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ERGODIC EQUIVALENCE RELATIONS, COHOMOLOGY, AND VON NEUMANN ALGEBRAS. I

BY

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*These papers are dedicated to George Mackey on his 60th birthday.**

ABSTRACT. Let $(X, \mathfrak{B})$ be a standard Borel space, $R \subset X \times X$ an equivalence relation $\in \mathfrak{B} \times \mathfrak{B}$. Assume each equivalence class is countable. *Theorem* 1: $\exists$ a countable group G of Borel isomorphisms of $(X, \mathfrak{B})$ so that $R = \{(x, gx): g \in G\}$. G is far from unique. However, notions like invariance and quasi-invariance and R-N derivatives of measures depend only on R, not the choice of G. We develop some of the ideas of Dye [1], [2] and Krieger [1]–[5] in a fashion explicitly avoiding any choice of G; we also show the connection with virtual groups. A notion of "module over R" is defined, and we axiomatize and develop a cohomology theory for R with coefficients in such a module. Surprising application (contained in Theorem 7): let α, β be rationally independent irrationals on the circle $\mathbf{T}$, and f Borel: $\mathbf{T} \to \mathbf{T}$. Then $\exists$ Borel g, h: $\mathbf{T} \to \mathbf{T}$ with $f(x) = (g(\alpha x)/g(x))(h(\beta x)/h(x))$ a.e. The notion of "skew product action" is generalized to our context, and provides a setting for a generalization of the Krieger invariant for the R-N derivative of an ergodic transformation: we define, for a cocycle c on R with values in the group A, a subgroup of A depending only on the cohomology class of c, and in Theorem 8 identify this with another subgroup, the "normalized proper range" of c, defined in terms of the skew action. See also Schmidt [1].

1. **Introduction.** This is the first of a series of two papers which will provide the details of the results announced in Feldman-Moore [1]. The first of these will be devoted more to a study of the equivalence relations and their cohomology, while the second will be devoted more to the application of these results and techniques to the study of von Neumann algebras.

Throughout, X will be a standard Borel space with σ-field $\mathfrak{B}$. If G is some countable group of Borel automorphisms of X we introduce the orbit equivalence relation of this action, namely $R_G = \{(x, y): \exists g \in G, y = g \cdot x\} \subset X \times X$. If μ is a σ-finite measure on X, we say that it is *quasi-invariant* if its null

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*By two of his students, one only virtual.

sets are invariant under the action of the group G. The starting point of this work is two papers by Dye [1] and [2], in which he considers largely the case when $G = Z$ (so he is really studying a single Borel automorphism), and when μ is finite and invariant. Dye introduces the notion of *weak equivalence* (or *orbit equivalence*) for two such Borel automorphisms, which means really that the equivalence relations R_G associated to the two actions are isomorphic (mod null sets). He proceeds to classify Borel automorphisms under this equivalence relation, and shows – for instance – that for ergodic automorphisms there is (surprisingly) only one equivalence class. He introduces a notion of *hyperfiniteness* for such countable group actions, and explores the connections with von Neumann algebras. This work was extended by Krieger in a series of papers, [1], [2], [3], [4], [5] and [6]; Krieger explores the non-measure-preserving case, and the rich and deep connections with the theory of type III factors. Many other workers have made contributions, especially Connes and Takesaki [1], [2], Dang-Ngoc-Nghiem [1], Hamachi, Oka and Osikawa [1], K. Schmidt [1]. There is some overlap of our work and the above, especially Schmidt [1].

Meanwhile, Mackey [2], [3] introduced a program of classifying ergodic group actions by mimicking the classification of *transitive* group actions. This led him to look at R_G for ergodic actions of general locally compact groups; he introduced what he called *ergodic groupoids*; any R_G is a *principal ergodic groupoid* (or, to use another of his terms, an *ergodic equivalence relation*).

Our point of view will likewise be to ignore the group G; we will look just at R_G, and axiomatize that object. Thus we will be considering equivalence relations $R \subset X \times X$ which are Borel subsets of the product, and such that each equivalence class is countable. Both Dye and Mackey noticed that certain notions which superficially seem to depend on the action of G, such as invariance of a measure, ergodicity, Radon-Nikodym derivatives, etc., really can all be described in terms of R_G alone; and we here define them for our axiomatized R. It turns out (somewhat surprisingly) that any such R is an R_G for some countable group. Thus, any such R, if ergodic, is a principal ergodic groupoid. Mackey has defined a notion of similarity for ergodic groupoids, and we compare this with Dye's weak equivalence.

The advantage of working with the relation R itself rather than with a group is that many constructions appear simpler, and their significance becomes clearer.

Further we shall develop a cohomology theory for relations, and for appropriately defined modules over relations. These were introduced in the virtual group context by Westman [1]. We are able to characterize these groups by a simple set of axioms, just as one can do for group cohomology. We describe a number of applications of the cohomology, and conclude this paper with a study of what we call the asymptotic range of a one-cocycle. This notion generalizes the Araki-Woods asymptotic ratio set [1] and the work of Krieger [5]. There is overlap here with Connes-Takesaki [1], [2] and with Hamachi-Oka-Osikawa [1], and some of the results were found earlier by Schmidt [1]. As noted, the second paper will apply these results to the construction and characterization of a certain class of von Neumann algebras.

2. **Countable group actions and equivalence relations.** Let X be a set and $R \subset X \times X$ an equivalence relation. We write $x \sim y$ for $(x, y) \in R$, and define $\pi_l(x, y) = x$, the left projection, and $\pi_r(x, y) = y$, the right projection of R. We let $\theta(x, y) = (y, x)$, the "flip", and note that $\theta^2 = \text{id}$. For any $x \in X$, $R(x) = \{y\colon (x, y) \in R\}$ is the equivalence class of x, and for a subset $A \subset X$, $R(A) = \bigcup\{R(x)\colon (x \in A)\}$, is called the saturation of X. The relation R will be called countable (finite) if $R(x)$ is countable (finite) for each x.

Now if X is in addition a standard Borel space with σ-field $\mathfrak{B}$, then we say that R is standard if R is a Borel subset of $X \times X$; that is, R is in the product σ-field $\mathfrak{B} \times \mathfrak{B}$. We write $\mathcal{C}$ for the restriction of $\mathfrak{B} \times \mathfrak{B}$ to R. The objects of study will be countable standard relations. It is important to notice in this case that π_l and π_r send Borel sets (in R) to Borel sets in X, since these maps are countable to one (cf. Kuratowski [1]). It follows that if A is a Borel set in X, then $R(A)$ is also a Borel set in X. Now if μ is a σ-finite measure on $(X, \mathfrak{B})$ with the property that $\mu(R(A)) = 0$ if $\mu(A) = 0$, then μ will be called *quasi-invariant* for R, and R will be called *nonsingular* with respect to μ. These notions depend only on the equivalence class of μ with respect to absolute continuity. In the presence of a measure one may also speak of R being *a.e. countable* (*finite*) and so on.

The example from which these definitions come is, of course, the following: Let G be a countable group acting on $(X, \mathfrak{B})$ as Borel automorphisms. We let $R_G = \{(x, g \cdot x)\colon x \in X, g \in G\}$. It is not hard to see that R_G is a countable standard equivalence relation. If μ is a σ-finite measure on $(X, \mathfrak{B})$, μ is quasi-invariant for R_G if and only if μ is quasi-invariant for G in the usual sense. The group G may or may not act freely (G is said to act freely on X if for each x the Borel map $(g, x) \mapsto g \cdot x$ of G into R is injective). The first result is that we have not enlarged the category of objects.

THEOREM 1. *If R is a countable standard equivalence relation on $(X, \mathfrak{B})$, then there is a countable group G of Borel automorphisms of X so that $R = R_G$.*

PROOF. Since $(R, \mathcal{C})$ is standard and π_l is countable to one, it follows from Kuratowski [1, §39, III, Corollary 5] that there are countably many Borel sets C_i in $\mathcal{C}$ forming a partition of R and so that π_l is injective on each C_i.

Actually, the reference above requires that the range and domain be complete separable metric spaces and that the function be continuous, but §37, II, Corollary 1 of the same book enables one to reduce to that case. It follows that the sets $\theta(C_i)$ form a partition of R and π_r is injective on each. So the sets $\{C_i \cap \theta(C_j), \forall i, j\}$ form a partition of R and both π_r and π_l are injective on each. Now let Δ be the diagonal in $X \times X$; then $R - \Delta$ is partitioned by Borel sets $\{D_k\}$, where the D_k are a relabeling of the nonvoid sets among $\{(C_j \cap \theta(C_j)) - \Delta\}$. But now D_j is the graph of a one-one function f_j whose domain is $\pi_l(D_j)$ and whose range is $\pi_r(D_j)$.

We wish to refine the partition $\{D_j\}$ of $R - \Delta$ yet further by writing each D_j as a union $\bigcup_k D_j^k$ of Borel subsets so that $\pi_l(D_j^k) \cup \pi_r(D_j^k) = \varnothing$ for all j and k. We may take $X \subset [0, 1]$ and let $\{P_i\}$ be a sequence of open rectangles in $[0, 1] \times [0, 1]$ whose union is all of $[0, 1] \times [0, 1]$ minus the diagonal. Such sequences obviously exist, and if $P_i = I_i \times J_i$ with open intervals I_i and J_i then $I_i \cap J_i = \varnothing$.

Now we simply let $D_j^k = D_j \cap P_k$. Since $\pi_l(D_j^k) \subset I_k$ and $\pi_r(D_j^k) \subset J_k$, we have the desired property. Finally let us relabel the sets D_j^k as a single sequence $\{E_i\}$.

We then define a function g_i from X to X by $g_i(x) = y$ if $(x, y) \in E_i$, $g_i(x) = y$ if $(y, x) \in E_i$ and $g_i(x) = x$ otherwise. The fact that $\pi_l(E_i) \cap \pi_r(E_i) = \varnothing$ and that both π_l and π_r are one-to-one on E_i assures that g_i is unambiguously and completely defined. The graph $\Gamma(g_i)$ of g_i is easily seen to be $E_i \cup \theta(E_i) \cup \Delta \cap (F_i \times F_i)$ where $F_i = X - (\pi_l(E_i) \cup \pi_r(E_i))$. This is a Borel set, so g_i is a Borel map. Since $g_i^2 = \mathrm{id}$, it is a Borel isomorphism. Now let G be the (countable) group generated by the g_i. Since $\Gamma(g_i) \subset R$ by inspection, $R_G \subset R$; but on the other hand, $\bigcup \Gamma(g_i) \supset R - \Delta$, so $R = R_G$ and we are done. $\square$

The result above is actually somewhat better than we need, for we will always have a quasi-invariant measure around and it would have been sufficient to produce a group G so that $R = R_G$ almost everywhere in an appropriate sense.

A much deeper question, and one which remains open, is whether one could always select the G of Theorem 1 so that it acts *freely*; or more conservatively, whether–given a quasi-invariant measure–one can find a *freely acting* G so that $\mu\{x: R(x) \neq R_G(x)\} = 0$.

For the remainder of this section R will be a nonsingular countable standard equivalence relation on $(X, \mathfrak{B}, \mu)$.

PROPOSITION 2.1. *If* $C \in \mathcal{C}$, *then* $\mu(\pi_r(C)) = 0 \Leftrightarrow \mu(\pi_l(C)) = 0$.

PROOF. We note that $\pi_l(C) \subset \pi_l(\pi_r^{-1}\pi_r(C)) = R(\pi_r(C))$. So if $\mu(\pi_r(C)) = 0$, it follows by nonsingularity that $\mu(R(\pi_r(C))) = 0$ and hence by the above,

$\mu(\pi_l(C)) = 0$. The opposite implication follows by symmetry. $\square$

In the following, $|S|$ will denote the cardinality of a set. The following theorem provides two measures on $(R, \mathcal{C})$ which will play a fundamental role in the sequel.

THEOREM 2. (a) *For any $C \in \mathcal{C}$, the function $x \mapsto |\pi_l^{-1}(x) \cap C|$ is Borel and the measure ν_l defined by*

$$\nu_l(C) = \int |\pi_l^{-1}(x) \cap C| d\mu(x)$$

is σ-finite; it will be called the left counting measure of μ.

(b) *The null sets of ν_l are exactly those $C \in \mathcal{C}$ such that $\mu(\pi_l(C)) = 0$.*

(c) *The right counting measure of μ defined analogously satisfies $\nu_r = \nu_l \circ \theta$, and we have $\nu_r \sim \nu_l$.*

PROOF. (a) As in the proof of Theorem 1, let $\{C_i\}$ be a partition of R into Borel sets so that π_l is injective on each C_i. Let f_j be the restriction of π_l to C_j and let $A_j = \pi_l(C_j)$. Then f_j is a Borel isomorphism of C_j onto A_j. If $\nu^j(C) = \mu(\pi_l(C_j \cap C))$, it is clear that ν^j is a σ-finite measure on $\mathcal{C}$. Notice that $|\pi_l^{-1}(x) \cap C| = \Sigma_j 1_C(x, f_j^{-1}(x))$ is a sum of Borel functions, and hence is Borel. We can integrate term by term to obtain

$$\nu_l(C) = \int |\pi_l^{-1}(x) \cap C| d\mu(x) = \sum_j \nu^j(C).$$

Thus ν_l is a countable sum of σ-finite measures with disjoint supports and hence is σ-finite.

(b) Let $C \subset R$; then $\nu_l(C) = 0 \Leftrightarrow |\pi_l^{-1}(x) \cap C| = 0$ for μ almost all $x \Leftrightarrow \mu(\pi_l(C)) = 0$, as $\pi_l(C) = \{x: |\pi_l^{-1}(x) \cap C| \neq 0\}$, and this establishes (b).

(c) It is clear that $\nu_l = \nu_r \circ \theta$, and the equivalence of ν_l and ν_r follows from Proposition 2.1. $\square$

We note that if μ' is equivalent to μ, and if ν_l' and ν_r' are the left and right counting measures for μ', then $d\nu_i'/d\nu_i = (d\mu'/d\mu) \circ \pi_i$ for $i = l$ or r. Now we define the *Radon-Nikodym derivative* of μ with respect to R; it will have the proper interpretation when $R = R_G$.

DEFINITION 2.1. The *Radon-Nikodym derivative* of μ with respect to R is the Borel function $D(x, y) = d\nu_l/d\nu_r(x, y)$ on R. It is unique up to null sets of $\nu_l \sim \nu_r$, and we say that μ is invariant if $D = 1$ a.e. Note that the Radon-Nikodym derivative D' of μ' with respect to R, where $\mu \sim \mu'$, has the form $D'(x, y) = g(x)^{-1}D(x, y)g(y)$ for a certain positive Borel function g on X.

We now show that D is what it should be when R is realized as an R_G.

DEFINITION 2.2. A *partial Borel isomorphism* on X will be a Borel isomorphism ϕ defined on some $A \in \mathfrak{B}$ with range some $B \in \mathfrak{B}$.

PROPOSITION 2.2. *If ϕ is a partial Borel isomorphism with $\Gamma(\phi)$ (the graph of ϕ) $\subset R$, then $\phi_*(\mu)$, viewed as a measure on the range of ϕ, is absolutely continuous with respect to μ there, and $d\phi_*(\mu)/d\mu(y) = D(\phi^{-1}(y), y)$ for a.e. y in the range of ϕ.*

PROOF. For $A \subset R(\phi)$,

$$\phi_*\mu(A) = \mu\big(\phi^{-1}(A)\big) = \mu\big(\pi_l\big(\Gamma(\phi) \cap \pi_r^{-1}(A)\big)\big)$$

$$= \nu_l\big(\Gamma(\phi) \cap \pi_r^{-1}(A)\big) = \int_B (d\nu_l/d\nu_r)d\nu_r$$

where $B = \Gamma(\phi) \cap \pi_r^{-1}(A)$. Since π_r is one-to-one on B with image A, and since ν_r on B projects under π_r to μ, the integral can be rewritten as

$$\mu\big(\phi^{-1}(A)\big) = \int_A (d\nu_l/d\nu_r)\big(\phi^{-1}(y), y\big)d\mu(y),$$

which gives the desired result. □

COROLLARY 1. *The following are equivalent*:

(a) *μ is invariant under R.*

(b) *μ is invariant under G for some G with $R = R_G$.*

(c) *μ is invariant under every partial Borel isomorphism with $\Gamma(\phi) \subset R$.*

COROLLARY 2. *There is a μ-null set N so that if $y \not\in N$, and $x \sim y \sim z$, we have $D(x, y)D(y, z) = D(x, z)$.*

PROOF. This is immediate if we use the fact that $R = R_G$ for some countable group, Proposition 2.2, and the functional equation satisfied by the Radon-Nikodym derivative of the product of two transformations. In fact the formula of the corollary is completely equivalent to that functional equation. Alternately, it is easy enough to construct a direct proof of this formula without any reference to a countable group. □

REMARK 1. One way of reformulating this equation is to observe that if we disintegrate the measure ν_l with respect to right projection, we obtain measures σ_y living on $R(y)$ so that

$$\nu_l\big(E \cap \pi_r^{-1}(F)\big) = \int_F \sigma_y(E)d\mu(y).$$

Of course, $R(y)$ is a countable set, so that σ_y is determined by the mass $\sigma_y(\{x\})$ $(x \sim y)$ which σ_y gives to singleton sets, and this is of course $D(x, y)$. The formula of Corollary 2 says that for $z \sim y$, σ_z is a multiple of σ_y, or more precisely, $\sigma_z = D(y, z)\sigma_y$.

REMARK 2. The formula of the corollary is, in the sense of §5, the cocycle identity in dimension one, so that D is a one-cocycle with values in the multiplicative positive reals. We have already observed that if D' is the

Radon-Nikodym derivative of μ' with respect to R, then $D'(x,y) = g(x)^{-1}D(x,y)g(y)$ for some g, and this in the language of cohomology means that D and D' are cohomologous. It also follows that μ has an equivalent invariant measure if and only if it is cohomologous to 1.

In the sequel we shall simply use ν to denote a measure on R equivalent to a counting measure. Moreover, we shall need later on a generalization of the counting measures introduced above. Specifically for $n > 0$, let R^n be the subset of $X^{n+1} = X \times \cdots \times X$ ($n+1$ times) consisting of all $n+1$ tuples $(x_0, x_2, \ldots, x_n)$ with $x_0 \sim x_1 \sim \cdots \sim x_n$. It is clear that R^n is a Borel subset of X^{n+1}, since $R = R^1$, a Borel subset of X^2. If $F \subset \{0, 1, \ldots, n\}$ is a subset of cardinality $m+1$, then the projection π_F of X^{n+1} onto X^{m+1} defined by $\pi_F(x_0, \ldots, x_n) = (x_i)_{(i \subset F)} \in X^m$ maps R^n onto R^m. The fiber in R^n over any point is countable as in the special case $n = 1$, where $\pi_F = \pi_l$ or π_r. The following result is established just as in Theorem 2.

PROPOSITION 2.3. *If C is a Borel set in R^n, then $|\pi_F^{-1}(u) \cap C|$ is a Borel function of $u \in R^m$, where π_F: $R^n \to R^m$. If λ^m is a σ-finite measure on R^m, then*

$$\nu(F, \lambda^m)(C) = \int |\pi_F^{-1}(u) \cap C| d\lambda^m(u)$$

defines a σ-finite measure on R^n whose equivalence class depends only on λ^m and F.

Now if we take $F = \{0\}$, so $R^0 = X$ and we take $\nu^m = \mu$, we obtain a measure ν^n on X^n. We are not so much interested at this point in ν^n itself, but rather in its equivalence class.

PROPOSITION 2.4. *If F is any subset of $\{0, 1, \ldots, n\}$ of cardinality $m+1$ and λ^m any measure on R^m equivalent to ν^m, then $\nu(F, \lambda^m)$ defined above is equivalent to ν^n.*

PROOF. By definition, the null sets of $\nu(F, \lambda^m)$ consist of all C such that $\pi_F^{-1}(u) \cap C = \varnothing$ for λ^m almost all $u \in R^m$. Let C be a $\nu(F, \lambda^m)$-null set, and let D be those $u \in R^m$ so that $\pi_F^{-1}(u) \cap C \neq \varnothing$; thus D is a null set and $\pi_F^{-1}(D) \supset C$. Now let G denote the singleton set of indices $\{0, 1, \ldots, n\}$ consisting of the first element of F, and let q be the surjection from R^m to X defined by projection to the first coordinate, so that we have $q \circ \pi_F = \pi_G$ on X^n. Now the null sets of $\lambda^m \sim \nu^m$ are by definition those N such that $q^{-1}(x) \cap N = \varnothing$ for μ ($= \nu^0$)-almost all $x \in X$. Thus there is a null set $E \subset X$ so that for $x \notin E$, $q^{-1}(x) \cap D = \varnothing$, which implies that $\varnothing = \pi_F^{-1} \circ q^{-1}(x) \cap \pi_F^{-1}(D) \supset \pi_G^{-1}(x) \cap C$. It follows immediately that C is a $\nu(G, \mu)$-null set, and it is not hard to show conversely that a $\nu(G, \mu)$-null set is a $\nu(F, \lambda^m)$-null set.

Thus to prove the result we are reduced to proving the proposition in case F is a singleton. Recall that ν^n is defined to be $\nu(H, \mu)$ where $H = \{0\}$ is the singleton consisting of the first coordinate. If $H = F$ we are done, and if not, $E = H \cup F$ is a two element set; we consider the projection π_E of R^n onto $R = R^1$. The equivalence of the measures $\nu(F, \mu)$ and $\nu(H, \mu)$ follows immediately from the fact that left and right counting measures ν_l and ν_r on R^1 are equivalent, since the null sets of $\nu(F, \mu)$ are those C such that $\nu_l\{u: |\pi_E^{-1}(u) \cap C| \neq 0\} = 0$, and the null sets of $\nu(H, \mu)$ are those C such that $\nu_r(u: |\pi_E^{-1}(u) \cap C| \neq 0\}) = 0$. The proposition is proved. $\square$

It follows that each R^n has a unique measure class represented by ν^n with the disintegration properties described by Proposition 2.4. When we say that some property holds for almost all $u \in R^n$, we shall always understand this measure (class).

3. Classification and decomposition. In this section we recall and summarize known facts about classification into von Neumann-Murray types and decomposition into ergodic pieces as they apply to our relations. For proofs the reader is referred to the article of Dang-Ngoc-Nghiem [1]. In the sequel, R will always be a countable standard equivalence relation on $(X, \mathfrak{B}, \mu)$, with μ quasi-invariant.

DEFINITION 3.1. Let R_j on $(X_j, \mathfrak{B}_j, \mu_j)$ be given.

(a) A Borel map $\phi: X_1 \to X_2$ will be called a *morphism* if $\phi_*(\mu_1) \prec \mu_2$ and $\phi \times \phi(R_1) \subset R_2$ up to a set of $(\nu_2)_l$ measure zero.

(b) Relations R_j on $(X_j, \mathfrak{B}_j, \mu_j)$ are isomorphic if there is a Borel isomorphism ϕ from almost all of X_1 to almost all of X_2 with $\phi_*(\mu_1) \sim \mu_2$ and $\phi(R_1(x)) = R_2(\phi(x))$ for μ_1 almost all x (or equivalently $\phi \times \phi(R_1) = R_2$ up to a set of $(\nu_2)_l$ measure zero).

DEFINITION 3.2. Given R on $(X, \mathfrak{B}, \mu)$, a set $A \in \mathfrak{B}$ is called *invariant* if $R(A) = A$ up to null sets. The invariant sets, denoted by $\mathfrak{I}(R)$, form a σ-subalgebra, and R is *ergodic* if $\mathfrak{I}(R)$ consists only of null or conull sets. A Borel function f is invariant if $f(x) = f(y)$ for almost all pairs (x, y). The algebra of such functions will also be denoted by $\mathfrak{I}(R)$.

If $B \in \mathfrak{B}$, then we can form the restriction of the relation to B, denoted by $R|_B$; it is by definition that relation on B given by $R \cap (B \times B)$. It is a nonsingular relation on $(B, \mathfrak{B}|_B, \mu|_B)$. If in addition B is invariant, then $\mathfrak{I}(R|_B) = \mathfrak{I}(R)|_B$. $R|_B$ is said to be a *summand* of R in this case, and R is the sum in an obvious sense of $R|_B$ and $R|_{X-B}$.

DEFINITION 3.3. An R on $(X, \mathfrak{B}, \mu)$ is said to be

(a) *of semifinite type* if $\exists \mu_0 \sim \mu$, with μ_0 invariant,

(b) *of finite type* if (a) holds for some μ_0 of finite mass,

(c) *of purely infinite type* if there is *no* nonzero invariant $\mu_0 \prec \mu$,

(d) *of properly infinite type* if there is no *finite* non-zero invariant $\mu_0 \prec \mu$.

The notion of a relation being of finite type in the sense above is of course quite different from its being finite in the sense of §2.

DEFINITION 3.4. A relation R is of type I_n ($n = 1, 2, \ldots, \infty$) if it is isomorphic to an R' on $(X', \mathfrak{B}', \mu')$ where $X' = S \times X''$ with $|S| = n$, $\mathfrak{B}' = 2^S \times \mathfrak{B}''$, $\mu' = (\text{counting measure}) \times \mu''$ with $(s, x'') \sim (t, y'') \Leftrightarrow x'' = y''$. R is said to be *discrete* or of *type* I if there is a partition of X into invariant sets X_n with $R|_{X_n}$ of type I_n. (These X_n are of course unique.) R is of *type* II if it is of semifinite type with no type I summand; of *type* II_1 if it is of type II and of finite type; and of *type* II_∞ if it is of type II with no finite summands. R is of *type* III if it is purely infinite type.

PROPOSITION 3.1. *If R is given, there is a decompositon of X into invariant sets, unique up to sets of measure zero, $X = \bigcup_n X^{\mathrm{I}_n} \cup X^{\mathrm{II}_1} \cup X^{\mathrm{II}_\infty} \cup X^{\mathrm{III}}$ so that $R|_{X^\alpha}$ is of type α.*

Of course, if R is ergodic, one and only one of the X^α's above will be nonnull. An ergodic relation of type I is isomorphic to $(S, 2^S, \mu_S)$ where $|S|$ is finite or countable and μ_S is a counting measure. Following Mackey, we say that R is *strictly ergodic* if it is ergodic and not type I. Note also that a relation R is finite in the sense of §2 iff R is of type I and X^{I_∞} is null.

Any equivalence relation can be decomposed as a continuous sum of ergodic ones. For simplicity we assume that that continuous decomposition was already given in the definition of a relation of type I. Consider a space $Z \times Y$, with σ-field $\mathfrak{B}_Z \times \mathfrak{B}_Y$ and let R_z for $z \in Z$ be an equivalence relation on $(Y, \mathfrak{B}_Y)$ with a quasi-invariant measure μ_z such that $\mu_z(A_z)$ is a Borel function of z for any Borel set $A \in \mathfrak{B}_Z \times \mathfrak{B}_Y$; here $A_z = \{y : (z, y) \in A\}$. Suppose that $R = \{((z, y), (z, y')) : (y, y') \in R_z\}$ is a Borel subset of $(Z \times Y) \times (Z \times Y)$. If $\tilde{\mu}$ is a measure on Z we can integrate the measures μ_z with respect to $\tilde{\mu}$ to obtain a measure μ on $Y \times Z$. Then R as defined is an equivalence relation on $(Z \times Y, \mathfrak{B}_Z \times \mathfrak{B}_Y, \mu)$ and may be regarded as the continuous sum of the R_z.

PROPOSITION 3.2. *If R' is given on $(X', \mathfrak{B}', \mu')$, it is equivalent to a relation R as defined above on $(Z \times Y, \mathfrak{B}_Z \times \mathfrak{B}_Y, \mu)$ so that any set in $\mathfrak{I}(R')$ is of the form $A \times Y$ up to a null set, and such that almost all of the R_z are ergodic. If $X^\alpha = Z^\alpha \times Y$ are the sets in the partition into pure types, then R_z is of type α for almost all $z \in Z^\alpha$.*

We come now to an important notion: comparison of two subsets of X with respect to an equivalence relation R. This is the analogue of comparison of projection in a von Neumann algebra.

DEFINITION 3.5. Given R on $(X, \mathfrak{B}, \mu)$ and $A, B \in \mathfrak{B}$, we say that A and B

are equivalent, or $A \sim B$ if there is a partial Borel isomorphism with domain A and range B and with $\Gamma(\phi) \subset R$ a.e.

One can characterize when two sets are equivalent, a result first proved in full generality by Dang-Ngoc-Nghiem [1]. We state the result only in the ergodic case, where it is simpler to state.

PROPOSITION 3.3 *Let R be ergodic on $(X, \mathfrak{B}, \mu)$ and $A, B \in \mathfrak{B}$. Then necessary and sufficient conditions for $A \sim B$ are as follows*:

(a) *if R is of type* I *or type* II *and μ_1 is the invariant measure, we must have* $\mu_1(A) = \mu_1(B)$;

(b) *If R is of type* III, *A and B are always equivariant if they are both of positive measure.*

REMARK. If $R = R_G$, then any partial Borel isomorphism can be written in terms of G as follows: Let $A_g = \{x\colon \phi(x) = gx\}$. Then the domain A of ϕ is the union of the A_g, and by replacing A_g by a suitable subset if G is not free, we can make them disjoint. Then $A \sim B$ if and only if there is a partition $A = \bigcup A_g$ of A such that $\{g(A_g)\}$ is a partition of B.

A useful addendum to Proposition 3.3 concerns the existence of Borel maps ϕ with $\Gamma(\phi) \subset R$, but not necessarily injective:

PROPOSITION 3.4. *If $A, B \subset X$ and $R(A) \subset R(B)$, then there exists a Borel map ϕ from A into B with* $\Gamma(\phi) \subset R$.

PROOF. The condition $R(A) \subset R(B)$ is clearly necessary for the existence of such a ϕ. If it is satisfied, then for each $x \in A$, its class $R(x)$ meets B. Then we write $R = R_G$ for a countable group G and number the elements g_i of G by the positive integers. We then define $\phi(x) = g_i(x)$ where i is the smallest index such that $g_i(x) \in B$. Since $\phi(x) = g_i(x)$ for $x \in A_i$ where $A = \bigcup A_i$ is a partition of A is into Borel sets, it is clear that ϕ is a Borel function. $\square$

We have already mentioned the notion of relativizing a relation to a subset $B \subset X$. Let $R(B)$ be the saturation of B, so that $R(B) \in \mathcal{I}(R)$ and $R|_{R(B)}$ is a summand of R.

PROPOSITION 3.5. (a) *$R|_B$ is ergodic if and only if $R|_{R(B)}$ is ergodic.*
(b) *The type* I, II *and* III *summands of $R|_B$ are* $B \cap X^\alpha$, $\alpha =$ I, II, III.

REMARK. The notion of relativizing an equivalence relation generalizes Kakutani's idea of the induced transformation [1]. Specifically, let $G = \mathbf{Z}$ be the group generated by the powers ϕ^n of a single conservative nonsingular transformation on $(X, \mathfrak{B}, \mu)$, let $R = R_G$, and let $A \subset X$ be of positive measure. If $x \in A$, let $n(x) = \inf\{n > 0\colon \phi^n(x) \in A\}$ and define $\phi_A(x) = \phi^{n(x)}(x)$. (Note that $n(x) < \infty$ a.e. as ϕ is conservative.) Now let $G(A)$ be the group consisting of the powers of ϕ_A. Then it is virtually obvious that $R|_A = R_{G(A)}$.

Suppose now that R_i are relations on $(X_i, \mathfrak{B}_i, \mu_i)$, $i = 1, 2$. We form $X = X_1 \times X_2$, equipped with the product σ-field $\mathfrak{B}_1 \times \mathfrak{B}_2$ and the product measure, and we let $R = R_1 \times R_2$ be the product of R_1 and R_2, specifically $(x_1, x_2) \sim (y_1, y_2)$ if and only if $x_1 \sim y_1$ and $x_2 \sim y_2$. If $R_i = R_{G(i)}$, then $R = R_{G(1)\times G(2)}$ with $G(1) \times G(2)$ operating naturally on the product.

PROPOSITION 3.6 *If R_1 and R_2 are ergodic, so is $R_1 \times R_2$.*

PROOF. Let $E = R(E)$ be an invariant set in $X_1 \times X_2$ and let E_x (E^y) its vertical (horizontal) section at x (y). Then as E_x and E^y are invariant sets for the ergodic relations R_1 and R_2 respectively, E_x and E^y are null and conull for each x and y. If p_1 and p_2 are the projections to X_1 and X_2, it follows that E is of the form $p_1^{-1}(E_1)$ a.e. and also of the form $p_2^{-1}(E_2)$ a.e. for $E_i \subset X_i$. Clearly the only way this can happen is for E_1 and E_2 to be simultaneously null or conull, and the result follows. $\square$

We will meet a generalization of this construction in a later section, but now let us turn to a different topic. If R is an ergodic relation on $(X, \mathfrak{B}, \mu)$ it has the algebraic structure of a principal groupoid, and is indeed a principal ergodic groupoid in the language of Mackey [2], and therefore defines a virtual group. Virtual groups are "similarity" classes of ergodic groupoids, but the notion of equivalence used to identify two groupoids is broader than our notion of isomorphism of relations. Thus it is natural to raise the question of when two of our equivalence relations define the same virtual group. The following theorem answers this question. We will use I_n to denote the ergodic equivalence relation of type I_n, $n = 1, \ldots, \infty$.

THEOREM 3. *For countable standard ergodic equivalence relations R_i on $(X_i, \mathfrak{B}_i, \mu_i)$, the following are equivalent*:

(i) *R_1 and R_2 define the same virtual group (or are "similar");*

(ii) *there are Borel sets of positive measure $E(i)$ so that the relativizations $R_1|_{E(1)}$ and $R_2|_{E(2)}$ are isomorphic.*

(iii) *$R_1 \times \mathrm{I}_\infty$ is isomorphic to $R_2 \times \mathrm{I}_\infty$.*

PROOF. We first prove (i) $\Rightarrow$ (ii); according to Mackey [2], similarity of R_1 and R_2 means that there are morphisms (Definition 3.1) ϕ_1 and ϕ_2 from X_1 to X_2 and from X_2 to X_1 such that $\phi_2\phi_1$ and $\phi_1\phi_2$ send almost all x into $R_1(x)$ and almost all y into $R_2(y)$. It follows that $\phi_1\phi_2$ and $\phi_2\phi_1$ are at most countable-to-one, and so the same holds for ϕ_1 and ϕ_2. Then just as in Theorem 1, we can find disjoint Borel sets $C_n \subset X_1$ such that ϕ_1 restricted to each C_n is an isomorphism and $\bigcup C_n = X_1$. Since $\phi(X_1 - N)$ has positive μ_2 measure in X_2 for any null set N in X_1, we take N to be the union of the C_n with $\mu_1(C_n) = 0$ and conclude that there exists some C_n with $\mu_1(C_n) > 0$ and

$\mu_2(\phi(C_n)) > 0$. If σ_1 is μ_1 restricted to C_n and σ_2 is μ_2 restricted to $\phi(C_n)$, and ψ is ϕ restricted to C_1, then it follows that $\psi_*(\phi_1) \prec \sigma_2$. Then $\phi(C_n)$ is the disjoint union $\phi(C_n) = \phi(E(1)) \cup \phi(D(1))$ of Borel sets such that $\psi_*(\sigma_1) \sim \sigma_2$ on $\phi(E(1))$ and $\psi_*(\sigma_1) = 0$ on $\phi(D(1))$. Then ϕ restricted to $E(1)$ is an isomorphism of $R_1|_{E(1)}$ onto $R_2|_{\phi(E(1))}$ and this establishes (ii).

Now suppose that (ii) holds. From the construction of $R \times \mathrm{I}_\infty$ it is clear that R is isomorphic to a relativization of $R \times \mathrm{I}_\infty$. Thus if (ii) holds, $R_1 \times \mathrm{I}_\infty$ and $R_2 \times \mathrm{I}_\infty$ have isomorphic relativizations, and are both of infinite type. So to prove (ii) $\Rightarrow$ (iii), it will suffice to assume that (ii) holds for R_1 and R_2 of infinite type and show then that R_1 is isomorphic to R_2. Now by Proposition 3.5(a) and (b), it follows that R_1 and R_2 are both of type I_∞ or II_∞ or III. If both are type I_∞ they are clearly isomorphic; if both are of type III it follows by Proposition 3.3(b) that R_1 is isomorphic to $R_1|_{E(1)}$ and that R_2 is isomorphic to $R_2|_{E(2)}$, and hence that R_1 is isomorphic to R_2. If both are of type II_∞ and if one of the sets $E(i)$ is of infinite invariant measure, so is the other; then by Proposition 3.3(a) R_i is isomorphic to $R_i|_{E(i)}$ and hence as above R_1 is isomorphic to R_2. Finally in case both sets $E(i)$ are of finite invariant measure, we can partition $X_1 = \bigcup C(i)$ and $X_2 = \bigcup D(i)$ with $C(1) = E(1)$, $D(1) = E(2)$ and so that $\mu_1(C(i)) = \mu_1(C(1))$ and $\mu_2(D(i)) = \mu_2(D(1))$. Then by Proposition 3.3(a), $R_1|_{C(i)}$ is isomorphic to $R_1|_{C(1)}$ and the same for R_2, and it follows at once that R_i is isomorphic to $R_i|_{E(i)} \times \mathrm{I}_\infty$. Since $R_1|_{E(1)}$ is isomorphic to $R_2|_{E(2)}$ it follows that R_1 is isomorphic to R_2 as desired.

Finally we establish that (iii) $\Rightarrow$ (i), and for this it suffices to note that R and $R \times \mathrm{I}_\infty$ define the same virtual group; for it is apparent that there are morphisms going in both directions having the desired properties as outlined at the beginning of the proof. Thus if $R_1 \times \mathrm{I}_\infty$ is isomorphic to $R_2 \times \mathrm{I}_\infty$, they define a fortiori the same virtual group, and by our comments above, so do R_1 and R_2. □

As a corollary of the theorem and its proof we obtain the following result.

COROLLARY. *If R_1 and R_2 are of infinite type and are Mackey equivalent via morphisms ϕ_1 and ϕ_2 from X_1 to X_2 and ϕ_2 from X_2 to X_1, then R_1 and R_2 are isomorphic by means of an isomorphism ϕ such that $\phi(x) \sim \phi_1(x)$ for almost all x, and $\phi^{-1}(y) \sim \phi_2(y)$ for almost all y.*

PROOF. If R_1 and R_2 are infinite, $R_i \cong R_i \times \mathrm{I}_\infty$ and if we trace through the argument, we see that the ϕ which the argument constructs has the desired property. □

4. Hyperfiniteness. In this section we shall investigate the notion of hyperfiniteness in the context of our equivalence relations. Again for the most part this recalls known facts.

DEFINITION 4.1. A relation R on $(X, \mathfrak{B}, \mu)$ is said to be *hyperfinite* if there are finite relations R_n with $R_n \subset R_{n+1}$ and $\bigcup R_n = R$ a.e.

We recall a result that is basically known.

PROPOSITION 4.1. *For a relation R on $(X, \mathfrak{B}, \mu)$ the following are equivalent*:

(a) *R is hyperfinite.*

(b) *$R = R_{\mathbf{Z}}$ a.e. for some action of $\mathbf{Z}$.*

(c) *There are periodic Borel isomorphisms ϕ_i with $\Gamma(\phi_i) \subset R$ and $R_i \uparrow R$ a.e. where R_i is the relation generated by the powers of ϕ_i.*

(d) *For any countable G with $R = R_G$ a.e., the following holds*: *given $g_1, \ldots, g_n$ in G and $\varepsilon > 0$, $\exists \phi_1, \ldots, \phi_n$ with $\Gamma(\phi_i) \subset R$ such that $\mu\{x \mid g_j(x) \neq \phi_j(x)\} < \varepsilon$ for $j = 1, \ldots, n$ and such that the ϕ_j generate a finite group.*

PROOF. The equivalence of (b), (c) and (d) are due to Dye [1] in the II_1 case and to Krieger [1] in the general case. Condition (d) is Dye's original definition of "approximate finiteness". We need only show that (a) is equivalent to the others. Given (c), it is clear that R_j is a finite relation and so (a) holds. Given (a), we construct for each of the finite relations R_j, from the definition of (a), a periodic transformation ϕ_j which generates it. We split X into a union of $X(n)$ where R_j is of type I_n. Then $X(n)$ looks like $\{0, 1, \ldots, n-1\} \times Y_n$ and $(i, y) \sim (k, y')$ if and only if $y = y'$. Then we define ϕ_j on X_n by $\phi_j(i, y) = (i + 1, y) \pmod n$.

PROPOSITION 4.2. (a) *Any type* I *relation is hyperfinite*;

(b) *if $R_1 \subset R_2 \subset \ldots$ are all hyperfinite, then $R = \bigcup_j R_j$ is also*;

(c) *if S and R are two relations on $(X, \mathfrak{B}, \mu)$ with $S \subset R$ and if R is hyperfinite, so is S*;

(d) *if R is hyperfinite on $(X, \mathfrak{B}, \mu)$ and $A \in \mathfrak{B}$, then $R|_A$ is hyperfinite.*

PROOF. (a) This is clear from the description of type I relations in §3. (b) We can write $R_j = R_{G(j)}$ for a countable group $G(j)$ so that $G(n) \subset G(n+1)$ (replace $G(n)$ by the group generated by $G(j)$, $j < n$). Then clearly $R = R_G$ where $G = G(j)$. Now condition (d) of Proposition 4.1 holds by hypothesis for each $G(j)$ and hence evidently also for G. Hence by that proposition, R is hyperfinite. (c) This is clear from Proposition 4.1(a) since if $R_n \uparrow R$ a.e. $R_n \cap S \uparrow S$ a.e. (d) As in (c) if $R_n \uparrow R$ a.e., then $R_n|_A \uparrow R|_A$ a.e. and $R_n|_A$ is finite if R_n is. $\square$

Dye established the following remarkable isomorphism theorem for II_1 hyperfinite group actions which was extended to the II_∞ ergodic case by Krieger.

PROPOSITION 4.3. *Let R_j on $(X_j, \mathfrak{B}_j, \mu_j)$ be hyperfinite. If both are of type I_n, II_1, or II_∞, they are isomorphic if and only if the measure algebras of $\mu_i|_{\mathfrak{I}(R_i)}$ are isomorphic.*

PROPOSITION 4.4. *If G is a countable abelian group then R_G is hyperfinite.*

PROOF. The II_1 case is due to Dye [2], the II_∞ ergodic case to Krieger [1], and the general case to Feldman and Lind [1].

Finally, it now appears that Connes and Krieger have established the much more general (and difficult) fact that R_G is hyperfinite for any countable solvable group. In the opposite direction, the following may be noted:

PROPOSITION 4.5. *If R_G is hyperfinite on $(X, \mathfrak{B}, \mu)$ and of finite type with G freely acting, then G is amenable.*

PROOF. This follows from Sakai [1, 4.4.21].

REMARK. This is false if the assumption that R is of finite type is dropped; A. Connes pointed this out in conversation with one of us, by a simple example.

Still open is the following question: Does a converse of Proposition 4.5 hold; that is, is R_G hyperfinite for amenable (not just solvable) G?

5. **Modules for equivalence relations.** We now turn to discussion of cohomology groups which can be associated to relations. The first order of business is to define a category of modules for a relation R. This is somewhat complicated by the fact that we wish to treat nonergodic relations as well as the ergodic ones. To see what the proper definition is, we imagine a pair $(x, y) \in R$ to be something like a group element, and so we should consider a group A (abelian or not) together with a map u from R into $\mathrm{Aut}(A)$ with the property that $u(x, z) = u(x, y)u(y, z)$ for all $x \sim y \sim z$. The proper kind of group to consider here is a polonais group (cf. Moore [3]) and the function u must be Borel in the obvious sense. Actually we also want to allow the group A to vary as one moves about in the space X so that we replace A by a family of polonais groups $\{A_x, x \in X\}$ subject to some regularity conditions summarized below.

DEFINITION 5.1. Let R be a relation on $(X, \mathfrak{B}, \mu)$. An R-module A or (A, u) will be specified by a countable partition $X = \bigcup X_n$ of X into R-invariant sets and the assignment of a polonais group A_n for each n together with a map u of R into $\bigcup_n \mathrm{Aut}(A_n)$ such that for $(x, y) \in R_n = R \cap X_n \times X_n$, $u(x, y) \in \mathrm{Aut}(A_n)$ and satisfies

(i) $u(x, y)u(y, z) = u(x, z)$, $x, y, z \in R_n$, and

(ii) $(x, y, a) \mapsto u(x, y) \cdot a$ is jointly Borel on $R_n \times A_n$ for each n. For $x \in X_n$ we write $A_n = A_x$.

Of course if R is ergodic, then one of the X_n is conull so that the module is equivalent in the sense described below to one with $X_n = X$ and $A_n = A$.

DEFINITION 5.2. An R-homomorphism ϕ of an R-module (A, u) into another one (B, v) consists of a continuous homomorphism ϕ_x of A_x into B_x

so that $\phi_x u(x, y) = v(x, y)\phi_y$ for almost all $(x, y) \in R$ (with respect to the measure of Theorem 2) and so that ϕ_x is a Borel function from A_n into B_m for $x \in X_n \cap Y_m$ where $\{X_n\}$ and $\{Y_m\}$ are the partitions defining A and B respectively.

Equivalence of modules is then defined in the obvious way; one may also define the notion of a closed submodule, the quotient modulo by a closed submodule, and the notion of a short exact sequence. Moreover, one may define finite Cartesian products of modules and countably infinite products provided that there is some fixed partition finer than all of those defining the factors. Finally if we are given a standard measure space (Y, σ) equipped with a partition $Y = \bigcup Y_n$ we can define the module $\tilde{U}(Y, A)$ in analogy with Moore [3]. Specifically, an R-module $\tilde{U}(Y, A)$ is to be defined by the same partition X_n of X as A and $\tilde{U}(Y, A)_n = U(Y_n, A_n)$, the group of equivalence classes of Borel maps from Y_n into A_n. For $(x_1, x_2) \in R_n$ the corresponding automorphism $v(x_1, x_2)$ of $U(Y_n, A_n)$ is defined by $(v(x_1, x_2)f)(y) = u(x_1, x_2)(f(y))$ for $y \in Y_n$.

We let $U(Y, A)$ denote the product $\prod_n U(Y_n, A_n)$ and view its elements $f = (f_n)$ as functions on Y into $\bigcup A_n$ with $f(y) = f_n(y)$ for $y \in Y_n$. We call $U(Y, A)$ the functions from Y to A.

We shall now construct a module of importance to us; this is to be an analogue of the regular representation with coefficients in a fixed polonais group A and will be denoted $I(A)$. Let X_n consist of the subset of X corresponding to the part of R of type I_n, or equivalently, the set of x such that $|R(x)| = n$. Then $X = \bigcup_{n=1}^{\infty} X_n$ (including ∞) is a partition of X. Let N_n be a set of n elements and let $(I(A))_n = U(N_n, A)$. For $x \in X_n$, we let $N_x = N_n$ and $I(A)_x = I(A)_n$.

PROPOSITION 5.1. *There exists a function ϕ_x, $x \in X_n$, from N_n into $R(x)$, which is a bijection for each x and such that $\phi_x(k)$ is a Borel function of x for each k.*

PROOF. This is more or less implicit in the proof of Theorem 1 above and we omit the details. □

Now for $x \sim y$, $\phi_x^{-1}\phi_y = p(x, y)$ is a bijection of N_n into itself and an element of the permutation group $P(N_n)$ of N_n. It is immediate that for $x \sim y \sim z$, we have $p(x, y)p(y, z) = p(x, z)$, so that p is a (nonabelian) cocycle in the language to be introduced in a moment. If we were to select another family of functions ψ_x, we would have $\psi_x = \phi_x c(x)$ where $c(x) \in P(N_n)$. Then the corresponding cocycle q associated to ψ_x would be given by $q(x, y) = c(x)^{-1}p(x, y)c(y)$ so that p would be changed only by a coboundary. Then p (or q) defines a unique cohomology class which will be called the fundamental class of R. We shall return to this once we have introduced cohomology formally. In any case, $p(x, y)$ defines an automorphism $i(x,y)$ of $I(A)_x$ by $(i(x,y)f)(k) = f(p(x,y)^{-1}k)$ for $k \in N_x$ and $f \in I(A)_x$. In the notation above, if we set $(s(x)f)(k) = f(c(x)k)$ we see that $s(x)i(x,y) = j(x,y)s(y)$ where $j(x,y)$ is the automorphism of $I(A)_x$ defined using $q(x,y)$ in place of $p(x,y)$. Thus $(I(A), i)$ and $(I(A), j)$ are equivalent modules in the sense of our definitions and $I(A)$ will denote this (unique) R module.

Suppose now that A additionally already has the structure of an R-module, with defining function u. Then we claim that A can be embedded in $I(A)$ as a closed submodule, just as in Moore [3]. To do this we must define a 1-1 map $t(x)$ from A_x to $I(A)_x$ with closed range satisfying $t(x)u(x,y) = i(x,y)t(y)$. One may verify that the specification $(t(x)a)(k) = u(\phi_x(k), x)a$ for $k \in N_x$ does the job. As in Moore [3], we let $U(A)$ be the cokernel of t so that we have a short exact sequence

$$0 \to A \to I(A) \to U(A) \to 0.$$

We now consider the category $\mathcal{P}(R)$ of abelian polonais R-modules, and our object is to attach cohomology groups $H^n(R, A)$ for $A \in \mathcal{P}(R)$, $n > 0$ analogous to Eilenberg-Mac Lane groups. These groups will contain interesting information about the relation R and about associated von Neumann algebras. We shall also define a cohomology set for nonabelian modules A for $n = 1$.

One cohomological object of interest that will appear often in the sequel is defined below.

DEFINITION 5.3. Let (A, u) be an R-module (abelian or not) defined by a partition (X_n); using the same partition, we define the group $U(X, A)$ as above. A function $f \in U(X, A)$ will be said to be R equivariant if $u(x,y) \cdot f(y) = f(x)$ holds for almost all pairs $(x, y) \in R$. The set of such functions will be denoted by A^R; it is obviously a closed subgroup of $U(X, A)$ and hence is a polonais group.

PROPOSITION 5.2. *For $A \in \mathcal{P}(R)$, the map $A \mapsto A^R$ is a left exact covariant functor from $\mathcal{P}(R)$ to polonais abelian groups.*

The routine proof is omitted.

The groups A^R will be the zero dimensional groups $H^0(R, A)$; we have already met the one dimensional groups informally above and we make this precise. Suppose that A is a (not necessarily) abelian R-module defined using $\bigcup_n X_n = X$. Let $C^1(R, A)$ be $U(R, A)$ where it is understood that the partition of R is $\{R_n\}$ with $R_n = X_n \times X_n \cap R$. This consists of all classes of Borel functions f from R into $U_n A_n$ with $f(x,y) \in A_n$ if $(x,y) \in R_n$. Let $Z^1(R, A)$ consist of those functions satisfying $f(x,z) = f(x,y)(u(x,y) \cdot f(y,z))$ for almost all triples (x, y, z), $x \sim y \sim z$, with respect to the measure

ν^2 on R^2 defined in Proposition 2.4. It follows from the disintegration properties of ν^2 in Proposition 2.4 that if $f = g$ a.e. and if f is in $Z^1(R, A)$ then g is also. We define an equivalence relation on $Z^1(R, A)$ by saying that $f \sim g$ if there is an element $b \in U(X, A)$ such that $f(x, y) = b(x)^{-1}g(x, y)(u(x, y) \cdot b(y))$ for a.e. (x, y). The set of equivalence classes is denoted by $H^1(R, A)$, the first cohomology set with coefficients in A. If A is abelian, $Z^1(R, A)$ is a polonais group, being closed in $U(R, A)$, and the subgroup $B^1(R, A) = \{b(x)^{-1}u(x, y) \cdot b(y), b \in U(X, A)\}$ of coboundaries. The quotient group is $H^1(R, A)$.

If $A = \mathbf{R}^+$ is the positive reals under multiplication, the Radon-Nikodym derivative of §2 is a cocycle. Also if we let $A_x = P(N_x)$ be the permutation group on the fixed set N_x of cardinality equal to the cardinality of $R(x)$, equipped with the topology of pointwise convergence, we have an R-module $P(N)$ with $u = 1$. Then the fundamental class defined in the first part of this section is an element of the set $H^1(R, P(N))$.

6. **Cohomology.** Rather than immediately introducing the cohomology groups we shall proceed axiomatically by lising some axioms that our cohomology groups should satisfy, and then prove a uniqueness theorem. The existence will then follow by a direct construction using an appropriate cochain complex.

DEFINITION 6.1. A cohomological functor on $\mathcal{P}(R)$ is a sequence of covariant functions $H^n(R,\cdot)$ on $\mathcal{P}(R)$ to abelian groups so that to each short exact sequence in $\mathcal{P}(R)$ there corresponds functorially a long exact sequence of cohomology (cf. Eilenberg-Mac Lane [1]) and such that

(1) $H^0(R, A) \cong A^R$ as defined above, and

(2) $H^n(R, I(A)) = 0$ for any A and $n > 1$.

PROPOSITION 6.1. *If H^n and $\tilde{H}^n$ are two cohomological functors on $\mathcal{P}(R)$, there are functorial isomorphisms $H^n(R, A) \cong \tilde{H}^n(R, A)$.*

PROOF. This follows easily just as in Moore [3], since all that argument needed was the embedding of A into $I(A)$ with the latter having trivial cohomology. $\square$

Recall that $U(A)$ was defined as the cokernel of the embedding $A \to I(A)$ defined in the last section. We now let $U^n(A) = U(\ldots U(A) \ldots)$ ($n - 1$ times). Then the following result allows one to reduce higher dimensional cohomology groups down to dimension one and is a consequence of the argument of the above proposition, just as in Moore [3].

PROPOSITION 6.2. *If $H^n(R,\cdot)$ is a cohomological functor, $H^n(R, A) \cong H^1(R, U^n(A))$.*

Our next task is to give an explicit construction of these groups. As we have already noted in §3, an ergodic R is a principal ergodic groupoid and hence defines a virtual group. Westman has introduced the cohomology groups defined by cochains in this context in Westman [1], but for completeness we give the definitions again as we shall need some specific properties of these cochains. We consider the set $R^n \subset X^{n+1}$ together with the measure (class) ν^n defined in §2. If $\bigcup X_m = X$ is a partition of X by invariant sets, R^n is always assumed to be partitioned by the sets $R^n \cap X_m \times \cdots \times X_m$. Thus if (A, u) is an abelian R module we let $C^n(R, A)$ be the group $U(R^n, A)$ of classes of Borel functions of R^n into A endowed with the usual topology in which it is a polonais group. We define a coboundary operator δ_{n-1} from C^{n-1} to C^n by

$$(\delta_{n-1}f)(x_0, \ldots, x_n) = u(x_0, x_1)\cdot f(x_1, \ldots, x_n) + \sum_{i=1}^{n} (-1)^i f(x_0, \ldots, \hat{x}_i, \ldots, x_n).$$

The disintegration properties of the measures ν^n described in Proposition 2.4 ensure that this formula makes sense modulo null sets. Moreover, each δ_n is a continuous homomorphism. It is evident that $C^n(R, A)$ is a complex. Let $Z^n(R, A)$, $B^n(R, A)$ and $H^n(R, A)$ be respectively the cocycles, the coboundaries, and the cohomology in dimension n. Of course if the action of R on A is trivial the formula for δ_n becomes especially simple and very reminiscent of topology.

If ϕ is a morphism of one R-module (A, u) into another R-module (B, v) then ϕ consists of a family of group homomorphisms $\phi(x)$ of continuous homomorphisms of A_x into B_x; recall that A_x and B_x are constant on each set of a partition of X into R-invariant Borel sets, so $A_x = A_y$ and $B_x = B_y$ if $x \sim y$; $\phi(x)$ satisfies the equation $\phi(x)u(x, y) = v(x, y)\phi(y)$. We now define a cochain map ϕ^n from $C^n(R, A)$ into $C^n(R, B)$ by $(\phi^n f)(x_0, \ldots, x_n) = \phi(x_0)f(x_0, \ldots, x_n)$. It is routine to verify that the ϕ^n are homomorphisms of complexes and so define natural induced homomorphisms ϕ^n on cohomology.

PROPOSITION 6.3. *The assignments $A \mapsto H^n(R, A)$ are functors of cohomological type and $H^0(R, A) = A^R$ as defined above.*

PROOF. The naturality of the maps is evident, and if $0 \to A' \to A \to A'' \to 0$ is a short exact sequence in $\mathcal{P}(R)$ it is easy to see as in Moore [3] that the corresponding three term sequence of cochain groups is a short exact sequence of polonais groups. The existence of a long exact sequence of cohomology is a standard construction in homological algebra.

For the second statement we observe that $B^0(R, A) = 0$ by definition and that $Z^0(R, A)$ consists of classes of Borel functions f from X into $U_n A_n$ with $f(X_n) \subset A_n$ (where $X = \bigcup X_n$ is a partition of X and $A_n = A_x$ for $x \in X_n$) such that $u(x_1, x_2)f(x_2) - f(x_1) = 0$ for almost all pairs x_1, x_2. But this is precisely the group A^R. □

PROPOSITION 6.4. *The group $H^1(R, A)$ coincides with all classes of Borel functions f: $R \to A$ satisfying $f(x,z) = u(x,y)\cdot f(y,z) + f(x,y)$ for a.e. x, y, z, modulo those of the form $f(x,y) = u(x,y)c(y) - c(x)$.*

Thus these groups coincide with the informal definition of $H^1(R, A)$ given at the beginning of this section for abelian A. Now in order to complete the circle of ideas, we establish a vanishing theorem which by Proposition 6.1 will establish existence and uniqueness of the cohomology groups.

PROPOSITION 6.5. *For any A, $H^n(R, I(A)) = 0$ for $n \geqslant 1$.*

Before proceeding with the proof of this fact let us first list a simple result which will simplify many calculations for us. If $X = \bigcup X_n$ is partitioned into R-invariant sets with $R_n = R \cap X_n \times X_n$ and if A is an R-module, then A defines in a natural way an R_n-module A_n. Now cohomology commutes with such decompositions:

PROPOSITION 6.6. *The group $H^n(R, A)$ is isomorphic to the product $\Pi_m H^n(R_m, A_m)$.*

PROOF. For each $m \geqslant 0$, the set $R^m \subset X^{m+1}$ is the disjoint union of the subsets $R^m = R^m \cap (X_n \times \cdots \times X_n)$ and the measure ν^m is the sum of the corresponding measures ν_n^m on each R_n^m. It follows at once that the cochain group $C^m(R, A) = U(R^m, A)$ is isomorphic to the product $\Pi_n U(R_n^m, A_n)$ by mapping an f into the sequence f_n of its restrictions to each R_n^m. This commutes with the coboundary operators and hence defines an isomorphism on cohomology. □

We now turn to the proof of Proposition 6.5. By Proposition 6.6, we may assume that $A = A_n$ is constant on X and that the cardinality of $R(x)$ is constant, say equal to n, and we let $N = N_n$ be a set of this cardinality. Then the group $I(A)_x$ is constant for $x \in X$ and equal to $U(N, A)$. The action of R is defined in terms of the fundamental cocycle $p \in Z^1(R, P(N))$ by $(u(x, y)f)(k) = f(p(x,y)^{-1}\cdot k)$, by $f \in U(N, A)$, where $p(x,y) = \phi(x)^{-1} \circ \phi(y)$ with $\phi(x)$ a bijection from N to $R(x)$.

Now if $F \in C^n(R, I(A))$, we define $\psi(F) = H \in C^{n-1}(R, I(A))$ by the formula

$$H(y_0, \ldots, y_{n-1})(k) = F(y_0, \ldots, y_{n-1}, \phi(x_0)(k))(k).$$

Note that this is well defined; for if $F = F'$ a.e. on R^n, then by Proposition 2.4 for almost all n-tuples $(y_0, \ldots, y_{n-1}) \in R^{n-1}$,

$$F(y_0, \ldots, y_{n-1}, a)(k) = F'(y_0, \ldots, y_{n-1}, a)(k)$$

for all $k \in N$ and $a \in R(y_0)$ since $N \times R(y_0)$ is countable. Thus H is a well-defined element in $C^{n-1}(R, I(A))$. It is a routine calculation that $\delta_{n-1}(H) = \pm F$ if $\delta_n(F) = 0$ and so $H^n(R, I(A)) = 0$. □

REMARK. This argument is quite simple because the countability of the relations is the analogue of discrete groups so that the "algebraic" argument applies directly and we do not need to use the more indirect and involved argument as in Theorem 4 of Moore [3] to take care of almost everywhere problems. For more general relations one would have to resort to such techniques.

We summarize what we have established.

THEOREM 4. *The functors $A \mapsto H^n(R, A)$ are the unique objects satisfying the conditions of Definition* 6.1.

7. **Some properties and applications.** We now investigate the properties of this cohomology theory, and the first item is the relation with group cohomology. Let $R = R_G$ come from a countable group G of Borel automorphisms of X, and let (A, u) be an R-module. Then we can define for each pair $(x, g) \in X \times G$ an element $a(g, x) \in \mathrm{Aut}(A_x)$ by $a(g, x) = u(g \cdot x, x)$. The cocycle property of u then becomes the familiar condition $a(gh, x) = a(g, h \cdot x)a(h, x)$. If conversely a is a function from $G \times X$ into $\bigcup_n \mathrm{Aut}(A_n)$ with $a(g, x) \in \mathrm{Aut}(A_n)$ for $x \in X_n$, with a satisfying this identity, we cannot quite recapture the structure of an R-module since G may have a nontrivial isotropy group G_x at x. It is easy to see that a comes from an R-module if and only if $a(g, x) = 1$ for $g \in G_x$ for almost all x. In particular, it always comes from an R-module if the action of G is free on X.

Suppose now that A is an $R = R_G$-module. We form the group $B = U(X, A)$ as per our conventions. Then G operates as a group of automorphisms of B by the prescription $(g \cdot F)(x) = a(g, g^{-1} \cdot x)F(g^{-1} \cdot x)$. Each automorphism $F \mapsto g \cdot F$ is continuous by the results in Moore [2], and as G is discrete, B is a polonais G-module in the language of Moore [3]. Moreover let ϕ be a morphism of one R-module (A_1, u_1) into another (A_2, u_2) so that ϕ consists of a family of continuous maps $\phi(x)$ from $(A_1)_x$ into $(A_2)_x$ satisfying $\phi(x)u_1(x, y) = u_2(x, y)\phi(y)$. If $B_i = U(X, A_i)$, we define a map $\tilde{\phi}$ of B_1 into B_2 by $(\tilde{\phi}F)(x) = \phi(x)F(x)$, and it is easy to see that this is a G-module homomorphism, and that $A \mapsto U(X, A)$ is an (exact) functor from R-modules to G-modules.

PROPOSITION 7.1. *The set B^G of G-invariants in $B = U(X, A)$ coincides with the group $A^R = H^0(R, A)$.*

PROOF. An element $F \in B$ is G-invariant provided that $a(g, g^{-1} \cdot x)F(g^{-1}$

$\cdot x) = F(x)$ for almost all x for each G. The map $(g, x) \mapsto (x, g^{-1} \cdot x)$ maps $G \times X$ onto R and carries (Haar measure) $\times \mu$ onto ν. Thus, if we put $y = g^{-1} \cdot x$, we see that $u(x,y)F(y) = F(x)$ for ν a.e. (x,y) and hence that $F \in A^R$. $\square$

The following result interprets our cohomology as Eilenberg-Mac Lane group cohomology, at least in some cases.

THEOREM 5. *If $R = R_G$ with G operating freely then $H^n(R, A) \cong H^n(G, U(X, A))$, the latter group being the Eilenberg-Mac Lane cohomology of the (discrete) group G.*

PROOF. We shall give two proofs of this fact. First note that, by virtue of Proposition 7.1 and the comments preceding it, $A \mapsto H^n(G, U(X, A))$ satisfies all the conditions of Definition 6.1 for functors of cohomological type except possibly for the vanishing axiom, even if G is not free. Thus by uniqueness, it suffices to show that $H^n(G, U(X, I^R(A))) = 0$ for $n > 1$ if G acts freely, where $I^R(A)$ is $I(A)$ formed using R. In that case each orbit $R(x)$ can be identified with G itself via the map $\phi(x)(g) = g \cdot x$, so that $I^R(A) \cong U(G, A)$ and $(u(x,y)F)(s) = F(g^{-1}s)$ where $s \cdot y = x$. By the Fubini theorem in Moore [3], $B = U(X, U(G, A)) \cong U(X \times G, A)$ and so the map $F \mapsto \tilde{F}$ of B onto B given by $(\tilde{F}(x))(g) = F(g^{-1} \cdot x)(g)$ is an isomorphism. After conjugating by this isomorphism the action of G on B has the form $((s \cdot F)(x))(g) = F(x)(s^{-1}g)$. Thus B as a G-module is $U(X, I^G(A))$ where $I^G(A)$ is the regular representation of G with coefficients in A as defined in Moore [4]. Now by Theorem 1 of Moore [4], $H^n(G, B) \cong U(X, H^n(G, I^G(A))) = 0$ and we are done.

The second argument is in some sense a computational version of this conceptual argument and consists of giving an isomorphism θ of the cochain group $C^n(R, A)$ onto $C^n(G, B)$. It may be verified that $H \mapsto \theta(H)$ where

$$\theta(H)(s_1, \dots, s_n)(x) = H\left(x, s_1^{-1} \cdot x, s_2^{-1} \cdot s_1^{-1} \cdot x, \dots, s_n^{-1} s_{n-1}^{-1} \dots s_1^{-1} \cdot x\right)$$

does the job. $\square$

The result is also valid for $n = 1$ and for nonabelian R-modules A. Indeed the cochain map above sets up the isomorphism of sets.

Now suppose we have a relation R on X and that $R = R_{G_1} = R_{G_2}$ for two freely acting groups G_1 and G_2. If A is any polonais abelian group $U(X, A) = B$ is endowed with the structure of a G_1-module and also that of a G_2-module. The following yields an isomorphism of cohomology which is very strange indeed from the point of view of group cohomology.

COROLLARY. *There are natural isomorphisms from $H^n(G_1, B)$ onto $H^n(G_2, B)$.*

A further consequence of this result is the following which is very much the same as a result of Westman in [1].

THEOREM 6. *If R is hyperfinite and A is an R-module, then $H^n(R, A) = (0)$ for $n \geq 2$.*

PROOF. By Proposition 6.6, and the fact that the relativization of a hyperfinite relation is hyperfinite, it suffices to show $H^n(R_m, A) = (0)$, $n \geq 2$, for a partition $X = \bigcup X_m$ of X into invariant sets with $R_m = R \cap X_m \times X_m$. In particular we can take $A_m = A$ constant and may assume that R is either of type I_m for some finite m, or that all equivalence classes are infinite. In the first case we shall prove what we want and somewhat more in Proposition 7.2. We consider the case of infinite classes. According to Proposition 4.2, $R = R_{\mathbf{Z}}$ for action of the integers. If this action fails to be free with a nonzero isotropy group $\mathbf{Z}_x$ on a set of positive measure, the equivalence classes $R(x)$ would be finite contrary to assumption. Thus Theorem 5 is applicable, and $H^n(R, A) \cong H^n(\mathbf{Z}, U(X, A)) = 0$ for $n \geq 2$ as $\mathbf{Z}$ is free. $\square$

This provides a necessary but not sufficient condition for an equivalence relation to be hyperfinite. To see that it is not sufficient, consider the free group on two generators embedded as a dense subgroup of the rotation group $X = SO(3)$. Then G acting on X by left translation gives a countable standard equivalence relation R on X, with Haar measure, which is in fact ergodic of type II_1. Again since G is a free group and acts freely, it follows that $H^n(R, A) = (0)$ for $n \geq 2$. But $R = R_G$ is not hyperfinite, for if it were, then by Proposition 4.5, G would have to be amenable, which is not.

Theorem 5 has other rather interesting and unexpected corollaries. For instance, let $u_1, \ldots, u_n$ be n commuting Borel automorphisms of a space X with quasi-invariant measure u. Then this n-tuple defines an action of the group $\mathbf{Z}^n$ and let us assume that this action is free, which means that the u_i's are independent in that the fixed points of $u_i^{r_1} \ldots u_n^{r_n}$ form a null set. If they are jointly ergodic this can be weakened to the conditon that there is no nontrivial relation $u_i^{r_1} \ldots u_n^{r_n} = 1$ holding a.e.

By Proposition 4.4, the relation R generated by this action is hyperfinite and so $H^m(R, A) = 0$ for $m \geq 2$, and then by Theorem 5, $H^m(\mathbf{Z}^n, U(X, A)) = 0$ for $m \geq 2$. Let us take $n = 2$ and $m = 2$; then the group above is calculable by spectral sequence methods; cf. Moore [1], [2]. Let $G = \mathbf{Z}^2$, $H =$ the subgroup generated by, say, u_1 and $K = G/H$. Then the spectral sequence produces a three term grading $S_0 \supset S_1 \supset S_2 \supset 0$ on $H^2(G, U(X, A))$; here S_2 is the image by inflation of $H^2(K, U(X, A))$, which is zero as $K \cong \mathbf{Z}$ is free; and S_1 is the kernel of the restriction map into $H^2(H, U(X, A)^k)$, which is also zero as $H \cong \mathbf{Z}$ is free. Finally there is an injection i of S_1/S_2 (which is S_0) into $H^1(K, H^1(H, U(X, A)))$, with range equal to the kernel of a map from this group into $H^3(K, U(X, A))$ which is again zero. The conclusion is that

$$(0) = H^2(G, U(X, A)) \cong H^1\big(K, H^1(H, U(X, A))\big),$$

which we can evaluate. Any $\alpha \in Z^1(H, U(X, A))$ is determined by its value at the generator u_1 of H and it is easy to see that $H^1(H, U(X, A))$ is isomorphic to $U(X, A)/T_1(A)$, where $T_1(A)$ consists of all functions from X to A of the form $f(u_1(x))/f(x)$ for $f \in U(X, A)$. Now the quotient group $K = G/H$ operates in a natural way, and if we identify u_2 with a generator of k, this generator operates by sending the coset $g(x)T_1(A)$ into $h(x)T_1(A)$, where $h(x) = g(u_2^{-1}(x))$. The same calculation above shows that $H^1(K, H^1(H, U(X, A)))$ is isomorphic to $U(X, A)/T_1(A)$ modulo cosets of the form $h(u_2(x))/h(x)T_1(A)$. Thus if we let $T_2(A)$ be the group of elements of the form $h(u_2(x))/h(x)$, we see that $H^1(K, H^1(H, U(X, A)))$ is isomorphic to $U(X, A)/T_1(A)T_2(A)$. We then obtain the following theorem.

THEOREM 7. *If u_1 and u_2 are commuting automorphisms of $(X, \mathfrak{B}, \mu)$ such that $u_1^{r_1}u_2^{r_2}$ has a fixed point set of measure zero unless $r_1 = r_2 = 0$, then any $f \in U(X, A)$ can be written in the form*

$$f(x) = \big(f_1(u_1(x))/f_1(x)\big)\big(f_2(u_2(x))/f_2(x)\big)$$

for some $f_1, f_2 \in U(X, A)$.

It is of course quite false that any f can be written as a quotient $g(u(x))/g(x)$ for a *single* u and a *single* g, and it is somewhat surprising that the result of Theorem 7 is true. If we take u_1 and u_2 to be independent irrational rotations with $X = \mathbf{T}$, the circle and with $X = \mathbf{T}$ or $\mathbf{R}$, we obtain very concrete and elementary statements that seem quite inaccessible by any other method.

We turn now to some other vanishing theorems which are related to the above.

PROPOSITION 7.2. *If R is type* I, *$H^n(R, A) = 0$ for $n \geqslant 1$ and any A (including, for $n = 1$, nonabelian A).*

PROOF. In view of Proposition 6.2, it suffices to prove the result for $n = 1$, in which case we may take it nonabelian. As usual, we may and shall assume that $A = A_n$ is constant and that R is uniformly of type I_n for some $n = 1, \ldots, \infty$. In this case it is clear from Proposition 3.2 that R is the relation associated to a free action of the cyclic group $\boldsymbol{Z}_n$, where for convenience of notation $\boldsymbol{Z}_\infty = \boldsymbol{Z}$. According to Proposition 3.2 we can write X up to null sets as $X = Y \times \boldsymbol{Z}_m$ with $\boldsymbol{Z}_m$ acting by translation on the second factor. The module structure is given by a Borel function a of $\boldsymbol{Z}_m \times (Y \times$

$\mathbf{Z}_m$) into Aut(A) satisfying the usual cocycle identity, which in view of the nature of the action has the form $a(g, y, s) = b(y, gs)b(y, s)^{-1}$ for some Borel function b on $Y \times \mathbf{Z}_m$. Then $\mathbf{Z}_m$ acts on $U(X) = U(Y \times \mathbf{Z}_m)$ by

$$(g \cdot F)(y, s) = b(y, s)b(y, g^{-1}s)^{-1}F(y, g^{-1}s),$$

and it is immediate that $U(X)$ is isomorphic to $U(X)$ with $\mathbf{Z}_m$ acting by $(g \circ F)(y, s) = F(y, g^{-1}s)$. Thus, as a $\mathbf{Z}_m$-module, $U(X)$ is isomorphic to $U(Y, I(A))$ where $I(A)$ is (as in Moore [3]) the regular representation of $\mathbf{Z}_m$ with coefficients in A. Since $H^1(\mathbf{Z}_m, I(A)) = (0)$ it follows that $H^1(\mathbf{Z}_m, U(Y, I(A))) = (0)$ by Theorem 1 of Moore [4]. $\square$

We have something of a converse to this result; recall that associated with a relation we defined the fundmanental class $\alpha \in H^1(R, P(N))$. Here $P(N)$ is the module defined by the partition of $X = \bigcup X_n$ where $X_n = \{x: |R(x)| = n\}$ and where $N_n = N_x$ ($x \in N_n$) is a fixed set of n elements and $P(N)_x = P(N_x)$, $x \in X_n$, is the permutation group on N_x.

PROPOSITION 7.3. *A relation R is type* I *if and only if its fundamental class is trivial.*

PROOF. If R is type I, α is trivial by the previous proposition. Suppose conversely that α is trivial. We may assume without loss of generality that all the classes are infinite, since there is nothing to prove for the components of type I_n ($n < \infty$). So let $N = \mathbf{Z}$ be the integers. The triviality of α means that we may select bijections $\phi(x)$, $x \in X$, of N onto $R(x)$ such that $\phi(x) = \phi(y)$ for almost all pairs $(x, y) \in R$. Let $Y = \{x: \phi(x)(0) = x\}$, which is evidently a Borel set. Then Y meets each class $R(x)$ in exactly one point for almost all x. The map $(n, y) \mapsto \phi(y)(n)$ from $\mathbf{Z} \times Y$ is an equivalence between the relation R_0 on $\mathbf{Z} \times Y$ given by $(n, y) \sim (n', y')$ iff $y = y'$ and the relation R on X. Since R_0 is type I by definition, we are done. $\square$

One could ask for converses to Proposition 7.2 of a slightly different kind; in particular it is appealing to conjecture that R is type I if and only if $H^1(R, T) = (0)$ where T is the circle group with trivial action. At any rate, one can show the following.

PROPOSITION 7.4. *R is type* I *if and only if $H^1(R, A) = 0$ for all abelian A.*

PROOF. First, we observe that every non-type I relation R contains a hyperfinite subrelation S which is not type I. This may be seen by the same argument which Dye [1] uses in the II case. Now, one can define the notion of induced module which constructs from each S-module A an R-module $I_S^R(A)$ when $R \supset S$. This is the analogue of induced modules from a subgroup to a group defined in Moore [3] and in case $S = \Delta(X)$, the diagonal of X, $I_\Delta^R(A)$ is simply $I(A)$ which we defined above. In addition one has a

Shapiro's lemma which says that $H^n(S, A) \cong H^n(R, I_S^R(A))$, just as in Moore [3]. As we do not need this construction, we shall not explore the matter further. Then if S is hyperfinite and non type I, it is easy to see that $H^1(S, T) \neq 0$ where T is the circle so that R has nontrivial cohomology. $\square$

In regard to the conjecture about $H^1(R, T)$, it reduces to showing that if R is a strictly ergodic relation (i.e., non type I) then $H^1(R, T) \neq 0$. If R is type III, this is true since R has a nontrivial Radon-Nikodym cocycle D in $H^1(R, \mathbb{R}^+)$ as defined in §2, and if $D^{i\alpha}$ is the image of this cocycle in $H^1(R, T)$ under the map $x \mapsto x^{i\alpha}$, it may be shown without too much trouble that $D^{i\alpha}$ is nontrivial for almost all α. If R is type II_∞, the question can be reduced by relativization (see Proposition 7.6 below) to the II_1 case, where we do not know the answer. As we shall see in the second paper, the assertion that $H^1(R, T) \neq 0$ means that certain II_1 factors have nontrivial outer automorphisms of a specified type.

We shall conclude this section with some complements concerning cohomology that will be of use later on. Let R_i be relations on $(X_i, \mathfrak{B}_i, \mu_i)$, let ϕ be a morphism of R_1 into R_2, and let (A_i, u_i) be R_i-modules and ψ a map of modules compatible with ϕ in that ψ is given by a Borel family of maps $\psi(x)$ from $(A_2)_{\phi(x)}$ into $(A_1)_x$ so that $\psi(x)u_2(\phi(x), \phi(y)) = u_1(x, y)\psi(y)$. We define a cochain map t^n: $C^n(R_2, A_2) \to C^n(R_1, A_1)$ by

$$(t^n f)(x_0, \ldots, x_n) = \psi(x_0) \cdot f(\phi(x_0), \ldots, \phi(x_n))$$

and we obtain induced maps on cohomology, also denoted by t^n. We are interested in the special case when $R = R_1 = R_2$ and $A = A_1 = A_2$ and $\psi = \mathrm{id}$ and when ϕ is not only a morphism but satisfies $\Gamma(\phi) \subset R$, so that in some sense it is an "inner" morphism. The following is the analogue of the well-known fact that inner automorphisms of a group operate trivially on cohomology.

PROPOSITION 7.5. *The induced map t^n on cohomology corresponding to ϕ is the identity in all dimensions.*

PROOF. In view of Proposition 6.2, $H^n(R, A)$ is isomorphic to $H^1(R, U^n(A))$ in a way that commutes with the maps t^n, so that it suffices to consider the case $n = 1$. But if $f \in Z^1(R, A)$, then

$$\begin{aligned} t^1(f)(x, y) &= f(\phi(x), \phi(y)) \\ &= f(\phi(x), x) f(x, y) f(y, \phi(y)) = b(x)^{-1} f(x, y) b(y) \end{aligned}$$

with $b(y) = f(y, \phi(y))$. Thus $t^1(f) \sim f$ and we are done. We note that the same argument works even for nonabelian A. $\square$

COROLLARY. *If ϕ is as above and A is a nonabelian R-module, the natural map t^1 of the set $H^1(R, A)$ to itself induced by ϕ is the identity map.*

Now if R is a relation on $(X, \mathfrak{B}, \mu)$ and if $B \subset X$, we have a relation $R|_B$ on B as discussed earlier. The injection map i: $B \to X$ is a morphism of $R|_B$ into B and if A is an R-module, $\psi(x) = \text{id}$ gives a consistent map of the module A into restriction of the module A to B, $A|_B$. We obtain then maps on cohomology $r^n(B)$ from $H^n(R, A)$ into $H^n(R|_B, A|_B)$, which we think of as restriction to B.

PROPOSITION 7.6. *The maps $r^n(B)$ are surjective in all dimensions, and if the saturation $R(B)$ of B is conull, they are bijective in all dimensions.*

PROOF. By Proposition 6.6 we can immediately reduce to the case when $R(B) = X$. Then by Proposition 3.4 there is a map ϕ from X into B with $\Gamma(\phi) \subset R$, and we let t^n be the induced maps on cohomology, $H^n(R|_B, A|_B)$ into $H^n(R, A)$. Now $i \circ \phi$ maps X into X and $\Gamma(i \circ \phi) \subset R$, and $\phi \circ i$ maps B into B and $\Gamma(\phi \circ i) \subset R|_B$. It follows by Proposition 7.5 that these maps induce the identity on cohomology and hence $r^n \circ t^n$ and $t^n \circ r^n$ are both the identity and so r^n is bijective for all n as desired. $\square$

In dealing with two dimensional cocycles in the second paper in this series, it will be quite convenient to work with a particular kind of cocycle representative; specifically we say that $a \in Z^2(R, A)$ with A a trivial R-module is *normalized* if $s(\cdot, \cdot, \cdot) = 1$ whenever two of three variables are the same.

PROPOSITION 7.7. *If A is divisible by two (i.e., if each A_x is), then any $s \in Z^2(R, A)$ is cohomologous to a normalized one.*

PROOF. If s is given, then the function $c(x, y) = s(x, y, x)^{-1}$ is a well-defined function on R as the map j: $(x, y) \mapsto (x, y, x)$ embeds $R = R^1$ into R^2 and the image $j_*(\nu^1)$ is absolutely continuous with respect to ν^2. The coboundary $\delta(c)$ of c evaluated at (x, x, x) is obviously equal to $s(x, x, x)^{-1}$, and therefore we may modify s by δc and assume that $s(x, x, x) = e$. Now notice that $\delta s(x, x, y, y) = e$; this holds for almost all pairs (x, y) by the same kind of reasoning as above. So we find that $s(x, y, y) = s(y, y, y) = 1$ for a.a. x, y. By a similar argument, we see $s(y, y, x) = e$ for a.a. x, y. Finally, the relation $\delta s(x, y, x, y) = e$ tells us that $s(x, y, x) = s(y, x, y) = c(x, y)$ is a symmetric function of two variables, and that $c(x, x) = e$. We may find for each x a Borel map r_x from A_x to A_x so that $r_x(a)^2 = a$ and $r_x(e) = e$ and so that r_x is constant on a countable partition of X into invariant sets. Then let $d(x, y) = r_x(c(x, y))$ and replace s by $t = s(\delta d)^{-1}$, noting that t vanishes when all three variables agree since $d(x, x) = e$. Then $\delta d(x, y, x) = d(y, x)d(x, y) = c(x, y) = s(x, y, x)$ and so $t(x, y, x) = e$ a.e. Then t is a normalized cocycle cohomologous to s. $\square$

PROPOSITION 7.8. *If s is a normalized cocycle, s is skew symmetric in its three variables.*

PROOF. The equation $\delta s(x,y,z,y) = 1$ and the fact that s is normalized yields the result that $s(x,z,y) = s(x,y,z)^{-1}$. Similarly we obtain the same result for transposing the first two variables and the result follows. $\square$

8. The Poincaré flow of a one-cocycle. We now turn to a most important and significant construction which has been discussed in the literature in many contexts; see for example Anzai [1] and Ambrose [1]. We are given a relation R on $(X, \mathcal{B}, \mu)$ and a separable locally compact group A, not necessarily abelian, and view it as a trivial R-module. If $c \in Z^1(R, A)$ we will associate to the pair (R, c) an action of the group A on a standard measure space Z, which generalizes the Poincaré suspension of an automorphism and the construction of a flow built under a function (cf. Ambrose [1]). In the context of virtual groups this construction has been introduced by Mackey [2].

We first form the product $X \times A$ and endow it with the measure $\sigma = \mu \times \lambda$ where λ is a finite measure equivalent to Haar measure on A, and then introduce an equivalence relation $R(c)$ on $X \times A$ by defining $(x, a) \sim (x', a')$ if $(x, x') \in R$ and $a' = ac(x, x')$. It is not hard to verify that this is a countable standard equivalence relation and that σ is a quasi-invariant measure. The point is that even though R is ergodic, $R(c)$ is usually not; let $\mathcal{G}(R(c))$ be the σ-field of invariant sets, and let M_0 be the image of $\mathcal{G}(R(c))$ in the measure algebra M of the measure σ. We note that A operates on $X \times A$ by $a(x, a') = (x, aa')$ and that these automorphisms are in the normalizer $N(R(c))$ of $R(c)$ and preserve the measure class of σ. It follows that A sends $\mathcal{G}(R(c))$ into itself and hence each $a \in A$ defines an automorphism of the σ-algebra M_0. It is clear that the action of A as a group of automorphisms of M_0 satisfies the conditions of the point realization theorem in Mackey [1], and hence there exists an essentially unique standard Borel space Z with measure τ and an action of A on (Z, τ) so that the action of A on the measure algebra $M(\tau)$ of τ is equivalent to the action of A on M_0.

DEFINITION 8.1. We call the above action of A on (Z, τ) the Poincaré flow $P(R, c)$ associated to R and c.

It is evident that the invariants in $M(\tau)$ of the Poincaré flow are the A-invariants in M_0, or equivalently the $R(c)$-invariant sets among the A-invariant sets in the measure algebra M. However, the A-invariant sets are evidently those of the form $E \times A \subset X \times A$ and such a set is $R(c)$-invariant if and only if E is an R-invariant set. Hence the A-invariants in $M(\tau)$ can be identified with the measure algebra of R-invariants, and this proves the first part of the following proposition.

PROPOSITION 8.1. *The Poincaré flow $P(R, c)$ is ergodic if and only if R is ergodic. Moreover, $P(R, c)$ depends only on the cohomology class of c.*

PROOF. If $b(x,y) = d(x)^{-1}c(x,y)d(y)$, the map $(x, a) \mapsto (x, ad(x))$ carries $R(c)$ onto $R(b)$ and commutes with left translation by A. The result follows immediately. $\square$

We shall now assume without further mention that R is ergodic. We now want to describe an important invariant of the pair (R, c), namely the *proper range* of c. If A is abelian there is not much problem, but if A is nonabelian, things are more involved. If $z \in Z$, the underlying space of the Poincaré flow, let $A(z)$ be the isotropy group of this point. Then $A(z) \in S(A)$, the space of all closed subgroups of A. Fell [1] has introduced a topology on $S(A)$ in which it is a compact metric space, and we observe that ϕ: $z \mapsto A(z)$ is a Borel map of Z into $S(A)$ (Auslander and Moore [1]). We transport τ, or rather its measure class, by ϕ to obtain a measure (class) $\rho = \phi_*(\tau)$ on $S(A)$. Since $S(A)$ is a topological transformation group for A with A acting by conjugation and since ϕ is equivariant, the measure ρ or rather its class is quasi-invariant and ergodic, an object which Mackey calls a quasi-orbit. We denote this quasi-orbit by $C(R, c)$ and call it the proper range of c. Now let $NS(A)$ denote the subset of $S(A)$ consisting of normal subgroups, i.e., the fixed points of the A action. Now if $H \in S(A)$, the subgroup $\bigcap\{aHa^{-1}: a \in A\} = \psi(H)$ is a normal subgroup, and it is not hard to see that ψ is a Borel map which is also evidently equivariant. Then $\psi_*(C(R, c))$ is a quasi-orbit for the (trivial) action of A on $NS(A)$ and we denote this by $NC(R, c)$, the normalized proper range. Of course if A is abelian, which is the case of prime interest, $NC(R, c) = C(R, c)$. Since a quasi-orbit for a trivial action must be concentrated on one point, we have the following result:

PROPOSITION 8.2. *The normalized proper range is a Dirac measure concentrated at a single point B in $NS(A)$; this subgroup will also be called the normalized proper range of c, and denoted by* npr(c), *or the proper range* pr(c) *of c if A is abelian.*

The following provides a characterization of npr(c) that will be essential later on.

PROPOSITION 8.3. *The subgroup* npr(c) *of A is the kernel of the natural action of A on the measure algebra $M(\tau)$ of the Poincaré flow, or equivalently* npr(c) $= \{a: f(x, ab) = f(x, b)$ *a.e.*$\}$ *for every $R(c)$ invariant function f on $X \times A$.*

PROOF. If a is in the kernel of the action of A on $M(\tau)$ so that $a \cdot E = E$ for all $E \in M(\tau)$, then evidently a fixes almost all points in any point realization of the action of A on $M(\tau)$. In particular $a \in A(z)$ for almost all $z \in Z$. By Fubini's theorem we may conclude that, for almost all z, $a \in A(b \cdot z) = bA(z)b^{-1}$ for almost all b. Some further argument shows that this implies that for almost all z, $a \in bA(z)b^{-1}$ for *all* b. Thus $a \in \bigcap_b bA(z)b^{-1} = \psi(A(z))$ for almost all z. But $\psi(A(z))$ is a.e. constant and equal to npr(c) by definition. Hence $a \in$ npr(c).

Conversely if $a \in$ npr(c), this says that $a \in A(z)$ for almost all z and hence that $a \cdot z = a$ for almost all z and hence that a acts trivially on the measure algebra $M(\tau)$ of (Z, τ), and we are done. The fact that the kernel of the action on $M(\tau)$ is the same as those a fixing all $R(c)$ invariant functions f (i.e., those that are constant on $R(c)$ equivalence classes) is clear.

It is our intention to identify the normalized proper range and hence the proper range itself for A abelian in terms of a more concrete and accessible object which we call the asymptotic range of c. This result extends results of Krieger [5] and Araki-Woods [1], and is closely related to the recent work of Connes and Takesaki [1], [2]. See also Hamachi, Oka and Osikawa [1]. Since we obtained this result we learned that K. Schmidt [1] has obtained the same result for abelian A and hyperfinite R. □

We recall that Araki and Woods introduced the notion of the asymptotic ratio set for the classification of infinite tensor products of finite type I factors. This invariant is really an invariant of ergodic theory associated with the natural abelian group operating on an infinite product of finite probability spaces which describes the failure of this action to be measure preserving. Krieger [5] then extended this invariant to the context of a single ergodic transformation T, to obtain the set $r_\infty(T) \subset [0, \infty)$. This set describes in some sense the essential range of the Radon-Nikodym derivatives of $T^n\mu$ with respect to μ. A moment's reflection on the definition shows that all that matters is the relation R generated by the powers of T, and that what is relevant is the Radon-Nikodym cocycle $D \in Z^1(R, \mathrm{R}^+)$. It is evident then that one should be able to define the asymptotic ratio set associated to any countable standard relation R and a cocycle $c \in Z^1(R, A)$ where A is a locally compact group, abelian or not. In view of its true nature, the terminology "asymptotic range" is more appropriate than "asymptotic ratio set". It turns out in fact that when phrased in terms of relations, the definition becomes simpler and its meaning (we feel) becomes more transparent. Recall that the essential range of a Borel map f from a measure space (X, μ) into a topological space is the smallest closed set F such that $f^{-1}(F^c)$ has complement of μ measure zero; call this set $\sigma(f)$.

DEFINITION 8.2. Let R be a relation on $(X, \mathfrak{B}, \mu)$ and let A be a locally compact trivial R-module and let $c \in Z^1(R, A)$. The asymptotic range $r^*(c)$ is $\bigcap_{\mu(B)>0}\sigma(c_B)$, where c_B is the restriction of c to B described in Proposition 7.6. If $\bar{A}$ is the one point compactification of A and $\bar{c}$ is c viewed as a function with values in $\bar{A}$, the extended asymptotic range is

$$r_\infty^*(c) = \bigcap_{\mu(B)>0} \sigma(\bar{c}_B).$$

Note that $r^*(c)$ and $r^*_\infty(c)$ are closed subsets of A and $\bar{A}$ respectively, and that $r^*(c) = r^*_\infty(c) \cap A$; the point $\infty \in \bar{A} - A$ belongs to $r^*_\infty(c)$ if and only if for each set B of positive measure c is essentially unbounded on $B \times B \cap R$. The following characterization of points in $r^*_\infty(c)$ will be useful.

PROPOSITION 8.4. *If $a \in \bar{A}$, then $a \in r^*_\infty(c)$ if and only if for every neighborhood U of a and every subset Y of X of positive measure the set $\{(x,y)$: $x, y \in Y, c(x,y) \in U\}$ has projection onto the first (second) coordinate equal to almost all of Y.*

PROOF. The argument below is evidently symmetric in X and Y, so it is enough to consider projection to the first coordinate. If $a \in r^*_\infty(c)$, then for all Y and all neighborhoods U of a, $\{(x,y)$: $x, y \in Y, c(x,y) \in U\}$ has positive measure. Since R is a countable relation, its projection to the first coordinate is a Borel set. If its complement in Y, say N, were of positive measure, then c restricted to $N \times N$ would miss U entirely, and so $a \notin \sigma(c_N)$, contrary to assumption.

Conversely if the condition of the proposition is satisfied, we must show that $c_*(\nu_Y)(U)$ has positive measure for each neighborhood U of a and each set Y of positive measure where $\nu_Y$is the measure ν restricted to $Y \times Y$. But the measure of the above set is the ν measure of $\{(x,y)$: $x, y \in Y, c(x,y) \in U\}$ and our hypothesis is that for almost all x the section of this set over x is nonvoid and hence has positive measure for the counting measure on $R(x)$. It follows that the set has positive ν measure by the definition of ν. $\square$

We start to establish properties of $r^*(c)$.

PROPOSITION 8.5. *The asymptotic range $r^*(c)$ is a closed subgroup of A; if $r^*(c)$ is noncompact then $\infty \in r^*_\infty(c)$.*

PROOF. The last statement is obvious since $r^*_\infty(c)$ is closed. We have to show that $r^*(c)$ is a subgroup since we know it is closed. First, if Y is a set of positive measure $\Delta(Y)$ the diagonal on Y is a set of positive ν measure. Moreover, $c(y,y) = e$ so that c_Y takes on the identity element e of A on a set of positive measure. Thus e is surely in the essential range of c_Y and hence in $r^*(c)$. Moreover, since $c(x,y) = c(y,x)^{-1}$ an easy argument shows that $a \in r^*(c)$ if and only if $a^{-1} \in r^*(c)$.

Now let $a, b \in r^*(c)$ and let W be a neighborhood of ab and $Y \subset X$ a set of positive measure. Choose neighborhoods U and V of a and b respectively so that $UV \subset W$ and by Proposition 8.4 select null sets N_1 and N_2 so that if $x \in Y - N_1$, there exists $y \in Y$ with $c(x,y) \in U$ and if $x \notin Y - N_2$, there exists $z \in Y$ with $c(x, Z_0) \in V$. Let $N_3 = Y \cap R(N_2)$ which is also a null set and let $N = N_1 \cup N_3$. Then if $x \in Y - N$, the y above with $c(x,y) \in U$ cannot belong to N_2 since $x \sim y$. Consequently we can use this y as the x in the second statement and find $z \in Y$ with $c(y, z) \in V$. Then $c(x, z) = c(x, y)c(y, z) \in UV \subset W$, and the criterion of Proposition 8.4 is satisfied so that $st \in r^*(c)$ and we are done. $\square$

Recall that we defined the map ψ from $S(A)$ the subgroups of A to $NS(A)$, the normal subgroups of A by $H \mapsto \bigcap\{aHa^{-1}: a \in A\}$. Our purpose is to find invariants that depend only on the cohomology class of the cocycle c. We note that if A is nonabelian, $r^*(c)$ will (trivially) depend on more than just the class of c; for $d(x,y) = ac(x,y)a^{-1}$ is eqivalent to c, and $r^*(d)$ is evidently $ar^*(c)a^{-1}$, and it is easy to construct examples where $r^*(c)$ is not a normal subgroup. The best that one could hope for is that conjugacy class $\{ar^*(c)a^{-1}: a \in A\}$ should be an invariant of the class of c, but we cannot establish this, and there are some doubts that it is true. We can, however, establish the weaker result that $\psi(r^*(c))$ does depend only on the class of c.

PROPOSITION 8.6. *The normal subgroup $\psi(r^*(c))$ depends only on the class of c, and hence if A is abelian $r^*(c)$ depends only on the class of c.*

PROOF. Let $d(x,y) = b(x)^{-1}c(x,y)b(y)$ be an equivalent cocycle and let $a \in \psi(r^*(c))$ so that $sas^{-1} \in r^*(c)$ for all $s \in A$. It will suffice to show that $a \in r^*(b)$. So let U be a neighborhood of a and $Y \subset X$ a set of positive measure. For each $s \in A$, there is an open neighborhood $O(s)$ of s, and a neighborhood $V(s)$ of sas^{-1} so that $O(s)^{-1}V(s)O(s) \subset U$. We can find a countable family s_i so that $O(s_i)$ covers A and then we can find Borel sets $O_i \subset O(s_i)$ so that $A = \bigcup_i O_i$ is a partition. Now let $Y_i = b^{-1}(O_i) \cap Y$ so that $Y = \bigcup_i Y_i$ disjointly. Then as $s_i a s_i^{-1} \in r^*(c)$ we may find, by Proposition 8.4, null sets $N_i \subset Y_i$ so that if $x \notin N_i$, there exists $y \in Y_i$ with $c(x,y) \in V(s_i)$. Then $N = \bigcup_i N_i$ is a null set, and if $x \in Y - N$, $x \in Y_i - N_i$ for some i and there is $y \in Y_i$ so that $c(x,y) \in V(s_i)$. But now $b(x)$ and $b(y)$ belong to $O_i \subset O(s_i)$ and so $d(x,y) = b(x)^{-1}c(x,y)b(y) \subset O_i^{-1}V(s_i)O_i \subset U$ and by Proposition 8.4, $a \in r^*(d)$ and we are done. $\square$

REMARK. It may also be seen, without much difficulty, that if $c \sim c'$, then $\infty \in r^*_\infty(c) \Leftrightarrow \infty \in r^*_\infty(c')$.

We are now ready to state and prove the main result of this section connecting the proper range and the asymptotic range.

THEOREM 8. *For any cocycle c we have $\psi(r^*(c)) = \mathrm{npr}(c)$, the normalized proper range, so that for abelian A, the asymptotic range $r^*(c)$ is the same as the proper range $r(c)$.*

PROOF. We shall first show that $\mathrm{npr}(c) \subset \psi(r^*(c))$. Thus suppose $a \notin \psi(r^*(c))$; we have to show that $a \notin \mathrm{npr}(c)$. We are given that there is some t such $tat^{-1} \notin r^*(c)$, and evidently we may assume for notational simplicity that $a \notin r^*(c)$. This means that there is some $Y \subset X$ with $\mu(Y) > 0$ so that

$a \notin \sigma(c_Y)$. But now there is, by Proposition 3.4, a Borel map ϕ of X onto Y so that $\Gamma(\phi) \subset R$. By the corollary to Proposition 7.5, the cocycle c is equivalent to the cocycle $d(x, y) = c(\phi(x), \phi(y))$ and d has the property that $a \notin \sigma(d)$, so we may and shall assume that c has this property.

We may then select a neighborhood U of the identity and if necessary modify c on a null set so that $c(x, y) \notin Ua$ for any $(x, y) \in R$. Now select a neighborhood V of the identity in A so that $V\bar{a}^1Va^{-1} \subset U$. Then $c(x, y) \notin V\bar{a}^1V$ and so $Vc(x, y) \cap aV = \varnothing$. Now let $W = X \times V \subset X \times A$ and let $S(W)$ be its saturation with respect to the equivalence relation $R(c)$. The above equation says that $S(W) \cap (X \times aV) = \varnothing$, but $X \times aV = a(W)$ in the Poincaré action and so $a(W) \cap S(W) = \varnothing$. Since $S(W) \supset W$, it follows that $S(W) \neq aS(W)$ and by Proposition 8.3, $a \notin \operatorname{npr}(c)$, and the first part of the proof is complete.

We now turn to the reverse inclusion $\psi(r^*(c)) \subset \operatorname{npr}(c)$. Thus if $s \in (r^*(c))$, we must show by Proposition 8.4 that left translation by s on $X \times A$ fixes every $R(c)$-invariant function f. We may clearly assume that f is bounded, and let us assume for the moment that f has the following continuity property: $a \mapsto f(\cdot, a)$ is a continuous map from A into the norm topology of $L^\infty(X)$. We first prove that translation by a fixes any function satisfying this additional property.

Let $\varepsilon > 0$ be given and $s \in r^*(c)$ and let A_0 be any compact subset of A. We may choose a larger compact set A_1 so that A_0 and $A_0 s$ are in the interior of A_1. Then since $b \mapsto f(\cdot, b)$ is uniformly continuous on A_1, there is a finite partition of A_1 into Borel sets, $A_1 = \bigcup_{i=1}^N A^i$ so that the norm variation of $f(\cdot, b)$ for $b \in A^i$ is less than $\varepsilon/4$. Now select $a_i \in A^i$ and define a new function $g(x, a) = f(x, a_i)$ if $a \in A^i$ and $g(x, a) = f(x, a)$ if $a \notin A_1$. It is clear that $|g - f|_\infty < \varepsilon/4$. Now for each i, we can find a finite partition P^i of the space X so that $x \mapsto g(x, a_i) = f(x, a_i)$ has variation less than $\varepsilon/4$ as x varies through each atom of the partition P^i. We find a common refinement P of all of these P^i and label its atoms $X_1, \ldots, X_n$. Then we know that $|g(x, a) - g(y, a)| < \varepsilon/4$ for $a \in A_1$ provided $x, y \in X_i$.

We shall now need the following fact.

PROPOSITION 8.7. *Suppose $s \in r^*(c)$ and that U is a neighborhood of s. Then there exists a Borel map ϕ from X to X with $\Gamma(\phi) \subset R$ so that $c(x, \phi(x)) \in U$ for almost all x.*

PROOF. Let $Z = \{(x, y) \in R\colon c(x, y) \in U\}$ so that, recalling that π_l is projection to the first coordinate, $\pi_l(Z)$ is almost all of X according to Proposition 8.4. By standard cross section theorems (cf. Theorem 1) we may find a Borel function ϕ' from X into Z so that $\pi_l(\phi'(x)) = x$. Then $\phi'(x) = (x, \phi(x))$ and ϕ is the desired Borel function. $\square$

Now we apply this as follows: let U be a neighborhood of our given $s \in r^*(c)$ which will be at our disposal, and let R_i be the relativization of R to the atoms X_i of the partition P. It is clear by the definitions that $s \in r^*(c_i)$ where c_i is the restriction of c to R_i, and so we may find by the proposition, functions ϕ_i from X_i to X_i with $\Gamma(\phi_i) \in R$, and $c(x, \phi_i(x)) \in U$. We piece these partial functions together to obtain a global function from X to X. We select the neighborhood U of s as follows; first, find a neighborhood V of the identity in A so that $|f(\cdot, a) - f(\cdot, av)|_\infty < \varepsilon/4$ for $v \in V$ and $a \in A_1$ and so that $A_0 sV \subset A_1$. We then let $U = sV$. For convenience, introduce the function $k(x, a) = 1$ for $a \in A_0$ and 0 otherwise, and use the notation $c \doteq d$ to mean $|c - d| < \varepsilon/4$.

Then we have

$$k(x, a)f(x, a) = k(x, a)f\big(\phi(x), ac(x, \phi(x))\big)$$

(as f is $R(c)$ invariant)

$$\doteq k(x, a)g\big(\phi(x), ac(x, \phi(x))\big)$$

(by the properties of g)

$$\doteq k(x, a)g\big(x, ac(x, \phi(x))\big)$$

(as x and $\phi(x)$ both belong to the same X_i, and $A_0 U \subset A_1$)

$$\doteq k(x, a)g(x, as)$$

(since $\phi(x, \phi(x)) \in U$, and hence $s^{-1}\phi(x, \phi(x)) \in s^{-1}U = V$)

$$\doteq k(x, b)f(x, as)$$

(again by the approximation properties of g). Now we have four approximations each to within $\varepsilon/4$, and so we have $|k(x, a)(f(x, a) - f(x, as))| < \varepsilon$ almost everywhere. Since it is true for every ε, $k(x, a)f(x, a) = k(x, a)f(x, as)$ holds almost everywhere. We exhaust A by a sequence of compact sets A_0 so we see that $f(x, a) = f(x, as)$ a.e. for each $s \in r^*(c)$ wherever f satisfies the continuity property above. Note that we want invariance by left translations by s rather than right translations, but the above is the first step toward that end.

Now let f be an arbitrary bounded Borel invariant under $R(c)$ and let θ be a continuous positive function on A with integral one with respect to right Haar measure and with support concentrated in a small neighborhood of the identity. We "smooth" f with θ to form

$$f_\theta(x, a) = \int f(x, ta)\theta(t)dt = \int f(x, t)\theta(ta^{-1})dt$$

where dt is right Haar measure on A. Since left translations commute with $R(c)$, it is clear that f_θ is again $R(c)$-invariant, and it is evident from the second integral formula above that f_θ satisfies our continuity condition. Now let θ_n be a sequence of such θ's with supports shrinking to the identity, and let f_n be the corresponding smoothed functions. It is elementary to show now that for any $g \in L_1(X \times A)$,

$$\int f_n(x, a)\, g(x, a) dxda \to \int f(x, a)\, g(x, a) dxda.$$

Then for any $s \in r^*(c)$, $f_n(x, as) = f_n(x, a)$ a.e. and it follows immediately that

$$\int f(x, as)\, g(x, a) dx\, da = \int f(x, a)\, g(x, a) dxda$$

and hence that $f(x, as) = f(x, a)$ a.e. This holds in particular for $s \in \psi(r^*(c))$ which is a normal subgroup of A, and since left invariance and right invariance by translations from a normal subgroup amount to the same thing, we see that $f(x, sa) = f(x, a)$ a.e. for $s \in \psi(r^*(c))$ and Theorem 8 is proved. $\square$

REMARKS. 1. It was shown by K. Schmidt for hyperfinite R and abelian A that $r_\infty^*(c) = 0 \Rightarrow c$ is a coboundary. His argument may easily be carried out for general R.

2. It has been noted by several mathematicians that if $A = \mathbb{R}$ and c is bounded in various senses, then c must be a coboundary. It follows from Remark 1 that if A is *any* abelian group with no nontrivial bounded subgroups, then c bounded $\Rightarrow c$ is a coboundary. Another amusing proof of this for the case $A = \mathbb{R}$ may be obtained by von Neumann algebra techniques; this will be carried out in the sequel to the present paper.

We close with the observation that Krieger in [6], using Poincaré flows, has in effect classified ergodic hyperfinite relations in a beautiful and far reaching generalization of Dye's classification in the II_1 hyperfinite case. More precisely: if R is strictly ergodic and infinite (i.e., is of type II_∞ or type III), he shows that the Poincaré flow $P(R, \log D)$ associated to the cocycle $\log D \in Z^1(R, \mathbb{R})$ is a complete isomorphism invariant for R. Of course, one does not know isomorphism (i.e., conjugacy) invariants for ergodic actions of the real line, but the equivalence of the classification schemes of these two objects is of great potential value.

The II_∞ case corresponds to the case when $P(R, \log D)$ is translation on the real line, and the subdivision of the type III relations into the III_λ categories can be defined by corresponding properties of the Poincaré flow. Namely, R is III_λ, $0 \leq \lambda \leq 1$, if the Poincaré flow is periodic of period $-\log(\lambda)$ where period ∞ means that it is free and nontransitive.

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