$, then

$$(T - \lambda I)(h, 0) = ((T_0 - \lambda I)h, T_1 h) = (0, 0);$$

therefore, $\mathcal{N}(T - \lambda I) \neq \{0\}$ and $\lambda \in \sigma_p(T)$.

(iii) Let $\lambda \in \sigma_p(T)$. Assume that $\mathcal{N}(T - \lambda I) \nsubseteq \mathcal{N}(A)$, hence there exists $h \neq 0$ such that $Th = \lambda h$ and $A^{1/2}h \neq 0$. Since T is an A-contraction, we obtain

$$|\lambda| \, \|A^{1/2}h\| = \|A^{1/2}Th\| \leq \|A^{1/2}h\|,$$

and so $|\lambda| \leq 1$. The proof is finished. $\square$

Corollary 4.2. *If T is an A-contraction on $\mathcal{H}$ such that $T|_{\mathcal{N}(A)}$ is quasinilpotent, then $\sigma_p(T)$ is a subset of the closed unit disc.*

Proof. If $T_2 = T|_{\mathcal{N}(A)}$ is quasinilpotent, then $\sigma(T_2) = \{0\}$. So, if $0 \neq \lambda \in \sigma_p(T)$, then $\lambda \notin \sigma_p(T_2)$, that is $(T_2 - \lambda I)k \neq 0$ for any $k \in \mathcal{N}(A)$, $k \neq 0$. This implies $\mathcal{N}(T - \lambda I) \nsubseteq \mathcal{N}(A)$, hence $|\lambda| \leq 1$ by Proposition 4.1 (iii). $\square$

Corollary 4.3. *If T is an A-isometry on $\mathcal{H}$ and $\mathcal{N}(A) = \{0\}$, then $\sigma_p(T)$ is a subset of the unit circle.*

Proof. It follows from the proof of assertion (iii) above. $\square$

Other spectral properties may be proved for m-quasi-isometries.

Theorem 4.4. *Let T be an m-quasi-isometry for $m \geq 2$ such that $C = T|_{\overline{\mathcal{R}(T)}}$ is injective. If either C is hyponormal, or C is a quasi-isometry, then any isolated point in the spectrum of T is an eigenvalue.*

Proof. We use the following matrix representation of T with respect to the decomposition $\mathcal{H} = \overline{\mathcal{R}(T)} \oplus \mathcal{N}(T^*)$:

$$T = \begin{pmatrix} C & S \\ 0 & 0 \end{pmatrix}. \tag{4.1}$$

Let $\lambda \in \sigma(T)$ be an isolated point. First assume $\lambda = 0$. If $0 \in \sigma(C)$, then the hypothesis on C implies $0 \in \sigma_p(C)$, which contradicts the injectivity of C. So $0 \notin \sigma(C)$ that is C is an invertible operator. This fact and the assumption $0 \in \sigma(T)$ ensure $\mathcal{N}(T^*) = \{0\}$. If $S \neq 0$ and $k \in \mathcal{N}(T^*)$ such that $Sk \neq 0$, then $h = -C^{-1}Sk \neq 0$ and $T(h, k) = (Ch + Sk, 0) = (0, 0)$. If $S = 0$, then, for $0 \neq k \in \mathcal{N}(T^*)$, there is $T(0, k) = (0, 0)$. In both cases, there follows $\mathcal{N}(T) \neq \{0\}$, hence $0 \in \sigma_p(T)$.

Now assume $\lambda \neq 0$. Then $\lambda \in \sigma(C)$ and, by the hypothesis on C, there is $\lambda \in \sigma_p(C)$. Therefore, $\mathcal{N}(C - \lambda I) \neq \{0\}$, and for $0 \neq h \in \mathcal{N}(C - \lambda I)$, we get $(T - \lambda I)(h, 0) = (0, 0)$. Consequently, $\lambda \in \sigma_p(T)$, which ends the proof. $\square$

The case of $m = 1$ for the previous theorem was proved in [11, 12]. In this case, $C = T|_{\overline{\mathcal{R}(T)}}$ is an isometry; therefore, C satisfies the hypothesis of Theorem 4.4.

In the case of $m = 2$, we derive the following

Corollary 4.5. *Let T be a 2-quasi-isometry on $\mathcal{H}$ such that $\overline{\mathcal{R}(T)} \cap \mathcal{N}(T) = \{0\}$. Then any isolated point in the spectrum of T is an eigenvalue. If, furthermore, $\mathcal{N}(T_2) = \{0\}$, then $\sigma_p(T)$ is a subset of the unit circle.*

Proof. It was proved in [2] that in this case ($m = 2$) the operator $C = T|_{\overline{\mathcal{R}(T)}}$ in (4.1) is a quasi-isometry. Therefore, one can apply the previous theorem to obtain the first assertion. The second statement follows from Corollary 4.3 because T is a $T^{*2}T^2$-isometry and $\mathcal{N}(T^2) = \{0\}$. $\square$

The case of $m = 1$ in the theorem below appeared in [11] and has inspired the study of a general case.

Theorem 4.6. *Let $T \neq 0$ be an m-quasi-isometry on $\mathcal{H}$ for $m \geq 1$. The following statements hold:*

(i) *$\lambda \in \sigma_p(T)$ implies $\overline{\lambda} \in \sigma_p(T^*)$;*
(ii) *$\lambda \in \sigma_a(T)$ implies $\overline{\lambda} \in \sigma_a(T^*)$;*
(iii) *$\sigma_a(T) - \{0\}$ is a subset of the unit circle;*
(iv) *the eigenspaces corresponding to distinct non-zero eigenvalues of T are mutually orthogonal.*

Proof. (i) Let $\lambda \in \sigma_p(T)$. Suppose that $\lambda = 0$ and $0 \notin \sigma_p(T^*)$. So $\mathcal{N}(T^*) = \{0\}$, that is T^* is injective. Since $T^{*(m+1)}T^{m+1} = T^{*m}T^m$, we infer $T^{*m}T^{m+1} = T^{*(m-1)}T^m$ or $T^{*(m+1)}T^m = T^{*m}T^{m-1}$. This also implies $T^{*m}T^m = T^{*(m-1)}T^{m-1}$, which means that T is an $(m-1)$-quasi-isometry. If $m - 1 > 1$, one may recursively prove that T is an isometry, which contradicts $0 \in \sigma_p(T)$. Thus $0 \in \sigma_p(T^*)$.

Assume now $\lambda \neq 0$ and let $0 \neq h \in \mathcal{H}$ be such that $Th = \lambda h$. Since T is an isometry on $\mathcal{R}(T^m)$, then $T^*T^{m+1}h = T^m h$ or $\lambda^{m+1}T^*h = \lambda^m h$, and also ($\lambda \neq 0$) $\lambda T^*h = h$. So λ^{-1} is an eigenvalue of T^*. As T (and so T^*) is a power bounded operator, there is $|\lambda| \leq 1$ and $|\lambda^{-1}| \leq 1$, hence $|\lambda| = 1$. Thus $\overline{\lambda} = \lambda^{-1} \in \sigma_p(T^*)$.

(ii) Let $\lambda \in \sigma_a(T)$. First suppose that $\lambda = 0$ and $0 \notin \sigma_a(T^*)$. Then for any vector $h \in \mathcal{H}$ with $\|h\| = 1$ there is $T^*h \neq 0$, and it follows that $\mathcal{N}(T^*) = \{0\}$. But, as above, T may be proved to be an isometry, which contradicts $0 \in \sigma_a(T)$. So $0 \in \sigma_a(T^*)$.

Next, we assume $\lambda \neq 0$. Let $h_n \in \mathcal{H}$ with $\|h_n\| = 1$ such that $(T - \lambda I)h_n \to 0$ $(n \to \infty)$. Then

$$\|(T^2 - \lambda^2 I)h_n\| \leq (|\lambda| + \|T\|)\|(T - \lambda I)h_n\| \to 0 \quad (n \to \infty),$$

and, assuming $(T^j - \lambda^j I)h_n \to 0 \ (n \to \infty)$ for some integer $j > 2$, we also obtain

$$(T^{j+1} - \lambda^{j+1} I)h_n = T^j(T - \lambda I)h_n + \lambda(T^j - \lambda^j I)h_n \to 0.$$

By induction it follows that $(T^p - \lambda^p I)h_n \to 0 \ (n \to \infty)$ for all $p \in \mathbb{N}$. Since $T^{*(m+1)}T^{m+1} = T^{*m}T^m$, we infer that

$$(\lambda^{m+1}T^{*(m+1)} - \lambda^m T^{*m})h_n = T^{*(m+1)}(\lambda^{m+1}I - T^{m+1})h_n +$$

$$+T^{*m}(T^m - \lambda^m)h_n \to 0 \quad (n \to \infty),$$

and as $\lambda \neq 0$, we get

$$(\lambda T^* - I)T^{*m}h_n \to 0 \quad (n \to \infty). \tag{4.2}$$

But

$$\langle T^{*m}h_n, h_n \rangle = \langle h_n, (T^m - \lambda^m I)h_n \rangle + \overline{\lambda}^m \to \overline{\lambda}^m \quad (n \to \infty),$$

whence it follows that $T^{*m}h_n$ does not converge to $0 \ (n \to \infty)$. So there exist an $\varepsilon_0 > 0$ and a subsequence $\{h_{n_j}\}$ of $\{h_n\}$ such that $\|T^{*m}h_{n_j}\| \geq \varepsilon_0$ for any j. We put

$$k_j = \|T^{*m}h_{n_j}\|^{-1}T^{*m}h_{n_j}.$$

Then $\|k_j\| = 1$ for any j, and using (4.2) we obtain

$$\|(\lambda T^* - I)k_j\| \leq \frac{1}{\varepsilon}\|(\lambda T^* - I)T^*h_{n_j}\| \to 0 \quad (j \to \infty).$$

This means that $\lambda^{-1} \in \sigma_a(T^*)$ hence $|\lambda| = 1$ and $\overline{\lambda} = \lambda^{-1} \in \sigma_a(T^*)$.

Assertion (iii) follows as a consequence of the proof of (ii).

(iv) Let $\lambda, \mu \in \sigma_p(T)$, $\lambda \neq \mu$, $\lambda, \mu \neq 0$. Choose $h, k \neq 0$ such that $Th = \lambda h$, $Tk = \mu k$. Then we obtain

$$0 = \langle T^{m+1}h, T^{m+1}k \rangle - \langle T^m h, T^m k \rangle = (\lambda^{m+1}\overline{\mu}^{m+1} - \lambda^m\overline{\mu}^m)\langle h, k \rangle =$$
$$= \lambda^m \overline{\mu}^m (\lambda\overline{\mu} - 1)\langle h, k \rangle,$$

and since $\lambda \neq \mu$ and $|\lambda| = |\mu| = 1$, it follows that $h \perp k$. Hence the subspaces $\mathcal{N}(T - \lambda I)$ and $\mathcal{N}(T - \mu I)$ are orthogonal, and the proof is finished. $\square$

The corresponding spectral properties for TT^*-isometries are included in the following

Theorem 4.7. *Let $T \neq 0$ be a TT^*-isometry on $\mathcal{H}$. Then the following statements hold:*

(i) $\lambda \in \sigma_p(T)$ *implies* $\overline{\lambda} \in \sigma_p(T^*)$;
(ii) $\lambda \in \sigma_a(T)$ *implies* $\overline{\lambda} \in \sigma_a(T^*)$;
(iii) $\sigma_a(T) - \{0\}$ *is a subset of the unit circle;*
(iv) *the isolated points of $\sigma(T)$ are eigenvalues.*

Proof. (i) Let $\lambda \in \sigma_p(T)$. Since $\mathcal{N}(T) = \mathcal{N}(T^*)$ (by Proposition 3.1 (ii)), if $\lambda = 0$, then clearly $0 \in \sigma_p(T^*)$. Assume $\lambda \neq 0$ and let $h \neq 0$ be such that $Th = \lambda h$. Then $T^*Th = \lambda T^*h$, and also $TT^*h = (T^*T)^2 h = \lambda T^*TT^*h$. If $k = TT^*h = 0$, then $T^*h = 0$, that is $Th = 0$ and so $h = 0$, a contradiction. Hence $k \neq 0$ and $T^*k = \lambda^{-1}k$; therefore, $\lambda^{-1} \in \sigma_p(T^*)$. Thus $|\lambda^{-1}| \leq 1$ and, as $|\lambda| \leq 1$, it follows that $\overline{\lambda} = \lambda^{-1} \in \sigma_p(T^*)$.

(ii) Suppose that $\lambda \in \sigma_a(T)$. If $\lambda = 0$, then there exists $\{h_n\} \subset \mathcal{H}$ with $\|h_n\| = 1$ for all n such that $Th_n \to 0$ $(n \to \infty)$. This yields $T^*Th_n \to 0$, or equivalently, $T^*h_n \to 0$, because T is a TT^*-isometry. Therefore, $0 \in \sigma_a(T^*)$.

Assume now that $\lambda \neq 0$ and let us choose h_n as above and so that $(T - \lambda I)h_n \to 0$ $(n \to \infty)$. This implies $((T^*T)^2 - \lambda T^*TT^*)h_n \to 0$, whence one obtains

$$(I - \lambda T^*)TT^*h_n \to 0 \quad (n \to \infty). \tag{4.3}$$

Since T is a TT^*-contraction (so $\|T\| \leq 1$) and $\|h_n\| = 1$, then

$$\|T^*h_n\| = \langle TT^*h_n, h_n \rangle \leq \|TT^*h_n\| = \|(T^*T)^2 h_n\| \leq$$

$$\leq \|T^*Th_n\| \leq \|Th_n\| \leq \|T^*Th_n\| = \|T^*h_n\|.$$

In fact all these inequalities become equalities. So, assuming $TT^*h_n \to 0$, that is $Th_n \to 0$, from the choice of h_n we infer $\lambda = 0$, a contradiction. Hence TT^*h_n does not converge to 0, and there exist $\varepsilon_0 > 0$ and a subsequence $\{h_{n_j}\}$ of $\{h_n\}$ such that $\|Th_{n_j}\| = \|TT^*h_{n_j}\| \geq \varepsilon_0$ for any j. Putting

$$k_j = \|Th_{n_j}\|^{-1}TT^*h_{n_j},$$

we get $\|k_j\| = 1$ and (by (4.3)) $(I - \lambda T^*)k_j \to 0$ $(j \to \infty)$. Thus we get $\lambda^{-1} \in \sigma_a(T^*)$; therefore, $|\lambda| = 1$ and $\overline{\lambda} \in \sigma_p(T^*)$. This proves (ii) and also (iii).

Since T is hyponormal by Theorem 3.3, assertion (iv) follows from the known result ([13]) that an isolated point in the spectrum of a hyponormal operator is an eigenvalue. This ends the proof. $\square$

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