TOKYO J. MATH.
VOL. 11, NO. 2, 1988

Ergodic Measure Preserving Transformations of Finite Type

Stanley EIGEN, Arshag HAJIAN and Yuji ITO

Northeastern University and Keio University

§1. Introduction.

In this paper we shall consider properties of ergodic measure preserving (e.m.p.) transformations T defined on an infinite (σ-finite) Lebesgue measure space $(X, \mathscr{B}, m)$. It is well known that, in general, properties of such transformations are quite different from those of e.m.p. transformations defined on a finite measure space. For example, if $m(X)<\infty$ then it is easy to show that any non-singular measurable transformation S defined on $(X, \mathscr{B}, m)$ satisfying $ST=TS$ must preserve the same measure m; this need not be the case if $m(X)=\infty$, see [8]. Furthermore, if $m(X)=\infty$, then the L^{∞}-point spectrum $\Lambda(T)$ can be uncountable; [1], [9], [12].

A distinguishing feature of e.m.p. transformations T defined on a σ-finite measure space is the fact that if $m(X)=\infty$ then T always admits weakly wandering (w.w.) sets of positive measure, and hence w.w. sequences; this is never the case if $m(X)<\infty$.

In [7], an example of an e.m.p. transformation T was constructed which possessed an exhaustive (exh.) w.w. sequence. Namely, an infinite sequence $\{n_i\}$ of integers for which there exists a measurable set W such that $T^{n_i}W\cap T^{n_j}W=\varnothing$ for $i\neq j$, and $\cup_i T^{n_i}W=X$. It was shown later in [10] that every e.m.p. transformation T defined on an infinite measure space admits an exh. w.w. sequence $\{n_i\}$. However, sets which are exh. and w.w. under such sequences may or may not be of finite measure. In [4] and [5] a class of e.m.p. transformations is constructed which admit exh. w.w. sequences $\{n_i\}$ for which the corresponding w.w. sets W must have finite measure; we designate these as transformations of finite type. We will show in Theorem 1 below that for any e.m.p. transformation T defined on an infinite measure space $(X, \mathscr{B}, m)$ sets which are exh. and w.w. for T under the same sequence $\{n_i\}$ must have the same measure, finite or infinite. Hence for a given e.m.p. transformation T defined on

Received March 3, 1988

an infinite measure space it makes sense to talk about exh. w.w. sequences of finite type or of infinite type. We shall see later that there are e.m.p. transformations which admit exh. w.w. sequences only of infinite type; we designate these as transformations of infinite type. In this article we shall study some properties of e.m.p. transformations of finite type; in particular, we shall show that those transformations of finite type belonging to the special class constructed in [4] behave in many respects like e.m.p. transformations defined on a finite measure space.

§2. Definitions and basic properties.

Let T be an e.m.p. transformation defined on an infinite Lebesgue measure space $(X, \mathscr{B}, m)$. All sets considered will be measurable, and sets of measure zero will be ignored in expressing relations among sets. An infinite sequence of integers $\{n_i\}$ is called an *exhausting* (exh.) sequence for T if there exists a set W of positive measure such that $\cup_i T^{n_i} W = X$; and $\{n_i\}$ is called a *weakly wandering* (w.w.) sequence for T if $T^{n_i} W \cap T^{n_j} W = \varnothing$ for $i \neq j$. The corresponding set W in each case will be called an *exhausting* (exh.) or *weakly wandering* (w.w.) set for T under $\{n_i\}$, respectively.

DEFINITION. Let T be an e.m.p. transformation defined on an infinite measure space. If there exists a set W of finite measure which is exh. w.w. for T under a sequence of integers $\{n_i\}$ then we say that the transformation T is of *finite type*; otherwise, T is of *infinite type*.

The following theorem establishes some characteristic properties of e.m.p. transformations of finite type.

THEOREM 1. *Let T be an e.m.p. transformation defined on an infinite measure space $(X, \mathscr{B}, m)$. Suppose $W \in \mathscr{B}$, with $m(W) < \infty$, is an exh. w.w. set for T under a sequence $\{n_i\}$, and let $V \in \mathscr{B}$. We consider the following statements*:

a) *V is exh. under $\{n_i\}$.*
b) *V is w.w. under $\{n_i\}$.*
c) $m(V) = m(W)$.

Then any two of the above statements together imply the third. For the implication a) *and* b) *together implying* c) *the condition that $m(W) < \infty$ is not necessary.*

Before proving Theorem 1 we state and prove the following lemma where we gather some properties of sets that are exh. or w.w.

LEMMA 1. *For an e.m.p. transformation T defined on an infinite measure space $(X, \mathscr{B}, m)$ the following statements hold:*
(i) *W w.w. under $\{n_i\}$ implies W w.w. under $\{-n_i\}$.*
(ii) *W w.w. under $\{n_i\}$ implies T^kW w.w. under $\{n_i\}$ for every $k\in \boldsymbol{Z}$.*
(iii) *W exh. under $\{n_i\}$ implies T^kW exh. under $\{n_i\}$ for every $k\in \boldsymbol{Z}$.*
(iv) *W w.w. under $\{n_i\}$ and V exh. under $\{n_i\}$ imply $m(W)\leqq m(V)$.*
(v) *W exh. w.w. under $\{n_i\}$ and $m(W)<\infty$ imply W exh. w.w. under $\{-n_i\}$.*

PROOF. (i), (ii), and (iii) follow from the definitions. Next we prove (iv). Since V is exh. under $\{n_i\}$, T is m.p., and W is w.w. under $\{-n_i\}$ we have

$$\begin{aligned} m(W)&=m(W\cap \bigcup_i T^{n_i}V)\leqq\sum_i m(W\cap T^{n_i}V)=\sum_i m(T^{-n_i}W\cap V)\\ &=m(\bigcup_i T^{-n_i}W\cap V)\leqq m(V)\,. \end{aligned}$$

To prove (v) it is enough to show that W is exh. for T under $\{-n_i\}$. For each $k\in \boldsymbol{Z}$ we let $W_k=T^kW$. Then, since W_k is exh.w.w. under $\{n_i\}$, T is m.p., and W is w.w. under $\{-n_i\}$, we have

$$\begin{aligned} m(W)&=m(W\cap \bigcup_i T^{n_i}W_k)=\sum_i m(W\cap T^{n_i}W_k)=\sum_i m(T^{-n_i}W\cap W_k)\\ &=m(\bigcup_i T^{-n_i}W\cap W_k)\leqq m(W_k)=m(W)<\infty\,. \end{aligned}$$

We conclude that $W_k=T^kW\subset \bigcup_i T^{-n_i}W$ for each $k\in \boldsymbol{Z}$, and since $\bigcup_k T^kW=X$ by the ergodicity of T, it follows that $\bigcup_i T^{-n_i}W=X$. □

REMARK 1. In order to emphasize the significance of (v) in Lemma 1 we note that, in general, if W is an exh. set for a transformation T under a sequence $\{n_i\}$ then it does not follow that W is exh. for T under $\{-n_i\}$. To show this in a simple case for a finite sequence, we let T be an e.m.p. transformation defined on a measure space $(X, \mathscr{B}, m)$ for which there exists a decomposition

$$X=\bigcup_{i=0}^{5}X_i\ (\text{disj})\,,\quad \text{with } T(X_j)=X_{j+1}\ (\text{mod } 6),\ 0\leqq j\leqq 5\,.$$

We let

$$W=X_0\cup X_2\cup X_5,\ n_0=0,\ n_1=1,\ n_2=4\,.$$

Then it is easy to see that

$$\bigcup_{i=0}^{2}T^{n_i}W=X\quad\text{while}\quad \bigcup_{i=0}^{2}T^{-n_i}W=X-X_3\neq X\,.$$

PROOF OF THEOREM 1. We suppose a) holds together with b) and do not assume that $m(W)<\infty$. From (iv) in Lemma 1 follows that $m(W)\leq m(V)$; we obtain the reverse inequality by interchanging the roles of V and W, and conclude c).

For the remainder of the proof we assume that $m(W)<\infty$. For each $k\in \boldsymbol{Z}$ we let $W_k=T^k W$. Then using Lemma 1, since W_k is exh. w.w. under $\{-n_i\}$ and T is m.p. we have

$$m(V)=m(\bigcup_i T^{-n_i}W_k\cap V)=\sum_i m(T^{-n_i}W_k\cap V)=\sum_i m(W_k\cap T^{n_i}V)\ . \tag{2.1}$$

Now we suppose b) holds together with c). Then since V is w.w. under $\{n_i\}$ and T is m.p., using (2.1) we get

$$m(V)=\sum_i m(W_k\cap T^{n_i}V)=m(W_k\cap \bigcup_i T^{n_i}V)\leqq m(W_k)=m(W)\ .$$

This says that $T^kW\subset \bigcup_i T^{n_i}V$ for each $k\in \boldsymbol{Z}$. From the ergodicity of T we conclude that V is exh. under $\{n_i\}$, and therefore a) holds.

Next, instead of b) we suppose a) holds together with c). Then since V is exh. under $\{n_i\}$ and T is m.p., using (1.1) we get

$$m(V)=\sum_i m(W_k\cap T^{n_i}V)\geqq m(W_k\cap \bigcup_i T^{n_i}V)=m(W_k)=m(W)\ .$$

This says that for each $k\in \boldsymbol{Z}$ and for $i\neq j$ the sets $T^{n_i}V$ and $T^{n_j}V$ do not intersect on the set W_k. From the ergodicity of T we conclude that V is w.w. under $\{n_i\}$, and therefore b) holds. □

We note that for an e.m.p. transformation T defined on an infinite measure space, in view of Theorem 1, it makes sense to talk of an exh. w.w. sequence $\{n_i\}$ of finite or infinite type depending on whether it accepts an exh. w.w. set of finite measure or not, respectively.

COROLLARY. *Let T be an e.m.p. transformation of finite type on an infinite measure space $(X, \mathscr{B}, m)$, then T admits only m.p. commutators. In other words, if S is a non-singular transformation such that $ST=TS$, then S preserves the same measure m.*

PROOF. For a set $V\in\mathscr{B}$ we let $m'(V)=m(SV)$. Then $m'(TV)=m(STV)=m(TSV)=m'(V)$ shows that m' is an invariant measure for T, and the non-singularity of S says that $m'\sim m$. Since T is ergodic it follows that there exists a constant $c>0$ such that $m(SV)=cm(V)$ for all $V\in\mathscr{B}$. If W is a set of finite measure which is exh. w.w. for T under a sequence $\{n_i\}$ then so is the set SW under the same sequence $\{n_i\}$.

Hence by Theorem 1, $m(SW)=m(W)$, which implies $c=1$, and therefore, S preserves m. $\square$

REMARK 2. In [8], examples of e.m.p. transformations T were constructed which commuted with transformations S satisfying $m(SV)=cm(V)$ for all $V\in\mathscr{B}$ and for some $c\neq 1$. Theorem 1 above shows that these transformations are necessarily of infinite type.

REMARK 3. In [2], J. Aaronson introduced for e.m.p. tansformations T, a set $\Delta(T)$ of "normalizing constants". Namely, $\Delta(T)$ is the set of all positive numbers c for which there exist an e.m.p. transformation U on some σ-finite measure space $(Y, \mathscr{F}, \mu)$ and two measurable maps Θ_1 and Θ_2 of Y onto X such that

$$\Theta_i U = T\Theta_i \quad \text{for } i=1, 2\ , \quad \text{and} \quad \mu\Theta_1^{-1}=m\ , \quad \mu\Theta_2^{-1}=cm\ .$$

It turns out that $\Delta(T)$ is a multiplicative subgroup of $\boldsymbol{R}^+$, the non-negative reals, and is an invariant for "similarity" of transformations; for details we refer the reader to [2]. The argument used in the proof of the corollary above shows that $\Delta(T)=\{1\}$ for any e.m.p. transformation T of finite type, and hence among other things, the so called Strong Law of Large Numbers is valid for transformations of finite type.

§3. Further properties of transformations of finite type.

In this section, we discuss additional properties of e.m.p. transformations of finite type that belong to a special class of transformations constructed in [4]. These transformations have a close connection to the direct sum decomposition of the integers. Below we describe these transformations and discuss some of their properties. Now we give a brief explanation of direct sum decompositions.

DIRECT SUM DECOMPOSITIONS. We denote by $\boldsymbol{N}=\{0, 1, 2, \cdots\}$ the set of all non-negative integers. By a direct sum decomposition of $\boldsymbol{N}$ we mean two infinite subsets $\boldsymbol{A}$ and $\boldsymbol{B}$ of $\boldsymbol{N}$ whose direct sum is $\boldsymbol{N}$, $\boldsymbol{A}\oplus\boldsymbol{B}=\boldsymbol{N}$; this means that every integer $n\in\boldsymbol{N}$ can be written uniquely as $n=a+b$ with $a\in\boldsymbol{A}$ and $b\in\boldsymbol{B}$. In this case we say that $\boldsymbol{B}$ is a complement of $\boldsymbol{A}$ in $\boldsymbol{N}$. It is possible to characterize the sets $\boldsymbol{A}$ and $\boldsymbol{B}$ of such a decomposition of $\boldsymbol{N}$ as IP sets associated with a sequence of positive integers $\{m_i|i\geqq 1\}$ where $m_i\geqq 2$ for all i; see [3], [4], [11]. Namely, we let $M_0=1$, and $M_k=\prod_{n=1}^{k} m_n$ for $k=1, 2, \cdots$. Then $\boldsymbol{A}$ is the IP set generated by the sequence

$$\{0, \underbrace{M_1, M_1, \cdots, M_1}_{m_2-1 \text{ times}}, \underbrace{M_3, M_3, \cdots, M_3}_{m_4-1 \text{ times}}, \cdots, \underbrace{M_{2n-1}, M_{2n-1}, \cdots, M_{2n-1}}_{m_{2n}-1 \text{ times}}, \cdots\},$$

and B is the IP set generated by the sequence

$$\{0, \underbrace{M_0, M_0, \cdots M_0}_{m_1-1 \text{ times}}, \underbrace{M_2, M_2, \cdots, M_2}_{m_3-1 \text{ times}}, \cdots, \underbrace{M_{2n}, M_{2n}, \cdots, M_{2n}}_{m_{2n+1}-1 \text{ times}}, \cdots\}.$$

Here by an IP set generated by a sequence $\{P_1, P_2, \cdots, P_k, \cdots\}$ we mean the set of all finite sums $\{P_{i_1}+P_{i_2}+\cdots+P_{i_j}; 1\leqq i_1<i_2<\cdots<i_j, j=1, 2, \cdots\}$, see [6]. Moreover, it is possible to discuss direct sum decompositions of Z, the set of all integers. However, the situation is quite a bit more complicated and we leave the discussion to a subsequent paper.

In [4], for each direct sum decomposition of N, $A\oplus B=N$, an infinite measure space $(X, \mathscr{B}, m)$, an e.m.p. transformation T, and a set $W\subset X$ are constructed such that the set W is exh. and w.w. for T under the sequence A, with $m(W)=1$. In the following example we give a slightly different description of these transformations from that given in [4].

EXAMPLE. Let $A\oplus B=N$ be a direct sum decomposition of the non-negative integers N, and let $\{m_i | i\geqq 1\}$ where $m_i\geqq 2$ for all i be the sequence of positive integers characterizing the sets A and B as described above. We let $M_0=1$ and $M_n=\prod_{i=1}^n m_i$. Now we shall construct an e.m.p. transformation T defined on a measure space $(X, \mathscr{B}, m)$ and show that there exists an exh. w.w. set W for T under the sequence A with $m(W)=1$. For each $n>0$ we consider the measure space $(\Omega_n, \mathscr{B}_n, \mu_n)$ where $\Omega_n=\{0, 1, 2, \cdots, m_{2n-1}-1\}$, $\mathscr{B}_n=$all subsets of Ω_n, and $\mu_n(p)=1/m_{2n-1}$ for $p\in\Omega_n$. We let

$$(W, \mathscr{B}, m)=\prod_{n=1}^{\infty}(\Omega_n, \mathscr{B}_n, \mu_n)$$

be the infinite direct product measure space, and define the transformation T_W on it as follows: T_W: $W\to W$ is the "adding machine" transformation; namely, if $w\in W$

$$w=(w_1, w_2, \cdots, w_{n-1}, w_n, w_{n+1}, \cdots),$$
$$w_i=m_{2i-1}-1 \quad \text{for } 1\leqq i\leqq n-1, \quad \text{and} \quad w_n<m_{2n-1}-1,$$

then

$$T_W w=(0, \cdots, 0, w_n+1, w_{n+1}, \cdots).$$

We define a sequence of integers $\{z_n(k) \mid 0\leqq k\leqq m_{2n-1}-1, n=1, 2, \cdots\}$ by

$$
\begin{aligned}
z_1(i) &= iM_0 && \text{for} \quad 0 \leqq i \leqq m_1 - 1 \,, \\
z_2(i) &= iM_2 && \text{for} \quad 0 \leqq i \leqq m_3 - 1 \,, \\
&\cdots\cdots \\
z_n(i) &= iM_{2n-2} && \text{for} \quad 0 \leqq i \leqq m_{2n-1} - 1, \; n > 2 \,.
\end{aligned}
\tag{3.1}
$$

Then it is easy to check that for each $n>0$,

$$
z_n(i) > z_n(i-1) + \sum_{j=1}^{n-1} z_j(m_{2j-1}-1) \,, \qquad 1 \leqq i \leqq m_{2n-1} - 1
\tag{3.2}
$$

is satisfied. We define a sequence of random variables $\{Z_n\}$ on W by

$$
Z_n(w) = z_n(w_n) \qquad \text{for each} \quad n \geqq 1, \, w \in W \,,
\tag{3.3}
$$

then $\{Z_n\}$ becomes an independent sequence of random variables on the probability space $(W, \mathscr{B}, m)$, and if we let

$$
h(w) = \sum_{n=1}^{\infty} \{Z_n(T_W w) - Z_n(w)\} \qquad \text{for} \quad w \in W \,,
$$

then in view of (3.2), one can show that $1 \leqq h(w) < \infty$ for all $w \in W$. Next we let $X = \{(w, k) \mid w \in W, \, 0 \leqq k < h(w)\}$ and extend the σ-algebra structure and the measure to X in the natural way; we denote these by the same letters $\mathscr{B}$ and m, respectively. Finally, on the measure space $(X, \mathscr{B}, m)$ we define the transformation $T: X \to X$ by

$$
T(w, k) = \begin{cases} (w, k+1) & \text{if} \quad k+1 < h(w) \\ (T_W w, 0) & \text{if} \quad k+1 = h(w) \,. \end{cases}
$$

Then T is an e.m.p transformation defined on the measure space $(X, \mathscr{B}, m)$, with W an exh. w.w. set for T under the sequence $\boldsymbol{A}$. In fact, T is isomorphic to the transformation constructed in [4] for the sequence $\boldsymbol{A}$. A similar construction produces the dual e.m.p. transformation S defined on the measure space $(Y, \mathscr{F}, \mu)$ and the exh. w.w. set V for S under $\boldsymbol{B}$, with $\mu(V) = 1$.

A further important property of the transformation T is,

$$
m(T^n W \cap W) > 0 \quad \text{if and only if} \quad n \in \boldsymbol{B} - \boldsymbol{B} \,;
$$

or equivalently,

$$
T^n W \cap W = \varnothing \quad \text{if and only if} \quad n \notin \boldsymbol{B} - \boldsymbol{B} \,.
$$

A similar property holds for the transformation S, the set V, and the sequence $\boldsymbol{A}$. From this it follows that

$$(T\times S)^n(W\times V)\cap(W\times V)=(T^nW\cap W)\times(S^nV\cap V)=\varnothing \quad \text{for all } n>0\ .$$

This means that the set $W\times V$ is a wandering set for the transformation $T\times S$ defined on the measure space $(X\times Y, \mathscr{B}\times\mathscr{F}, m\times\mu)$. For $k\in \boldsymbol{Z}$ we let $W_k=T^kW$ and $V_k=S^kV$; the same argument as above shows that the sets $W_k\times V$ and $W\times V_k$ are also wandering for the transformation $T\times S$. We note that, as was observed in Remark 3 above, $\Delta(T)=\Delta(S)$ for these transformations; however, using the facts mentioned above, we prove the following:

THEOREM 2. *For a direct sum decomposition of the non-negative integers* $\boldsymbol{N}$, $\boldsymbol{A}\oplus\boldsymbol{B}=\boldsymbol{N}$, *consider the transformations* T *and* S *as described above. Then* T *and* S *are not similar* (in the sense of J. Aaronson [2].)

PROOF. Suppose T defined on the measure space $(X, \mathscr{B}, m)$ and S defined on the measure space $(Y, \mathscr{F}, \mu)$ are similar, then there exists a σ-finite measure ν defined on $(X\times Y, \mathscr{B}\times\mathscr{F}, m\times\mu)$ with the following properties: $\nu(\pi_1^{-1}E)=m(E)$ for all $E\in\mathscr{B}$, and $\nu(\pi_2^{-1}F)=\mu(F)$ for all $F\in\mathscr{F}$, and the transformation $T\times S$ is a conservative e.m.p. with respect to ν, where π_1: $X\times Y\to X$ and π_2: $X\times Y\to Y$ are the coordinate projections. This implies that $T\times S$ cannot have any wandering sets of positive measure, and thus $\nu(W\times S^kV)=0$ for all k. But then

$$0<m(W)=\nu(\pi_1^{-1}W)=\nu(W\times Y)\leqq\sum_k \nu(W\times S^kV)=0\ ,$$

a contradiction. □

REMARK 4. It is not known if there exist e.m.p. transformations of finite type which admit only exh. w.w. sequences of finite type. For the above class in general, and in particular for the transformation T described above, which is of finite type, we observe that T possesses an exh. w.w. sequence of infinite type as well.

We let A_1 be the IP set generated by the sequence

$$\{0, \underbrace{M_3, M_3, \cdots, M_3}_{m_4-1 \text{ times}}, \underbrace{M_7, M_7, \cdots, M_7}_{m_8-1 \text{ times}}, \cdots, \underbrace{M_{4n-1}, M_{4n-1}, \cdots, M_{4n-1}}_{m_{4n}-1 \text{ times}}, \cdots\}\ ,$$

and let A_2 be the IP set generated by the sequence

$$\{0, \underbrace{M_1, M_1, \cdots, M_1}_{m_2-1 \text{ times}}, \underbrace{M_5, M_5, \cdots, M_5}_{m_6-1 \text{ times}}, \cdots, \underbrace{M_{4n+1}, M_{4n+1}, \cdots, M_{4n+1}}_{m_{4n+2}-1 \text{ times}}, \cdots\}\ .$$

Then it is easy to see that $\boldsymbol{A}=\boldsymbol{A}_1\oplus\boldsymbol{A}_2$; from this it follows that if we let

$$W^* = \bigcup_{k \in A_2} T^k W ,$$

then $m(W^*) = \infty$, and W^* is exh. and w.w. for T under A_1. This says that A_1 is an exh. w.w. sequence of infinite type for T. A similar argument can be repeated for the dual transformation S.

Finally, we consider the L^∞-point spectrum of the transformation T.

We say that a number λ belongs to $\Lambda(T)$, the L^∞-point spectrum of an e.m.p. transformation T on $(X, \mathscr{B}, m)$, if there exists a function $f \in L^\infty(X, \mathscr{B}, m)$, not identically equal to 0, such that $f(Tx) = e^{2\pi i\lambda} f(x)$, a.e.

It is easy to see that $\Lambda(T)$ is an additive subgroup of $[0, 1)$ (mod 1), and it can be shown that it is a Borel subset of $[0, 1)$ and has Lebesgue measure 0. If $m(X) < \infty$, then $\Lambda(T)$ coincides with the set of all eigenvalues of the unitary operator U_T on $L^2(X, \mathscr{B}, m)$ defined by $U_T f(x) = f(Tx)$, and therefore, it is at most countable. However, in case $m(X) = \infty$, $\Lambda(T)$ can be uncountable; in fact, it can even have any arbitrary Hausdorff dimension β $(0 \leqq \beta \leqq 1)$, see [1], [9]. We will show in what follows that if the integers $\{m_i \mid i \geqq 1\}$ associated with the decomposition $\boldsymbol{A} \oplus \boldsymbol{B} = \boldsymbol{N}$ are bounded, then for the corresponding transformations T and S constructed as above, the sets $\Lambda(T)$ and $\Lambda(S)$ coincide and are countable.

In order to determine $\Lambda(T)$ for the transformation T constructed above, we note that from the results in [9], [12] it follows that a number $\lambda \in [0, 1)$ belongs to $\Lambda(T)$ if and only if

$$\sum_{n=1}^{\infty} \mathrm{Var}(\langle \lambda Z_n \rangle) < \infty \ , \tag{3.4}$$

where Z_n is the random variable defined in (3.3), Var means the variance of a random variable, and for a real number μ

$$\langle \mu \rangle = \begin{cases} \{\mu\} & \text{if } \{\mu\} \, (= \text{the fractional part of } \mu) < 1/2 \\ \{\mu\} - 1 & \text{if } \{\mu\} > 1/2 \ . \end{cases}$$

If we now assume that the integers m_i's are bounded so that $\min_{i \geqq 1} \mu_i(0) > 0$, then we can easily show that the condition (3.4) is equivalent to

$$\sum_{n=1}^{\infty} \sum_{i=1}^{m_{2n-1}-1} \|\lambda z_n(i)\|^2 \mu_n(i) < \infty \ , \tag{3.5}$$

where the $z_n(i)$'s are the numbers defined in (3.1), and for a real number μ

$$\|\mu\| = |\langle \mu \rangle| = \min\{|\mu - k| \ ; \ k \in \boldsymbol{Z}\} \ .$$

Substituting the values of $z_n(i)$ and $\mu_n(i)$ into (3.5), we obtain

$$(3.6) \qquad \lambda \in \Lambda(T) \quad \text{if and only if} \quad \sum_{n=1}^{\infty} \frac{1}{m_{2n-1}} \sum_{i=1}^{m_{2n-1}-1} \|\lambda i M_{2n-2}\|^2 < \infty .$$

It is well known that any number $\lambda \in [0, 1)$ has an expansion of the form

$$(3.7) \qquad \lambda = \sum_{k=1}^{\infty} \frac{\varepsilon_k}{M_k} = \sum_{k=1}^{\infty} \frac{\varepsilon_k}{m_1 m_2 \cdots m_k} ,$$

where for each k, $0 \leqq \varepsilon_k \leqq m_k - 1$. We call a number λ $\{m_k\}$-adic rational if it has an expansion of the form (3.7) with $\varepsilon_k = 0$ for all $k \geqq p$, for some p. All $\{m_k\}$-adic rational numbers have another expansion of the form (3.7) with $\varepsilon_k = m_k - 1$ for all $k \geqq q$, for some q, but for all other numbers the expansion of the form (3.7) is unique. Whenever λ is an $\{m_k\}$-adic rational, we adopt the expansion of the form (3.7) with $\varepsilon_k = 0$ for all $k \geqq p$, for some p. When $\lambda \in [0, 1)$ is expanded in the form (3.7), we have for all i $(1 \leqq i \leqq m_{2n-1} - 1)$,

$$\|\lambda i M_{2n-2}\| = \left\| \sum_{k=2n-1}^{\infty} \frac{i \varepsilon_k}{m_{2n-1} m_{2n} \cdots m_k} \right\| ,$$

since

$$\lambda i M_{2n-2} = \text{integer} + \sum_{k=2n-1}^{\infty} \frac{i \varepsilon_k M_{2n-2}}{M_k} .$$

Therefore, if λ is an $\{m_k\}$-adic rational, then for all sufficiently large n, $\|\lambda i M_{2n-2}\| = 0$ for all i $(1 \leqq i \leqq m_{2n-1} - 1)$; thus by (3.6), such a λ belongs to the set $\Lambda(T)$.

If, on the other hand, $\varepsilon_k = \varepsilon_k(\lambda) \neq 0$ for infinitely many k's in the expansion of λ in the form (3.7), then one of the following three cases must occur:

(i) $\varepsilon_{2n-1} \neq 0$, $\varepsilon_{2n} \neq m_{2n} - 1$ for infinitely many n's.

(ii) $\varepsilon_{2n-1} \neq 0$, $\varepsilon_{2n} = m_{2n} - 1$, $\varepsilon_{2n+1} \neq m_{2n+1} - 1$ for infinitely many n's.

(iii) $\varepsilon_{2n-1} = 0$, $\varepsilon_{2n} \neq 0$, $\varepsilon_{2n+1} = 0$ for infinitely many n's.

We can show that $\|\sum_{k=2n-1}^{\infty} \varepsilon_k/(m_{2n-1} m_{2n} \cdots m_k)\|$ is bounded below by $1/(m_{2n-1} m_{2n})$ for infinitely many n's in the case (i) or (iii), and bounded below by $1/(m_{2n-1} m_{2n} m_{2n+1})$ for infinitely many n's in case (ii).

Therefore, if we denote by K an upper bound for $\{m_k\}$, then we have in the sum appearing in the condition (3.6) infinitely many terms which are at least $1/K^7$, whenever λ is not an $\{m_k\}$-adic rational number. We conclude that any of these numbers does not belong to $\Lambda(T)$.

We summarize this in the following theorem.

THEOREM 3. *For a direct sum decomposition of the non-negative integers* N, $A \oplus B = N$, *let* T *be the transformation constructed as above. Assume that the set of integers* $\{m_i \mid i \geqq 1\}$ *characterizing the decomposition is bounded. Then the set* $\Lambda(T)$, *the* L^∞*-point spectrum of* T, *is precisely the set of all* $\{m_k\}$*-adic rationals in* $[0, 1)$.

Thus, all such transformations T have countable L^∞-point spectrum. Arguments similar to the one used above show that for the dual transformation S associated with the decomposition $A \oplus B = N$, $\Lambda(S)$ is also precisely the set of all $\{m_k\}$-adic rationals in $[0, 1)$, and therefore, $\Lambda(T) = \Lambda(S)$.

This shows that dissimilar transformations can have identical L^∞-point spectrum. We note that the dissimilarity of the transformations T and S precludes the existence of isomorphism between T and S.

Finally, we note that when $m_i = 2$ for all $i \geqq 1$, the transformation T is the one constructed by A. Hajian and S. Kakutani in [7], and for this T the L^∞-point spectrum $\Lambda(T)$ is precisely the set of all dyadic rationals in $[0, 1)$.

References

[1] J. AARONSON, The eigenvalues of non-singular transformations, Israel J. of Math., **45** (1983), 297–312.

[2] J. AARONSON, The intrinsic normalizing constants of transformations preserving infinite measure, J. d'Analyse Math., **49** (1988), 239–271.

[3] N. G. DE BRUIJN, On bases for the set of integers, Publ. Math. Debrecen, **1** (1950), 232–242.

[4] S. EIGEN and A. HAJIAN, Sequences of integers and ergodic transformations, to appear in Advances in Math.

[5] S. EIGEN and A. HAJIAN, A characterization of exhaustive weakly wandering sequences for non-singular transformations, Comm. Math. St. Pauli., **36** (1987), 227–233.

[6] H. FURSTENBERG, *Recurrence in Ergodic Theory and Combinatorial Number Theory*, Princeton University Press, 1981.

[7] A. HAJIAN and S. KAKUTANI, An example of an e.m.p. transformation defined on an infinite measure space, Lecture Notes in Math., **160** (1970), 45–52, Springer-Verlag.

[8] A. HAJIAN, Y. ITO and S. KAKUTANI, Invariant measures and orbits of dissipative transformations, Advances in Math., **9** (1972), 52–65.

[9] Y. ITO, T. KAMAE and I. SHIOKAWA, Point spectrum and Hausdorff dimension, *Number Theory and Combinatorics, Japan 1984*, edited by J. Akiyama et al., World Scientific Publishing Co., 1985, 209–227.

[10] L. JONES and U. KRENGEL, On transformations without finite invariant measures, Advances in Math., **12** (1974), 275–295.

[11] C. T. LONG, Additive theorems for sets of integers, Pacific J. Math., **23** (1967), 107–112.

[12] M. OSIKAWA, Point spectra of non-singular flows, Publ. Res. Inst. Math. Sci. Kyoto Univ., **13** (1977), 167–172.

Present Address:
STANLEY EIGEN and ARSHAG HAJIAN
DEPARTMENT OF MATHEMATICS, NORTHEASTERN UNIVERSITY
BOSTON, MA 02115, U.S.A.

YUJI ITO
DEPARTMENT OF MATHEMATICS, FACULTY OF SCIENCE AND TECHNOLOGY, KEIO UNIVERSITY
HIYOSHI, KOHOKU-KU, YOKOHAMA 223, JAPAN