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ERGODIC CONTROL OF MULTIDIMENSIONAL DIFFUSIONS I.  
THE EXISTENCE RESULTS

by

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ABSTRACT

The existence of optimal stable Markov relaxed controls for the ergodic control of multidimensional diffusions is established by direct probabilistic methods based on a characterization of a.s. limit sets of empirical measures. The optimality of the above is established in the strong (i.e., almost sure) sense among all admissible controls under very general conditions.

KEY WORDS

Ergodic control, Markov controls, optimal controls, empirical measures, invariant probability measures

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## I. INTRODUCTION

The 'ergodic' or 'long run average cost' control problem for multidimensional diffusions is one of the few classical problems of stochastic control that still eludes a completely satisfactory treatment. The problem can be formulated as follows: Let  $U$  be a compact metric space called the control set. Let  $X(\cdot)$  be an  $R^n$ -valued controlled diffusion process on some probability space satisfying the stochastic differential equation

$$dX(t) = m(X(t), u(t))dt + \sigma(X(t))dW(t), X(0) = X_0, \quad (1.1)$$

for  $t \geq 0$ , where

- (i)  $m(\cdot, \cdot) = [m_1(\cdot, \cdot), \dots, m_n(\cdot, \cdot)]^T: R^n \times U \rightarrow R^n$  is continuous and satisfies for all  $x, y \in R^n, u \in U$ ,

$$\|m(x, u) - m(y, u)\| \leq K \|x - y\|$$

$$\|m(x, u)\| \leq K$$

for some constant  $K > 0$ .

- (ii)  $\sigma(\cdot) = [[\sigma_{ij}(\cdot)]]: R^n \rightarrow R^{n \times n}$  satisfies for  $x, y \in R^n$ ,

$$\|\sigma(x) - \sigma(y)\| \leq K \|x - y\|, \quad \|\sigma(x)\| \leq K$$

$$\|\sigma^T x\|^2 \geq \lambda \|x\|^2 \quad (\text{uniform ellipticity})$$

for some constants  $\lambda > 0, K > 0$ ,

- (iii)  $X_0$  is a prescribed random variable,  
 (iv)  $W(\cdot) = [W_1(\cdot), \dots, W_n(\cdot)]^T$  is a standard  $n$ -dimensional Wiener process independent of  $X_0$ , and,  
 (v)  $u(\cdot)$  is a  $U$ -valued process with measurable sample paths satisfying the following 'nonanticipativity' condition: For

and  $t \geq s \geq y \geq 0$ ,  $W(t) - W(s)$  is independent of  $u(y)$ .

A process  $u(\cdot)$  as above will be called an admissible control. Of special interest is the case when  $u(\cdot) = v(X(\cdot))$  for some measurable  $v: \mathbb{R}^n \rightarrow U$ . In this case, (1.1) will have a strong solution [29] implying in particular that  $u(\cdot)$  is admissible.  $X(\cdot)$  will then be a homogeneous Markov process. Hence we call such a  $u(\cdot)$  or, by abuse of terminology, the function  $v$  itself, a Markov control. A Markov control will be said to be stable if the corresponding process is positive recurrent and thus has a unique invariant measure. (The uniqueness is ensured by our uniform ellipticity condition. See, e.g. [6], [18] or [28], Ch. 30-32). If  $u(\cdot) = v(X(\cdot), \cdot)$  for some measurable  $v: \mathbb{R}^n \times \mathbb{R}^+ \rightarrow U$ , the corresponding process will also be a Markov process, albeit not a homogeneous one. Call such a  $u(\cdot)$  or again, by abuse of terminology, the map  $v$  itself, an inhomogeneous Markov control. The admissibility of these once again follows from the existence of strong solutions for the corresponding s.d.e. as in [29].

Let  $c: \mathbb{R}^n \times U \rightarrow \mathbb{R}$  be a continuous function called the cost function. We assume that

$$c(\cdot, \cdot) \geq -K \tag{1.2}$$

for some constant  $K$ . In the ergodic control problem, one typically seeks to minimize

$$\limsup_{t \rightarrow \infty} \frac{1}{t} \int_0^t E[c(X(s), u(s))] ds \tag{1.3}$$

or a.s. minimize

$$\limsup_{t \rightarrow \infty} \frac{1}{t} \int_0^t c(X(s), u(s)) ds \quad (1.4)$$

over all admissible controls. An admissible control is said to be optimal in the mean if it minimizes (1.3) and a.s. optimal if it a.s. minimizes (1.4). The primary aims of the ergodic control problem are:

- (i) to show the existence of a stable Markov control which is optimal in an appropriate sense (cf. above definitions of optimality), and,
- (ii) to characterize the same via the dynamic programming equation (the 'Hamilton-Jacobi-Bellman' equation).

The first attempt in this direction is perhaps [24], Ch. VI, where a one dimensional compact state space was considered. Subsequent works considered the multidimensional case as well. An extensive survey of these appears in [25]. Here, we shall briefly recall the focus of some recent works. The traditional approach to this problem, inherited from earlier developments in discrete time and discrete state space situations, is to start with the Hamilton-Jacobi-Bellman equation and arrive at an existence result for optimal stable Markov control using this equation, the equation itself being approached by a 'vanishing discount' limit argument from the corresponding H.J.B. equation for the infinite horizon discounted cost control problem. The most recent development in this direction is [27] where the H.J.B. equation is studied under a condition on the gradient of the cost. Another recent work [12] also focuses on the H.J.B. equation, but

treats it as a limiting case of finite horizon problems instead of discounted cost problems on infinite time horizon. The only direct proof of existence of an optimal stable Markov control by probabilistic compactness arguments seems to be [21], which also considers the corresponding maximum principle.

These works share one or more of the following limitations:

- (a) Optimality in the mean and not a.s. optimality is considered.
- (b) Optimality is established only within the class of Markov controls and not with respect to all admissible controls.
- (c) The system model is often more restrictive than the above, e.g. it is sometimes assumed that  $\sigma =$  the identity matrix and  $m(x,u) = u$ .
- (d) Either a blanket stability assumption is imposed or a condition on the cost function which penalizes instability is assumed.

It is clear that some condition on cost or stability must be necessary to give the desired existence of an optimal stable Markov control. For example, consider the case

$$c(x,u) = \exp(-\|x\|^2).$$

Then the cost of any stable Markov control is a.s. positive while that of an unstable Markov control is a.s. zero, making the latter optimal.

In this paper, we extend the approach of [7], [8], [11], to multidimensional diffusions. In the one dimensional case, this was partially done in [5], [9]. These works, however, use many specificities of the one dimensional case in a crucial manner. Here we address only the

first of the two issues mentioned above viz. the existence of stable optimal Markov controls, thus subsuming the results of [9]. The second issue viz. the dynamic programming equations will be treated in a subsequent publication [15]. The advantages of our approach are the following:

- (1) a.s. optimality (as opposed to optimality in the mean) of a stable Markov control is established in the class of all admissible controls.
- (2) The approach has a more probabilistic flavour than the previous ones and brings out certain features of the problem (e.g., asymptotics for the empirical measures) not apparent in the latter.

The main disadvantage of our approach is that we have to work with the larger class of relaxed controls. This means that we assume  $U$  to be of the form  $P(V)$  = the space of probability measures on some compact metric space  $V$  with the topology of weak convergence and  $c, m$  to be of the form

$$c(x,u) = \int_V \bar{c}(x,y)u(dy), \quad m_i(x,u) = \int_V \bar{m}_i(x,y)u(dy), \quad 1 \leq i \leq n$$

for some  $\bar{c}: R^n \times V \rightarrow R$  and  $\bar{m}: R^n \times V \rightarrow R^n$ ,  $\bar{m}(\cdot, \cdot) = [\bar{m}_1(\cdot, \cdot), \dots, \bar{m}_n(\cdot, \cdot)]$ , which satisfy the same hypotheses as  $c, m$  resp., but with  $V$  replacing  $U$ . Note that any  $V$ -valued process  $v(\cdot)$  can be identified with a  $U$ -valued process  $u(\cdot)$  defined by  $u(t) =$  the Dirac measure at  $v(t)$  for  $t \geq 0$ . Thus relaxed controls subsume controls in the ordinary sense. In fact, if  $c$  has no explicit control dependence and  $m(x,U)$  is convex for each  $x$ , each relaxed control can be identified with a control in the ordinary sense by a

straightforward application of the selection theorem in Lemma 1.1 [3], as was pointed out in [9]. In [5], it was shown in the one dimensional case that the dynamic programming equations allow one to do away with the relaxed control framework. Analogous development in the multidimensional case will be reported in [15].

The use of relaxed controls is tantamount to compactifying the space of control trajectories in a certain precise sense. A nice exposition of this can be found in [2], Section 1.9, pp. 31-36. The concept of relaxed controls was first introduced in deterministic control theory in [31]. Its use in stochastic control dates back to [14].

For a stable markov control  $v$ , we shall denote by  $\eta_v$  the corresponding unique invariant probability measure for  $X(\cdot)$ . We assume throughout this paper that at least one stable Markov control  $v$  exists such that

$$\int c(x, v(x)) \eta_v(dx) \leq \infty.$$

Thus

$$\alpha = \inf_{v \text{ stable Markov}} \int c(x, v(x)) \eta_v(dx) \quad (1.5)$$

is well-defined. We shall prove our existence result under two sets of assumptions. In the first one, we assume that  $c$  is near-monotone in the sense that it satisfies

$$\liminf_{\|x\| \rightarrow \infty} \inf_{u \in U} c(x,u) > \alpha \quad (1.5)$$

The terminology is suggested by the fact that (1.5) is always satisfied when  $c(x,u) = k(\|x\|)$  for a monotone increasing  $k: \mathbb{R}^+ \rightarrow \mathbb{R}$ . Such costs discourage unstable behaviour for obvious reasons and arise often in practice.

The second case we shall consider is a Liapunov-type stability condition the details of which are left to Section III. For the time being, we only mention that in particular it implies the stability of all Markov controls.

The plan of the paper is as follows: Section II establishes a characterization of a.s. limit sets for empirical measures of the joint state and control process along the lines of [9]. This leads to the existence result in the near-monotone case. Section III gives a full statement of the Liapunov condition mentioned above and uses it to prove certain moment bounds for a class of stopping times to be defined later, which in turn implies that all Markov controls are stable and the set of their invariant probability measures is compact in  $P(\mathbb{R}^n)$ . ( $P(S)$  will always denote the space of probability measures on a Polish space  $S$  with the topology of weak convergence.) Section IV proves the existence of an optimal stable Markov controls under the conditions of Section III.



## II. EXISTENCE IN THE NEAR-MONOTONE CASE

The key result of this section is Lemma 2.2, which characterizes the a.s. limit sets of the process of empirical measures we are about to define. This immediately leads to the desired existence result for a near-monotone cost (Theorem 2.1).

Let  $\bar{R}^n = R^n \cup \{\infty\}$  be the one point compactification of  $R^n$  and let  $H = \{Ax \in B \mid A, B \text{ Borel subsets of } \bar{R}^n, V \text{ resp.}\}$ . For  $t \geq 0$ , define the empirical measure  $\nu_t$  on  $H$  by

$$\nu_t(Ax \in B) = \frac{1}{t} \int_0^t I\{X(s) \in A\} u(s, B) ds$$

for  $X(\cdot)$ ,  $u(\cdot)$  as in (1.1), with

$$u(s, B) = \int_B du(s), \quad B \in V.$$

For each fixed sample point and fixed  $t$ ,  $\nu_t$  extends uniquely to a  $\mu_t \in P(\bar{R}^n \times V)$ . This defines the process of empirical measures  $\mu_t$ ,  $t \geq 0$ , taking values in  $P(\bar{R}^n \times V)$ . Since the latter is a compact space (because  $\bar{R}^n \times V$  is compact),  $\{\mu_t\}$  converges to a sample point dependent compact subset of  $P(\bar{R}^n \times V)$  as  $t \rightarrow \infty$ .

Each  $\eta \in P(\bar{R}^n \times V)$  can be decomposed as

$$\eta(A) = \delta(\eta) \eta'(A \cap (R^n \times V)) + (1 - \delta(\eta)) \eta''(A \cap (\{\infty\} \times V)) \quad (2.1)$$

for  $A$  Borel in  $\bar{R}^n \times V$ , where  $\delta(\eta) \in [0, 1]$ ,  $\eta' \in P(R^n \times V)$  and  $\eta'' \in P(\{\infty\} \times V)$ . This

decomposition can be rendered unique by imposing a fixed choice of  $\eta' \in P(\mathbb{R}^n \times V)$  (resp.  $\eta'' \in P(\{\infty\} \times V)$ ) when  $\delta(\eta) = 0$  (resp. 1). Disintegrate  $\eta'$  as follows:

$$\int_{\mathbb{R}^n \times V} f(x,y) \eta'(dx,dy) = \int_{\mathbb{R}^n} \int_V f(x,y) v_\eta(x,dy) \eta^*(dx) \quad (2.2)$$

for all bounded continuous  $f: \mathbb{R}^n \times V \rightarrow \mathbb{R}$ , where  $\eta^*$  is the image of  $\eta'$  under the projection  $\mathbb{R}^n \times V \rightarrow \mathbb{R}^n$  and  $v_\eta(x, \cdot) \in U$  for  $x \in \mathbb{R}^n$  is the regular conditional law. Then the map  $x \rightarrow v_\eta(x, \cdot): \mathbb{R}^n \rightarrow U$  can be identified with a Markov control which we also denote by  $v_\eta$  (i.e.,  $v_\eta(x) \in U$  is defined by  $v_\eta(x) = v_\eta(x, \cdot)$ , the r.h.s. defined as above.) Note that this  $v_\eta$  is defined only  $\eta^*$ -a.s. We pick any one representative of this a.s. - equivalence class. Throughout this paper, this choice of a representative is immaterial wherever the above decomposition is used.

Thus we have associated with  $\eta \in P(\overline{\mathbb{R}^n \times V})$ , the objects  $\delta(\eta) \in [0,1]$ ,  $\eta' \in P(\mathbb{R}^n \times V)$ ,  $\eta'' \in P(\{\infty\} \times V)$ ,  $\eta^* \in P(\mathbb{R}^n)$ ,  $v_\eta: \mathbb{R}^n \rightarrow U$  a Markov control. If in addition  $v = v_\eta$  is stable, we also have its unique invariant probability measure  $\eta_V$ . This notation plays an important role in what follows.

Let  $C_0^2 =$  the Banach space of twice continuously differentiable maps  $\mathbb{R}^n \rightarrow \mathbb{R}$  which, along with their first and second partial derivatives vanish at infinity, with the norm

$$\|f\| = \sup_x |f(x)| + \sum_{i=1}^n \sup_x \left| \frac{\partial f}{\partial x_i}(x) \right| + \sum_{i,j=1}^n \sup_x \left| \frac{\partial^2 f}{\partial x_i \partial x_j}(x) \right|.$$

For any  $f \in C_0^2$ , let

$$(Lf)(x, u) = \sum_{i=1}^n \frac{\partial f}{\partial x_i}(x) \bar{m}_i(x, u) + \frac{1}{2} \sum_{i, j, k=1}^n \sigma_{ik}(x) \sigma_{jk}(x) \frac{\partial^2 f}{\partial x_i \partial x_j}(x)$$

and for any Markov control  $v$ ,

$$(L_v f)(x) = \int_V (Lf)(x, y) v(x, dy)$$

where the meaning of the right hand side is obvious.

Let  $G$  be a countable dense subset of  $C_0^2$ . Then  $G$  is also countable dense in  $C_0 = \{f \in C(\mathbb{R}^n) \mid \lim_{\|x\| \rightarrow \infty} f(x) = 0\}$  with supremum norm. In particular, this implies that it is a convergence determining class and hence a separating class for  $P(\mathbb{R}^n)$  (i.e.,  $\int f d\mu_n \rightarrow \int f d\mu_\infty$  for  $f \in G$ ,  $\{\mu_n, n=1, 2, \dots, \infty\} \subset P(\mathbb{R}^n)$ , implies  $\mu_n \rightarrow \mu_\infty$  in  $P(\mathbb{R}^n)$  and  $\int f d\mu = \int f d\nu$  for  $f \in G$ ,  $\mu, \nu \in P(\mathbb{R}^n)$  implies  $\mu = \nu$ ).

Lemma 2.1. If  $\nu \in P(\mathbb{R}^n)$  satisfies

$$\int L_v f d\nu = 0 \text{ for } f \in G \tag{2.3}$$

for some Markov control  $v$ , then  $\nu = \eta_v$ . (Recall that  $\eta_v$  is the unique invariant probability measure under  $v$ , whose stability is thus a part of the conclusion.)

Proof. This follows in a straightforward manner from Theorem 9.19, pp. 252-253, [13], and the density of  $G$  in  $C_0^2$ .

Lemma 2.2. Outside a set of zero probability, each limit point  $\nu$  of  $\{\nu_t\}$  for which  $\delta(\nu) > 0$ , satisfies

$$\nu^* = \eta_{\nu}^{\eta}. \quad (2.4)$$

Remarks. Note that we do not claim pathwise tightness of  $\{\nu_t\}$ , which would correspond to  $\delta(\nu) = 1$  a.s. This cannot be true in general, e.g. for an unstable Markov control. Thus we must allow for the possibility  $\delta(\nu) < 1$ , which necessitates the compactification of the state space as done above.

Proof. For  $f \in G$ , Ito's formula gives

$$\begin{aligned} f(X(t)) - f(X(0)) &= \int_0^t \int_V Lf(X(s), y) u(s, dy) ds \\ &+ \int_0^t \langle \nabla f(X(s)), \sigma(X(s)) dW(s) \rangle \end{aligned} \quad (2.5)$$

By standard time change arguments (See, e.g., Sect. 6.1 of [13] or Sections 3.1, 4.4 of [17]), the stochastic integral term above can be shown to be of the form  $B(\tau_t)$  for a standard Brownian motion  $B(\cdot)$  and a process of time change  $\tau$  satisfying

$$\limsup_{t \rightarrow \infty} \tau_t/t < \infty \text{ a.s.}$$

Since

$$\lim_{t \rightarrow \infty} \frac{B(\tau_t)}{\tau_t} = 0 \text{ a.s. on } \{\lim_{t \rightarrow \infty} \tau_t = \infty\}$$

$$\text{and } < \infty \text{ a.s. on } \{\lim_{t \rightarrow \infty} \tau_t < \infty\},$$

we have

$$\lim_{t \rightarrow \infty} \frac{B(\tau_t)}{t} = 0 \text{ a.s.}$$

Hence

$$\lim_{t \rightarrow \infty} \frac{1}{t} \int_0^t \int_V Lf(X(s), y) u(s, dy) ds = \lim_{t \rightarrow \infty} \int Lf d\mu_t = 0 \text{ a.s.}$$

Since  $G$  is countable, we can find a set  $N$  of zero probability outside which the above limit holds for all  $f \in G$ . Then outside  $N$ , each limit point  $\nu$  of  $\{\mu_t\}$  with  $\delta(\nu) > 0$  must satisfy

$$\int Lf d\nu = 0 \text{ for } f \in G.$$

The claim follows from Lemma 2.1.

Q.E.D.

Lemma 2.3. Under a stable Markov control  $v$ ,

$$\lim_{t \rightarrow \infty} \frac{1}{t} \int_0^t c(X(s), v(X(s))) ds = \int c(x, v(x)) \eta_v(dx).$$

See [6] for a proof using the ergodic theorem.

Lemma 2.4. For a near-monotone  $c$ , there exists a stable Markov control  $v$  such that

$$\int c(x, v(x)) \eta_v(dx) = \alpha.$$

Proof. Let  $\{v_n\}$  be a sequence of stable Markov controls such that

$$\int c(x, v_n(x)) \eta_{v_n}(dx) \downarrow \alpha.$$

Define  $p_n \in P(\bar{\mathbb{R}}^n \times V)$  by

$$\int_{\bar{\mathbb{R}}^n \times V} f(x, y) p_n(dx, dy) = \int_{\mathbb{R}^n \times V} f(x, y) v_n(x, dy) \eta_{v_n}(dx)$$

for bounded continuous  $f: \bar{\mathbb{R}}^n \times V \rightarrow \mathbb{R}$ . Let  $p_\infty$  be a limit point of  $\{p_n\}$  and let

$$v_\infty = v_{p_\infty}.$$

For  $f \in G$ , we have

$$\int_{V_n} L_{V_n} f d\eta_{V_n} = \int L f dp_n = 0, \quad n=1,2,\dots$$

Letting  $n \rightarrow \infty$  along an appropriate subsequence,

$$\int L f p_\infty = 0.$$

By Lemma 2.1 and the decomposition (2.1),

$$p_\infty^* = \eta_{V_\infty} \quad \text{if } \delta(p_\infty) > 0.$$

Now, the near-monotonicity of  $c$  implies that for some  $\varepsilon > 0$ ,

$$\liminf_{\|x\| \rightarrow \infty} \inf_{u \in V} \bar{c}(x,u) > \alpha + \varepsilon.$$

Using this, one can construct continuous maps  $c^m: \bar{R}^n \times V \rightarrow R$ ,  $m \geq 1$ , such that

$$c^m(\infty, u) = \alpha + \varepsilon, \quad m \geq 1,$$

$$c^m(x, u) \uparrow \bar{c}(x, u) \quad \text{on } R^n \times V.$$

Thus

$$\int \bar{c} dp'_n \geq \int c^m dp_n.$$

Lettings  $n \rightarrow \infty$ ,

$$\lim \int \bar{c} dp'_n = \alpha \geq (\int c^m dp'_\infty) \delta(p_\infty) + (1 - \delta(p_\infty))(\alpha + \varepsilon)$$

Letting  $m \rightarrow \infty$  on the right hand side,

$$\alpha \geq (\int \bar{c} dp'_\infty) \delta(p_\infty) + (1 - \delta(p_\infty))(\alpha + \varepsilon)$$

If  $\delta(p_\infty) > 0$ ,

$$\int \bar{c} dp'_\infty = \int c(x, v_\infty(x)) \eta_{v_\infty}(dx) \geq \alpha$$

by the definition of  $\alpha$ . Hence we must have  $\delta(p_\infty) = 1$  and

$$\int \bar{c} dp'_\infty = \int c(x, v_\infty(x)) \eta_{v_\infty}(dx) = \alpha. \quad \text{Q.E.D.}$$

As remarked earlier,  $v_\infty$  is defined  $p_\infty^*$  - a.s. and it does not matter which representative we pick.

**Theorem 2.1.** For a near-monotone  $c$ , there exists a stable a.s. optimal Markov control.



Proof. Using Lemma 2.2 and arguments similar to those employed in the proof of the above lemma, one can show that

$$\liminf_{t \rightarrow \infty} \frac{1}{t} \int_0^t c(X(s), u(s)) ds \geq \alpha \quad \text{a.s.}$$

The claim now follows from Lemmas 2.3., 2.4.

Q.E.D.

### III. TIGHTNESS OF INVARIANT PROBABILITY MEASURES

In this and the next section, we study the situation where the near-monotonicity condition on the cost is dropped, but instead we impose a Liapunov-type stability condition which among other things, will be shown to imply that all the Markov controls are stable and their invariant probability measures form a compact set in  $P(R^n)$ . This, in fact, is the principal result of this section (Theorem 3.1, Cor. 3.2), the proof of the existence of an a.s. optimal Markov control being left to Section IV.

Before we give a precise statement of this condition, we mention the following technical lemma:

Lemma 3.1. Let  $X_0 = x \in R^n$ ,  $t > 0$ ,  $u(\cdot)$  an admissible control. Then the law of  $X(t)$  has a density  $p(t, x, \cdot)$  with respect to the Lebesgue measure on  $R^n$ , satisfying

$$c_1 \exp(-c_2 \|x-y\|/t) \leq p(t, x, y) \leq c_3 \exp(-c_4 \|x-y\|^2/t) \quad (3.1)$$

for some constants  $c_i > 0$ ,  $i=1,2,3,4$ , independent of  $x, t, u(\cdot)$ .

Proof. If  $u(\cdot)$  is an inhomogeneous Markov control, this is precisely the estimate of [1]. For arbitrary  $u(\cdot)$ , the law of  $X(t)$  is the same as that under some inhomogeneous Markov control by the results of [10] and we are done. Q.E.D.

The Liapunov-type condition we use is the following:

Assumption A. There exists a twice continuously differentiable function  $w: \mathbb{R}^n \rightarrow \mathbb{R}$  satisfying:

$$(i) \quad \lim_{\|x\| \rightarrow \infty} w(x) = +\infty \text{ uniformly in } \|x\|, \quad (3.2)$$

(ii) there exist  $a > 0$ ,  $\varepsilon_0 > 0$  such that whenever  $\|x\| > a$ ,

$$Lw(x, u) < -\varepsilon_0 \text{ for all } u \in U, \quad (3.3)$$

$$\|\nabla w\|^2 > \varepsilon_0, \quad (3.4)$$

$$(iii) \quad \int_0^T \int_{\mathbb{R}^n} \|\sigma^T(x) \nabla w(x)\|^2 \exp(-c_4 \|x-y\|^2/t) dx dt < \infty, \quad \forall T > 0, \quad (3.5)$$

where  $c_4$  is as in Lemma 3.1.

Remarks. (a) (3.5) is a mild technical condition that ensures (by virtue of Lemma 3.1) that the stochastic integral

$$\int_0^T \langle \nabla w(X(t)), \sigma(X(t)) dW(t) \rangle, \quad T > 0,$$

is always well-defined.

(b) We have chosen the above formulation of a Liapunov-type condition because it is easily stated and still quite general. Other variants are possible (see, e.g., [21] for one). For the general theory of stochastic Liapunov functions, see [19]. The key consequence of the above assumption for our purposes is Lemma 3.2 below. Thus any condition that implies Lemma 3.2 will suffice. In fact, the crudeness of estimates used in proving the

lemma shows that there is ample scope for improvement.

(c) As an example, consider  $n=1$ ,  $\sigma(\cdot) \equiv 1$ ,  $m(x,u) \leq -\varepsilon$  for  $x$  sufficiently large and  $\geq \varepsilon$  for  $-x$  sufficiently large for some  $\varepsilon > 0$ . Then  $w(x) = x^2$  will do the job.

Let  $B_1, B_2 \subset \mathbb{R}^n$  be concentric balls centered at zero with radii  $r_1, r_2$  and boundaries  $\delta B_1, \delta B_2$  resp., where we choose  $r_2 > r_1 > a$  such that for some  $a_1 > 0$ ,  $\{x \mid |w(x)| \leq a_1\}$  is nonempty and contained in  $B_1$ . Let  $a_2 = \max_{x \in \delta B_2} |w(x)|$  and  $a_3 = a_1 - a_2$ .

Lemma 3.2. Let  $X_0 = x \in \delta B_2$  and  $\tau = \inf\{t \geq 0 \mid X(t) \in \delta B_1\}$ . Then

$$\sup E[\tau^2] < \infty \tag{3.6}$$

where the supremum is over all  $x \in \delta B_2$  and all admissible  $u(\cdot)$ .

Proof. For  $t > 0$ ,

$$\begin{aligned} P(\tau \geq t) &= P\left(\min_{s \in [0, t]} W(X(s)) \geq a_1, \tau \geq t\right) \\ &\leq P\left(\min_{y \in [0, t]} \int_0^y \langle \nabla w(X(s)), \sigma(X(s)) dW(s) \rangle \geq a_3 + \varepsilon_0 t\right) \end{aligned}$$

by (3.3). Using the random time change argument we used earlier,

$$\int_0^t \langle \nabla w(X(s)), \sigma(X(s)) dW(s) \rangle = B(\xi(t))$$

for a standard Brownian motion  $B(\cdot)$  with

$$\xi(t) = \int_0^t \|\sigma^T(X(s))\nabla w(X(s))\|^2 ds \geq \lambda \varepsilon_0 t.$$

(Recall that  $\lambda$  is the ellipticity constant for  $\sigma\sigma^T$ .) Thus

$$\begin{aligned} P(\tau \geq t) &\leq P(B(\lambda \varepsilon_0 t) \geq \varepsilon_0 t + a_3) \\ &= (2\pi\lambda \varepsilon_0 t)^{-1/2} \int_{a_3 + \varepsilon_0 t}^{\infty} \exp(-y^2/2\lambda \varepsilon_0 t) dy \end{aligned}$$

It is not hard to verify from this that

$$\int_0^{\infty} tP(\tau \geq t) dt < K < \infty$$

where the constant  $K$  is independent of the choice of  $x$  in  $\delta B_2$  and of  $u(\cdot)$ .

The claim follows.

Q.E.D.

Now take  $X_0 = x \in \bar{B}_2$  and define  $\tau' = \inf\{t \geq 0 \mid X(t) \in \delta B_2\}$ . We have the following companion result to the above, which, however, does not need Assumption A.

Lemma 3.3.

$$\sup E[(\tau')^2] < \infty \tag{3.7}$$

where the supremum is over  $x \in \bar{B}_2$  and admissible  $u(\cdot)$ .

In order to prove this result, we need another technical lemma, Lemma

3.4 below, which will also be useful elsewhere in this paper. Let  $\{F_t\}$  denote the natural filtration of  $X(\cdot)$ .

Lemma 3.4. For any  $\{F_t\}$ -stopping time  $\tau$ , the regular conditional law of  $X(\tau+\cdot)$  given  $F_\tau$  on  $\{\tau < \infty\}$  is a.s. the law of a controlled diffusion of the same type as (1.1).

Proof. The results of [30] (See Theorem 4.3 and the final comments on page 632) allow us to assume without any loss of generality that  $\{F_t\}$  is the canonical filtration on  $C([0, \infty); \mathbb{R}^n)$  and  $u(\cdot)$  is of the form

$$u(t) = G(t, X(\cdot))$$

for some measurable  $G: [0, \infty) \times C([0, \infty); \mathbb{R}^d) \rightarrow U$  which is progressively measurable with respect to  $\{F_t\}$ . By Lemma 1.3.3, pp. 33, of [26], a version of the regular conditional law of  $X(\tau+\cdot)$  given  $F_\tau$  on  $\{\tau < \infty\}$  will be a.s. given by the law of a controlled diffusion  $X(\cdot)$  as in (1.1), but with initial condition  $X(\tau)$  and control  $\bar{u}(\cdot)$  given by  $\bar{u}(t) = G(\tau+t, X(\cdot))$  with  $\tau$  and the restriction of  $X(\cdot)$  to  $[0, \tau]$  being held fixed as parameters. Q.E.D.

From here on,  $M_i(S)$ ,  $S \subset \mathbb{R}^n$ ,  $i=1,2$ , will denote the set of  $X(\cdot)$  as in (1.1) under Markov/arbitrary admissible controls resp. with initial law supported in  $S$ .

Proof of Lemma 3.3. By the results of [10], the law of  $X(t)$  for any  $t > 0$  coincides with that under some inhomogeneous Markov control and thus by the uniform ellipticity assumption on  $\sigma\sigma^T$ , is absolutely continuous with respect

to the Lebesgue measure (Recall (3.1).) Let  $X(\cdot) \in M_2(\bar{B}_2)$ ,  $\tau = \inf\{t \geq 0 | X(t) \in \bar{B}_2\}$ . Then for  $t > 0$ ,

$$P(\tau=t) \leq P(X(t) \in \delta B_2) = 0$$

and thus  $P(\tau=t) = 0$ . Fix  $t > 0$ . Let  $\{X^n(\cdot)\}$  be a sequence in  $M_2(\bar{B}_2)$  such that if  $\{\tau^n\}$  denote the corresponding first exit times from  $\bar{B}_2$ ,

$$P(\tau^n > t) \uparrow \sup_{X(\cdot) \in M_2(\bar{B}_2)} P(\tau > t).$$

As in the proof of Theorem 3.1, [20], one can argue that  $X^n(\cdot) \rightarrow X^\infty(\cdot)$  in law along a subsequence (denoted  $\{n\}$  again by abuse of notation) where  $X^\infty(\cdot) \in M_2(\bar{B}_2)$ . (The only difference with Theorem 3.1 of [20] is the varying initial law. This can, however, be easily accommodated since the initial laws are supported on  $\bar{B}_2$  and hence are tight.) By Skorohod's theorem ([17], pp. 9), we may assume that this convergence is a.s. on a common probability space. (See [20] for an analogous argument.) Let  $\tau^\infty = \inf\{t \geq 0 | X^\infty(t) \in \bar{B}_2\}$  and  $\bar{\tau} = \inf\{t \geq 0 | X^\infty(t) \in \delta B_2\}$ . Path continuity of  $\{X^n(\cdot)\}$  and simple geometric considerations show that for any sample point, any limit point of  $\{\tau^n\}$  in  $[0, \infty]$  must lie in  $[\bar{\tau}, \tau^\infty]$ . By our uniform ellipticity condition on  $\sigma\sigma^T$ ,  $\bar{\tau} = \tau^\infty$  a.s. Thus  $\tau^n \rightarrow \tau^\infty$  a.s. Since  $P(\tau^\infty=t) = 0$ ,  $P(\tau^n > t) \rightarrow P(\tau^\infty > t)$ . Since

$$P(\tau^\infty > t) \leq P(X^\infty(t) \in \bar{B}_2) < 1,$$

we have

$$\beta = \sup_{X(\cdot) \in M_2(\bar{B}_2)} P(\tau > t) < 1.$$

Hence for  $X(\cdot) \in M_2(\delta B_1) \subset M_2(\bar{B}_2)$ ,

$$\begin{aligned} P(\tau > nt) &= E[I\{\tau > nt\}] \\ &= E[E[I\{\tau > nt\} / F_{(n-1)t} | I\{\tau > (n-1)t\}]] \\ &\leq \beta E[I\{\tau > (n-1)t\}] \end{aligned}$$

by Lemma 3.4. Iterating the argument,

$$P(\tau > nt) \leq \beta^n$$

The rest is easy.

Q.E.D.

Define the extended real-valued stopping times

$$\tau_1 = \inf\{t \geq 0 \mid X(t) \in \delta B_1\} \quad (3.8)$$

$$\xi_n = \inf\{t \geq \tau_n \mid X(t) \in \delta B_2\} \quad (3.9)$$

$$\tau_{n+1} = \inf\{t \geq \xi_n \mid X(t) \in \delta B_1\} \quad (3.10)$$

for  $n=1,2,\dots$ , where as usual the quantity on the left is set equal to  $+\infty$  if the set on the right is empty.



Let  $v$  be a Markov control and  $X(\cdot)$  the corresponding process with initial law supported on  $\delta B_1$ . By the above three lemmas,  $E[\xi_i], E[\tau_i] < \infty$  for all  $i$  with  $\tau_1=0$ . Then  $X(\tau_i), i=1,2,\dots$ , is a  $\delta B_1$ -valued Markov chain having a unique invariant probability measure (say,  $q$ ) as argued in [18].

Corollary 3.1. The measure  $\eta \in P(R^n)$  defined by

$$\int f d\eta = E\left[\int_0^{\tau_2} f(X(t))dt\right]/E[\tau_2], \quad f \in C_b(R^n),$$

with the law of  $X(0) = q$ , coincides with  $\eta_v$ . (In particular,  $v$  is stable).

For a proof, see [18].

Let  $\{v_n\}$  be a sequence of Markov controls and  $X^n(\cdot)$  the corresponding diffusions as in (1.1) for some initial laws and suppose that  $X^n(\cdot) \rightarrow X^\infty(\cdot)$  in law for some process  $X^\infty(\cdot)$ .

Lemma 3.5.  $X^\infty(\cdot)$  is a diffusion satisfying (1.1) for some Markov control.

Proof. Let  $T_{s,t}^n, t \geq s$ , denote the transition semigroup for  $X^n(\cdot)$ ,  $n \geq 1$ . Let  $f \in C^2(R^n)$  with compact support and  $g \in C_b(R^n \times R^n \times \dots \times R^n)$  ( $m$  times) for some  $m \geq 1$ . Then for  $t \geq s \geq t_m \geq t_{m-1} \geq \dots \geq t_1 \geq 0$ ,  $E[(f(X^n(t)) - T_{s,t}^n f(X^n(s)))g(X^n(t_1), \dots, X^n(t_m))] = 0, n=1,2,\dots$ . For each  $n$ ,  $T_{s,t}^n f(\cdot)$  satisfies the appropriate backward Kolmogorov equation. From standard p.d.c. theory (See [22], Ch. III, or [32], pp. 133-134), it follows that  $T_{s,t}^n f(\cdot), n=1,2,\dots$ , are equicontinuous. Since they are clearly bounded, they form a sequentially precompact set in  $C(R^n)$  with the topology of

uniform convergence on compacts. Let  $T_{s,t}f(\cdot)$  be a limit point of the same in  $C(\mathbb{R}^n)$ . Passing to the limit in the above as  $n \rightarrow \infty$ , it is easily seen (e.g., using Skorohod's theorem) that  $E\{(f(X^\omega(t)) - T_{s,t}f(X^\omega(s)))g(X^\omega(t_1), \dots, X^\omega(t_m))\} = 0$ . Since  $f, g, \{t_i\}$  were arbitrary, a standard argument using the monotone class theorem establishes the Markov property of  $X^\omega(\cdot)$ . By Theorem 3.1 of [20],  $X^\omega(\cdot)$  satisfies (1.1) for some  $u(\cdot)$ . Argue as in [15], pp. 184-5, to conclude that  $u(\cdot)$  must be on the form  $u(\cdot) = v(X^\omega(\cdot), \cdot)$  for some measurable map  $v: \mathbb{R}^n \times \mathbb{R}^+ \rightarrow U$ . Since  $T_{s,t}^\omega f$  depends on  $t, s$  only through  $t-s$  for each  $f$  and  $n=1, 2, \dots$ , the same must be true for  $T_{s,t}f$  in view of the above limiting argument. It follows that  $X^\omega(\cdot)$  is a time-homogeneous Markov process and hence  $u(\cdot)$  is in fact a Markov control. QED

Theorem 3.1. The set  $\{\eta_v \mid v \text{ Markov control}\}$  is compact in  $P(\mathbb{R}^n)$ .

Proof. Let  $\{v_n\}$  be a sequence of Markov controls and  $X^n(\cdot)$  the corresponding diffusions whose initial laws will soon be specified. Define  $\{\tau_1^n\}, \{\xi_1^n\}$  as in (3.8)-(3.10) correspondingly. Let  $q^n$  be the unique invariant probability measure for the chain  $\{X^n(\tau_1^n)\}$ . Set the law of  $X^n(0)$  equal to  $q^n$  for each  $n=1, 2, \dots$ . Argue as in the proof of Theorem 3.1, [19], to conclude that  $X^n(\cdot) \rightarrow X^\omega(\cdot)$  in law along a subsequence, denoted  $\{n\}$  again by abuse of notation. By Lemma 4.5,  $X^\omega(\cdot)$  satisfies (1.1) for some Markov control  $v_\omega$ . Invoke Skorohod's theorem as before to assume that the above convergence is a.s. on a common probability space. Define  $\{\tau_1^\omega\}, \{\xi_1^\omega\}$  as in (3.8) - (3.10) for  $X^\omega(\cdot)$ . By arguments similar to those used to prove  $\tau_1^n \rightarrow$

$\tau^\infty$  a.s. in the proof of Lemma 3.3, we can inductively prove that

$$\tau_i^n \rightarrow \tau_i^\infty \text{ a.s.}, \quad \xi_i^n \rightarrow \xi_i^\infty \text{ a.s. for all } i. \quad (3.11)$$

Thus

$$X^n(\tau_i^n) \rightarrow X^\infty(\tau_i^\infty) \text{ a.s.}$$

$$\int_{\tau_i^n}^{\tau_{i+1}^n} f(X^n(s)) ds \rightarrow \int_{\tau_i^\infty}^{\tau_{i+1}^\infty} f(X^\infty(s)) ds \text{ a.s. for all } i \quad (3.12)$$

where  $f \in C_b(\mathbb{R}^n)$ . By Lemmas 3.2, 3.3,

$$\sup_n E[(\tau_2^n)^2] < \infty \quad (3.13)$$

and hence  $\{\tau_2^n, n \geq 1\}$  are uniformly integrable. Thus

$$E[\tau_2^n] \rightarrow E[\tau_2^\infty] \quad (3.14)$$

$$E\left[\int_0^{\tau_2^n} f(X^n(s)) ds\right] \rightarrow E\left[\int_0^{\tau_2^\infty} f(X^\infty(s)) ds\right], \quad f \in C_b(\mathbb{R}^n)$$

by (3.11), (3.12). By Corollary 3.1,

$$\int f d\eta_{v_n} = \frac{E\left[\int_0^{\tau_2^n} f(X^n(s)) ds\right]}{E[\tau_2^n]}$$

$$\frac{E\left[\int_0^{\tau_2^\infty} f(X^\infty(s)) ds\right]}{E[\tau_2^\infty]} \tag{3.15}$$

Since  $\tau_1^n = 0$  a.s.  $n=1,2,\dots$ ,  $\tau_1^\infty=0$  a.s. since for each  $n=1,2,\dots$ ,  $\{X^n(\tau_i), i=1,2,\dots\}$  are identically distributed, it follows that  $\{X^\infty(\tau_i), i=1,2,\dots\}$  are identically distributed. Thus the initial law of  $X^\infty(\cdot)$  equals the unique invariant probability measure for the chain  $\{X^\infty(\tau_i), i=1,2,\dots\}$ . Hence by Corollary 3.1, the right hand side of (3.15) equals  $\int f d\eta_v$ . Thus  $\eta_v \rightarrow \eta_v$  in  $P(R^N)$ . The claim follows. QED

Corollary 3.2. There exists a Markov control  $v$  such that

$$\int c(x, v(s)) \eta_v(dx) = \alpha.$$

Proof. Pick  $\{v_n\}$  above so that

$$\int c(x, v_n(x)) \eta_{v_n}(dx) \downarrow \alpha.$$

Define  $p_n \in P(R^N \times V)$ ,  $n=1,2,\dots$ , by

$$\int f(x,y)p_n(dx,dy) = \iint f(x,y)v_n(x,dy)\eta_{v_n}(dx), f \in C_b(\mathbb{R}^n \times V).$$

Since  $V$  is compact, the above theorem implies that  $\{p_n\}$  is tight in  $P(\mathbb{R}^n \times V)$  and hence converges along a subsequence (denoted  $n$  again) to some  $p_\infty \in P(\mathbb{R}^n \times V)$ . Argue as in the proof of Lemma 2.4 to conclude that  $p_\infty$  is of the form

$$p_\infty(dx,dy) = \eta_v(dx)v(x,dy)$$

for some Markov control  $v$ . Then

$$\int c(x,v(x))\eta_v(dx) = \alpha$$

follows from Fatou's lemma and the definitions of  $\alpha$ .

Q.E.D.

## IV. EXISTENCE OF AN OPTIMAL MARKOV CONTROL UNDER ASSUMPTION A.

In this section, we shall show that the Markov control in the statement of Corollary 3.2 is a.s. optimal. Before we get down to the main result (Theorem 4.1), we shall collect together a few minor consequences of the foregoing that will be used later.

Lemma 4.1.  $\{E[\tau_2] | X(\cdot) \in M_1(\delta B_1)\}$  is bounded from above and bounded away from zero from below.

Proof. The upper bound follows from Lemmas 3.2-3.4 in an obvious manner. An argument similar to that leading to (3.14) can be employed to show the rest. Q.E.D.

Lemma 4.2. The set of probability measures  $\eta$  defined by

$$\int_{\mathbb{R}^n} f d\eta = E\left[\int_0^{\tau_2} f(X(t)) dt\right] / E[\tau_2], \quad f \in C_b(\mathbb{R}^n),$$

for  $X(\cdot) \in M_1(\delta B_1)$  is tight in  $P(\mathbb{R}^n)$ .

Proof. This can be proved the same way as Theorem 3.1 by showing that each sequence has a subsequence that converges in  $P(\mathbb{R}^n)$ . Q.E.D.

Let  $\{f_n\}$  be a collection of smooth maps  $\mathbb{R}^n \rightarrow [0,1]$  such that  $f_n(x) = 0$  for  $\|x\| \leq n$  and  $=1$  for  $\|x\| \geq n+1$ .

Lemma 4.3. For any  $\varepsilon \geq 0$ , there exists  $N_\varepsilon \geq 1$  such that for all  $n \geq N_\varepsilon$  and

$n$

$$X(\cdot) \in M_1(\delta B_1),$$

$$E\left[\int_0^{\tau_2} f_n(X(s))ds\right] < \varepsilon.$$

Proof. Let  $Y(\cdot) = X(\xi_1 + \cdot)$ . Let  $\beta_m$  denote the first exit time from  $\{x \mid \|x\| \leq m\} \setminus B_1$  where  $m$  is any integer sufficiently large so that  $\|x\| < m$  for  $x \in B_2$ . (We do not specify for which process, leaving that to depend on the context for economy of notation). Consider the control problem for  $\bar{X}(\cdot) \in M_2(\delta B_2)$  with the cost

$$E\left[\int_0^{\beta_m} f_n(\bar{X}(s))ds\right]$$

for some  $n, m$ . By the results of [4], Section IV.3, pp. 150-155, an optimal Markov control exists for this problem. Thus

$$E\left[\int_0^{\beta_m} f_n(Y(s))ds\right] \leq \sup_{\bar{X}(\cdot) \in M_1(\delta B_2)} E\left[\int_0^{\beta_m} f_n(\bar{X}(s))ds\right].$$

For large  $n$ ,  $f_n = 0$  on  $B_2$  and hence the above is the same as

$$E\left[\int_0^{\gamma_m} f_n(X(s))ds\right] \leq \sup_{\bar{X}(\cdot) \in M_1(\delta B_1)} E\left[\int_0^{\bar{\gamma}_m} f_n(\bar{X}(s))ds\right]$$

where  $\gamma_m$  (resp.  $\bar{\gamma}_m$ ) =  $\inf\{t \geq \xi_1 \mid X(t) \text{ (resp. } \bar{X}(t)) \in \{x \mid \|x\| \leq m\} \setminus B_1\}$ . Since

$\gamma_m \uparrow \tau_2$  a.s., we have

$$E\left[\int_0^{\tau_2} f_n(X(s))ds\right] \leq \sup_{\bar{X}(\cdot) \in M_1(\delta B_1)} E\left[\int_0^{\tau_2} f_n(\bar{X}(s))ds\right] \leq \varepsilon. \quad (4.1)$$

for  $n$  sufficiently large, by virtue of Lemmas 4.1, 4.2.

Q.E.D.

Lemma 4.4. The set  $\{E[\tau_2] | X(\cdot) \in M_2(\delta B_1)\}$  is bounded from above and bounded away from zero from below.

Proof. The first claim is proved by the same arguments that imply the first half of Lemma 4.1. The second claim follows by arguments similar to those used to prove a similar claim for  $M_1(\delta B_1)$  in Lemma 4.1 with the following change: One considers a sequence  $\{X^n(\cdot)\}$  in  $M_2(\delta B_1)$  instead of  $M_1(\delta B_1)$ , with initial laws arbitrary in  $P(\delta B_1)$ .

Q.E.D.

We can now prove the main result of this section:

Theorem 4.1. There exists an a.s. optimal Markov control.

Proof. Let  $X(\cdot)$  be as in (1.1). By Lemmas 3.2-3.4,  $\tau_i < \infty$  a.s. for all  $i$ . Thus for  $\{f_n\}$  as in Lemma 4.3,



$$\limsup_{t \rightarrow \infty} \frac{1}{t} \int_0^t f_n(X(s)) ds = \limsup_{m \rightarrow \infty} \frac{\sum_{i=1}^m \int_{\tau_i}^{\tau_{i+1}} f_n(X(s)) ds}{\sum_{i=1}^m (\tau_{i+1} - \tau_i)} \quad (4.2)$$

By Lemmas 3.4 and 4.3, for any  $\varepsilon > 0$ , there exists  $N_\varepsilon \geq 1$  such that for all  $n \geq N_\varepsilon$ ,  $i \geq 1$ ,

$$E\left[\int_{\tau_i}^{\tau_{i+1}} f_n(X(s)) ds / \mathcal{F}_{\tau_i}\right] < \varepsilon \quad \text{a.s.} \quad (4.3)$$

By Lemmas 3.2-3.4,

$$\sup_i E[(\tau_{i+1} - \tau_i)^2] < \infty.$$

Hence one can use the strong law of large numbers for square-integrable martingales ([23], pp.53) to conclude that

$$\lim_{n \rightarrow \infty} \frac{1}{n} \sum_{i=1}^n \left[ \int_{\tau_i}^{\tau_{i+1}} f_n(X(s)) ds - E\left[\int_{\tau_i}^{\tau_{i+1}} f_n(X(s)) ds / \mathcal{F}_{\tau_i}\right] \right] = 0 \quad \text{a.s.} \quad (4.4)$$

$$\lim_{n \rightarrow \infty} \frac{1}{n} \sum_{i=1}^n [(\tau_{i+1} - \tau_i) - E[(\tau_{i+1} - \tau_i) / \mathcal{F}_{\tau_i}]] = 0 \quad \text{a.s.} \quad (4.5)$$

From Lemmas 3.4, 4.4 and (4.2)-(4.5) above, we conclude that

$$\limsup_{t \rightarrow \infty} \frac{1}{t} \int_0^t f_n(X(s)) ds < C\varepsilon \quad \text{a.s.}$$

for  $n$  large enough, with some constant  $C$  independent of  $n$ . Recalling the definition of  $\{f_n\}$ , it is easily deduced from this that in the set-up of Lemma 2.2,  $\delta(y) = 1$  for all limit points  $y$  of  $\{y_t\}$  outside a set of zero probability. The claim now follows as in the proof of Theorem 2.1 in view of Corollary 3.2. Q.E.D.

Remarks. Let  $n=1$ . Pick Markov controls  $v_1, v_2$  such that  $m(x, v_1(x)) = \max m(x, v)$ ,  $m(x, v_2(x)) = \min m(x, v)$ . Our conditions on  $m$  and the selection theorem of Lemma 1, [3], guarantee the existence of  $v_1, v_2$  as above. Let  $X(\cdot)$  be as in (1.1) for some admissible control  $u(\cdot)$  and  $X_1(\cdot), X_2(\cdot)$  be the diffusions controlled by  $v_1, v_2$  resp. with the same initial condition as  $X(\cdot)$ . (Recall that a strong solution to Markov-controlled (1.1) exists [29]. Thus we can construct  $X(\cdot), X_1(\cdot), X_2(\cdot)$  on the same probability space.) By the well-known comparison theorem for one dimensional Ito processes ([17], pp. 352-355), it follows that outside a set  $N'$  of zero probability,

$$X_2(t) \leq X(t) \leq X_1(t) \text{ for all } t \geq 0. \quad (4.4)$$

Suppose we assume that  $v_1, v_2$  are stable. Then (4.4) implies in a straightforward manner that

- (i) all Markov controls are stable,
- (ii)  $\delta(\mu)$  in Lemma 2.2 can always be taken to be 1 outside  $NUN'$  (N as in Lemma 2.2),
- (iii)  $H = \{\eta_v | v \text{ Markov}\}$  is compact.

Thus in the one dimensional case, we have the conclusion of Theorem 4.1 under a seemingly more general set-up than that of Assumption A.

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