

# Ergodic property of stable-like Markov chains

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## Abstract

A stable-like Markov chain is a time-homogeneous Markov chain on the real line with the transition kernel  $p(x, dy) = f_x(y - x)dy$ , where the density functions  $f_x(y)$ , for large  $|y|$ , have a power-law decay with exponent  $\alpha(x) + 1$ , where  $\alpha(x) \in (0, 2)$ . In this paper, under a certain uniformity condition on the density functions  $f_x(y)$  and additional mild drift conditions, we give sufficient conditions for recurrence in the case when  $0 < \liminf_{|x| \rightarrow \infty} \alpha(x)$ , sufficient conditions for transience in the case when  $\limsup_{|x| \rightarrow \infty} \alpha(x) < 2$  and sufficient conditions for ergodicity in the case when  $0 < \inf\{\alpha(x) : x \in \mathbb{R}\}$ . As a special case of these results, we give a new proof for the recurrence and transience property of a symmetric  $\alpha$ -stable random walk on  $\mathbb{R}$  with the index of stability  $\alpha \neq 1$ .

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## 1 Introduction

Let  $(\Omega, \mathcal{F}, \mathbb{P})$  be a probability space and let  $\{J_n\}_{n \geq 1}$  be a sequence of i.i.d. random variables on  $(\Omega, \mathcal{F}, \mathbb{P})$  taking values in  $\mathbb{R}^d$ ,  $d \geq 1$ . Let us define  $X_n := \sum_{i=1}^n J_i$  and  $X_0 := 0$ . The sequence  $\{X_n\}_{n \geq 0}$  is called a *random walk* with jumps  $\{J_n\}_{n \geq 1}$ . The random walk  $\{X_n\}_{n \geq 0}$  is said to be *recurrent* if

$$\sum_{n=1}^{\infty} \mathbb{P}(|X_n| < a) = \infty \quad \text{for every } a > 0,$$

and *transient* if

$$\sum_{n=1}^{\infty} \mathbb{P}(|X_n| < a) < \infty \quad \text{for every } a > 0.$$

It is well known that every random walk is either recurrent or transient (see [Dur10, Theorem 4.2.6]). Further, recall that an  $\mathbb{R}^d$ -valued random variable  $S$  is said to have *stable distribution* if, for any  $n \in \mathbb{N}$ , there are  $a_n > 0$  and  $b_n \in \mathbb{R}^d$ , such that

$$S_1 + \dots + S_n \stackrel{d}{=} a_n S + b_n,$$

where  $S_1, \dots, S_n$  are independent copies of  $S$  and  $\stackrel{d}{=}$  denotes equality in distribution. It turns out that  $a_n = n^{\frac{1}{\alpha}}$  for some  $\alpha \in (0, 2]$  which is called the index of stability (see [ST94, Definition 1.1.4 and Corollary 2.1.3]). The case when  $\alpha = 2$  corresponds to the Gaussian random variable. A random walk  $\{X_n\}_{n \geq 0}$  is said to be *stable* if the random variable  $J_1$  has stable distribution. In the case  $d = 1$ , every stable distribution is characterized by four parameters: the stability parameter  $\alpha \in (0, 2]$ , the skewness parameter  $\beta \in [-1, 1]$ , the scale parameter  $\gamma \in (0, \infty)$  and the shift parameter  $\delta \in \mathbb{R}$  (see [ST94, Definition 1.1.6]). Using the notation from [ST94], we denote one-dimensional stable distributions by  $S_\alpha(\beta, \gamma, \delta)$ . For symmetric one-dimensional stable distributions, that is, for  $S_\alpha(0, \gamma, 0)$  (see [ST94, Property 1.2.5]), we write  $S_\alpha S$ . It is well known that a  $S_\alpha S$  random walk is recurrent if and only if  $\alpha \geq 1$  (see the discussion after [Dur10, Lemma 4.2.12]). For the occurrence of stable distributions in applications (for example, biology, economy, engineering, physics, etc.) we refer the reader to [Fel71] and [Zol86]. The concept of stable-like Markov chains, as a natural generalization of  $S_\alpha(0, \gamma, \delta)$  random walks, has been introduced in [San13b] in the way that the stability, scale and shift parameters of the jump distribution depend on the current position of the process. In the same paper, the author also analyzes structural properties and provides sufficient conditions for recurrence and transience of such processes. In this paper, we generalize the results presented in [San13b] and we also discuss ergodic property of such processes.

From now on, let us denote by  $\mathcal{B}(\mathbb{R}^d)$  the Borel  $\sigma$ -algebra on  $\mathbb{R}^d$ ,  $d \geq 1$ , by  $\lambda(\cdot)$  the Lebesgue measure on  $\mathcal{B}(\mathbb{R}^d)$  and for arbitrary  $B \in \mathcal{B}(\mathbb{R}^d)$  and  $x \in \mathbb{R}^d$  we define  $B - x := \{y - x : y \in B\}$ . Furthermore, for  $f, g : \mathbb{R} \rightarrow \mathbb{R}$ , let us introduce the notation  $f(y) \sim g(y)$ , as  $y \rightarrow y_0$ , for  $\lim_{y \rightarrow y_0} f(y)/g(y) = 1$ , where  $y_0 \in [-\infty, \infty]$ . Recall that if  $f(y)$  is the density function of a  $S_\alpha(0, \gamma, \delta)$  distribution, where  $\alpha \in (0, 2)$ ,  $\gamma \in (0, \infty)$  and  $\delta \in \mathbb{R}$ , then

$$f(y) \sim c_\alpha |y|^{-\alpha-1},$$

for  $|y| \rightarrow \infty$ , where  $c_1 = \gamma/2$  and  $c_\alpha = \gamma \Gamma(\alpha + 1) \sin(\pi\alpha/2) / \pi$ , for  $\alpha \neq 1$  (see [ST94, Property 1.2.15]).

Now, we recall the definition of stable-like Markov chains introduced in [San13b]. Let  $\alpha : \mathbb{R} \rightarrow (0, 2)$  be an arbitrary function and let  $\{f_x : x \in \mathbb{R}\}$  be a family of density functions on  $\mathbb{R}$  and  $c : \mathbb{R} \rightarrow (0, \infty)$  such that

(C1)  $x \mapsto f_x(y)$  is a Borel measurable function for all  $y \in \mathbb{R}$

(C2)  $f_x(y) \sim c(x)|y|^{-\alpha(x)-1}$ , for  $|y| \rightarrow \infty$ , for all  $x \in \mathbb{R}$

(C3) there exists  $k_0 > 0$  such that

$$\lim_{|y| \rightarrow \infty} \sup_{x \in [-k_0, k_0]^c} \left| f_x(y) \frac{|y|^{\alpha(x)+1}}{c(x)} - 1 \right| = 0$$

(C4)  $\inf_{x \in C} c(x) > 0$  for every compact set  $C \subseteq [-k_0, k_0]^c$

(C5) there exists  $l_0 > 0$  such that for every compact set  $C \subseteq [-l_0, l_0]^c$  with  $\lambda(C) > 0$ , we have

$$\inf_{x \in [-k_0, k_0]} \int_{C-x} f_x(y) dy > 0.$$

Let us define a Markov chain  $\{X_n\}_{n \geq 0}$  on  $\mathbb{R}$  by the following transition kernel

$$p(x, dy) := f_x(y - x) dy. \tag{1.1}$$

The chain jumps from the state  $x$  with the transition density  $f_x(y - x)$  with the power-law decay with exponent  $\alpha(x) + 1$ , and this jump distribution depends only on the current state  $x$ . Transition densities  $\{f_x : x \in \mathbb{R}\}$  are asymptotically equivalent to the densities of  $S_\alpha(0, \gamma, \delta)$  distributions, and such Markov chain is called a *stable-like Markov chain*.

Conditions (C1)-(C5) are needed to control the jumps of  $\{X_n\}_{n \geq 0}$ . They are crucial in deriving certain structural properties of stable-like chains such as irreducibility and aperiodicity and in identifying the class of “singletons” for such chains. These properties are essential in finding sufficient conditions for recurrence, transience and ergodicity. We refer the reader to [San13b] for more details about conditions (C1)-(C5).

A concrete application of stable-like processes in geophysics is given in [Dit99]. Shortly, paleoclimatic records from ice core show that the climate of the last glacial period experienced rapid transitions between two climatic states and the triggering mechanism for climate changes is random fluctuations of the atmospheric forcing on the ocean circulation. To describe this stochastic climate dynamics the Langevin equation  $dy = -(dU/dy)dt + dN$  is used. The variable  $y$  represents the climate state associated with the pole ward heat transport. The first term on the right hand side represents the dynamics of the ocean circulation, where  $U$  is the climate potential which describes the multi state character of the climate system, and the second term on the right hand side is a noise term which represents the atmospheric forcing on the climate state (wind stress, heating and water transport). From the data from the Greenland Ice Core Project, in [Dit99] has been deduced that this noise has an  $\alpha$ -stable component which depends on the position (climate state). This suggests that the noise should be modeled by a stable-like process.

The aim of this paper is to investigate a long-time behavior of stable-like chains, that is, as in the random walk case, to find conditions for recurrence, transience and ergodicity of stable-like chains in terms of the functions  $\alpha(x)$  and  $c(x)$ . To the best of our knowledge, all methods used in establishing conditions for recurrence and transience in the random walk case are based on the i.i.d. property of random walk jumps, that is, laws of large numbers (Chung-Fuchs theorem), central limit theorems, characteristic functions approach (Stone-Ornstein formula), etc. (see [Dur10, Theorems 4.2.7, 4.2.8 and 4.2.9]). Although we deal with distributions similar to  $S_\alpha(\beta, \gamma, \delta)$  distributions, it is not clear if these methods can be used in the case of the non-constant functions  $\alpha(x)$  and  $c(x)$ . Special cases of this problem have been considered in [RF78], [San12] and [San13b]. In [RF78] the authors consider the countable state space  $\mathbb{Z}$  and the function  $\alpha(x)$  is a two-valued step function which takes one value on negative integers and the other one on nonnegative integers. In [San12], the author considers two special cases of stable-like chains, as in [RF78], the case when the function  $\alpha(x)$  is of the form

$$\alpha(x) = \begin{cases} \alpha, & x < 0 \\ \beta, & x \geq 0, \end{cases}$$

and the case when  $\alpha(x)$  is periodic function. In the first case, it has been proved that the corresponding stable-like chain is recurrent if and only if  $\alpha + \beta \geq 2$ , while in the second case, under the assumption  $\lambda(\{x : \alpha(x) = \alpha_0 := \inf\{\alpha(y) : y \in \mathbb{R}\}\}) > 0$ , it has been proved that the corresponding stable-like chain is recurrent if and only if  $\alpha_0 \geq 1$ . In [San13b], it has been proved that  $\liminf_{|x| \rightarrow \infty} \alpha(x) > 1$ , under an additional mild drift condition, implies recurrence of the corresponding stable-like chain, and  $\limsup_{|x| \rightarrow \infty} \alpha(x) < 1$ , under an additional mild drift condition, implies the transience of the corresponding stable-like chain. In this paper, we generalize these results. More precisely, we give sufficient conditions for recurrence in the case when  $0 < \liminf_{|x| \rightarrow \infty} \alpha(x)$  and sufficient conditions for transience in the case when  $\limsup_{|x| \rightarrow \infty} \alpha(x) < 2$ . Further, we also discuss the ergodic property of stable-like chains and give sufficient conditions for ergodicity in the case when  $0 < \inf\{\alpha(x) : x \in \mathbb{R}\}$ . Let us also mention that if we allow  $\alpha(x) \in (0, \infty)$ , the recurrence

property of the corresponding stable-like chain in the case when  $\liminf_{|x| \rightarrow \infty} \alpha(x) > 2$  has been covered in [MAY95]. For the continuous-time version of stable-like chains and their recurrence and transience property we refer the reader to [Böt11], [Fra06, Fra07], [SW13], [San13a] and [San14].

Before stating the main results of this paper, we recall relevant definitions of recurrence, transience and ergodicity.

**Definition 1.1.** Let  $\{Y_n\}_{n \geq 0}$  be a Markov chain on  $(\mathbb{R}^d, \mathcal{B}(\mathbb{R}^d))$ . The chain  $\{Y_n\}_{n \geq 0}$  is called

- (i) Lebesgue irreducible if  $\lambda(B) > 0$  implies  $\sum_{n=1}^{\infty} \mathbb{P}(Y_n \in B | Y_0 = x) > 0$  for all  $x \in \mathbb{R}^d$ .
- (ii) Recurrent if it is Lebesgue irreducible and if  $\lambda(B) > 0$  implies  $\sum_{n=1}^{\infty} \mathbb{P}(Y_n \in B | Y_0 = x) = \infty$  for all  $x \in \mathbb{R}^d$ .
- (iii) Harris recurrent if it is Lebesgue irreducible and if  $\lambda(B) > 0$  implies  $\mathbb{P}^x(\tau_B < \infty) = 1$  for all  $x \in \mathbb{R}^d$ , where  $\tau_B := \min\{n \in \mathbb{N} : Y_n \in B\}$ .
- (iv) Transient if it is Lebesgue irreducible and if there exists a countable covering of  $\mathbb{R}^d$  with sets  $\{B_j\}_{j \in \mathbb{N}} \subseteq \mathcal{B}(\mathbb{R}^d)$ , such that for each  $j \in \mathbb{N}$  there is a finite constant  $M_j \geq 0$  such that  $\sum_{n=1}^{\infty} \mathbb{P}(Y_n \in B_j | Y_0 = x) \leq M_j$  holds for all  $x \in \mathbb{R}^d$ .

Note that the Lebesgue irreducibility of  $S_\alpha(0, \gamma, \delta)$  random walks is trivially satisfied, and the Lebesgue irreducibility of general stable-like chains has been shown in [San13b, Proposition 2.1]. Hence, according to [MT93, Theorem 8.3.4], every stable-like chain is either recurrent or transient. Further, clearly, every Harris recurrent chain is recurrent but in general, these two properties are not equivalent. They differ on a set of Lebesgue measure zero (see [MT93, Theorem 9.1.5]). In the case of stable-like chains, these two properties are equivalent (see [San13b, Proposition 5.3]). For further structural properties of stable-like chains we refer the reader to [San13b]. A  $\sigma$ -finite measure  $\pi(\cdot)$  on  $\mathcal{B}(\mathbb{R}^d)$  is called an *invariant measure* for a Markov chain  $\{Y_n\}_{n \geq 0}$  on  $(\mathbb{R}^d, \mathcal{B}(\mathbb{R}^d))$  if

$$\pi(B) = \int_{\mathbb{R}} \mathbb{P}(Y_1 \in B | Y_0 = x) \pi(dx)$$

holds for all  $B \in \mathcal{B}(\mathbb{R}^d)$ .

**Definition 1.2.** A Markov chain  $\{Y_n\}_{n \geq 0}$  on  $(\mathbb{R}^d, \mathcal{B}(\mathbb{R}^d))$  is called ergodic if an invariant probability measure  $\pi(\cdot)$  exists and if

$$\lim_{n \rightarrow \infty} \|\mathbb{P}(Y_n \in \cdot | Y_0 = x) - \pi(\cdot)\| = 0$$

holds for all  $x \in \mathbb{R}^d$ , where  $\|\cdot\|$  denotes the total variation norm on the space of signed measures.

Now, let us state the main results of this paper. The following three constants will appear in the statements of the main results. For  $\alpha \in (0, 2)$  let

$$R_1(\alpha) := \frac{-\pi \operatorname{ctg}\left(\frac{\pi\alpha}{2}\right)}{\alpha},$$

for  $\alpha \in (0, 2)$  and  $\beta \in (0, 1] \cap (0, \alpha)$  let

$$R_2(\alpha, \beta) := - \sum_{n=1}^{\infty} \binom{\beta}{2n} \frac{2}{2n - \alpha} + \frac{2}{\alpha} - \frac{{}_2F_1(-\beta, \alpha - \beta, 1 + \alpha - \beta; -1) + {}_2F_1(-\beta, \alpha - \beta, 1 + \alpha - \beta; 1)}{\alpha - \beta}$$

and for  $\alpha \in (0, 2)$  and  $\beta \in (0, 1)$  let

$$T(\alpha, \beta) := \sum_{n=1}^{\infty} \binom{-\beta}{2n} \frac{2}{2n - \alpha} - \frac{2}{\alpha} + \frac{{}_2F_1(\beta, \alpha + \beta, 1 + \alpha + \beta; 1) + {}_2F_1(\beta, \alpha + \beta, 1 + \alpha + \beta; -1)}{\alpha + \beta},$$

where  $\binom{z}{n}$  is the binomial coefficient and  ${}_2F_1(a, b, c; z)$  is the Gauss hypergeometric function (see Section 3 for the definition of this function). Clearly, the constant  $R_1(\alpha)$ , as a function of  $\alpha \in (0, 2)$ , is strictly increasing and it satisfies  $\lim_{\alpha \rightarrow 0} R_1(\alpha) = -\infty$ ,  $R_1(1) = 0$  and  $\lim_{\alpha \rightarrow 2} R_1(\alpha) = \infty$ . The constant  $R_2(\alpha, \beta)$ , as a function of  $\alpha \in (\beta, 2)$  for fixed  $\beta \in (0, 1)$ , is strictly increasing and it satisfies  $\lim_{\alpha \rightarrow \beta} R_2(\alpha, \beta) = -\infty$ ,  $R_2(1 + \beta, \beta) = 0$  and  $\lim_{\alpha \rightarrow 2} R_2(\alpha, \beta) = \infty$  (see the proof of Theorem 1.3). Finally, the constant  $T(\alpha, \beta)$ , as a function of  $\alpha \in (0, 2)$  for fixed  $\beta \in (0, 1)$ , is strictly increasing and it satisfies  $\lim_{\alpha \rightarrow 0} T(\alpha, \beta) = -\infty$ ,  $T(1 - \beta, \beta) = 0$  and  $\lim_{\alpha \rightarrow 2} T(\alpha, \beta) = \infty$  (see the proof of Theorem 1.4).

**Theorem 1.3.** *Let  $\alpha : \mathbb{R} \rightarrow (0, 2)$  be an arbitrary function such that*

$$0 < \alpha := \liminf_{|x| \rightarrow \infty} \alpha(x)$$

and let  $\beta \in (0, 1] \cap (0, \alpha)$  be arbitrary. Furthermore, let  $\{f_x : x \in \mathbb{R}\}$  be a family of density functions on  $\mathbb{R}$  and  $c : \mathbb{R} \rightarrow (0, \infty)$  which satisfy conditions (C1)-(C5) and such that

$$\limsup_{\delta \rightarrow 0} \limsup_{|x| \rightarrow \infty} \frac{|x|^{\alpha(x)}}{c(x)} \int_{-\delta|x|}^{\delta|x|} \log \left( 1 + \operatorname{sgn}(x) \frac{y}{1 + |x|} \right) f_x(y) dy < R_1(\alpha) \quad (1.2)$$

or

$$\limsup_{\delta \rightarrow 0} \limsup_{|x| \rightarrow \infty} \frac{|x|^{\alpha(x)}}{c(x)} \int_{-\delta|x|}^{\delta|x|} \left( \left( 1 + \operatorname{sgn}(x) \frac{y}{|x|} \right)^{\beta} - 1 \right) f_x(y) dy < R_2(\alpha, \beta). \quad (1.3)$$

Then the stable-like Markov chain  $\{X_n\}_{n \geq 0}$  is recurrent.

**Theorem 1.4.** *Let  $\alpha : \mathbb{R} \rightarrow (0, 2)$  be an arbitrary function such that*

$$\limsup_{|x| \rightarrow \infty} \alpha(x) =: \alpha < 2$$

and let  $\beta \in (0, 1)$  be arbitrary. Furthermore, let  $\{f_x : x \in \mathbb{R}\}$  be a family of density functions on  $\mathbb{R}$  and  $c : \mathbb{R} \rightarrow (0, \infty)$  which satisfy conditions (C1)-(C5) and such that

$$\liminf_{\delta \rightarrow 0} \liminf_{|x| \rightarrow \infty} \frac{|x|^{\alpha(x)}}{c(x)} \int_{-\delta|x|}^{\delta|x|} \left( 1 - \left( 1 + \operatorname{sgn}(x) \frac{y}{1 + |x|} \right)^{-\beta} \right) f_x(y) dy > T(\alpha, \beta). \quad (1.4)$$

Then the stable-like Markov chain  $\{X_n\}_{n \geq 0}$  is transient.

**Theorem 1.5.** *Let  $\alpha : \mathbb{R} \rightarrow (0, 2)$  be an arbitrary function such that*

$$0 < \inf\{\alpha(x) : x \in \mathbb{R}\},$$

let  $\alpha := \liminf_{|x| \rightarrow \infty} \alpha(x)$  and let  $\beta \in (0, 1] \cap (0, \inf\{\alpha(x) : x \in \mathbb{R}\})$  be arbitrary. Furthermore, let  $\{f_x : x \in \mathbb{R}\}$  be a family of density functions on  $\mathbb{R}$  and  $c : \mathbb{R} \rightarrow (0, \infty)$  which satisfy conditions (C1)-(C5) and such that

$$\limsup_{d \rightarrow 0} \limsup_{\delta \rightarrow 0} \limsup_{|x| \rightarrow \infty} \frac{|x|^{\alpha(x)}}{c(x)} \left( \int_{-\delta|x|}^{\delta|x|} \log \left( 1 + \operatorname{sgn}(x) \frac{y}{1 + |x|} \right) f_x(y) dy + d \right) < R_1(\alpha) \quad (1.5)$$

or

$$\limsup_{d \rightarrow 0} \limsup_{\delta \rightarrow 0} \limsup_{|x| \rightarrow \infty} \frac{|x|^{\alpha(x)}}{c(x)} \left( \int_{-\delta|x|}^{\delta|x|} \left( \left( 1 + \operatorname{sgn}(x) \frac{y}{|x|} \right)^\beta - 1 \right) f_x(y) dy + d|x|^{-\beta} \right) < R_2(\alpha, \beta). \quad (1.6)$$

Then the stable-like Markov chain  $\{X_n\}_{n \geq 0}$  is ergodic.

Conditions (1.2), (1.3), (1.4), (1.5) and (1.6) are needed to control small jumps of  $\{X_n\}_{n \geq 0}$ . Big jumps are controlled with condition (C3). If we assume that

$$\limsup_{|x| \rightarrow \infty} \alpha(x) < 2 \quad \text{and} \quad \lim_{|x| \rightarrow \infty} c(x)|x|^{2-\alpha(x)} = \infty,$$

then conditions (1.2) and (1.3) are equivalent to

$$\limsup_{\delta \rightarrow 0} \limsup_{|x| \rightarrow \infty} \operatorname{sgn}(x) \frac{|x|^{\alpha(x)-1}}{c(x)} \int_{-\delta|x|}^{\delta|x|} y f_x(y) dy < R_1(\alpha), \quad (1.7)$$

condition (1.4) is equivalent to

$$\liminf_{\delta \rightarrow 0} \liminf_{|x| \rightarrow \infty} \operatorname{sgn}(x) \frac{|x|^{\alpha(x)-1}}{c(x)} \int_{-\delta|x|}^{\delta|x|} y f_x(y) dy > R_1(\alpha) \quad (1.8)$$

and conditions (1.5) and (1.6) are equivalent to

$$\limsup_{d \rightarrow 0} \limsup_{\delta \rightarrow 0} \limsup_{|x| \rightarrow \infty} \frac{|x|^{\alpha(x)-1}}{c(x)} \left( \operatorname{sgn}(x) \int_{-\delta|x|}^{\delta|x|} y f_x(y) dy + d|x| \right) < R_1(\alpha) \quad (1.9)$$

and

$$\limsup_{\beta \rightarrow 0} \limsup_{d \rightarrow 0} \limsup_{\delta \rightarrow 0} \limsup_{|x| \rightarrow \infty} \frac{|x|^{\alpha(x)-1}}{c(x)} \left( \operatorname{sgn}(x) \int_{-\delta|x|}^{\delta|x|} y f_x(y) dy + \frac{d|x|^{-\beta+1}}{\beta} \right) < R_1(\alpha), \quad (1.10)$$

respectively. In addition, if  $\liminf_{|x| \rightarrow \infty} \alpha(x) > 1$ , then the term  $\int_{-\delta|x|}^{\delta|x|} y f_x(y) dy$  in (1.7), (1.8), (1.9) and (1.10) can be replaced by  $\mathbb{E}[X_1 - X_0 | X_0 = x] = \int_{\mathbb{R}} y f_x(y) dy$ . Hence, conditions (1.2), (1.3), (1.5) and (1.6) actually say that when the stable-like chain  $\{X_n\}_{n \geq 0}$  has moved far away from the origin it cannot have strong tendency to move further from the origin, while condition (1.4) says that the stable-like chain  $\{X_n\}_{n \geq 0}$  has a permanent tendency of moving away from the origin. See Section 4 for the proof of the above equivalences and further discussion about conditions (1.2), (1.3), (1.4), (1.5) and (1.6).

As a simple consequence of these results we get a new proof for the well-known recurrence and transience property of  $S\alpha S$  random walks.

**Corollary 1.6.** *A  $S\alpha S$  random walk with  $1 < \alpha \leq 2$  is recurrent. A  $S_\alpha(0, \gamma, \delta)$  random walk with  $0 < \alpha < 1$  and arbitrary shift  $\delta \in \mathbb{R}$  is transient.*

Note that Theorem 1.3 and condition (1.7), that is, conditions (1.2) and (1.3), do not imply the recurrence property of SIS random walk since, in this case, the left-hand side and the right-hand side in (1.7) are equal to zero. A simple example of the application of Theorems 1.3, 1.4 and 1.5 in the non-random walk case is the following. Let  $\alpha : \mathbb{R} \rightarrow (0, 2)$ ,  $\gamma : \mathbb{R} \rightarrow (0, \infty)$  and  $\delta : \mathbb{R} \rightarrow \mathbb{R}$  be arbitrary Borel measurable functions which take finitely many values. Then, the Markov chain  $\{X_n\}_{n \geq 0}$  with  $S_{\alpha(x)}(0, \gamma(x), \delta(x))$  jumps satisfies conditions (C1)-(C5), that is, it is a stable-like chain. Now, by Theorems 1.3, 1.4 and 1.5,

- (i) if  $\inf\{\alpha(x) : x \in \mathbb{R}\} > 1$ ,  $\sup\{\delta(x) : x > 0\} \leq 0$  and  $\inf\{\delta(x) : x < 0\} \geq 0$ , then, from (1.7),  $\{X_n\}_{n \geq 0}$  is recurrent.
- (ii) if  $\sup\{\alpha(x) : x \in \mathbb{R}\} < 1$ , then, from (1.8),  $\{X_n\}_{n \geq 0}$  is transient.
- (iii) if  $\inf\{\alpha(x) : x \in \mathbb{R}\} > 1$ ,  $\sup\{\delta(x) : x > 0\} < 0$  and  $\inf\{\delta(x) : x < 0\} > 0$ , then, from (1.6) where we take  $\beta = 1$ ,  $\{X_n\}_{n \geq 0}$  is ergodic.

Now, we explain our strategy of proving the main results. The proofs of Theorems 1.3, 1.4 and 1.5 are based on the *Foster-Lyapunov drift criteria* (see [MT93, Theorems 8.4.2, 8.4.3 and 13.0.1]). These criteria are based on finding an appropriate “distance” function  $V(x)$  (positive and unbounded in the recurrence case, positive and bounded in the transience case and positive and finite in the ergodic case) and a compact set  $C \subseteq \mathbb{R}$ , such that  $\Delta V(x) := \mathbb{E}[V(X_1) - V(X_0) | X_0 = x] \leq 0$ , in the recurrent case,  $\Delta V(x) \geq 0$ , in the transient case, and  $\Delta V(x) \leq -d$ , for some  $d > 0$ , in the ergodic case, for every  $x \in C^c$ . The idea is to find test functions  $V(x)$  such that the associated level sets  $C_V(r) := \{y : V(y) \leq r\}$  are “singletons” and such that  $C_V(r) \uparrow \mathbb{R}$ , for  $r \rightarrow \infty$ , in the cases of recurrence and ergodicity, and  $C_V(r) \uparrow \mathbb{R}$ , for  $r \rightarrow 1$ , in the case of transience. In the recurrent case, for the test function we take  $\log(1 + |x|)$  and  $|x|^\beta$ , where  $\beta \in (0, 1] \cap (0, \liminf_{|x| \rightarrow \infty} \alpha(x))$  is arbitrary. In the transient case, for the test function we take  $V(x) = 1 - (1 + |x|)^{-\beta}$ , where  $\beta \in (0, 1)$  is arbitrary, and in the ergodic case we take again  $\log(1 + |x|)$  and  $|x|^\beta$ , where  $\beta \in (0, 1] \cap (0, \inf\{\alpha(x) : x \in \mathbb{R}\})$  is arbitrary. Now, by proving that

$$\limsup_{|x| \rightarrow \infty} \frac{|x|^{\alpha(x)}}{c(x)} \Delta V(x) < 0 \quad \text{and} \quad \limsup_{|x| \rightarrow \infty} \frac{|x|^{\alpha(x) - \beta}}{c(x)} \Delta V(x) < 0$$

in the recurrent case,

$$\liminf_{|x| \rightarrow \infty} \frac{|x|^{\alpha(x) + \beta}}{c(x)} \Delta V(x) > 0$$

in the transient case and

$$\limsup_{|x| \rightarrow \infty} \frac{|x|^{\alpha(x)}}{c(x)} (\Delta V(x) + d) < 0 \quad \text{and} \quad \limsup_{|x| \rightarrow \infty} \frac{|x|^{\alpha(x) - \beta}}{c(x)} (\Delta V(x) + d) < 0,$$

for some  $d > 0$ , in the ergodic case, the proofs of Theorems 1.3, 1.4 and 1.5 are accomplished.

Let us remark that a similar approach, using similar test functions, can be found in [Lam60], [MAY95] and [San13b] in the discrete-time case and in [ST97], [Wan08] and [San13a] in the continuous-time case.

The paper is organized as follows. In Section 2 we discuss several structural properties of stable-like chains which will be crucial in finding sufficient conditions for recurrence, transience and ergodicity. In Section 3, using the Foster-Lyapunov drift criteria, we give the proofs of Theorems 1.3, 1.4 and 1.5. Finally, in Section 4, we discuss conditions (1.2), (1.3), (1.4), (1.5) and (1.6) and some consequences of the main results.

Throughout the paper we use the following notation. We write  $\mathbb{Z}_+$  and  $\mathbb{Z}_-$  for nonnegative and nonpositive integers, respectively. For  $x, y \in \mathbb{R}$  let  $x \wedge y := \min\{x, y\}$  and  $x \vee y := \max\{x, y\}$ . Furthermore,  $\{X_n\}_{n \geq 0}$  will denote a stable-like Markov chain given by (1.1) with transition densities satisfying conditions (C1)-(C5), while  $\{Y_n\}_{n \geq 0}$  will denote an arbitrary Markov chain on  $(\mathbb{R}^d, \mathcal{B}(\mathbb{R}^d))$  given by the transition kernel  $p(x, B)$ . For  $x \in \mathbb{R}^d$ ,  $B \in \mathcal{B}(\mathbb{R}^d)$  and  $n \in \mathbb{N}$  let  $p^n(x, B) := \mathbb{P}^x(Y_n \in B) := \mathbb{P}(Y_n \in B | Y_0 = x)$ .

## 2 Preliminary and auxiliary results

In this section, we discuss several structural properties of stable-like chains which are crucial for deriving sufficient conditions for recurrence, transience and ergodicity.

First, recall that, according to [MT93, Theorem 10.4.9], if  $\{Y_n\}_{n \geq 0}$  is a recurrent chain, then it possesses a unique (up to constant multiples) invariant measure. If the invariant measure is finite, then it may be normalized to a probability measure. The chain  $\{Y_n\}_{n \geq 0}$  is called a *positive chain* if it admits an invariant probability measure. Otherwise, it is called a *null chain*. In the case of stable-like chains, from [MT93, Theorems 11.5.1 and 11.5.2], we have the following result.

**Theorem 2.1.** *Let  $\alpha : \mathbb{R} \rightarrow (0, 2)$  be an arbitrary function such that*

$$\limsup_{|x| \rightarrow \infty} \alpha(x) =: \alpha < 2$$

*and let  $\beta \in (0, 1] \cap (0, \inf\{\alpha(x) : x \in \mathbb{R}\})$  be arbitrary. Furthermore, let  $\{f_x : x \in \mathbb{R}\}$  be a family of density functions on  $\mathbb{R}$  and  $c : \mathbb{R} \rightarrow (0, \infty)$  which satisfy conditions (C1)-(C5) and such that*

$$\liminf_{\delta \rightarrow 0} \liminf_{|x| \rightarrow \infty} \frac{|x|^{\alpha(x)}}{c(x)} \int_{-\delta|x|}^{\delta|x|} \log \left( 1 + \operatorname{sgn}(x) \frac{y}{1 + |x|} \right) f_x(y) dy > R_1(\alpha)$$

*or*

$$\liminf_{\delta \rightarrow 0} \liminf_{|x| \rightarrow \infty} \frac{|x|^{\alpha(x)}}{c(x)} \int_{-\delta|x|}^{\delta|x|} \left( \left( 1 + \operatorname{sgn}(x) \frac{y}{|x|} \right)^\beta - 1 \right) f_x(y) dy > R_2(\alpha, \beta)$$

*holds. Then the stable-like Markov chain  $\{X_n\}_{n \geq 0}$  is a null chain, provided it is recurrent.*

The proof of this statement completely follows the proof of Theorem 1.3, hence we omit it here. Recall that every positive chain must be recurrent (see [MT93, Proposition 10.1.1]) and a random walk is never positive since it cannot possess a finite invariant measure (see [Sat99, Exercise 29.6]).

Next, recall that a Markov chain  $\{Y_n\}_{n \geq 0}$  is called *aperiodic* if there does not exist a partition  $\mathbb{R}^d = P_1 \cup \dots \cup P_m$  for some  $m \geq 2$ ,  $P_1, \dots, P_m \in \mathcal{B}(\mathbb{R}^d)$ , such that  $p(x, P_{i+1}) = 1$  for all  $x \in P_i$  and all  $1 \leq i \leq m - 1$  and  $p(x, P_1) = 1$  for all  $x \in P_m$ . The aperiodicity of stable-like chains has been shown in [San13b, Proposition 2.4]. Now, as in the countable state case, one would expect that every positive recurrent and aperiodic chain is automatically ergodic, but in general this is not true. The recurrence property is too weak. We need Harris recurrence (see [MT93, Theorem 13.2.5]). Hence, every positive Harris recurrent and aperiodic chain is ergodic (see [MT93, Theorem 13.3.3]). In the case of stable-like chains we have even more. According to [MT93, Proposition 10.1.1], in the case of stable-like chains these two properties coincide. Thus, in order to check ergodicity of stable-like chains it suffices to check that the corresponding invariant measure is finite, and, by [MT93, Theorem 13.0.1], sufficient conditions for the finiteness of the corresponding invariant measure have been given in Theorem 1.5.

Further, as a simple consequence of Theorems 1.3 and 1.5, [MT93, Theorems 9.2.2, 17.0.1 and 18.3.2] and [San13b, Proposition 5.2], we get the following additional long-time properties of stable-like chains.

**Corollary 2.2.** *(i) Under assumptions of Theorem 1.3, for every initial position  $x \in \mathbb{R}$  and every covering  $\{O_n\}_{n \in \mathbb{N}}$  of  $\mathbb{R}$  by open bounded sets we have*

$$\mathbb{P}^x \left( \bigcap_{n=1}^{\infty} \left\{ \sum_{k=1}^{\infty} 1_{\{X_k \in O_n\}} < \infty \right\} \right) = 0.$$



In other words, for every initial position  $x \in \mathbb{R}$  the event  $\{X_n \in C \text{ for any compact set } C \subseteq \mathbb{R} \text{ and all } n \in \mathbb{N} \text{ sufficiently large}\}$  has probability 0.

(ii) Under assumptions of Theorem 1.5, for every initial position  $x \in \mathbb{R}$  and every  $\varepsilon > 0$  there exists a compact set  $C \subseteq \mathbb{R}$ , such that

$$\liminf_{n \rightarrow \infty} \mathbb{P}^x(X_n \in C) \geq 1 - \varepsilon \quad \text{and} \quad \liminf_{n \rightarrow \infty} \mathbb{E}^x \left[ \frac{1}{n} \sum_{k=1}^n 1_{\{X_k \in C\}} \right] \geq 1 - \varepsilon.$$

We end this section with the following observation. Assume that  $\{Y_n\}_{n \geq 0}$  is an ergodic Markov chain with invariant measure  $\pi(\cdot)$ . Then, clearly,

$$\lim_{n \rightarrow \infty} \mathbb{E}^x[f(Y_n)] = \int_{\mathbb{R}^d} f(y) \pi(dy) =: \pi(f)$$

holds for all  $x \in \mathbb{R}^d$  and all bounded Borel measurable functions  $f(y)$ . In what follows, we extend this convergence to a wider class of functions. For any Borel measurable function  $f(y) \geq 1$  and any signed measure  $\mu(\cdot)$  on  $\mathcal{B}(\mathbb{R}^d)$  we write

$$\|\mu\|_f := \sup_{|g| \leq f} |\mu(g)|.$$

A Markov chain  $\{Y_n\}_{n \geq 0}$  is called *f-ergodic* if it is positive Harris recurrent with the invariant probability measure  $\pi(\cdot)$ , if  $\pi(f) < \infty$ , and if

$$\lim_{n \rightarrow \infty} \|p^n(x, \cdot) - \pi(\cdot)\|_f = 0$$

holds for all  $x \in \mathbb{R}^d$ . Note that  $\|\cdot\|_1 = \|\cdot\|$ . Hence, *f-ergodicity* implies ergodicity. Now, from [MT93, Theorem 14.0.1], we have the following.

**Theorem 2.3.** *Let  $\alpha : \mathbb{R} \rightarrow (0, 2)$  be an arbitrary function such that*

$$0 < \inf\{\alpha(x) : x \in \mathbb{R}\},$$

*let  $\alpha := \liminf_{|x| \rightarrow \infty} \alpha(x)$  and let  $\beta \in (0, 1] \cap (0, \inf\{\alpha(x) : x \in \mathbb{R}\})$  be arbitrary. Furthermore, let  $\{f_x : x \in \mathbb{R}\}$  be a family of density functions on  $\mathbb{R}$  and  $c : \mathbb{R} \rightarrow (0, \infty)$  which satisfy conditions (C1)-(C5) and such that*

$$\limsup_{d \rightarrow 0} \limsup_{\delta \rightarrow 0} \limsup_{|x| \rightarrow \infty} \frac{|x|^{\alpha(x)}}{c(x)} \left( \int_{-\delta|x|}^{\delta|x|} \log \left( 1 + \operatorname{sgn}(x) \frac{y}{1+|x|} \right) f_x(y) dy + d g(x) \right) < R_1(\alpha) \quad (2.1)$$

or

$$\limsup_{d \rightarrow 0} \limsup_{\delta \rightarrow 0} \limsup_{|x| \rightarrow \infty} \frac{|x|^{\alpha(x)}}{c(x)} \left( \int_{-\delta|x|}^{\delta|x|} \left( \left( 1 + \operatorname{sgn}(x) \frac{y}{|x|} \right)^\beta - 1 \right) f_x(y) dy + d (g(x))^{-\beta} \right) < R_2(\alpha, \beta), \quad (2.2)$$

for some Borel measurable function  $g(x) \geq 1$ . Then the stable-like Markov chain  $\{X_n\}_{n \geq 0}$  is *f-ergodic* for every Borel measurable function  $f(x) \geq 1$  such that  $f(x) \leq g(x)$ .

### 3 Proof of the main results

In this section we give the proofs of Theorems 1.3, 1.4, 1.5 and 2.3 and Corollary 1.6. Before the proofs, we recall several special functions we need. The Gamma function is defined by the formula

$$\Gamma(z) := \int_0^\infty t^{z-1} e^{-t} dt, \quad z \in \mathbb{C}, \operatorname{Re}(z) > 0,$$

and it can be analytically continued on  $\mathbb{C} \setminus \mathbb{Z}_-$ . The Digamma function is a function defined by  $\Psi(z) := \Gamma'(z)/\Gamma(z)$ , for  $z \in \mathbb{C} \setminus \mathbb{Z}_-$ , and it satisfies the following properties:

(i)

$$\Psi(1+z) = -\gamma + \sum_{n=1}^{\infty} \frac{z}{n(n+z)}, \quad (3.1)$$

where  $\gamma$  is Euler's number;

(ii)

$$\Psi(1+z) = \Psi(z) + \frac{1}{z}; \quad (3.2)$$

(iii)

$$\Psi(2z) = \frac{1}{2}\Psi(z) + \frac{1}{2}\Psi\left(z + \frac{1}{2}\right) + \log 2; \quad (3.3)$$

(iv)

$$\Psi(1-z) = \Psi(z) + \pi \operatorname{ctg}(\pi z). \quad (3.4)$$

The Gauss hypergeometric function is defined by the formula

$${}_2F_1(a, b, c; z) := \sum_{n=0}^{\infty} \frac{(a)_n (b)_n}{(c)_n} \frac{z^n}{n!}, \quad (3.5)$$

for  $a, b, c, z \in \mathbb{C}$ ,  $c \notin \mathbb{Z}_-$ , where for  $w \in \mathbb{C}$  and  $n \in \mathbb{Z}_+$ ,  $(w)_n$  is defined by

$$(w)_0 = 1 \quad \text{and} \quad (w)_n = w(w+1) \cdots (w+n-1).$$

The series (3.5) absolutely converges on  $|z| < 1$ , absolutely converges on  $|z| \leq 1$  when  $\operatorname{Re}(c-a-b) > 0$ , conditionally converges on  $|z| \leq 1$ , except for  $z = 1$ , when  $-1 < \operatorname{Re}(c-b-a) \leq 0$  and diverges when  $\operatorname{Re}(c-b-a) \leq -1$ . In the case when  $\operatorname{Re}(c) > \operatorname{Re}(b) > 0$ , it can be analytically continued on  $\mathbb{C} \setminus (1, \infty)$  by the formula

$${}_2F_1(a, b, c; z) = \frac{\Gamma(c)}{\Gamma(b)\Gamma(c-b)} \int_0^1 t^{b-1} (1-t)^{c-b-1} (1-tz)^{-a} dt. \quad (3.6)$$

For further properties of the Gamma function, Digamma function and hypergeometric functions see [AS84, Chapters 6 and 15].

*Proof of Theorem 1.3.* In [San13b, Theorem 1.3] it has been proved that if  $1 < \alpha := \liminf_{|x| \rightarrow \infty} \alpha(x)$  and if

$$\begin{aligned} & \limsup_{\delta \rightarrow 0} \limsup_{|x| \rightarrow \infty} \frac{|x|^{\alpha(x)}}{c(x)} \int_{-\delta|x|}^{\delta|x|} \log \left( 1 + \operatorname{sgn}(x) \frac{y}{1+|x|} \right) f_x(y) dy \\ & < \sum_{i=1}^{\infty} \frac{1}{i(2i-\alpha)} - \frac{\log 2}{\alpha} - \frac{1}{2\alpha} \left( \Psi\left(\frac{\alpha+1}{2}\right) - \Psi\left(\frac{\alpha}{2}\right) \right) + \frac{\gamma}{\alpha} + \frac{\Psi(\alpha)}{\alpha}, \end{aligned} \quad (3.7)$$

then the corresponding stable-like chain is recurrent. Further, by assuming (3.7) and performing completely the same proof as in [San13b, Theorem 1.3], the same conclusion also remains true in the case when  $0 < \alpha := \liminf_{|x| \rightarrow \infty} \alpha(x)$ . Finally, from (3.1), (3.3) and (3.4), the right-hand side in (3.7) equals  $R_1(\alpha)$ , that is, (3.7) becomes (1.2). Thus, the first claim follows.

To prove the second claim, we divide the proof into four steps.

**Step 1.** In the first step we explain our strategy of the proof. Let  $\beta \in (0, 1] \cap (0, \liminf_{|x| \rightarrow \infty} \alpha(x))$  be arbitrary and let us define the function  $V : \mathbb{R} \rightarrow [0, \infty)$  by the formula

$$V(x) := |x|^\beta.$$

Clearly, the level set  $C_V(r) := \{y : V(y) \leq r\}$  is a bounded set for every  $r > 0$ . Thus, by [San13b, Proposition 2.6] and [MT93, Theorem 8.4.3], it suffices to show that there exists  $r_0 > 0$  such that

$$\int_{\mathbb{R}} p(x, dy)V(y) - V(x) \leq 0$$

for all  $x \in C_V^c(r_0)$ . Next, since  $C_V(r) \uparrow \mathbb{R}$ , for  $r \rightarrow \infty$ , it is enough to show that

$$\limsup_{|x| \rightarrow \infty} \frac{|x|^{\alpha(x) - \beta}}{c(x)} \left( \int_{\mathbb{R}} p(x, dy)V(y) - V(x) \right) < 0.$$

We have

$$\begin{aligned} |x|^{-\beta} \left( \int_{\mathbb{R}} p(x, dy)V(y) - V(x) \right) &= |x|^{-\beta} \left( \int_{\mathbb{R}} f_x(y)V(y+x)dy - V(x) \right) \\ &= \int_{\{y+x>0\}} \left( \left( \frac{x+y}{|x|} \right)^\beta - 1 \right) f_x(y)dy + \int_{\{y+x<0\}} \left( \left( \frac{-x-y}{|x|} \right)^\beta - 1 \right) f_x(y)dy. \end{aligned} \quad (3.8)$$

**Step 2.** In the second step we find an appropriate upper bound for the first summand in (3.8). For any  $x > 0$  we have

$$\int_{\{y+x>0\}} \left( \left( \frac{x+y}{x} \right)^\beta - 1 \right) f_x(y)dy = \int_{\{y+x>0\}} \left( \left( 1 + \frac{y}{x} \right)^\beta - 1 \right) f_x(y)dy.$$

Let  $0 < \delta < 1$  be arbitrary. By restricting the function  $(1+t)^\beta - 1$  to the intervals  $(-1, -\delta)$ ,  $[-\delta, \delta]$ ,  $(\delta, 1)$  and  $[1, \infty)$ , and using its Binomial series, that is,

$$(1+t)^\beta - 1 = \sum_{i=1}^{\infty} \binom{\beta}{i} t^i,$$

for  $t \in (-1, 1)$ , where

$$\binom{\beta}{i} = \frac{\beta(\beta-1)\cdots(\beta-i+1)}{i!},$$

from Fubini's theorem we get

$$\begin{aligned} &\int_{\{y+x>0\}} \left( \left( \frac{x+y}{x} \right)^\beta - 1 \right) f_x(y)dy \\ &= \sum_{i=1}^{\infty} \binom{\beta}{i} \frac{1}{x^i} \int_{\{-x<y<-\delta x\} \cap \{y+x>0\}} y^i f_x(y)dy + \int_{\{-\delta x \leq y \leq \delta x\} \cap \{y+x>0\}} \left( \left( 1 + \frac{y}{x} \right)^\beta - 1 \right) f_x(y)dy \\ &\quad + \sum_{i=1}^{\infty} \binom{\beta}{i} \frac{1}{x^i} \int_{\{\delta x < y < x\} \cap \{y+x>0\}} y^i f_x(y)dy + \int_{\{y \geq x\} \cap \{y+x>0\}} \left( \left( 1 + \frac{y}{x} \right)^\beta - 1 \right) f_x(y)dy. \end{aligned}$$

Further, we have

$$\begin{aligned}
& \int_{\{y+x>0\}} \left( \left( \frac{x+y}{x} \right)^\beta - 1 \right) f_x(y) dy \\
&= \sum_{i=1}^{\infty} \binom{\beta}{i} \frac{1}{x^i} \int_{\{-x<y<-\delta x\}} y^i f_x(y) dy + \int_{\{-\delta x \leq y \leq \delta x\}} \left( \left( 1 + \frac{y}{x} \right)^\beta - 1 \right) f_x(y) dy \\
&+ \sum_{i=1}^{\infty} \binom{\beta}{i} \frac{1}{x^i} \int_{\{\delta x < y < x\}} y^i f_x(y) dy + \int_{\{y \geq x\}} \left( \left( 1 + \frac{y}{x} \right)^\beta - 1 \right) f_x(y) dy.
\end{aligned}$$

Let us put

$$\begin{aligned}
U_1^\delta(x) &:= -\frac{\beta}{x} \int_{\{\delta x < y < x\}} y f_x(-y) dy + \frac{\beta}{x} \int_{\{\delta x < y < x\}} y f_x(y) dy \\
U_2^\delta(x) &:= \sum_{i=2}^{\infty} \binom{\beta}{i} \frac{(-1)^i}{x^i} \int_{\{\delta x < y < x\}} y^i f_x(-y) dy + \sum_{i=2}^{\infty} \binom{\beta}{i} \frac{1}{x^i} \int_{\{\delta x < y < x\}} y^i f_x(y) dy, \\
U_3^\delta(x) &:= \int_{\{-\delta x \leq y \leq \delta x\}} \left( \left( 1 + \frac{y}{x} \right)^\beta - 1 \right) f_x(y) dy \quad \text{and} \\
U_4(x) &:= \int_{\{y \geq x\}} \left( \left( 1 + \frac{y}{x} \right)^\beta - 1 \right) f_x(y) dy,
\end{aligned}$$

for  $0 < \delta < 1$ . Hence, we find

$$\int_{\{y+x>0\}} \left( \left( \frac{x+y}{x} \right)^\beta - 1 \right) f_x(y) dy = U_1^\delta(x) + U_2^\delta(x) + U_3^\delta(x) + U_4(x). \quad (3.9)$$

Here comes the crucial step where condition (C3) is needed. In the above terms, by (C3), we can replace all the density functions  $f_x(y)$  by the functions  $c(x)|y|^{-\alpha(x)-1}$  and find a more operable upper bound in (3.9). Let  $0 < \varepsilon < 1$  be arbitrary. Then, by (C3), there exists  $y_\varepsilon \geq 1$ , such that for all  $|y| \geq y_\varepsilon$

$$\left| f_x(y) \frac{|y|^{\alpha(x)+1}}{c(x)} - 1 \right| < \varepsilon,$$

for all  $x \in [-k_0, k_0]^c$  (recall that the constant  $k_0$  is defined in condition (C3)). Let  $x > (k_0 \vee y_\varepsilon / \delta)$ . By a simple computation, we have

$$U_1^\delta(x) < \frac{2\varepsilon c(x)\beta}{(\alpha(x)-1)x^{\alpha(x)}} \left( \delta^{-\alpha(x)+1} - 1 \right),$$

in the case when  $\alpha(x) \neq 1$ , and

$$U_1^\delta(x) < \frac{2\varepsilon c(x)\beta}{x} \log \left( \frac{1}{\delta} \right),$$

in the case when  $\alpha(x) = 1$ . Let us denote the right hand side in the above inequalities by  $U_1^{\delta, \varepsilon}(x)$ . Further, we have

$$\begin{aligned}
U_2^\delta(x) &< \frac{c(x)}{x^{\alpha(x)}} \sum_{i=2}^{\infty} \binom{\beta}{i} \frac{1 + (-1)^i - 2\varepsilon(-1)^i}{i - \alpha(x)} \left( 1 - \delta^{i-\alpha(x)} \right) =: U_2^{\delta, \varepsilon}(x) \quad \text{and} \\
U_4(x) &< (1 + \varepsilon)c(x) \int_x^\infty \left( \left( 1 + \frac{y}{x} \right)^\beta - 1 \right) \frac{dy}{y^{\alpha(x)+1}} =: U_4^\varepsilon(x).
\end{aligned}$$

Hence, from (3.9), we get

$$\int_{\{y+x>0\}} \left( \left( \frac{x+y}{x} \right)^\beta - 1 \right) f_x(y) dy < U_1^{\delta,\varepsilon}(x) + U_2^{\delta,\varepsilon}(x) + U_3^\delta(x) + U_4^\varepsilon(x). \quad (3.10)$$

**Step 3.** In the third step we find an appropriate upper bound for the second summand in (3.8). Let  $x > (k_0 \vee y_\varepsilon/\delta)$ . Then, again by (C3),

$$\begin{aligned} & \int_{\{y+x<0\}} \left( \left( \frac{-x-y}{x} \right)^\beta - 1 \right) f_x(y) dy \\ & < c(x)(1-\varepsilon) \int_x^{2x} \left( \left( -1 + \frac{y}{x} \right)^\beta - 1 \right) \frac{dy}{y^{\alpha(x)+1}} + c(x)(1+\varepsilon) \int_{2x}^\infty \left( \left( -1 + \frac{y}{x} \right)^\beta - 1 \right) \frac{dy}{y^{\alpha(x)+1}} \\ & = c(x)(1-\varepsilon) \int_x^\infty \left( \left( -1 + \frac{y}{x} \right)^\beta - 1 \right) \frac{dy}{y^{\alpha(x)+1}} + 2\varepsilon c(x) \int_{2x}^\infty \left( \left( -1 + \frac{y}{x} \right)^\beta - 1 \right) \frac{dy}{y^{\alpha(x)+1}}. \end{aligned}$$

Note that in the first inequality we make a change of variables  $y \mapsto -y$ . Let us put

$$\begin{aligned} U_5^\varepsilon(x) & := c(x)(1-\varepsilon) \int_x^\infty \left( \left( -1 + \frac{y}{x} \right)^\beta - 1 \right) \frac{dy}{y^{\alpha(x)+1}} \\ & \quad + 2\varepsilon c(x) \int_{2x}^\infty \left( \left( -1 + \frac{y}{x} \right)^\beta - 1 \right) \frac{dy}{y^{\alpha(x)+1}}. \end{aligned}$$

We have

$$\int_{\{y+x<0\}} \left( \left( -1 + \frac{y}{x} \right)^\beta - 1 \right) f_x(y) dy < U_5^\varepsilon(x). \quad (3.11)$$

**Step 4.** In the fourth step we prove

$$\limsup_{x \rightarrow \infty} \frac{x^{\alpha(x)-\beta}}{c(x)} \left( \int_{\mathbb{R}} p(x, dy) V(y) - V(x) \right) < 0.$$

By combining (3.8), (3.9), (3.10) and (3.11) we have

$$x^{-\beta} \left( \int_{\mathbb{R}} p(x, dy) V(y) - V(x) \right) < U_1^{\delta,\varepsilon}(x) + U_2^{\delta,\varepsilon}(x) + U_3^\delta(x) + U_4^\varepsilon(x) + U_5^\varepsilon(x). \quad (3.12)$$

In the rest of the fourth step we prove

$$\begin{aligned} & \limsup_{x \rightarrow \infty} \frac{x^{\alpha(x)-\beta}}{c(x)} \left( \int_{\mathbb{R}} p(x, dy) V(y) - V(x) \right) \\ & < \limsup_{\delta \rightarrow 0} \limsup_{\varepsilon \rightarrow 0} \limsup_{x \rightarrow \infty} \frac{x^{\alpha(x)}}{c(x)} U_1^{\delta,\varepsilon}(x) + \limsup_{\delta \rightarrow 0} \limsup_{\varepsilon \rightarrow 0} \limsup_{x \rightarrow \infty} \frac{x^{\alpha(x)}}{c(x)} U_2^{\delta,\varepsilon}(x) \\ & \quad + \limsup_{\varepsilon \rightarrow 0} \limsup_{x \rightarrow \infty} \frac{x^{\alpha(x)}}{c(x)} (U_4^\varepsilon(x) + U_5^\varepsilon(x)) + R_2(\alpha, \beta) \leq 0. \end{aligned}$$

Recall that  $0 < \alpha = \liminf_{|x| \rightarrow \infty} \alpha(x)$ ,

$$R_2(\alpha, \beta) = - \sum_{n=1}^{\infty} \binom{\beta}{2n} \frac{2}{2n-\alpha} + \frac{2}{\alpha} - \frac{{}_2F_1(-\beta, \alpha-\beta, 1+\alpha-\beta; -1) + {}_2F_1(-\beta, \alpha-\beta, 1+\alpha-\beta; 1)}{\alpha-\beta}$$

and

$$\limsup_{\delta \rightarrow 0} \limsup_{x \rightarrow \infty} \frac{x^{\alpha(x)}}{c(x)} U_3^\delta(x) < R_2(\alpha, \beta)$$

(assumption (1.3)). Clearly,

$$\limsup_{\delta \rightarrow 0} \limsup_{\varepsilon \rightarrow 0} \limsup_{x \rightarrow \infty} \frac{x^{\alpha(x)}}{c(x)} U_1^{\delta, \varepsilon}(x) = 0. \quad (3.13)$$

Further, by the dominated convergence theorem, we have

$$\begin{aligned} & \limsup_{\delta \rightarrow 0} \limsup_{\varepsilon \rightarrow 0} \limsup_{x \rightarrow \infty} \frac{x^{\alpha(x)}}{c(x)} U_2^{\delta, \varepsilon}(x) \\ &= \limsup_{\delta \rightarrow 0} \limsup_{\varepsilon \rightarrow 0} \limsup_{x \rightarrow \infty} \sum_{i=2}^{\infty} \binom{\beta}{i} \frac{1 + (-1)^i - 2\varepsilon(-1)^i}{i - \alpha(x)} (1 - \delta^{i-\alpha(x)}) \\ &= \limsup_{x \rightarrow \infty} \sum_{i=1}^{\infty} \binom{\beta}{2i} \frac{2}{2i - \alpha(x)} \\ &= \sum_{i=1}^{\infty} \binom{\beta}{2i} \frac{2}{2i - \alpha}. \end{aligned} \quad (3.14)$$

Now, let us compute

$$\limsup_{\varepsilon \rightarrow 0} \limsup_{x \rightarrow \infty} \frac{x^{\alpha(x)}}{c(x)} (U_4^\varepsilon(x) + U_5^\varepsilon(x)).$$

By (3.6), we have

$$\begin{aligned} & \limsup_{\varepsilon \rightarrow 0} \limsup_{x \rightarrow \infty} \frac{x^{\alpha(x)}}{c(x)} (U_4^\varepsilon(x) + U_5^\varepsilon(x)) \\ &= \limsup_{\varepsilon \rightarrow 0} \limsup_{x \rightarrow \infty} \left[ (1 + \varepsilon) \left( -\frac{1}{\alpha(x)} + \frac{{}_2F_1(-\beta, \alpha(x) - \beta, 1 + \alpha(x) - \beta; -1)}{\alpha(x) - \beta} \right) \right. \\ & \quad + (1 - \varepsilon) \left( -\frac{1}{\alpha(x)} + \frac{{}_2F_1(-\beta, \alpha(x) - \beta, 1 + \alpha(x) - \beta; 1)}{\alpha(x) - \beta} \right) \\ & \quad \left. + 2\varepsilon \left( -\frac{1}{\alpha(x)2^{\alpha(x)}} - \frac{{}_2F_1(-\beta, \alpha(x) - \beta, 1 + \alpha(x) - \beta; \frac{1}{2})}{(\beta - \alpha(x))2^{\beta-\alpha(x)}} \right) \right] \\ &= \limsup_{x \rightarrow \infty} \left( -\frac{2}{\alpha(x)} + \frac{{}_2F_1(-\beta, \alpha(x) - \beta, 1 + \alpha(x) - \beta; -1) + {}_2F_1(-\beta, \alpha(x) - \beta, 1 + \alpha(x) - \beta; 1)}{\alpha(x) - \beta} \right) \\ &= -\frac{2}{\alpha} + \frac{{}_2F_1(-\beta, \alpha - \beta, 1 + \alpha - \beta; -1) + {}_2F_1(-\beta, \alpha - \beta, 1 + \alpha - \beta; 1)}{\alpha - \beta}, \end{aligned} \quad (3.15)$$

where in the second equality we use (3.5) and the fact that all terms are bounded, and in the last equality we use the fact that the function

$$x \mapsto -\frac{2}{x} + \frac{{}_2F_1(-\beta, x - \beta, 1 + x - \beta; -1) + {}_2F_1(-\beta, x - \beta, 1 + x - \beta; 1)}{x - \beta}$$

is decreasing on  $(\beta, 2)$ . Now, by combining (1.3), (3.12), (3.13), (3.14) and (3.15) we have

$$\limsup_{x \rightarrow \infty} \frac{x^{\alpha(x)-\beta}}{c(x)} \left( \int_{\mathbb{R}} p(x, dy) V(y) - V(x) \right) < 0.$$

The case when  $x < 0$  is treated in the same way. Therefore, we have proved the desired result.  $\square$

*Proof of Theorem 1.4.* We proceed similarly as in the proof of Theorem 1.3. The proof is divided into four steps.

**Step 1.** In the first step we explain our strategy of the proof. Let  $\beta \in (0, 1)$  be arbitrary and let us define the function  $V : \mathbb{R} \rightarrow [0, \infty)$  by the formula

$$V(x) := 1 - (1 + |x|)^{-\beta}.$$

Clearly, the sets  $C_V(r) := \{y : V(y) \leq r\}$  and  $C_V^c(r)$  have positive Lebesgue measure for every  $0 < r < 1$ . Thus, by [MT93, Theorem 8.4.2], it suffices to show that there exists  $0 < r_0 < 1$  such that

$$\int_{\mathbb{R}} p(x, dy)V(y) - V(x) \geq 0$$

for all  $x \in C_V^c(r_0)$ . Next, since  $C_V(r) \uparrow \mathbb{R}$  for  $r \rightarrow 1$ , it is enough to show that

$$\liminf_{|x| \rightarrow \infty} \frac{(1 + |x|)^{\alpha(x) + \beta}}{c(x)} \left( \int_{\mathbb{R}} p(x, dy)V(y) - V(x) \right) > 0.$$

We have

$$\begin{aligned} (1 + |x|)^\beta \left( \int_{\mathbb{R}} p(x, dy)V(y) - V(x) \right) &= (1 + |x|)^\beta \left( \int_{\mathbb{R}} f_x(y)V(y+x)dy - V(x) \right) \\ &= \int_{\{y+x>0\}} \left( 1 - \left( \frac{1+x+y}{1+|x|} \right)^{-\beta} \right) f_x(y)dy + \int_{\{y+x<0\}} \left( 1 - \left( \frac{1-x-y}{1+|x|} \right)^{-\beta} \right) f_x(y)dy. \end{aligned} \quad (3.16)$$

**Step 2.** In the second step we find an appropriate lower bound for the first summand in (3.16). For any  $x > 0$  we have

$$\int_{\{y+x>0\}} \left( 1 - \left( \frac{1+x+y}{1+x} \right)^{-\beta} \right) f_x(y)dy = \int_{\{y+x>0\}} \left( 1 - \left( 1 + \frac{y}{1+x} \right)^{-\beta} \right) f_x(y)dy.$$

Let  $0 < \delta < 1$  be arbitrary. By restricting the function  $1 - (1+t)^{-\beta}$  to the intervals  $(-1, -\delta)$ ,  $[-\delta, \delta]$ ,  $(\delta, 1)$  and  $[1, \infty)$ , and using its Binomial series, that is,

$$1 - (1+t)^{-\beta} = - \sum_{i=1}^{\infty} \binom{-\beta}{i} t^i,$$

for  $t \in (-1, 1)$ , from Fubini's theorem we get

$$\begin{aligned} &\int_{\{y+x>0\}} \left( 1 - \left( \frac{1+x+y}{1+x} \right)^{-\beta} \right) f_x(y)dy \\ &= - \sum_{i=1}^{\infty} \binom{-\beta}{i} \frac{1}{(1+x)^i} \int_{\{-1-x < y < -\delta(1+x)\} \cap \{y+x>0\}} y^i f_x(y)dy \\ &\quad + \int_{\{-\delta(1+x) \leq y \leq \delta(1+x)\} \cap \{y+x>0\}} \left( 1 - \left( 1 + \frac{y}{1+x} \right)^{-\beta} \right) f_x(y)dy \\ &\quad - \sum_{i=1}^{\infty} \binom{-\beta}{i} \frac{1}{(1+x)^i} \int_{\{\delta(1+x) < y < 1+x\} \cap \{y+x>0\}} y^i f_x(y)dy \\ &\quad + \int_{\{y \geq 1+x\} \cap \{y+x>0\}} \left( 1 - \left( 1 + \frac{y}{1+x} \right)^{-\beta} \right) f_x(y)dy. \end{aligned}$$

Furthermore, by taking  $x > \delta/(1 - \delta)$  we get

$$\begin{aligned}
& \int_{\{y+x>0\}} \left(1 - \left(\frac{1+x+y}{1+x}\right)^{-\beta}\right) f_x(y) dy \\
&= - \sum_{i=1}^{\infty} \binom{-\beta}{i} \frac{1}{(1+x)^i} \int_{\{-x<y<-\delta(1+x)\}} y^i f_x(y) dy \\
&\quad + \int_{\{-\delta(1+x)\leq y\leq\delta(1+x)\}} \left(1 - \left(1 + \frac{y}{1+x}\right)^{-\beta}\right) f_x(y) dy \\
&\quad - \sum_{i=1}^{\infty} \binom{-\beta}{i} \frac{1}{(1+x)^i} \int_{\{\delta(1+x)<y<1+x\}} y^i f_x(y) dy \\
&\quad + \int_{\{y\geq 1+x\}} \left(1 - \left(1 + \frac{y}{1+x}\right)^{-\beta}\right) f_x(y) dy.
\end{aligned}$$

Let us put

$$\begin{aligned}
U_1^\delta(x) &:= -\frac{\beta}{1+x} \int_{\{\delta(1+x)<y<x\}} y f_x(-y) dy + \frac{\beta}{1+x} \int_{\{\delta(1+x)<y<1+x\}} y f_x(y) dy \\
U_2^\delta(x) &:= -\sum_{i=2}^{\infty} \binom{-\beta}{i} \frac{(-1)^i}{(1+x)^i} \int_{\{\delta(1+x)<y<x\}} y^i f_x(-y) dy \\
&\quad - \sum_{i=2}^{\infty} \binom{-\beta}{i} \frac{1}{(1+x)^i} \int_{\{\delta(1+x)<y<1+x\}} y^i f_x(y) dy, \\
U_3^\delta(x) &:= \int_{\{-\delta(1+x)\leq y\leq\delta(1+x)\}} \left(1 - \left(1 + \frac{y}{1+x}\right)^{-\beta}\right) f_x(y) dy \quad \text{and} \\
U_4(x) &:= \int_{\{y\geq 1+x\}} \left(1 - \left(1 + \frac{y}{1+x}\right)^{-\beta}\right) f_x(y) dy,
\end{aligned}$$

for  $0 < \delta < 1$  and  $x > \delta/(1 - \delta)$ . Hence, we find

$$\int_{\{y+x>0\}} \left(1 - \left(\frac{1+x+y}{1+x}\right)^{-\beta}\right) f_x(y) dy = U_1^\delta(x) + U_2^\delta(x) + U_3^\delta(x) + U_4(x). \quad (3.17)$$

Now, we apply (C3) and find a more operable lower bound in (3.17). Let  $0 < \varepsilon < 1$  be arbitrary. Then, by (C3), there exists  $y_\varepsilon \geq 1$ , such that for all  $|y| \geq y_\varepsilon$

$$\left| f_x(y) \frac{|y|^{\alpha(x)+1}}{c(x)} - 1 \right| < \varepsilon,$$

for all  $x \in [-k_0, k_0]^c$  (recall that the constant  $k_0$  is defined in condition (C3)). Let  $x > (k_0 \vee (y_\varepsilon - \delta))/\delta \vee \delta/(1 - \delta)$ . By a simple computation, we have

$$\begin{aligned}
U_1^\delta(x) &> -\frac{(1+\varepsilon)c(x)\beta}{(\alpha(x)-1)(1+x)^{\alpha(x)}} \left( \delta^{-\alpha(x)+1} - \left(\frac{x}{1+x}\right)^{-\alpha(x)+1} \right) \\
&\quad + \frac{(1-\varepsilon)c(x)\beta}{(\alpha(x)-1)(1+x)^{\alpha(x)}} \frac{\delta - \delta^{\alpha(x)}}{\delta^{\alpha(x)}},
\end{aligned}$$



in the case when  $\alpha(x) \neq 1$ , and

$$U_1^\delta(x) > -\frac{(1+\varepsilon)c(x)\beta}{1+x} \log\left(\frac{x}{\delta(1+x)}\right) + \frac{(1-\varepsilon)c(x)\beta}{1+x} \log\left(\frac{1}{\delta}\right),$$

in the case when  $\alpha(x) = 1$ . Let us denote the right hand side in the above inequalities by  $U_1^{\delta,\varepsilon}(x)$ . Further, we have

$$\begin{aligned} U_2^\delta(x) &> -\frac{(1+\varepsilon)c(x)}{(1+x)^{\alpha(x)}} \sum_{i=2}^{\infty} \binom{-\beta}{i} \frac{(-1)^i}{i-\alpha(x)} \left( \left(\frac{x}{1+x}\right)^{i-\alpha(x)} - \delta^{i-\alpha(x)} \right) \\ &\quad - \frac{c(x)}{(1+x)^{\alpha(x)}} \sum_{i=2}^{\infty} \binom{-\beta}{i} \frac{1+(-1)^i \varepsilon \delta^{\alpha(x)} - \delta^i}{i-\alpha(x) \delta^{\alpha(x)}} =: U_2^{\delta,\varepsilon}(x) \quad \text{and} \\ U_4(x) &> (1-\varepsilon)c(x) \int_{1+x}^{\infty} \left( 1 - \left(1 + \frac{y}{1+x}\right)^{-\beta} \right) \frac{1}{y^{\alpha(x)+1}} dy =: U_4^\varepsilon(x). \end{aligned}$$

Hence, from (3.17), we get

$$\int_{\{y+x>0\}} \left( 1 - \left(\frac{1+x+y}{1+x}\right)^{-\beta} \right) f_x(y) dy > U_1^{\delta,\varepsilon}(x) + U_2^{\delta,\varepsilon}(x) + U_3^\delta(x) + U_4^\varepsilon(x). \quad (3.18)$$

**Step 3.** In the third step we find an appropriate lower bound for the second summand in (3.16). Let  $x > (k_0 \vee (y_\varepsilon - \delta)/\delta \vee \delta/(1-\delta))$ . Then, again by (C3), we have

$$\begin{aligned} &\int_{\{y+x<0\}} \left( 1 - \left(\frac{1-x-y}{1+x}\right)^{-\beta} \right) f_x(y) dy \\ &> c(x)(1+\varepsilon) \int_x^{2x} \left( 1 - \left(\frac{1-x+y}{1+x}\right)^{-\beta} \right) \frac{1}{|y|^{\alpha(x)+1}} dy \\ &\quad + c(x)(1-\varepsilon) \int_{2x}^{\infty} \left( 1 - \left(\frac{1-x+y}{1+x}\right)^{-\beta} \right) \frac{1}{|y|^{\alpha(x)+1}} dy \\ &= c(x)(1+\varepsilon) \int_x^{\infty} \left( 1 - \left(\frac{1-x+y}{1+x}\right)^{-\beta} \right) \frac{1}{|y|^{\alpha(x)+1}} dy \\ &\quad - 2\varepsilon c(x) \int_{2x}^{\infty} \left( 1 - \left(\frac{1-x+y}{1+x}\right)^{-\beta} \right) \frac{1}{|y|^{\alpha(x)+1}} dy. \end{aligned}$$

Note that in the first inequality we make a change of variables  $y \mapsto -y$ . Let us put

$$\begin{aligned} U_5^\varepsilon(x) &:= c(x)(1+\varepsilon) \int_x^{\infty} \left( 1 - \left(\frac{1-x+y}{1+x}\right)^{-\beta} \right) \frac{1}{|y|^{\alpha(x)+1}} dy \\ &\quad - 2\varepsilon c(x) \int_{2x}^{\infty} \left( 1 - \left(\frac{1-x+y}{1+x}\right)^{-\beta} \right) \frac{1}{|y|^{\alpha(x)+1}} dy. \end{aligned}$$

We have

$$\int_{\{y+x<0\}} \left( 1 - \left(\frac{1-x-y}{1+x}\right)^{-\beta} \right) f_x(y) dy > U_5^\varepsilon(x). \quad (3.19)$$

**Step 4.** In the fourth step we prove

$$\liminf_{x \rightarrow \infty} \frac{(1+x)^{\alpha(x)+\beta}}{c(x)} \left( \int_{\mathbb{R}} p(x, dy) V(y) - V(x) \right) > 0.$$

By combining (3.16), (3.17), (3.18) and (3.19) we have

$$(1+x)^\beta \left( \int_{\mathbb{R}} p(x, dy) V(y) - V(x) \right) > U_1^{\delta, \varepsilon}(x) + U_2^{\delta, \varepsilon}(x) + U_3^\delta(x) + U_4^\varepsilon(x) + U_5^\varepsilon(x). \quad (3.20)$$

In the rest of the fourth step we prove

$$\begin{aligned} & \liminf_{x \rightarrow \infty} \frac{(1+x)^{\alpha(x)+\beta}}{c(x)} \left( \int_{\mathbb{R}} p(x, dy) V(y) - V(x) \right) \\ & > \liminf_{\delta \rightarrow 0} \liminf_{\varepsilon \rightarrow 0} \liminf_{x \rightarrow \infty} \frac{(1+x)^{\alpha(x)}}{c(x)} U_1^{\delta, \varepsilon}(x) + \liminf_{\delta \rightarrow 0} \liminf_{\varepsilon \rightarrow 0} \liminf_{x \rightarrow \infty} \frac{(1+x)^{\alpha(x)}}{c(x)} U_2^{\delta, \varepsilon}(x) \\ & \quad + \liminf_{\varepsilon \rightarrow 0} \liminf_{x \rightarrow \infty} \frac{(1+x)^{\alpha(x)}}{c(x)} (U_4^\varepsilon(x) + U_5^\varepsilon(x)) + T(\alpha, \beta) \geq 0. \end{aligned}$$

Recall that  $\limsup_{|x| \rightarrow \infty} \alpha(x) =: \alpha < 2$ ,

$$T(\alpha, \beta) = \sum_{n=1}^{\infty} \binom{-\beta}{2n} \frac{2}{2n-\alpha} - \frac{2}{\alpha} + \frac{{}_2F_1(\beta, \alpha + \beta, 1 + \alpha + \beta; 1) + {}_2F_1(\beta, \alpha + \beta, 1 + \alpha + \beta; -1)}{\alpha + \beta}$$

and

$$\liminf_{\delta \rightarrow 0} \liminf_{x \rightarrow \infty} \frac{(1+x)^{\alpha(x)}}{c(x)} U_3^\delta(x) > T(\alpha, \beta)$$

(assumption (1.4)). Clearly,

$$\liminf_{\delta \rightarrow 0} \liminf_{\varepsilon \rightarrow 0} \liminf_{x \rightarrow \infty} \frac{(1+x)^{\alpha(x)}}{c(x)} U_1^{\delta, \varepsilon}(x) = 0. \quad (3.21)$$

Further, by the dominated convergence theorem, we have

$$\begin{aligned} & \liminf_{\delta \rightarrow 0} \liminf_{\varepsilon \rightarrow 0} \liminf_{x \rightarrow \infty} \frac{(1+x)^{\alpha(x)}}{c(x)} U_2^{\delta, \varepsilon}(x) \\ & = \liminf_{\delta \rightarrow 0} \liminf_{\varepsilon \rightarrow 0} \liminf_{x \rightarrow \infty} \left[ - (1+\varepsilon) \sum_{i=2}^{\infty} \binom{-\beta}{i} \frac{(-1)^i}{i-\alpha(x)} \left( \left( \frac{x}{1+x} \right)^{i-\alpha(x)} - \delta^{i-\alpha(x)} \right) \right. \\ & \quad \left. - \sum_{i=2}^{\infty} \left( \binom{-\beta}{i} \frac{1 + (-1)^i \varepsilon \delta^{\alpha(x)} - \delta^i}{i-\alpha(x) \delta^{\alpha(x)}} \right) \right] \\ & = \liminf_{\delta \rightarrow 0} \liminf_{\varepsilon \rightarrow 0} \liminf_{x \rightarrow \infty} \left[ - \sum_{i=2}^{\infty} \binom{-\beta}{i} \frac{(-1)^i \left( \frac{x}{1+x} \right)^{i-\alpha(x)} - (-1)^i \delta^{i-\alpha(x)} + 1 - \delta^{i-\alpha(x)}}{i-\alpha(x)} \right. \\ & \quad \left. - \varepsilon \sum_{i=2}^{\infty} \binom{-\beta}{i} \frac{(-1)^i \left( \frac{x}{1+x} \right)^{i-\alpha(x)} + (-1)^i - 2(-1)^i \delta^{i-\alpha(x)}}{i-\alpha(x)} \right] \end{aligned}$$

$$\begin{aligned}
&= \liminf_{\delta \rightarrow 0} \liminf_{x \rightarrow \infty} - \sum_{i=2}^{\infty} \binom{-\beta}{i} \frac{(-1)^i \left(\frac{x}{1+x}\right)^{i-\alpha(x)} - (-1)^i \delta^{i-\alpha(x)} + 1 - \delta^{i-\alpha(x)}}{i - \alpha(x)} \\
&= \liminf_{x \rightarrow \infty} - \sum_{i=2}^{\infty} \binom{-\beta}{i} \frac{(-1)^i \left(\frac{x}{1+x}\right)^{i-\alpha(x)} + 1}{i - \alpha(x)} \\
&= - \sum_{i=1}^{\infty} \binom{-\beta}{2i} \frac{2}{2i - \alpha}. \tag{3.22}
\end{aligned}$$

Now, let us compute

$$\liminf_{\varepsilon \rightarrow 0} \liminf_{x \rightarrow \infty} \frac{(1+x)^{\alpha(x)}}{c(x)} (U_4^\varepsilon(x) + U_5^\varepsilon(x)).$$

By (3.6), we have

$$\begin{aligned}
&\liminf_{\varepsilon \rightarrow 0} \liminf_{x \rightarrow \infty} \frac{(1+x)^{\alpha(x)}}{c(x)} (U_4^\varepsilon(x) + U_5^\varepsilon(x)) \\
&= \liminf_{\varepsilon \rightarrow 0} \liminf_{x \rightarrow \infty} \left[ (1-\varepsilon) \left( \frac{1}{\alpha(x)} - \frac{{}_2F_1(\beta, \alpha(x) + \beta, 1 + \alpha(x) + \beta; -1)}{\alpha(x) + \beta} \right) \right. \\
&\quad + (1+\varepsilon) \left( \frac{(x+1)^{\alpha(x)}}{\alpha(x)x^{\alpha(x)}} - \frac{(x+1)^{\alpha(x)+\beta} {}_2F_1(\beta, \alpha(x) + \beta, 1 + \alpha(x) + \beta; \frac{x-1}{x})}{x^{\alpha(x)+\beta}(\alpha(x) + \beta)} \right) \\
&\quad \left. - 2\varepsilon \left( \frac{(x+1)^{\alpha(x)}}{\alpha(x)(2x)^{\alpha(x)}} - \frac{(x+1)^{\alpha(x)+\beta} {}_2F_1(\beta, \alpha(x) + \beta, 1 + \alpha(x) + \beta; \frac{x-1}{2x})}{x^{\alpha(x)+\beta}(\alpha(x) + \beta)2^{\alpha(x)+\beta}} \right) \right] \\
&= \liminf_{x \rightarrow \infty} \left( \frac{2}{\alpha(x)} - \frac{{}_2F_1(\beta, \alpha(x) + \beta, 1 + \alpha(x) + \beta; -1) + {}_2F_1(\beta, \alpha(x) + \beta, 1 + \alpha(x) + \beta; 1)}{\alpha(x) + \beta} \right) \\
&= \frac{2}{\alpha} - \frac{{}_2F_1(\beta, \alpha + \beta, 1 + \alpha + \beta; -1) + {}_2F_1(\beta, \alpha + \beta, 1 + \alpha + \beta; 1)}{\alpha + \beta}, \tag{3.23}
\end{aligned}$$

where in the second equality we use (3.5) and the fact that all terms are bounded, and in the last equality we use the fact that the function

$$x \mapsto \frac{2}{x} - \frac{{}_2F_1(\beta, x + \beta, 1 + x + \beta; -1) + {}_2F_1(\beta, x + \beta, 1 + x + \beta; 1)}{x + \beta}$$

is decreasing on  $(0, 2)$ . Now, by combining (1.4), (3.20), (3.21), (3.22) and (3.23) we have

$$\liminf_{x \rightarrow \infty} \frac{(1+x)^{\alpha(x)+\beta}}{c(x)} \left( \int_{\mathbb{R}} p(x, dy) V(y) - V(x) \right) > 0.$$

The case when  $x < 0$  is treated in the same way. Therefore, we have proved the desired result.  $\square$

*Proof of Theorem 1.5.* In order to prove the theorem, according to [MT93, Theorems 8.4.3 and 13.0.1], Theorem 1.3 and [San13b, Proposition 2.6], it is enough to prove that there exists a Borel measurable function  $V : \mathbb{R} \rightarrow [0, \infty)$  such that corresponding level sets  $C_V(r) := \{y : V(y) \leq r\}$  are bounded for every  $r > 0$  and there exist  $r_0 > 0$  and  $d > 0$ , such that

$$\int_{\mathbb{R}} p(x, dy) V(y) - V(x) \leq -d$$

for all  $x \in C_V^c(r_0)$  and

$$\sup \left\{ \left| \int_{\mathbb{R}} p(x, dy) V(y) - V(x) \right| : x \in C_V(r_0) \right\} < \infty.$$

By assumption,  $0 < \inf\{\alpha(x) : x \in \mathbb{R}\}$ . Let  $\beta \in (0, 1] \cap (0, \inf\{\alpha(x) : x \in \mathbb{R}\})$  be arbitrary and for the test function let us again take

$$V(x) := \log(1 + |x|) \quad \text{and} \quad V(x) := |x|^\beta.$$

Since  $\beta < \inf\{\alpha(x) : x \in \mathbb{R}\}$ , it easy to see that

$$\sup \left\{ \left| \int_{\mathbb{R}} p(x, dy) V(y) - V(x) \right| : x \in C \right\} < \infty$$

for every bounded set  $C \subseteq \mathbb{R}$ . Hence, since  $C_V(r) \uparrow \mathbb{R}$  for  $r \rightarrow \infty$ , it is enough to show that

$$\limsup_{|x| \rightarrow \infty} \frac{|x|^{\alpha(x)}}{c(x)} \left( \int_{\mathbb{R}} p(x, dy) V(y) - V(x) + d \right) < 0$$

and

$$\limsup_{|x| \rightarrow \infty} \frac{|x|^{\alpha(x)-\beta}}{c(x)} \left( \int_{\mathbb{R}} p(x, dy) V(y) - V(x) + d \right) < 0,$$

for some  $d > 0$ , respectively. Now, by performing completely the same computations as in Theorem 1.3 and using (1.5) and (1.6), the desired result follows.  $\square$

*Proof of Corollary 1.6.* In the case when  $\alpha \neq 2$ , the claim easily follows from Theorems 1.3 and 1.4 and conditions (1.7) and (1.8), while, in the case when  $\alpha = 2$ , the claim follows from [MT93, Proposition 8.5.4].  $\square$

*Proof of Theorem 2.3.* The claim of Theorem 2.3 trivially follows from [MT93, Theorem 14.0.1], (2.1), (2.2) and Theorem 1.5 by replacing the constant 1 by an arbitrary Borel measurable function  $g(x) \geq 1$  which satisfies (2.1) or (2.2).  $\square$

## 4 Some remarks on the main results

We start this section with the argumentation of various versions of conditions (1.2), (1.3), (1.4), (1.5) and (1.6) given in Theorems 1.3, 1.4 and 1.5. First, we show that under

$$\limsup_{|x| \rightarrow \infty} \alpha(x) < 2 \quad \text{and} \quad \lim_{|x| \rightarrow \infty} c(x) |x|^{2-\alpha(x)} = \infty$$

conditions (1.2) and (1.3) are equivalent to (1.7). Indeed, from the elementary inequalities

$$t - ct^2 \leq \log(1 + t) \leq t \quad \text{and} \quad \beta t - dt^2 \leq (1 + t)^\beta - 1 \leq \beta t,$$

for  $|t|$  small enough, where  $c > 1/2$  and  $d > \beta(1 - \beta)/2$  are arbitrary, it follows

$$\begin{aligned} & \operatorname{sgn}(x) \frac{|x|^{\alpha(x)}}{c(x)(1 + |x|)} \int_{-\delta|x|}^{\delta|x|} y f_x(y) dy - c \frac{|x|^{\alpha(x)}}{c(x)(1 + |x|)^2} \int_{-\delta|x|}^{\delta|x|} y^2 f_x(y) dy \\ & \leq \frac{|x|^{\alpha(x)}}{c(x)} \int_{-\delta|x|}^{\delta|x|} \log \left( 1 + \operatorname{sgn}(x) \frac{y}{1 + |x|} \right) f_x(y) dy \\ & \leq \operatorname{sgn}(x) \frac{|x|^{\alpha(x)}}{c(x)(1 + |x|)} \int_{-\delta|x|}^{\delta|x|} y f_x(y) dy \end{aligned}$$

and

$$\begin{aligned}
& \beta \operatorname{sgn}(x) \frac{|x|^{\alpha(x)-1}}{c(x)} \int_{-\delta|x|}^{\delta|x|} y f_x(y) dy - d \frac{|x|^{\alpha(x)-2}}{c(x)} \int_{-\delta|x|}^{\delta|x|} y^2 f_x(y) dy \\
& \leq \frac{|x|^{\alpha(x)}}{c(x)} \int_{-\delta|x|}^{\delta|x|} \left( \left( 1 + \operatorname{sgn}(x) \frac{y}{|x|} \right)^\beta - 1 \right) f_x(y) dy \\
& \leq \beta \operatorname{sgn}(x) \frac{|x|^{\alpha(x)-1}}{c(x)} \int_{-\delta|x|}^{\delta|x|} y f_x(y) dy,
\end{aligned}$$

for  $|x|$  large enough. Further, let  $\varepsilon > 0$  be arbitrary. Then, by (C3), there exists  $y_\varepsilon > 0$  such that

$$\begin{aligned}
& \frac{|x|^{\alpha(x)-2}}{c(x)} \int_{-\delta|x|}^{\delta|x|} y^2 f_x(y) dy \\
& \leq \frac{|x|^{\alpha(x)-2}}{c(x)} \int_{-y_\varepsilon}^{y_\varepsilon} y^2 f_x(y) dy + 2(1 + \varepsilon) \left( \frac{\delta^{2-\alpha(x)}}{2 - \alpha(x)} - \frac{|x|^{\alpha(x)-2}}{2 - \alpha(x)} y_\varepsilon^{2-\alpha(x)} \right),
\end{aligned}$$

for  $|x|$  large enough. Now, by taking  $\limsup_{\delta \rightarrow 0} \limsup_{\varepsilon \rightarrow 0} \limsup_{|x| \rightarrow \infty}$ , it follows that

$$\limsup_{|x| \rightarrow \infty} \frac{|x|^{\alpha(x)-2}}{c(x)} \int_{-\delta|x|}^{\delta|x|} y^2 f_x(y) dy = 0.$$

Hence, conditions (1.2) and (1.3) are equivalent to

$$\limsup_{\delta \rightarrow 0} \limsup_{|x| \rightarrow \infty} \operatorname{sgn}(x) \frac{|x|^{\alpha(x)-1}}{c(x)} \int_{-\delta|x|}^{\delta|x|} y f_x(y) dy < R_1(\alpha) \quad (4.1)$$

and

$$\limsup_{\delta \rightarrow 0} \limsup_{|x| \rightarrow \infty} \operatorname{sgn}(x) \frac{|x|^{\alpha(x)-1}}{c(x)} \int_{-\delta|x|}^{\delta|x|} y f_x(y) dy < \frac{R_2(\alpha, \beta)}{\beta}, \quad (4.2)$$

respectively. Further, it can be proved that the function  $\beta \mapsto R_2(\alpha, \beta)/\beta$  is strictly decreasing. Thus, in (4.2), we choose  $\beta$  close to zero. From (3.1), (3.2) and (3.4), we have

$$\lim_{\beta \rightarrow 0} \frac{R_2(\alpha, \beta)}{\beta} = R_1(\alpha),$$

which proves the desired result. Next, the assumption  $\liminf_{|x| \rightarrow \infty} \alpha(x) > 1$  implies that the transition densities  $f_x(y)$  have finite first moment for all  $|x|$  large enough. Therefore, in order to prove that (1.7) is equivalent to the following drift condition

$$\limsup_{|x| \rightarrow \infty} \operatorname{sgn}(x) \frac{|x|^{\alpha(x)-1}}{c(x)} \mathbb{E}[X_1 - X_0 | X_0 = x] < R_1(\alpha) \quad (4.3)$$

it suffices to prove that

$$\limsup_{\delta \rightarrow 0} \limsup_{|x| \rightarrow \infty} \operatorname{sgn}(x) \frac{|x|^{\alpha(x)-1}}{c(x)} \left( \int_{-\infty}^{-\delta|x|} y f_x(y) dy + \int_{\delta|x|}^{\infty} y f_x(y) dy \right) = 0.$$

But this fact can again be easily verified by using (C3).

Further, by completely the same arguments as above, it is easy to check that, under

$$\limsup_{|x| \rightarrow \infty} \alpha(x) < 2 \quad \text{and} \quad \lim_{|x| \rightarrow \infty} c(x)|x|^{2-\alpha(x)} = \infty,$$

conditions (1.5) and (1.6) are equivalent to (1.9) and (1.10), respectively. Again, by having finite first moments, that is, under  $\liminf_{|x| \rightarrow \infty} \alpha(x) > 1$ , conditions (1.9) and (1.10) are equivalent to the following drift conditions

$$\limsup_{d \rightarrow 0} \limsup_{|x| \rightarrow \infty} \frac{|x|^{\alpha(x)-1}}{c(x)} (\operatorname{sgn}(x)\mathbb{E}[X_1 - X_0|X_0 = x] + d|x|) < R_1(\alpha) \quad (4.4)$$

and

$$\limsup_{\beta \rightarrow 0} \limsup_{d \rightarrow 0} \limsup_{|x| \rightarrow \infty} \frac{|x|^{\alpha(x)-1}}{c(x)} \left( \operatorname{sgn}(x)\mathbb{E}[X_1 - X_0|X_0 = x] + \frac{d|x|^{-\beta+1}}{\beta} \right) < R_1(\alpha), \quad (4.5)$$

respectively.

Finally, it remains to justify various versions of condition (1.4). Under the assumption

$$\lim_{|x| \rightarrow \infty} c(x)|x|^{2-\alpha(x)} = \infty$$

(recall that  $\limsup_{|x| \rightarrow \infty} \alpha(x) < 2$  is assumed in Theorem 1.4) and from the elementary inequality

$$\beta t - ct^2 \leq 1 - (1+t)^{-\beta} \leq \beta t,$$

for  $|t|$  small enough, where  $c > \beta(\beta+1)/2$  is arbitrary, by completely the same arguments as above, it follows that (1.4) is equivalent to

$$\liminf_{\delta \rightarrow 0} \liminf_{|x| \rightarrow \infty} \operatorname{sgn}(x) \frac{|x|^{\alpha(x)-1}}{c(x)} \int_{-\delta|x|}^{\delta|x|} y f_x(y) dy > \frac{T(\alpha, \beta)}{\beta}. \quad (4.6)$$

Further, it can be proved that the function  $\beta \mapsto T(\alpha, \beta)/\beta$  is strictly increasing. According to this, we choose  $\beta$  close to zero. Again, from (3.1), (3.2) and (3.4), we have

$$\lim_{\beta \rightarrow 0} \frac{T(\alpha, \beta)}{\beta} = R_1(\alpha),$$

hence (1.4) is equivalent to (1.8). Again, by assuming  $1 < \liminf_{|x| \rightarrow \infty} \alpha(x)$ , (1.4) is equivalent to the following drift condition

$$\liminf_{|x| \rightarrow \infty} \operatorname{sgn}(x) \frac{|x|^{\alpha(x)-1}}{c(x)} \mathbb{E}[X_1 - X_0|X_0 = x] > R_1(\alpha). \quad (4.7)$$

At the end, let us assume that  $\limsup_{|x| \rightarrow \infty} \alpha(x) < 1$ . Then, it is not hard to see that Theorem 1.4 holds true under condition

$$\liminf_{\delta \rightarrow 0} \liminf_{|x| \rightarrow \infty} \frac{\alpha(x)|x|^{\alpha(x)}}{c(x)} \int_{-\delta|x|}^{\delta|x|} \left( 1 - \left( 1 + \operatorname{sgn}(x) \frac{y}{1+|x|} \right)^{-\beta} \right) f_x(y) dy > \alpha T(\alpha, \beta). \quad (4.8)$$

Further, condition (4.8) is equivalent to

$$\liminf_{a \rightarrow \infty} \liminf_{|x| \rightarrow \infty} \frac{\alpha(x)|x|^{\alpha(x)}}{c(x)} \int_{-a}^a \left( 1 - \left( 1 + \operatorname{sgn}(x) \frac{y}{1+|x|} \right)^{-\beta} \right) f_x(y) dy > \alpha T(\alpha, \beta). \quad (4.9)$$

Indeed, let  $0 < \varepsilon < 1$  and  $0 < \delta < 1$  be arbitrary. Then, by (C3) and (3.6), there exists  $y_\varepsilon > 0$  such that

$$\begin{aligned} & (1 - \operatorname{sgn}(x)\varepsilon)|x|^{\alpha(x)} \left[ \frac{1}{y_\varepsilon^{\alpha(x)}} - \frac{1}{\delta^{\alpha(x)}|x|^{\alpha(x)}} + \frac{{}_2F_1\left(-\alpha(x), \beta, 1 - \alpha(x), -\operatorname{sgn}(x) \frac{\delta|x|}{1+|x|}\right)}{\delta^{\alpha(x)}|x|^{\alpha(x)}} \right. \\ & \quad \left. - \frac{{}_2F_1\left(-\alpha(x), \beta, 1 - \alpha(x), -\operatorname{sgn}(x) \frac{y_\varepsilon}{1+|x|}\right)}{y_\varepsilon^{\alpha(x)}} \right] + (1 + \operatorname{sgn}(x)\varepsilon)|x|^{\alpha(x)} \left[ \frac{1}{y_\varepsilon^{\alpha(x)}} - \frac{1}{\delta^{\alpha(x)}|x|^{\alpha(x)}} \right. \\ & \quad \left. + \frac{{}_2F_1\left(-\alpha(x), \beta, 1 - \alpha(x), \operatorname{sgn}(x) \frac{\delta|x|}{1+|x|}\right)}{\delta^{\alpha(x)}|x|^{\alpha(x)}} - \frac{{}_2F_1\left(-\alpha(x), \beta, 1 - \alpha(x), \operatorname{sgn}(x) \frac{y_\varepsilon}{1+|x|}\right)}{y_\varepsilon^{\alpha(x)}} \right] \\ & \quad + \frac{|x|^{\alpha(x)}\alpha(x)}{c(x)} \int_{-y_\varepsilon}^{y_\varepsilon} \left( 1 - \left( 1 + \operatorname{sgn}(x) \frac{y}{1+|x|} \right)^{-\beta} \right) f_x(y) dy \\ & \leq \frac{|x|^{\alpha(x)}\alpha(x)}{c(x)} \int_{-\delta|x|}^{\delta|x|} \left( 1 - \left( 1 + \operatorname{sgn}(x) \frac{y}{1+|x|} \right)^{-\beta} \right) f_x(y) dy, \end{aligned}$$

holds for all  $|x|$  large enough. In the similar way we get a similar upper bound for the left-hand side term in (4.8). Now, since  $\limsup_{|x| \rightarrow \infty} \alpha(x) < 1$ , by letting  $\liminf_{\delta \rightarrow 0} \liminf_{\varepsilon \rightarrow 0} \liminf_{y_\varepsilon \rightarrow \infty} \liminf_{|x| \rightarrow \infty}$  and applying (3.5), the desired result follows. Further, from the concavity of the function  $x \mapsto x^\beta$  (recall that  $\beta \in (0, 1)$ ), we have

$$\begin{aligned} & \frac{|x|^{\alpha(x)}\alpha(x)}{c(x)} \int_{-a}^a \left( 1 - \left( 1 + \operatorname{sgn}(x) \frac{y}{1+|x|} \right)^{-\beta} \right) f_x(y) dy \\ & \geq \frac{|x|^{\alpha(x)}\alpha(x)}{c(x)} \frac{(1+|x|-a)^\beta - (1+|x|)^\beta}{(1+|x|-a)^\beta} \int_{-a}^a f_x(y) dy \\ & \geq -\frac{a\beta\alpha(x)|x|^{\alpha(x)}}{c(x)(1+|x|-a)}. \end{aligned}$$

Similarly,

$$\frac{|x|^{\alpha(x)}\alpha(x)}{c(x)} \int_{-a}^a \left( 1 - \left( 1 + \operatorname{sgn}(x) \frac{y}{1+|x|} \right)^{-\beta} \right) f_x(y) dy \leq \frac{a\beta\alpha(x)|x|^{\alpha(x)}(1+|x|)^{\beta-1}}{c(x)(1+a+|x|)^\beta}.$$

Thus, by taking  $0 < \beta < 1 - \alpha = 1 - \limsup_{|x| \rightarrow \infty} \alpha(x)$  and letting  $\liminf_{a \rightarrow \infty} \liminf_{|x| \rightarrow \infty}$ , we get that condition (4.8) follows from

$$\lim_{|x| \rightarrow \infty} \frac{\alpha(x)|x|^{\alpha(x)-1}}{c(x)} = 0 \quad (4.10)$$

(note that in this case  $T(\alpha, \beta) < 0$ ). Essentially, condition (4.10) says that the scaling function  $c(x)$  cannot decrease too fast (recall that  $\sup\{c(x) : x \in \mathbb{R}\} < \infty$  (see [San13b])). Otherwise, we could enter in the recurrence regime.

At the end, note that all conclusions, methods and proofs given in this paper can also be carried out in the discrete state space  $\mathbb{Z}$ . Note that in this case conditions (C1)-(C5) are reduced just to conditions (C2) and (C3), since compact sets are replaced by finite sets. Therefore, we deal with a Markov chain  $\{X_n^d\}_{n \geq 0}$  on  $\mathbb{Z}$  given by the transition kernel

$$p_{ij} := f_i(j - i),$$

for  $i, j \in \mathbb{Z}$ , where  $\{f_i : i \in \mathbb{Z}\}$  is a family of probability functions which satisfies the following conditions:

**(CD1)**  $f_i(j) \sim c(i)|j|^{-\alpha(i)-1}$ , for  $|j| \rightarrow \infty$ , for every  $i \in \mathbb{Z}$ ;

**(CD2)** there exists  $k_0 \in \mathbb{N}$  such that

$$\lim_{|j| \rightarrow \infty} \sup_{i \in \{-k_0, \dots, k_0\}^c} \left| f_i(j) \frac{|j|^{\alpha(i)+1}}{c(i)} - 1 \right| = 0.$$

Functions  $\alpha : \mathbb{Z} \rightarrow (0, 2)$  and  $c : \mathbb{Z} \rightarrow (0, \infty)$  are arbitrary given functions. The proofs and assumptions of Theorems 1.3, 1.4, 1.5, 2.1 and 2.3 in the discrete case remain the same as in the continuous case because we can switch from sums to integrals due to the tail behavior of transition jumps.

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