254. *Ergodic Properties of Piecewise Linear Transformations*

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1. Introduction. After the work of Rényi [1], ergodic properties of β-expansions of real numbers have been studied in [2]–[4]. In this paper we generalize these results for a class of expansions, called piecewise linear expansions, which includes β-expansions as special cases.

Let $\bar{\beta}=(\beta_0,\beta_1,\cdots,\beta_N), N\geqq 1$, be a $(N+1)$-tuple of positive number such that $0<\theta\equiv\beta_N(1-\sum_{k=0}^{N-1}(1/\beta_R))\leqq 1$.

We denote the set of all $(N+1)$-tuples by $V(N+1)$. For each $\bar{\beta}\in V(N+1)$, we define a corresponding function $f(t)$ by

$$f(t)=\begin{cases}\dfrac{t}{\beta_0}, & 0\leqq t\leqq 1,\\ f(K)+\dfrac{t-k}{\beta_k}, & k<t\leqq k+1,\ (k=1,2,\cdots,N+1),\\ & N<t\leqq N+\theta,\ (k=N),\\ 1, & t>N+\theta.\end{cases}$$

The function $f(t)$ satisfies the Rényi's conditions [1]. Thus every real number x has the f-expansion

$$x=a_0(x)+f(a_1(x)+f(a_2(x)+\cdots)\cdots),$$

where the digits $a_n(x), n=0,1,\cdots$, and the remainders

$$T^n x=f(a_n(x)+f(a_{n+1}(x)+\cdots)\cdots),\qquad n=0,1,\cdots,$$

are defined by the following recursive relations: $a_0(x)=[x], T^0x=\{x\}$, $T^{n+1}x=\{f^{-1}(T^nx)\}, a_{n+1}(x)=[f^{-1}(T^nx)]$, $n=0,1,\cdots$, where $[z]$ denotes the integral part and $\{z\}$ the fractional part of the real number z and f^{-1} is the inverse function of f.

This f-expansion is called a *piecewise linear expansion induced by $\bar{\beta}$ or simply $\bar{\beta}$-expansion*, and the transformation $Tx=\{f^{-1}(x)\}$, $0\leqq x<1$, is called a *piecewise linear transformation induced by $\bar{\beta}$*. By definition, T is a many to one transformation of $[0,1)$ onto itself and nonsingular with respect to the Lebesgue measure m.

For the number 1, we define, especially, $a_0(1)=0$ and $T^01=1$. Then $\bar{\beta}\in V(N+1)$ is said to be *periodic* if the $\bar{\beta}$-expansion of 1 has a recurrent tail, and *rational* if the $\bar{\beta}$-expansion of 1 has a zero tail. The *order of a rational* $\bar{\beta}$ is the minimum integer r such that $a_n(1)=0$ for all $n>r+1$.

2. Invariant measures. Lemma 1. *Let T be a piecewise linear transformation induced by $\bar{\beta}\in V(N+1)$ and μ a finite measure equivalent to the Lebesgue measure m. Then μ is T-invariant if and only if*

$$h(x)=\sum_{k=0}^{N} f'(k+x)h(f(k+x))dx, \ a.e.$$

where $h(x)$ is the Radon-Nikodym derivative of μ.

Proof. For any $t \in [0,1)$, we have

$$\mu(T^{-1}[0,1))=\int_0^t \sum_{k=0}^{N} f'(k+x)h(f(k+x))dx.$$

The lemma is an immediate conclusion of this fact.

For any $\bar{\beta} \in V(N+1)$, we define a function

$$h(x)=\sum_{n=0}^{\infty} \frac{C_n(x)}{\beta_{a_0(1)}\beta_{a_1(1)}\cdots\beta_{a_n(1)}},$$

where $\beta_{a_0(1)}=$ and $C_n(x)$ is the characteristic function of the interval $[0, T^n 1)$.

Theorem 1. *Let T be a piecewise linear transformation induced by $\bar{\beta}$ and put $\mu(A)=\int_A h(x)dx$ for any measurable set A. Then μ is finite T-invariant measure equivalent to the Lebesgue measure.*

Proof. First we prove that

$$(1) \qquad \sum_{k=0}^{N} f'(k+x)C_n(f(k+x))=f(a_{n+1}(1))+\frac{C_{n+1}(x)}{\beta_{a_{n+1}(1)}}$$

If $f(x)>T^n 1$, then (1) is trivial. Thus it suffices to prove (1) when there exists an integer k such that $f(k+x)<T^n 1$. There are two possibilities: (i) there exists k such that $f(k+x)<T^n x<f(k+x)$, (ii) there exists k such that $f(k+1)\leqq T^n 1\leqq f(k+1+x)$. In the case (i) $a_{n+1}(1)=k$, $C_{n+1}(x)=1$, and in the case (ii) $a_{n+1}(1)=k+1, C_{n+1}(X)=0$. As a result, we get (1). Furthermore, by the piecewise linearity of f, we have

$$(2) \qquad 1=\sum_{n=0}^{\infty} \frac{f(a_{n+1}(1))}{\beta_{a_0(1)}\beta_{a_1(1)}\cdots\beta_{a_n(1)}}.$$

Therefore, we have

$$\begin{aligned}
&\sum_{k=0}^{N} f'(k+x)h(f(k+x))\\
&=\sum_{n=0}^{\infty} \frac{1}{\beta_{a_0(1)}\beta_{a_1(1)}\cdots\beta_{a_n(1)}} \sum_{k=0}^{N} f'(k+x)C_n(f(k+x))\\
&=\sum_{n=0}^{\infty} \frac{C_{n+1}(x)}{\beta_{a_0(1)}\beta_{a_1(1)}\cdots\beta_{a_{n+1}(1)}}+\sum_{n=0}^{\infty} \frac{f(a_{n+1}(1))}{\beta_{a_0(1)}\beta_{a_1(1)}\cdots\beta_{a_n(1)}} \qquad \text{(by (1))}\\
&=h(x) \qquad \text{(by (2))}.
\end{aligned}$$

this and Lemma 1 imply the theorem.

Corollary 1. *$h(x)$ is a decreasing jump function which satisfies* $1=h(1)\leqq h(x)\leqq h(0)<\infty$, a.e.

Corollary 2. *$h(x)$ is a step function with a finite number of steps if and only if $\bar{\beta}$ is periodic. Especially $h(x)=1$ if and only if $\bar{\beta}$ is rational of order* 0.

In what follows we shall investigate the transformation T with the normalized invariant measure $p(\cdot)=\mu(\cdot)/\mu([0,1))$.

3. Exactness. A measure preserving transformation T on a Lebesgue space $(X, \boldsymbol{B}, P)$ is said to be exact if $\bigcap_{n=0}^{\infty} T^{-n}\boldsymbol{B}=\{X, \emptyset\}$.

Rohlin's criterion [4]. Let $\boldsymbol{U}$ be a countable system of sets of positive measure on X such that the finite unions of pairwise disjoint sets $A \in \boldsymbol{U}$ form an ensemble everywhere dense in $\boldsymbol{B}$. If there exists a positive integer-valued function $n(A), A \in \boldsymbol{U}$, and a positive number q such that $P(T^{n(A)}A)=1, A \in \boldsymbol{U}$, and

$$(4) \qquad P(T^{n(A)}E) \leqq q\frac{P(E)}{P(A)},$$

for all measurable set $E \subset A$ with measurable image $T^{n(A)}E$, then T is exact.

Theorem 2. *Every piecewise linear transformation is exact.*

Proof. The proof is based on the Rohlin's criterion. Let $\bar{\beta} \in V(N+1)$, be given arbitrary and let us denote by ξ a partition of $(0,1)$ into subintervals generated by the points $f(k), k=1,2,\cdots,N$. We set $\boldsymbol{U}_n=\{A \in T^{-(n-1)}\xi\,;\, TA \in T^{-(n-2)}\}$, $n=1,2,\cdots$, $\boldsymbol{U}=\bigcup_{n=1}^{\infty}\boldsymbol{U}_n$ and $n(A)=n$ if $A \in \boldsymbol{U}_n$. Then, the density and the relation $P(T^{n(A)}A)=1$, $A \in \boldsymbol{U}$, are obviously satisfied. We must prove that there exists a constant $q=q(\bar{\beta})$ satisfying the inequality (4). For any $A \in \boldsymbol{U}$, there exists a sequence of digits $(a_1(A),\cdots,a_n(A))$ which is admissible in the $\bar{\beta}$-expansion such that $A=(a_1(x)=a_1(A),\cdots,a_n(x)=a_n(A))$. Since T is picewise linear, we have $m(T^{n(A)}E)=\beta_{a_1(A)}\cdots\beta_{a_n(A)}m(E)=m(E)/m(A)$, for any $E \in \boldsymbol{B}$ in A. By this relation and Corollary 1, we obtain $P(T^{n(A)}E) \leqq h(0)^2\mu([0,1))(P(E)/P(A))$. Thus we may set $q=h(0)^2\mu([0,1))$.

4. Markov properties. Let $x=(a_1(x), a_2(x), \cdots)$ be a $\bar{\beta}$-expansion of a real number $x, 0<x<1$, then $Tx=(a_2(x), a_3(x), \cdots)$, that is, T is a shift transformation of the stochastic process $(a_1(x), a_2(x), \cdots)$, $0<x<1$, with a finite number of states. Since P is T-invariant the process is stationary.

Theorem 3. *Let $\bar{\beta}$ be rational of order r, then T is a stationary r-ple Markov chain. $r=0$ implies the independency of the process.*

Lemma 2. *Let $\bar{\beta}$ be rational of order r and let n be any non-negative integer. Then for any sequence of digits $(c_1, c_2, \cdots, c_{n+r})$ which is admissible in the $\bar{\beta}$-expansion, we have*

$$(5) \qquad m((c_{n+1}, c_{n+2}, \cdots, c_{n+r}))=\beta_{c_1}\beta_{c_2}\cdots\beta_{c_n}m((c_1, c_2, \cdots, c_{n+r}))$$

where $(c_1, c_2, \cdots, c_k)=(a_1(x)=c_1, a_2(x)=c_2, \cdots, a_k(x)=c_k)$.

Proof. If $n=0$, then the relation (5) is trivial. Let $n \geqq 1$. We suppose that (5) holds for $n-1$. Then we have

$$m((c_2, c_3, \cdots, c_{n+r}))=\beta_{c_2}\beta_{c_3}\cdots\beta_{c_n}m((c_{n+1}, c_{n+2}, \cdots, c_{n+r})).$$

Therefore, we must prove

$$(6) \qquad m((c_2, c_3, \cdots, c_{n+r}))=\beta_{c_1}m((c_1, c_2, \cdots, c_{n+r}))$$

for any admissible sequence $(c_1, c_2, \cdots, c_{n+r})$. Here (6) holds obviously for $c_1=0, 1, \cdots, N-1$. Thus it remains to show that (6) holds for $c_1=N$. To do this it suffices to prove

$$(c_2, c_3, \cdots, c_{n+r})\subset[0, T1). \tag{7}$$

Since $\bar{\beta}$ is rational of order r, $T1$ is an endpoint of an interval $(c_1', c_2', \cdots, c_k')$ of length $k\geqq r$. So we have

$$(c_2, c_3, \cdots, c_{n+r})\subset[0, T1) \quad \text{or} \quad (c_2, c_3, \cdots, c_{n+r})\subset[T1, 1).$$

But the last relation contradicts the admissibility of the sequence $(N, c_2, \cdots, c_{n+1})$. Thus we have the relation (7). By induction the lemma is proved.

Proof of Theorem 1. By Lemma 2, we have

$$m(a_{n+r+1}(x)=c_{n+r+1}\,;\, a_1(x)=c_1, \cdots, a_{n+r}(x)=c_{n+r})$$
$$=\frac{m((c_{n+1}, c_{n+2}, \cdots, c_{n+r+1}))}{m((c_{n+1}, c_{n+2}, \cdots, c_{n+r}))}=Q(c_{n+1}, \cdots, c_{n+r+1})$$

where Q is a constant which depends only on the admissible sequence $(c_{n+1}, c_{n+2}, \cdots, c_{n+r+1})$. Since $\bar{\beta}$ is rational of order r, $h(x)$ is constant on every interval $(c_1, c_2, \cdots, c_k)$ of length $k\geqq r$. Then, we have

$$P(a_{n+r+1}(x)=c_{n+r+1}\,;\, a_1(x)=c_1, \cdots, a_{n+r}(x)=c_{n+r})=Q(c_{n+1}, \cdots, c_{n+r+1})$$

for any admissible sequence $(c_1, c_2, \cdots c_{n+r+1})$.

References

[1] Rényi, A.: Representations for real numbers and their ergodic properties. Acta Math. Acad. Sci. Hung., **8**, 477–493 (1957).

[2] Parry, W.: On the β-expansions of real numbers. Acta Math. Sci. Hung., **11**, 401–416 (1960).

[3] Cigler, J.: Ziffenverteilung in ϑ-adischen Büchen. Math. Zeit., **75**, 8–13 (1961).

[4] Rohlin, V. A.: Exact endomorphisms of a Lebesgue space. Izv Akad. Nauk SSSR, **25**, 499–530 (1961). Amer. Math. Soc. Transl., **39** (2), 1–36 (1964).