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# ERGODIC STATES IN A NON COMMUTATIVE ERGODIC THEORY

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## §1 - Introduction

A number of recent papers [1] [2] [3] [4] [5] [6] [6a] concern themselves with "asymptotically abelian systems" i.e. pairs of $C^*$-algebra $\mathfrak{A}$ and a locally compact group $G$ together with a homomorphism $g \to \alpha_g$ of $G$ into the automorphism group of $\mathfrak{A}$ such that one has an "asymptotic abelian property" : the commutator

$$\alpha_g(A) \cdot B - B \cdot \alpha_g(A)$$

tends to zero for $g \in G$ tending to infinity for all elements $A, B \in \mathfrak{A}$ (there are different ways of stating this condition corresponding to different choices of topologies - more general conditions can also be stated for general non topological - groups). The consideration of such asymptotically abelian systems originates in algebraic field theory [10] [11] where the $C^*$-algebra is that of quasi-local observables (i.e. bounded observables performed within bounded space-time regions together with their norm limits). The group $G$ corresponds to some invariant group of the physical theory (e.g. spatial translations), and the condition of asymptotic abelianness expresses the physical requirement that observables performed far away from each other should in the limit not mutually influence themselves and are therefore quantum-mechanically described by commuting operators . In this context a physically important and mathematically basic problem is the investigation of invariant states over the systems $\{\mathfrak{A}, \alpha\}$ i.e. states $\Phi$ over $\mathfrak{A}$ such that $\Phi(\alpha_g(A)) = \Phi(A)$ for all $A \in \mathfrak{A}$ and $g \in G$ : those are clearly candidates for the mathematical description of physical equilibrium states since the latter are homogeneous (invariant under space translations).

Meanwhile it has been increasingly realized that from a mathematical point of view asymptotically abelian systems lead to a non commutative generalization of standard (commutative) ergodic theory : indeed if we take the $C^*$-algebra $\mathcal{A}$ to be abelian and thus isomorphic to an algebra of continuous functions on some locally compact space $X$ , the (supposedly strongly continuous) homomorphism of $G$ into the automorphism group of $\mathcal{A}$ becomes a homomorphism into homeomorphisms of $X$ , asymptotic abelianness being automaticaly realized. A state over $\mathcal{A}$ invariant under $G$ will then be represented by a bounded Radon measure $\mu$ over $X$ invariant under these homeomorphisms . We thus obtain a special case of the usual setting of ergodic theory - the specialization consisting in that the measure $\mu$ is bounded and the Borel structure of $X$ stems from a locally compact topology (the fact that $\mu$ is invariant rather than quasi-invariant is inessential and would be removed if we considered covariant representations rather than invariant states ) . The latter restrictions are however accompanied by a gain in flexibility in varying the (quasi)invariant measure $\mu$ for a given system $\{\mathcal{A}, \alpha\}$ .

The papers quoted above have revealed that basic theorems of standard ergodic theory still hold in the enlarged non commutative frame . The original results of [1] , [2] and [3] have been generalized in several respects . In [5] ,[6a] and Part I of [6] , results are given for an arbitrary group G . The second part of [6] on the other hand gives further results generalizing those of [2] for the case of a locally compact amenable group i.e. a group possessing invariant means . In this paper we study the more general "$\mathcal{M}$ - abelian systems", a notion defined in terms of the mean described in [14] by Godement and we investigate a non commutative ergodic theory in this enlarged frame . One should however not conclude from our results that amenable groups are irrelevant to the study of asymptotically abelian systems : it is indeed because the present paper is essentially confined to the study of the "vacuum theory" (i.e. representations generated by an invariant vector ) that we can work with Godement's mean , since the latter is only defined on linear combinations of functions of positive type on G .

The paper comprises four sections :Section 2 presents results on Godement's mean $\mathcal{M}$ as well as a mild extension theorem needed in Section 4 . Section 3 presents a generalization of the mean ergodic theorem treating not only invariant vectors, but also vectors transforming under finite dimensional representations of the group G . Section 4 defines Godement's mean for algebraic elements modulated by positive type functions on G and studies general features of ergodic states, emphasis being put on the relationship to Mackey's theory of imprimitivity systems [17] . The paper concludes mentioning a classification of ergodic states and some properties of the $E_{II}$ states generalizing results in [2] .

## §2 - Properties of Godement's mean

In this section we collect known results mainly extracted from [13] and [14]. We consider an arbitrary group G and denote by $\mathcal{B}(G)$ the set of bounded complex functions on G. $\mathcal{B}(G)$ is a $C^*$-algebra under linear combinations of functions, the ordinary function product, the complex conjugation $f \to \bar{f}$ of functions and the sup norm $\| \|_\infty$. If G is locally compact we denote by $\mathcal{C}(G)$ and $\mathcal{C}_0(G)$ the sub $C^*$-algebras of $\mathcal{B}(G)$ consisting of continuous functions on G respectively bounded and vanishing at infinity. We denote by $\mathcal{P}(G)$ the set of continuous positive-type functions on G i.e. functions such that

$$\sum_{i,j=1}^{n} \bar{\alpha}_i \, \alpha_j \, f(g_i^{-1} \, g_j) \geq 0$$

for arbitrary $g_i \in G$ and complex constants $\alpha_i$, $i = 1, 2, \ldots n$. Since $\mathcal{P}(G)$ consists of bounded functions and is closed under complex conjugation the set $\mathcal{U}(G)$ of complex linear combinations of elements of $\mathcal{P}(G)$ is a sub-$*$-algebra of $\mathcal{B}(G)$. Its closure in $\mathcal{B}(G)$ ($C^*$-completion) will be denoted by $\bar{\mathcal{U}}(G)$. It is well known (see, for instance [15] §13.4.5) that $\mathcal{U}(G)$ (resp. $\mathcal{P}(G)$) consists of all coefficients (resp. positive coefficients) of arbitrary unitary representations $g \in G \to U_g$ of G i.e. functions of the type $g \in G \to (\varphi|U_g|\psi)$ (resp. $(\varphi|U_g|\varphi)$) where $\varphi$ and $\psi$ are vectors in the representations Hilbert space of U. The fact that $\mathcal{P}(G)$ is closed under multiplication and complex conjugation stems from the existence of tensor products and conjugates of representations ([15]§13.4.9). Since those operations performed on finite dimensional representations again lead to finite-dimensional representations, the norm-limits of coefficients of the latter, called <u>almost periodic</u>

functions on $G$, form an uniformly closed sub-$*$-algebra $AP(G)$ of $\mathcal{U}(G)$. $AP(G)$ can be identified with the $C^*$-algebra of all continuous complex functions on a compact group $\bar{G}$ whose continuous irreducible representations are one-to-one with the finite-dimensional irreducible representations of $G$. $G$ is homomorphically and densely mapped into $\bar{G}$ ( in general in a many-to-one way) and the above mentioned identification of almost periodic functions on $G$ with continuous functions on $\bar{G}$ is obtained by continuous extension of the former to $\bar{G}$ ( for these facts see for instance [15]§16 ). The mean $\mathcal{M}$ defined in [14] by Godement is roughly speaking an average process on elements $f \in \mathcal{U}(G)$ which filters out the almost periodic functions.

We first introduce a convenient notation for left and right translations of functions on $G$. We define

$$\begin{aligned} \{\delta_s * f\}(g) &= f(s^{-1} g) \\ \{f * \varepsilon_s\}(g) &= f(g s^{-1}) \end{aligned} \qquad , \qquad f \in \mathcal{B}(G) , g \in G ,$$

where the notation is a reminder of the fact that translations are the same as convolutions by Dirac measures. If we denote by $\Gamma$ (resp. $\Gamma'$) the convex hulls of $\delta_s$, $s \in G$ (resp. $\varepsilon_s$, $s \in G$), the sets $\overline{\Gamma * f}$ and $\overline{f * \Gamma'}$, $f \in \mathcal{B}(G)$, will thus denote the uniformly closed convex hulls of left resp. right translates of the function $f$ ( note that for $\mu \in \Gamma$, $\nu \in \Gamma'$ we have $\|\mu * f\|_\infty \leq \|f\|_\infty$ and $\|f * \nu\|_\infty \leq \|f\|_\infty$ ). We are now ready to state the results which we shall need.

(a) Let $\mathcal{F}$ be the set of $f \in \mathcal{B}(G)$ such that the closed convex hulls $\overline{\Gamma * f}$ and $\overline{f * \Gamma'}$ both contain a complex constant. This constant is then unique and the same for both . Denoting it by $\mathcal{M}(f)$ one has the properties

(i) for all $f \in \mathcal{F}$ $|\mathcal{M}(f)| \leq \|f\|_\infty$ . $\mathcal{F}$ is a closed subset of $\mathcal{B}(G)$.

(ii) for all $f \in \mathcal{F}$ , $g \in G$ and complex constants $\alpha$ one has $\bar{f}$ , $\alpha f$ , $\delta_g * f$ , $f * \varepsilon_g \in \mathcal{F}$ and $\mathcal{M}(\bar{f}) = \overline{\mathcal{M}(f)}$ , $\mathcal{M}(\alpha f) = \alpha\mathcal{M}(f)$, $\mathcal{M}(\delta_g * f) = \mathcal{M}(f * \varepsilon_g) = \mathcal{M}(f)$

(iii) $f \in$ , $f \geq 0$ imply $\mathcal{M}(f) \geq 0$ .

Proof : Let $a \in \overline{f * \Gamma} \cap \mathbb{C}$ and $b \in \overline{f * \Gamma'} \cap \mathbb{C}$ ( $\mathbb{C}$ denotes the set of complex numbers). For each $\varepsilon > 0$ there are elements $\mu \in \Gamma$ and $\nu \in \Gamma'$ such that

$$\| \mu * f - a \|_\infty \leq \varepsilon \quad \text{and} \quad f * \quad - b\|_\infty \leq \varepsilon$$

Thus, since $a * \nu = a$ and $\mu * b = b$ ,

$$\| \mu * f * \nu - a \|_\infty \leq \varepsilon \quad \text{and} \quad \| \mu * f * \nu - b \|_\infty \leq \varepsilon ,$$

whence $|a - b| \leq 2\varepsilon$ and thus $a = b$ since $\varepsilon$ is arbitrary . The other properties are obvious .

The existence and properties of $\mathcal{M}$ on $\bar{U}(G)$ will furnish the main technical tool in the three subsequent sections . We merely quote the two following results due to Godement to whom the reader is refered for proofs ( [14] §§23, 24).

(b) One has $\bar{U}(G) \leq \mathcal{F}$ . $\mathcal{M}$ is a two-sided translation invariant, unit norm positive linear form over the $C^*$-algebra $\bar{U}(G) \subseteq \mathcal{B}(G)$ .

The precise way in which almost periodic functions on $G$ are "filtered out" by Godement's mean $\mathcal{M}$ is expressed by :

(c) Let $g \in \bar{\mathcal{U}}(G)$ ; $\varphi$ admits a unique decomposition , $\varphi = \varphi_1 + \varphi_2$ with $\varphi_1$ , $\varphi_2 \in \bar{\mathcal{U}}(G)$ , $\varphi_1 \in AP(G)$ and $\mathcal{M}(|\varphi_2|^2)=0$. If $\varphi \in \mathcal{U}(G)$ and $\varphi(g) = (\psi_1|U_g|\psi_2)$ is a realization of $\varphi$ as a coefficient of the unitary representation $U$ of $G$ on the Hilbert space $\mathcal{H}$ , $\psi_1$ , $\psi_2 \in \mathcal{H}$ , and if $E_1$ is the smallest projection in $\mathcal{H}$ containing all projections on finite-dimensional subspaces invariant under $U$ with $E_2 = I - E_1$ , we have

$$\varphi_1(g) = (\psi_1 \,|U_g E_1|\psi_2)$$

$$\varphi_2(g) = (\psi_1 \,|U_g E_2|\psi_2)$$

Consequently the following are equivalent for $\varphi \in \mathcal{P}(G)$ :
(i) $U^{(\varphi)}$ does not contain any finite dimensional subrepresentation of $G$ other than the trivial one of zero dimension ; (ii) $\mathcal{M}(|\varphi|^2) = 0$ ; (iii) $\mathcal{M}(|\varphi|)$

The equivalence of (ii) and (iii) is a special case of the known fact that, for a positive element $A = |\varphi|$ of a $C^*$-algebra (here $\bar{\mathcal{U}}(G)$ ) and for an arbitrary state $\mathcal{M}$ , $\mathcal{M}(A) = 0$ is equivalent to $\mathcal{M}(A^2) = 0$ . Indeed Schwartz's inequality yields $\mathcal{M}(A)^2 \leq \mathcal{M}(1) \cdot \mathcal{M}(A^2)$ and $\mathcal{M}(A^2)^2 \leq \mathcal{M}(A) \cdot \mathcal{M}(A^3)$.

The above mentioned isomorphism between $AP(G)$ and $\mathcal{C}(\bar{G})$ (the $C^*$-algebr of continuous functions of the compact group $\bar{G}$ ) together with the uniqueness of normalized Haar measure on $\bar{G}$ (a two-sided invariant mormalized state over $\mathcal{C}(\bar{G})$ immediately entail :

(d) For any $f \in AP(G)$ one has

$$\mathcal{M}(f) = \int_{\bar{G}} f(\bar{g})\, dm(\bar{g})$$

where $f$ denotes on the right side the unique continuous extension of $f \in AP(G)$ to $\bar{G}$ and $dm$ is the normalized Haar measure on $\bar{G}$.

The next result serves to establish the connection between the present work and Part II of [6] :

(e) If the group $G$ is amenable each left (or right) invariant mean $\eta$ over reduces to $\mathcal{M}$ on $\mathfrak{F}$.

<u>Proof</u> : Let $f \in \mathfrak{F}$ and choose $\mu \in \Gamma(\nu \in \Gamma')$ such that $\| \mu * f - \mathcal{M}(f)\|_\infty \le \varepsilon$ $(\|f * \nu - \mathcal{M}(f)\|_\infty \le \varepsilon)$. It follows that

$|\eta(\mu * f - \mathcal{M}(f)| = |\eta(f) - \mathcal{M}(f)| \le \varepsilon \quad (|\eta(f * \nu - \mathcal{M}(f))| = |\eta(f) - \mathcal{M}(f)| \le \varepsilon)$

whence $\eta(f) = \mathcal{M}(f)$ since $\varepsilon$ is arbitrary.

## § 3 - A generalized mean ergodic theorem

This section describes Godement's means of group representations modulated by elements of $\mathcal{U}(G)$ . We first give the

Lemma 1 . Let $G$ be a group and $g \in G \to U_g$ an unitary representation of $G$ in a Hilbert space $\mathcal{H}$ . For each $f \in \mathcal{U}(G)$ there is a unique bounded operator $M(fU)$ such that, for all $\psi_1, \psi_2 \in \mathcal{H}$

$$(1) \qquad (\psi_1 | M(fU) | \psi_2) = \mathcal{M}\{ f(\hat{g}) \, (\psi_1 | U_{\hat{g}} | \psi_2)\}$$

(we write $\mathcal{M}(\varphi(\hat{g}))$ for $\mathcal{M}(\varphi)$ , $\hat{g}$ indicating a dummy variable ). This operator $M(fU)$ is of norm not exceeding $\| f \|_\infty$ and lies in the Von Neumann algebra generated by the $U_g$ , $g \in G$ .

Proof : (See [6] Lemma 4 ) The right hand side of (1) makes sense by (b) of Section 1 , and since $\mathcal{M}$ is a normalized state over $\overline{\mathcal{U}}(G)$, it is linear in $\psi_2$ , conjugate-linear in $\psi_1$ and of modulus not exceeding $\|f\|_\infty \cdot \|\psi_1\| \cdot \|\psi_2\|$ ; whence the existence of a unique $M(fU)$ satisfying (1) and of norm $\leq \| f \|_\infty$ . Further we have, for any bounded operator $T$ on $\mathcal{H}$

$$\begin{aligned}(\psi_1 | TM(fU) - M(fU)T | \psi_2) &= (T^*\psi_1 | \mathcal{M}(fU) | \psi_2) - (\psi_1 | \mathcal{M}(fU) | T\psi_2) \\ &= \mathcal{M}\{f(\hat{g})\{(T^*\psi_1 | U_{\hat{g}} | \psi_2) - (\psi_1 | U_{\hat{g}} | T\psi_2)\}\} \\ &= \mathcal{M}\{f(\hat{g}) \, (\psi_1 | TU_{\hat{g}} - U_{\hat{g}}T | \psi_2)\}\end{aligned}$$

Thus $M(fU)$ is contained in the bicommutant of all $U_g$ , $g \in G$ .

The next theorem gives an explicit description of the operators $M(fU)$ .

<u>Theorem 1</u>. Let $G$ , $U$ and $\mathcal{H}$ be as in the preceding Lemma and denote by $E$ the smallest (orthogonal) projector in $\mathcal{H}$ whose range contains all the finite dimensional subspaces of $\mathcal{H}$ invariant under $U$ and by $U^E$ the restriction of $U$ to $E\mathcal{H}$ .

(i) If $f \in \mathcal{U}(G)$ is the coefficient of a unitary representation of $G$ disjoint from the representation $U^E$ , $M(fU) = 0$ .

(ii) Let

$$\text{(2)} \qquad \begin{cases} E\mathcal{H} = \sum\limits_{\sigma \in \Sigma} \mathcal{H}^{(\sigma)} \otimes \mathcal{H}'^{(\sigma)} \\ U^E_g = \sum\limits_{\sigma \in \Sigma}^{\oplus} U^{(\sigma)}_g \otimes I'^{(\sigma)} \end{cases}$$

be the factorial decomposition of $U^E$ , with $\Sigma$ the set of equivalence classes of irreducible representations contained in $U^E$ , $U^{(\sigma)}$ an element of the class $\sigma$ acting in the finite dimensional Hilbert space $\mathcal{H}^{(\sigma)}$ , $\mathcal{H}'^{(\sigma)}$ a Hilbert space of dimensionality equal to the multiplicity of $\sigma$ , and $I'^{(\sigma)}$ the unit operator in $\mathcal{H}'^{(\sigma)}$ . If $\varphi^{(\sigma)}$ and $\psi^{(\sigma)}$ denote arbitrary vectors in $\mathcal{H}^{(\sigma)}$ we have

$$\text{(3)} \qquad M\{(\varphi^{(\sigma)}|U^{(\sigma)}_{\hat{g}}|\psi^{(\sigma)})U_{\hat{g}}\} = d^{-1}_{(\sigma)}\{|\varphi^{(\sigma)})(\psi^{(\sigma)}| \otimes I'^{(\sigma)}\}E^{(\sigma)}$$

where $E^{(\sigma)}$ is the projector in $\mathcal{H}$ on the subspace $\mathcal{H}^{(\sigma)} \otimes \mathcal{H}'^{(\sigma)}$ and $d_{(\sigma)}$ is the dimension of $\sigma$ . In particular

$$\text{(4)} \qquad M(U) = E_0 ,$$

the projector onto vectors of $\mathcal{H}$ invariant under $U$ .

Proof : Let $U^{(\alpha)}$ and $U^{(\beta)}$ be two finite dimensional irreductible unitary representations of G in the respective Hilbert spaces $\mathcal{H}^{(\alpha)}$ and $\mathcal{H}^{(\beta)}$ and take $\xi, \eta \in \mathcal{H}^{(\alpha)}$ and $x,y \in \mathcal{H}^{(\beta)}$. It follows from (d) in Section 1 and from the orthogonality relations on the compact group $\bar{G}$ (see [15] § 15.2.5; contrary to Dixmier we take scalar products conjugate-linear in the first and linear in the second vector ) :

$$\mathcal{M}\{(\xi|U_g^{(\alpha)}|\eta)(x\, U_g^{(\beta)}|y)\} \begin{cases} 0 & \text{if } U^{(\alpha)} \text{ and } U^{(\beta)} \text{ are disjoint} \\ d_{(\alpha)}^{-1}(x|\xi)(\eta|y) & \text{if } U^{(\alpha)} = U^{(\beta)} \end{cases}$$

with $d_{(\alpha)}$ = dimension of $U^{(\alpha)}$ .

Proof of (i) : Let $f(g) = (u|U'_g|v)$ correspond to a unitary representation U' og G in the Hilbert space $\mathcal{H}'$ disjoint from the representation $U^E$ ; let $F_1$ be the smallest projector in $\mathcal{H}'$ whose range contains all finite dimensional subspaces ; and write $u_1 = F_1 u$ , $v_1 = F_1 v$ , $u_2 = u - u_1$ , $v_2 = v - v_1$ and $f_i(g) = (u_i|U'_g|v_i)$ ;$i = 1,2$ . On the other hand set $E^\perp = I - E$ . We have, with obvious notations

$$M(fU) = M(f_1U^E) + M(f_2U^E) + M(fU^{E^\perp})$$

The first term vanishes because of the above orthogonality relations since the finite dimensional representations $U^E$ and $U'^{F_1}$ are disjoint. The two other terms however vanish by (c) of Section 1 due to the fact that the representation $U^{E^\perp}$ and $U'^{F_2}$ have no non vanishing finite dimensional subrepresentations (we remind that $\mathcal{M}(|h|^2) = 0$ for a $h \in \mathcal{V}(G)$ implies $\mathcal{M}(\varphi h) = 0$ for all $\varphi \in \mathcal{V}(G)$).

Proof of (ii) : in order to establish formula (3) we have to evaluate the quantity

$$(5) \qquad \mathcal{M}\{\overline{(\varphi^{(\sigma)}|U_{\hat{g}}^{(\sigma)}|\psi^{(\sigma)})}\ (u|U_{\hat{g}}|v)\} \qquad , \quad u\ ,\ v \in \mathcal{H}$$

Let us write

$$u = E^{(\sigma)}u + \sum_{\substack{\sigma' \in \Sigma \\ \sigma' \neq \sigma}} E^{(\sigma')}u + E^{\perp}u$$

and analogously for $v$ . We have, correspondingly

$$(u|U_g|v) = (E^{(\sigma)}u|U_g^{(\sigma)}|E^{(\sigma)}v) + \sum_{\sigma' \neq \sigma}(E^{(\sigma')}u|U_g^{(\sigma')}|E^{(\sigma')}v) + (E^{\perp}u|U_g^{E^{\perp}}|E^{\perp}v)$$

where the first term is the only one giving a non vanishing contribution to (5) as shown by (i) applied to the representations $U^{E^{\perp}}$ and $U^{(\sigma')}$ , $\sigma' \neq \sigma$ .

Writing

$$\begin{cases} E^{(\sigma)}u = \sum_{i=1}^{p} x_i \otimes x_i' \\ E^{(\sigma)}v = \sum_{j=1}^{q} y_j \otimes y_j' \end{cases} \quad , x_i\ ,\ y_i \in \mathcal{H}^{(\sigma)}\ ,\ x_i'\ ,\ y_i' \in \mathcal{H}'^{(\sigma)}$$

we then have

$$(E^{(\sigma)}u|U_g^{(\sigma)}|E^{(\sigma)}v) = \sum_{i=1}^{p} \sum_{j=1}^{q} (x_i|U_g^{(\sigma)}|y_j)(x_i'|y_j')$$

and thus using the above orthogonality relations

$$\mathcal{M}\{\overline{(\varphi^{(\sigma)}|U_{\hat{g}}^{(\sigma)}|\psi^{(\sigma)})}(u|U_g|v)\} = \sum_{i=1}^{p} \sum_{j=1}^{q} d_{(\sigma)}^{-1}(x_i|\varphi^{(\sigma)})(\psi^{(\sigma)}|y_j)(x_i'|y_j')$$

$$= \left( \sum_{i=1}^{p} x_i \otimes x_i' \,\middle|\, |\varphi^{(\sigma)})(\psi^{(\sigma)}| \otimes I'^{(\sigma)} \,\middle|\, \sum_{j=1}^{q} y_j \otimes y_j' \right),$$

whence formula (3) .

## §4 - Representations of $\mathcal{M}$-abelian systems generated by an invariant vector

Definition 1 . Let $\{\mathcal{A},\alpha\}$ be a pair of $C^*$-algebra $\mathcal{A}$ and a homomorphic mapping $g \in G \to \alpha_g$ of a group $G$ into the automorphism group of $\mathcal{A}$ . To each state $\Phi$ over $\mathcal{A}$ we attach the following subset of $\mathcal{B}(G)$ :

$$S_\Phi = \{g \in G \to \Phi(A_1[\alpha_g(A),B]A_2) \quad | \quad A, B, A_1, A_2 \in \mathcal{A}\}$$

The system $\{\mathcal{A},\alpha\}$ is called $\mathcal{M}$- abelian whenever for each state $\Phi$ over $\mathcal{A}$ invariant under $G$ (i.e. such that $\Phi(\alpha_g(A)) = \Phi(A)$ for all $A \in \mathcal{A}$ and $g \in G$ ) and each $f \in S_\Phi$ we have $|f| \in \mathfrak{F}$ and

$$\mathcal{M}(|f|) = 0$$

$\{\mathcal{A},\alpha\}$ is called weakly asymptotically abelian (cf. [6] Definition 1 ) whenever

a) $G$ is a locally compact non compact topological group

b) $\alpha$ is strongly continuous (i.e. $\alpha_g(A)$ is norm continuous in $g$ for each $A \in \mathcal{A}$ )

c) $S_\Phi \in \mathcal{C}_0(G)$ for each $G$ - invariant state over $\mathcal{A}$ .

Notation . In addition to the preceding notation, given a state $\Phi$ over $\mathcal{A}$ invariant under $G$ we will denote by $\pi_\Phi$, $U^\Phi$ the respective $*$-representation of $\mathcal{A}$ , unitary representation of $G$ (both acting in the Hilbert space $\mathcal{H}_\Phi$ ) and cyclic vector $\Omega_\Phi$ of $\mathcal{H}_\Phi$ (cyclicity is with respect to $\pi_\Phi(\mathcal{A})$) determined by (cf. [15] , 2.12.11 ) :

$$(6)\quad \begin{cases} (\Omega_\Phi|\pi_\Phi(A)|\Omega_\Phi) = \Phi(A) \\ \pi_\Phi(\alpha_g(A)) = U_g^\Phi\, \pi_\Phi(A)\, U_{g^{-1}}^\Phi \qquad , A \in \mathcal{A} \; , g \in G \\ U_g^\Phi\, \Omega_\Phi = \Omega_\Phi \end{cases}$$

Furthermore we will denote by $E_{\Omega_\Phi}$ the projector in $\mathcal{H}_\Phi$ onto the one - dimensional subspace generated by the vector $\Omega_\Phi$ ; by $E_0^\Phi$ the projector in $\mathcal{H}_\Phi$ with range $\{\psi \in \mathcal{H}_\Phi | U_g^\Phi \psi = \psi$ for all $g \in G\}$; by $E^\Phi$ the smallest projector in $\mathcal{H}_\Phi$ whose range contains all finite dimensional subspaces of $\mathcal{H}_\Phi$ invariant under $U^\Phi$ . Finally the Von Neumann ring generated by the $\pi_\Phi(A)$ , $A \in \mathcal{A}$ and $U_g^\Phi$ , $g \in G$ will be denoted by $\mathcal{R}_\Phi$ and its commutant by $\mathcal{R}_\Phi'$ .

We note that, since $f \in \mathcal{E}_0(G)$ implies that $f \in \mathcal{E}$ and $\mathcal{M}(f) = 0$ , weak asymptotic abelianness implies $\mathcal{M}$-abelianness . The latter, more general, concept is interesting in situations where $\alpha_g(A)$ and $B$ tend to commute only for $g \to \infty$ along certain directions ( possibly depending upon $A$ and $B$ ) .

We now give

<u>Lemma 2</u> . Let $\{\mathcal{A},\alpha\}$ be an $\mathcal{M}$-abelian system and $\Phi$ a state over $\mathcal{A}$ invariant under $G$ . The set of functions

$$F_{f,\psi_1,\psi_2,A} \; : \; g \in G \to f(g)\, (\psi_1|\pi_\Phi(\alpha_g(A))|\psi_2)$$

where $f$ runs through $\breve{U}(G)$, $\psi_1$ and $\psi_2$ through $\mathcal{H}_\Phi$ and $A$ through $\mathcal{A}$ , is contained in $\mathcal{E}$ . Furthermore $\mathcal{M}\{F_{f,\psi_1,\psi_2,A}\}$ is linear in $f$ , $\psi_2$ and $A$ and conjugate-linear in $\psi$ .

<u>Proof</u> : Since $\Omega_\Phi$ is cyclic under $\pi_\Phi(A)$ and since $\mathcal{E}$ is uniformly closed it suffices to prove that the set of functions

$$G_{f,A_1,A_2,A} \quad : \quad g \to f(g) \ \Phi(A_1 \alpha_g(A) A_2) \ ,$$

where $A_1$ , $A_2$ and $A$ run through $\mathfrak{A}$, is contained in $\mathfrak{F}$ and that $\mathcal{M}(G_{f,A_1,A_2,A})$ is linear in $f$ , $A_1$ , $A_2$ and $A$ . For the proof we notice that

$$G_{f,A_1,A_2,A}(g) = f_1(g) + f(g) f_2(g)$$

where

$$\begin{cases} f_1(g) = f(g)(\Omega_\Phi | \pi_\Phi(A_1 A_2) U_g^\Phi \ \pi_\Phi(A) | \Omega_\Phi) \\ f_2(g) = \Phi(A_1 [\alpha_g(A), \ A_2] \ ) \end{cases}$$

with $f_1 \in \mathcal{U}(G)$ and $f_2 \in S_\Phi$ . By Definition 1 , given $\varepsilon > 0$ , we can choose $\mu \in \Gamma$ such that

$$\| \mu * |f_2| \ \|_\infty \leq \varepsilon \ .$$

We then choose a $\mu' \in \Gamma$ such that

$$\| \mu' * \mu * f_1 - \mathcal{M}(f_1) \|_\infty \leq \varepsilon \ .$$

It follows

$$\| \mu' * \mu * G_{f,A_1,A_2,A} - \mathcal{M}(f_1) \|_\infty$$

$$\leq \|\mu' * \mu * f_1 - \mathcal{M}(f_1)\|_\infty + \|\mu' * \mu * (ff_2)\|_\infty \leq \varepsilon \ (1 + \| f \|_\infty)$$

since the second term on the right hand side is majorized by $\| \mu * (ff_2)\|_\infty \leq \|f\|_\infty \ \|\mu * |f_2| \ \|_\infty$ .(*) A similar argument for right

(*) Note that we have proven that $|f| \in \mathfrak{F}, \mathcal{M}(|f|) = 0$ and $f_2 \in \mathcal{B}(G)$ imply that $ff_2 \in \mathfrak{F}$ and $\mathcal{M}(ff_2) = 0$ ; whence easily follows, incidentally, that $\mathcal{M}(|f|^2) = 0 \iff \mathcal{M}(|f|) = 0$ .

translations shows that $G_{f,A_1,A_2,A} \in \mathcal{E}$ with

$$(7) \qquad \mathcal{M}(G_{f,A_1,A_2,A}) = \mathcal{M}\{f(\hat{g})(\Omega_\Phi|\pi_\Phi(A_1A_2)U^\Phi_{\hat{g}}\pi_\Phi(A)|\Omega_\Phi\} ,$$

whence the result .

<u>Lemma 3</u> . Let $\{\mathfrak{A},\alpha\}$ be an $\mathcal{M}$-abelian system and $\Phi$ a state over $\mathfrak{A}$ invariant under $G$ with the notation of Definition 1 . To each $f \in \overline{\mathcal{U}}(G)$ there exists a linear norm-continuous mapping $M_f$ from $\mathfrak{A}$ to $\pi(\mathfrak{A})'' \cap \pi(\mathfrak{A})'$ determined by

$$(8) \qquad (\psi_1|M_f(A)|\psi_2) = \mathcal{M}\{f(\hat{g})(\psi_1|\pi(\alpha_{\hat{g}}(A))|\psi_2)\} , \qquad \psi_1, \psi_2 \in \mathcal{H}, \; A \in \mathfrak{A}$$

The correspondance $f \in \overline{\mathcal{U}}(G) \to M_f$ is linear, positive ($M_f(A) \geq 0$ if $f \geq 0$ and $A \geq 0$ ) , bounded ($\|M_f(A)\| \leq \|f\|_\infty \cdot \|A\|$ ) and such that $M_f = M_{f_1}$ if $f = f_1 + f_2$ is the decomposition of §2 , (c) ;

$$(9) \qquad M_{\bar{f}}(A) = M_f(A^*)^* \qquad , A \in \mathfrak{A}$$

and

$$(10) \qquad \begin{cases} M_f \circ \alpha_s = M_f^s \\ U_s M_f(A) U_s^{-1} = M_{f_s}(A) \end{cases} \qquad s \in G , A \in \mathfrak{A}$$

where $f_s(g) = f(s^{-1}g)$ and $f^s(g) = f(g\,s^{-1})$ , $g$ , $s \in G$ . In particular if we take for $f$ the constant function equal to $1$ , we get a norm decreasing positive linear map $M_1$ from $\mathfrak{A}$ to $\pi(\mathfrak{A})'' \cap \pi(\mathfrak{A})' \cap U_G'$ ( we denote by $U_G'$ the commutator of the set $\{U_g | g \in G\}$ ). The $M_f(A)$ , $A \in \mathfrak{A}$ , $f \in \overline{\mathcal{U}}(G)$ , furthermore satisfy for each $B \in \pi(\mathfrak{A})''$ the relation

$$(11) \qquad M_f(A)\, B\, E_o = B\, M(f\,U)\pi(A)\, E_o$$

through which they are determined owing to cyclicity of $\Omega$ for $\pi(\mathcal{A})$. In particular

$$(12) \qquad M_1(A)\, E_o = E_o\, \pi(A)\, E_o$$

so that $M_1$ coincides with the restriction to $\mathcal{A}$ of the mapping M of [6], Theorem 1 (see formula (7) above).

Proof : (along the lines of [6] Lemma 1). **The right hand side of [8] exists** and is linear in $\psi_2$, f and A and conjugate-linear in $\psi_2$ by virtue of Lemma 2. Furthermore its module does not exceed $\|f\|_\infty \cdot \|A\| \cdot \|\psi_1\| \cdot \|\psi_2\|$, therefore we have a unique bounded operator $M_f(A)$ satisfying (8) ; and $M_f(A)$ depends linearly upon f and A and has a norm not exceeding $\|f\|_\infty \cdot \|A\|$. As in the proof of Lemma 1 in Section 3 we show that, for each bounded operator T on $\mathcal{H}$ and all $\psi_1, \psi_2 \in \mathcal{H}$

$$(\psi_1 | T M_f(A) - M_f(A) T | \psi_2) = \mathcal{M}\{f(\hat{g})(\psi_1 | T\pi(\alpha_{\hat{g}}(A)) - \pi(\alpha_{\hat{g}}(A)) T | \psi_2)\ .$$

If T commutes with $\pi(\mathcal{A})$ this is also the case for $M_f(A)$, therefore $M_f(A) \in \pi(\mathcal{A})''$. On the other hand for $T = \pi(B)$, $B \in \mathcal{A}$, the right-hand term vanishes according to Definition 1 and thus $M_f(A) \in \pi(\mathcal{A})'$. Finally the properties (9) and (10) easily follow from the real character and translation invariance of $\mathcal{M}$ : we have, for $\psi_1, \psi_2 \in \mathcal{H}$

$$(\psi_1 | M_{\overline{f}}(A) | \psi_2) = \mathcal{M}\{\overline{f}(g)(\psi_1 | \pi(\alpha_{\hat{g}}(A)) | \psi_2)\} = \overline{\mathcal{M}\{f(\overline{g})(\psi_2 | \pi(\alpha_{\hat{g}}(A^*)) | \psi_1)\}}$$

$$= \overline{(\psi_2 | M_f(A^*) | \psi_1)} = (\psi_1 | M_f(A^*)^* | \psi_2)$$

and

$$(\psi_1 | U_s M_f(A) U_s^{-1} | \psi_2) = (U_s^* \psi_1 | M_f(A) | U_s^{-1} \psi_2) = \mathcal{M}\{f(\hat{g})(U_s^* \psi_1 | \pi(\alpha_{\hat{g}}(A)) | U_s^{-1} \psi_2)\}$$

$$= \mathcal{M}\{f(\hat{g})(\psi_1 | \pi(\alpha_{s\hat{g}}(A)) | \psi_2)\} = \mathcal{M}\{f(s^{-1}\hat{g})(\psi_1 | \pi(\psi_{\hat{g}}(A)) | \psi_2)\}.$$

As for property (11), it suffices to verify it for $B = I$ since $M_1(A) \in \pi(\mathcal{O})'$. Verification is immediate using Definition (8) and taking account of (4). Property (12) then follows immediately from (4). More generally if $f \in G \to \chi_i(g)$, $i = 1, 2$, are one-dimensional unitary representations of $G$ we have

$$(13) \qquad M_{\bar{\chi}_1}(A)\, E_{\chi_2} = E_{\chi_1 \chi_2}\, \pi(A)\, E_{\chi_2}$$

where $E_{\chi_2}$ is the projector in $\mathcal{H}$ with range $\{\psi \in \mathcal{H} \,|\, U_g \psi = \chi_2(g)\psi \,, g \in G\}$ and analogously for $E_{\chi_1 \chi_2}$.

<u>Remark</u> : We note that, in the case of weak asymptotic abelianness, the Lemma can be proved without using the fact that the representation $\pi_\Phi$ of $\mathcal{O}$ was generated by the invariant state $\Phi$. All we need for the proof is the existence of a covariant representation of the weakly asymptotically abelian system $\{\mathcal{O}, \alpha\}$ (i.e. a pair of $*$ - representation $\pi$ of $\mathcal{O}$ and a continuous unitary representation $U$ of $G$ -both in the Hilbert space $\mathcal{H}$ - satisfying

$$U_g\, \pi(A)\, U_g^{-1} = \pi(\alpha_g(A)) \qquad , A \in \mathcal{O}, \; g \in G \;)$$

such that in addition the subspace of vectors of $\mathcal{H}$ invariant under $U$ is cyclic for $\pi(\mathcal{O})$ in $\mathcal{H}$. If the group $G$ is amenable with right or left invariant mean $\eta$ the construction above with $\eta$ instead of $\mathcal{M}$ can be applied to the universal representation of $\mathcal{O}$ (direct sum of all cyclic representations cf. [6] Lemma 1 ) yielding a linear mapping $A \in \mathcal{O} \to \tilde{\eta}(f_{\hat{g}}\, \alpha_{\hat{g}}(A))$ of norm

$\leq \|f\|_\infty$ from $\mathfrak{A}$ to the center of its Von Neumann enveloping algebra $\mathfrak{A}^{**}$. The above mapping $M_f$ can then be obtained by composition with the representation $\pi$ extended to $\mathfrak{A}^{**}$ in the standard manner ([15], 12.1.5):

(14) $$M_f(A) = \pi(\tilde{\eta}(f_{\hat{g}}\alpha_{\hat{g}}(A)))$$

Theorem 2. Let $\{\mathfrak{A},\alpha\}$ be an $\mathcal{M}$-abelian system and $\Phi$ a state over $\mathfrak{A}$ invariant under $G$ with the notation of Def. 1. We then have that

a) the set of operators $E_0^\Phi \pi_\Phi(A) E_0^\Phi$ is abelian (and consequently $E_0^\Phi \mathcal{R}_\Phi E_0^\Phi$ is a maximal abelian Von Neumann ring).

b) the Von Neumann ring $\mathcal{R}'_\Phi$ is abelian and isomorphic to $E_0^\Phi \mathcal{R}_\Phi E_0^\Phi$.

c) the following conditions are equivalent

(i) $E_{\Omega_\Phi} = E_0$ (uniqueness of the "vacuum")

(ii) for all $A \in \mathfrak{A}$ $\mathcal{M}\{\Phi(A^*\alpha_{\hat{g}}(A)) - |\Phi(A)|^2\} = 0$

(iii) for all $A \in \mathfrak{A}$ and $\psi_1, \psi_2 \in \mathcal{H}_\Phi$

$$\mathcal{M}\{(\psi_1|\pi_\Phi(\alpha_{\hat{g}}(A))|\psi_2) - \Phi(A)(\psi_1|\psi_2)\} = 0$$

(weak clustering property).

(iv) $M_1(A)$ is a multiple of the identity (equal to $\Phi(A).I$) for all $A \in \mathfrak{A}$

(v) the set of operators $\pi_\Phi(\mathfrak{A}) \cup \{U_g^\Phi \mid g \in G\}$ is irreducible (or $\mathcal{R}_\Phi$ consists of all bounded operators on $\mathcal{H}_\Phi$).

(vi) $\mathcal{R}_\Phi$ is a factor

(vii) $\Phi$ is an extremal element of the convex set of states over $\mathfrak{A}$ invariant under $G$.

We do not give the proof of this theorem for which we refer the reader to refs [1] through [6]. We limit ourselves to noticing that a) immediately results from the $\mathcal{M}$-abelianness condition

$$\mathcal{M}\{(\psi_1|\pi_\Phi([\alpha_{\hat{g}}(A)\ ,\ B])|\ \psi_2)\ =\ 0 \qquad ,\ \psi_1,\psi_2 \in \mathcal{H}_\Phi,\ A,B,\ \in \mathcal{Cl}\ ,$$

taking account of expression (4) .

Invariant states $\Phi$ satisfying the equivalent condition (i) through (vii) are called <u>extremal invariant</u> or <u>ergodic states</u> (<u>E - states</u>) .

The abelian character of $E_0^\Phi\ \pi_\Phi(\mathcal{Cl})\ E_0^\Phi$ together with the cyclicity of $\Omega_\Phi$ imply [6] that the commutant $\mathcal{R}'_\Phi$ of the Von Neumann algebra generated by the $\pi_\Phi(A)$ and $U_g^\Phi$ , $A \in \mathcal{Cl}$ , $g \in G$ , is abelian ; and furthermore that the mapping M from $\mathcal{R}_\Phi$ to $\mathcal{R}'_\Phi$ determined by

$$(15) \qquad M(T)\ E_0^\Phi\ =\ E_0^\Phi\ T\ E_0^\Phi \qquad ,\ T \in \mathcal{R}_\Phi$$

is normal and onto (consequently $E_0^\Phi\ \mathcal{R}_\Phi\ E_0^\Phi$ and $\mathcal{R}'_\Phi$ are isomorphic as Von Neumann algebras ) .

<u>Theorem 3</u> . Let $\{\mathcal{Cl},\alpha\}$ be an $\mathcal{M}$-abelian system and $\Phi$ a state over $\mathcal{Cl}$ invariant under G. With again the notation of Definition 1 the following are equivalent

(i) $E_{\Omega_\Phi}\ =\ E^\Phi$

(ii) for all $A \in \mathcal{Cl}$ , $\mathcal{M}\{|\Phi(A^*\alpha_{\hat{g}}(A))\ -\ |\Phi(A)|^2|\} =\ 0$

(iii) for all $A \in \mathcal{Cl}$ and $\psi_1$ , $\psi_2 \in \mathcal{H}_\Phi$ ,

$$\mathcal{M}\{|(\psi_1|\pi_\Phi(\alpha_{\hat{g}}(A))|\psi_2)\ -\ \Phi(A)(\psi_1|\psi_2)|\}\ =\ 0$$

<u>Proof</u> : The implication (iii)$\Rightarrow$(ii) is trivial . Further (ii)$\Rightarrow$(i) as a consequence of (c) of Section 2 taking account of the cyclicity of $\Omega_\Phi$ and

the relation

$$\Phi(A^{*}\alpha_{g}(A)) - |\Phi(A)|^{2} = (\pi_{\Phi}(A)\Omega_{\Phi}|U_{g}^{\Phi}\; E_{\Omega_{\Phi}}^{\perp}\; |\pi_{\Phi}(A)\Omega_{\Phi})$$

where $E_{\Omega_{\Phi}}^{\perp} = I - E_{\Omega_{\Phi}}$ . In order to prove the implication (i)$\Rightarrow$(iii) it suffices to show that , as a result of (i), the function

$$f : g \in G \rightarrow |\Phi(A_{1}\alpha_{g}(A)A_{2}) - \Phi(A)\Phi(A_{1}A_{2})|$$

is contained in $\mathcal{F}$ for all $A_1, A_2, A \in \mathcal{Q}$ and that moreover $\mathcal{M}(f) = 0$ .
We have

$$f(g) \leq |\Phi(A_{1}[\alpha_{g}(A),A_{2}])| + |(\pi_{\Phi}(A_{2}^{*}A_{1}^{*})\Omega_{\Phi}|U_{g}^{\Phi}\, E_{\Omega_{\Phi}}^{\perp}\pi_{\Phi}(A)\Omega_{\Phi})|$$

and (iii) follows from (c) of Section 2 using Lemma 2 and $\mathcal{M}$-abelianness of $\{\mathcal{Q},\alpha\}$ .

The invariant states $\Phi$ satisfying the equivalent condition (i) to (iii) are obviously ergodic states . We will call them <u>weakly mixing states</u> or <u>$E_I$ - states</u> see Definition 2 below) because they are a generalization of the weakly mixing states of standard (commutative) ergodic theory .

<u>Theorem 4</u> . Let $\{\mathcal{Q},\alpha\}$ be an $\mathcal{M}$-abelian system and let $\Phi$ be an ergodic state over $\mathcal{Q}$ with the notation of Definition 1 leaving out for shortness the subscripts and superscripts $\Phi$ . We adopt also the notation of Theorem 1 ( see formula (2) in Section 2) for the decomposition of $U^{E}$ into factors . Furthermore we set, for $A \in \mathcal{Q}$, $\sigma \in \Sigma$ and $\varphi, \psi \in \mathcal{H}^{(\sigma)}$

$$(16) \qquad M_{\varphi,\psi}^{(\sigma)}(A) = M_{\bar{f}}(A) \quad \text{where} \quad f(g) = (\varphi|U_{g}^{(\sigma)}|\psi) .$$

Then

(i) One has $U_g M^{(\sigma)}_{\varphi,\psi}(A)\, U_g^{-1} = M^{(\sigma)}_{U_g^{(\sigma)}\varphi,\psi}(A)$ , $g \in G$

(ii) The vectors $M^{(\sigma)}_{\varphi,\psi}(A)\Omega$ belong to the subspace $\mathscr{H}^{(\sigma)} \otimes \mathscr{H}'^{(\sigma)}$ . More precisely , if $\{\varphi_k\}$ is an orthonormal base of $\mathscr{H}^{(\sigma)}$ , the $M^{(\sigma)}_{\varphi_k,\psi}(A)\Omega$ deliver for appropriately chosen $A$ , $\psi$ an orthonormal base of a subspace of $\mathscr{H}$ equivalent to $\mathscr{H}^{(\sigma)}$ .

(iii) For each pair $\sigma$, $\sigma' \in \Sigma$ the representation $U^{(\sigma)} \otimes U^{(\sigma')}$ is not disjoint from $U$ .

(iv) For each $\sigma \in \Sigma$ one has $\bar{\sigma} \in \Sigma$ where $U^{(\bar{\sigma})}$ is the conjugate of the the representation $U^{(\sigma)}$ .

Proof : Property (i) immediately results from (10) applied to the definition (16) . From it follows that the $M^{(\sigma)}_{\varphi_k,\psi}(A)\Omega$ are contained in $\mathscr{H}^{(\sigma)} \otimes \mathscr{H}'^{(\sigma)}$ and span a subspace of $\mathscr{H}$ equivalent to $\mathscr{H}^{(\sigma)}$ if they are linearly independent . However one gets from (11)

$$M_{\bar{f}}(A)\Omega = M(\bar{f}\, U)\pi(A)\Omega$$

and, specializing to the choice of $f$ in (16), by use of (3) .,

$$(17) \qquad M^{(\sigma)}_{\varphi,\psi}(A)\Omega = d^{-1}_{(\sigma)}\{|\varphi)(\psi| \otimes I'^{(\sigma)}\}\; E^{(\sigma)}\pi(A)\Omega$$

so that one has

$$(18) \qquad (M^{(\sigma)}_{\varphi_k,\psi}(A)\Omega \mid M^{(\sigma)}_{\varphi_l,\psi}(A)\Omega) = \delta_{kl}\; d^{-2}_{(\sigma)}\|\psi\|^2 \cdot \|(E_\psi \otimes I'^{(\sigma)})E^{(\sigma)}\pi(A)\Omega\|^2$$

where $E_\psi$ is the projector onto $\psi$ in $\mathscr{H}^{(\sigma)}$ . Thus the $M^{(\sigma)}_{\varphi_k,\psi}(A)$ are mutually orthogonal and of common length. By the cyclicity of $\Omega$ , the latter can be chosen equal to unity for an appropriate choice of $A$ and $\psi$ , whence (iii) . Due to irreducibility of $\pi(\mathfrak{A}) \cup U_G$ the vectors $M^{(\sigma)}_{\varphi_k,\psi}(A)\Omega$ , $k = 1, 2, \dots d_{(\sigma)}$ , then generate a subspace cyclic for $\pi(\mathfrak{A})$ in $\mathcal{H}$ and thus, if we choose a coefficient $f \in \mathcal{P}(G)$ of representation $U^{(\sigma')}$ , $\sigma' \in \Sigma$ , and a $B \in \mathfrak{A}$ such that $M_{\bar{f}'}(B)\Omega \neq 0$ we have at least one $k$ for which $M_{\bar{f}'}(B)\, M^{(\sigma)}_{\varphi_k,\psi}(A)\Omega \neq 0$ . The non-disjointness of $U^{(\sigma)} \otimes U^{(\sigma')}$ and $U$ then results from the fact that, for a finite dimensional group representation, any quotient representation is equivalent to a subrepresentation .

Finally let $f$ be a coefficient of the representation $U^{(\sigma)}$, $\sigma \in \Sigma$ and $A$ an element of $\mathfrak{A}$ such that $M_{\bar{f}}(A)\Omega \neq 0$ . We have $(\Omega|M_{\bar{f}}(A)|\Omega)=(M_f(A^*)\Omega|\Omega) \neq 0$ (cf. (9)) and thus $M_f(A^*)\Omega \neq 0$ . But $M_f(A^*)\Omega \in \mathcal{H}^{(\bar{\sigma})} \otimes \mathcal{H}'^{(\bar{\sigma})}$ so that $\bar{\sigma} \in \Sigma$ .

The rest of this section handles the more special situation where the system $\{\mathfrak{A},\alpha\}$ is supposed to be weakly asymptotically abelian and discusses further the relationship with the standard ergodic theory . In this connexion, the next theorem will allow us to apply Mackey's results [16] by reduction to a commutative $C^*$-algebra .

<u>Theorem 5</u> . Let $\{\mathfrak{A},\alpha\}$ be a weakly asymptotically abelian system with a non compact group $G$ and $\Phi$ a state over $\mathfrak{A}$ invariant under $G$ with the notation of the preceding Theorem . We denote further by $N$ the stabilizer of $E$ i.e. the intersection of the kernels of all finite-dimensional subrepresentations of $U$ . If we assume the quotient group $\mathcal{G} = G / N$ to be either compact or connected (which is true if $G$ itself is connected )

the set of operators $E\,\pi(A)\,E$ , $E \in \mathfrak{E}$ , is abelian .

<u>Proof</u> : $N$ is obviously a closed invariant subgroup of $G$ , so that $\mathcal{G} = G/N$ is a topological group . By definition, the direct sum of all finite dimensional representations of $\mathcal{G}$ is faithful . Thus, by [15] , 16.4.6 , $\mathcal{G}$ is a direct product $K_1 \times T$ with $K_1$ compact and $T = R^n$ , n integer $\geq 0$ ($R^0$ is the group with one element ; if $n = 0$ , $\mathcal{G}$ is compact ). Let $\tau$ be a canonical homomorphism of $G$ onto $\mathcal{G}$ , set $G' = \tau^{-1}(T)$ and , for a continuous character $\chi \in \hat{T}$ of the group $T$ , let $E_\chi$ be the projector with range $\{\psi \in \mathcal{H} | U_{g'}\psi = \chi(\tau(g'))\psi$ for all $g' \in G'\}$ . The first step in the proof is to show that $E_\chi \mathcal{H}$ is stable under $U$ and that

$$E = \sum_{\chi \in \hat{T}} E_\chi . \tag{19}$$

Now $G'$ is obviously a closed invariant subgroup of $G$ with $N$ an invariant subgroup of $G'$ . Each character $\chi \in \hat{T}$ defines a one-dimensional unitary representation $g' \to \chi(\tau(g'))$ of $G'$ , different characters yielding different representations so that $F = \sum_{\chi \in \hat{T}} E_\chi$ is a sum of orthogonal projectors. Clearly $E \leq F$ . Take now a $\psi \in E_\chi \mathcal{H}$ . For arbitrary $g \in G$ , $g' \in G'$ we have $g^{-1} g' g \in G'$ and thus $U_{g^{-1}g'g}\psi = \chi(\tau(g^{-1}g'\,g))\psi = \chi(\tau(g'))\psi$ because $\tau(g') \in T$ is in the center of $\mathcal{G}$ . Thus $U_{g'} U_g \psi = \chi(\tau(g')U_g\psi$ , showing that $E_\chi \mathcal{H}$ is invariant under $U$ . Let $U_\chi = U E_\chi$ be the corresponding subrepresentation of $U$ . Obviously $U_\chi(n) = \chi(\tau(n)) = I$ for $n \in N$ so that $U_\chi = V_\chi \circ \tau$ where $V_\chi$ is a unitary representation of $\mathcal{G}$ . The kernel $J$ of $V_\chi$ contains all elements $kt \in \mathcal{G}$ with $k \in K$ , $t \in T$ and $\chi(t) = 1$ . Therefore $\mathcal{G}/J$ is compact , $V_\chi$ (and also $U_\chi$ ) is a sum of finite-dimensional representations and $E_\chi \leq E$ whence $F \leq E$ ; (19) is established (*).

Thus to prove the Theorem it is enough to show that, for $\chi', \chi'' \in \hat{T}$ and $A_1, A_2 \in \mathfrak{A}$

(20) $$E_{\chi'}\pi(A_1)E\pi(A_2)E_{\chi''} = E_{\chi'}\pi(A_2)E\pi(A_1)E_{\chi''}$$

whereby the left hand term is equal to

(21) $$\sum_{\chi\in\hat{T}} E_{\chi'}\pi(A_1)E_{\chi}\pi(A_2)E_{\chi''}$$

Now since $G/G' = (G/N)/(G'/N) = \mathcal{G}/T = K_1$ is compact and $G$ is non compact, by Proposition 1 p.31 of [19], $G'$ must be non compact and the system $\{\mathfrak{A}, \alpha'\}$, where $\alpha'$ is the restriction of $\alpha$ to $G'$, is weakly asymptotically abelian. Therefore if we denote by $M'_f(A)$, $f \in B(G')$, the means defined for the group $G'$ as was done in (8) for the group $G$, we have by (13), for $\chi_1, \chi_2 \in \hat{T}$ and $A \in \mathfrak{A}$

$$M'_{\overline{\chi}_1}(A)\, E_{\chi_2} = E_{\chi_1\chi_2}\, \pi(A)\, E_{\chi_2}$$

$$E_{\chi_2}\, M'_{\chi_1}(A) = E_{\chi_2}\pi(A)\, E_{\chi_1\chi_2}$$

where $\chi_1, \chi_2$ denote also the functions $\chi_1 \circ \tau$, $\chi_2 \circ \tau$ on $G'$. Thus (21) can be written

$$\sum_{\chi\in\hat{T}} E_{\chi'}\pi(A)M'_{\overline{\chi}\chi''}(A_2)E_{\chi''} = \sum_{\chi\in\hat{T}} E_{\chi'}\, M_{\overline{\chi}\chi''}\, (A_2)\, \pi(A_1)\, E_{\chi''}$$

$$= \sum_{\chi\in\hat{T}} E_{\chi'}\, \pi(A_2)\, E_{\chi'\overline{\chi}\chi''}\, \pi(A_1)\, E_{\chi''}$$

$$= \sum_{\chi\in\hat{T}} E_{\chi'}\, \pi(A_2)\, E_{\chi}\, \pi(A_1)\, E_{\chi''}$$

(*) At this point the proof of the theorem is deduced to the abelian case for which a proof was given by G. Gallavotti and D. Ruelle (private communication ).

thus proving (20) q.e.d.

Remark . We note that in the preceding proof the fact that the representation $\pi$ of $\mathfrak{A}$ was generated by the invariant state $\Phi$ was not actually used . All we need for the proof is the existence of a covariant representation of the weakly asymptotically abelian system $\{\mathfrak{A},\alpha\}$ such that the subspace $E\mathcal{H}$ is cyclic for $\pi(\mathfrak{A})$ in $\mathcal{H}$ .

We will now see that Theorem 4 renders available for the study of covariant representation of weakly asymptotically abelian systems Mackey's theory of imprimitivity systems . Since we work in a $C^*$-algebra frame we have in fact to deal with the more special case in which Borel structures are provided by locally compact topologies .

Lemma 4 . Let $\mathfrak{A}$ be a $C^*$-algebra , $g \to \alpha_g$ a homomorphic mapping of the locally compact group $G$ into the homomorphism group of $\mathfrak{A}$ such that $g \to \alpha_g(A)$ is norm continuous in $g$ for each $A \in \mathfrak{A}$ and $(\pi,U)$ a covariant representation of the system $\{\mathfrak{A},\alpha\}$ in a Hilbert space $\mathcal{H}$ . Let $E$ be a projector in the commutant of the set $\{U_g \mid g \in G\}$ such that $E\,\pi\,(\mathfrak{A})E$ is abelian . We denote by $\mathfrak{M}$ the commutative $C^*$-algebra of operators on $E\mathcal{H}$ generated by the $E\,\pi\,(A)\,E$ , $A \in \mathfrak{A}$ , by $\mathcal{Y}$ its spectrum and by $M \leftrightarrow \hat{M}$ the Gelfand isomorphism :

$$(22) \qquad \hat{M}(s) = \langle s\ ,\ M\rangle \qquad , \ s \in \mathcal{Y}\ ,\ M \in \mathfrak{M}\ ,\ \hat{M} \in \mathcal{C}_0(\mathcal{Y})\ .$$

(i) If we set $U_g^E = E\,U_g\,E$ and define

$$(23) \qquad \alpha_g(M) = U_g^E\,M\,U_{g^{-1}}^E \qquad , \ M \in \mathfrak{M}\ ,\ \ g \in G$$

we get a homomorphism $g \to \alpha_g$ of $G$ into the automorphism group $\mathfrak{M}$ such that $g \to \alpha_g(M)$ is norm continuous in $g \in G$ for each $M \in \mathcal{A}$

(ii) The dual action of $G$ on $\mathcal{T}$ given by

(24) $$< [g]s \, , M > \; = \; <s \, , \alpha_{g^{-1}}(M) > \qquad , s \in \mathcal{T}, \quad g \in G$$

defines a homomorphism $g \to [g]$ of $G$ into the homeomorphism group of $\mathcal{T}$ such that $(s,g) \to [g]$ is continuous from $\mathcal{T} \times G$ to $\mathcal{T}$.

(iii) Let us call $P$ the unique regular spectral measure $P$ on $\mathcal{T}$ such that, for all $M \in \mathfrak{M}$

(25) $$M = \int \hat{M}(s) \, dP(s)$$

The support of $P$ is the whole $\mathcal{T}$ and we have, for all Borel subsets $\Delta$ of $\mathcal{T}$

(26) $$U^E_g \, P(\Delta) \, U^E_{g^{-1}} = P([g]\Delta) \, .$$

$\{U^E \, , P \, , g \to [g]\}$ is called the <u>system of imprimitivity attached to $E$</u>.

If the set of operators $\pi(A), A \in \mathcal{A}$ and $U_g$, $g \in G$ is irreducible in $\mathcal{H}$, this system is ergodic. The converse is true if $E\mathcal{H}$ is cyclic for $\pi(\mathcal{A})$ in $\mathcal{H}$ and invariant under $\pi(\mathcal{A})' \cap U'_G$.

(iv) Let the covariant representation $(\pi,U)$ be generated as above by a state $\Phi$ over $\mathcal{A}$ invariant under $G$ with corresponding cyclic invariant vector $\Omega \in E\mathcal{H}$. The Hilbert space $E\mathcal{H}$ can then be identified with $L_2(\mathcal{T},\mu)$ where the bounded G-invariant Radon measure $\mu$ on $\mathcal{T}$ is defined by

(27) $< \mu, \hat{M} > = (\Omega|M|\Omega)$ , $\hat{M} \in \mathcal{C}_0(\mathcal{T})$

with the elements of $\mathfrak{M}$ acting multiplicatively

(28) $\{M \psi\}(s) = \hat{M}(s)\psi(s)$ , $\psi \in L_2(\mathcal{T}, \mu), M \in \mathfrak{M}, s \in \mathcal{T}$,

whilst the group acts by "shifts of the variable"

(29) $\{U^E_g \psi\}(s) = \psi([g^{-1}]\, s)$ , $\psi \in L_2(\mathcal{T}, \mu)$, $g \in G$, $s \in \mathcal{T}$.

The spectral measure P is then of unit multiplicity : we have

(30) $\{P(\Delta)\psi\}(s) = \chi_\Delta(s)\psi(s)$ , $\psi \in L_2(\mathcal{T}, \mu)$ , $s \in \mathcal{T}$

where $\chi_\Delta$ is the characteristic function of the Borel subset $\Delta$ of $\mathcal{T}$.

Proof : $\alpha_g$ defined by (23) is such that $\alpha_g(\prod_{i=1}^{n} E\,\Pi(A_i)E = \prod_{i=1}^{n} E\,\Pi(\alpha_g(A_i))E$, $A_i \in \mathcal{A}$. Thus, for M a polynomial of elements of the type $E\,\Pi(A)E$, $A \in \mathcal{A}$, and, by density, for a general element of $\mathfrak{M}$ , $M \to \alpha_g(M)$ is $*$-homomorphic and norm continuous in $g$ (the latter property stems from the assumed norm continuity of $\alpha_g(A)$ in $g$ for $A \in \mathcal{A}$). The mapping $g \to [g]$ defined in (24) is evidently homomorphic ; and $[g]$ is continuous in the $*$-weak topology as the transposed of a continuous operator , and id therefore a homeomorphism. For $s$ , $s' \in \mathcal{T}$ and $M_i \in \mathfrak{M}$, $i = 1,2,\dots n$ , we have; on the other hand

$$|<[g]s,M_i> - <[g']s',M_i>| \leq$$

$$\leq |<[g]s,M_i> - <[g]s',M_i>| + |<[g]s',M_i> - <[g']s',M_i>| \leq$$

$$\leq |<s,\alpha_{g^{-1}}(M_i)> \; < s', \alpha_{g^{-1}}(M_i)> | + ||\alpha_{g^{-1}}(M_i) - \alpha_{g'^{-1}}(M_i)||$$

The condition that the first term be less than $\frac{\varepsilon}{2}$ amounts to choosing $s'$ in a $*$-weak neighbourhood of $s$ and the second term is less than $\frac{\varepsilon}{2}$ for $g'$ in a neighbourhood of $g$ ; thus we have the continuity of $(g,s) \to gs$ . The uniqueness of the regular spectral measure P yielding (26) is well known ($*$-representations of abelian $C^*$-algebras are one-to-one with regular spectral measures on their spectrum). If the support of the spectral measure P was smaller than $\mathcal{T}$, one could find an $\hat{M} \neq 0$ vanishing on $\mathcal{T}$ so that $M = 0$ , a contradiction . Relations (26) is obtained by setting $\alpha_g(M)$ for M in $\mathcal{M}$ and using (23) and (24). Let us next denote by $\mathcal{R}$ the Von Neumann algebra generated by the $\pi(A)$, $A \in \mathcal{A}$ and $U_g$, $g \in G$ ; and by $\mathcal{R}_E$ the Von Neumann algebra of operators on $E\mathcal{H}$ generated by $\mathcal{M}$ and the $U_g^E$ , $g \in G$. Since E commutes with the $U_g$ , $\mathcal{R}_E$ contains $E\mathcal{R}E$ . Therefore if $\mathcal{R}$ is irreducible in $\mathcal{H}$ , the same holds for $E\mathcal{R}E$ , and a fortiori for $\mathcal{R}_E$ ( or equivalently for the system of $P(\Delta)$ and $U_g^E$) in $E\mathcal{H}$ .Conversely if $E \subset \mathcal{R}$ and $\mathcal{R}_E$ is irreducible, $(\mathcal{R}_E)' = (\mathcal{R}')_E$ (*) reduces to the scalars. But if $E\mathcal{H}$ is cyclic for $\pi(\mathcal{A})$ it separates $\mathcal{R}'$ and thus $\mathcal{R}'$ also reduces to the scalars and $\mathcal{R}$ is irreducible. Finally, if $\pi$ is generated by an invariant state $\Phi$ , cyclicity of the corresponding vector $\Omega \in E\mathcal{H}$ for $\pi(\mathcal{A})$ in $\mathcal{H}$ entails cyclicity of $\Omega$ for the commutative $C^*$-algebra $\mathcal{M}$ in $E\mathcal{H}$ . The Segal-Gelfand construction applied to the state (24) over $\mathcal{M}$ then shows in the familiar way (cf.[21]§17.4) that $E\mathcal{H}$ is isomorphic to $L_2(\mathcal{T},\mu)$ with the property (27). On the other han for $M \in \mathcal{M}$ and $g \in G$ we have by (23) $U_g^E M\Omega = \alpha_g(M)\Omega$ whence for the corresponding element $\hat{M} \in \mathcal{C}_0(\mathcal{T}) \subset L_2(\mathcal{T}, \mu)$ using (24),

$$\{U_g^E\}\hat{M}\ (s) = \widehat{\alpha_g(M)}\ (s) = M([g^{-1}]\ s)$$

(*) Proposition 1 Chapt. I §2 of [20] .

On the other hand the spectral measure (30) in our case evidently fulfills (26), completing the proof of our Lemma .

Theorem 4 in the setting of the appended remark together with the previous Lemma allow a direct application of Theorem 2 in [16] to give

Theorem 6 . Let $(\pi, U)$ be an ergodic covariant representation of the weakly asymptotically abelian system $\{\mathfrak{A}, \alpha\}$ with a non compact separable G (ergodicity means irreducibility of the system $\pi(A)$ , $A \in \mathfrak{A}$ and $U_g$ , $g \in G$ ) and let $E$ , $\mathcal{G}$ be as in Theorem 5 with $\Sigma$ the set of irreducible finite-dimensional components of U acting on Hilbert spaces $\mathcal{H}^{(\sigma)}$ . Denote furthermore by K the compact group obtained by taking the closure of $\varphi(G)$ , where $\varphi$ is the homomorphic continuous map $g \in G \to \{U_g^{(\sigma)}; \sigma \in \Sigma\}$ into the product of unitary groups in all $\mathcal{H}^{(\sigma)}$ , $\sigma \in \Sigma$ . Then there exists a closed subgroup H of K , a unitary representation L of H and a unitary map W of $E\mathcal{H}$ onto the Hilbert space of the representation $U^L$ of K induced by L such that

(i) $W B W^{-1} = B^L$

(ii) $W U_g^E W^{-1} = U^L_{\varphi(g)}$ for all $g \in G$

where $B^L$ and B are the complete boolean algebrasof projections determined respectively by the canonical imprimitivity system of $U^L$ and the imprimitivit system attached to E .

We conclude this section by noticing that the classification of ergodic states given in [2] can be generalized to the case of non abelian groups in the following manner .

Definition 2 . Let $\{\mathcal{A},\alpha\}$ be a weakly asymptotically abelian system and $\Phi$ state invariant under $G$ with the notation of Definition 1 . Let $N_\varphi$ be the kernel of the representation $g \to U_g^\Phi E^\Phi$ and $\mathcal{G}_\Phi$ the quotient group $\mathcal{G}_\Phi = G/N_\Phi$ . We distinguish the following three classes of ergodic states

(i) $\Phi$ is called an $E_I$-state (or a weakly mixing state , cf. Theorem 3 above) whenever the only finite dimensional subrepresentation of $U_\Phi$ is the one-dimensional subrepresentation spanned by the invariant vector $\Omega_\Phi$ i.e. $\mathcal{G}_\Phi = G/N_\Phi$ is the group with one element .

(ii) $\Phi$ is called an $E_{II}$-state if it is not an $E_I$-state and if the quotient group $\mathcal{G}_\Phi = G/N_\Phi$ is compact . In other terms an $E_{II}$-state is an $E_{II}$-state is an ergodic state for which $\mathcal{G}_\Phi$ is compact and contains more than one element .

(iii) $\Phi$ is called $E_{III}$-state whenever $\mathcal{G}_\Phi$ is not compact .

Remarks .

We add some remark on $E_{II}$-states over a weakly asymptotically abelian system $\{\mathcal{A},\alpha\}$ ; the proofs are either immediate or easily deduced from what preceeds and the literature .

1°) If $\Phi$ is an $E_{II}$-state it follows from (19) that

$$E^\Phi \mathcal{H}_\Phi = \{\psi/\psi \in \mathcal{H}_\Phi \ , \ U_g\psi = \psi \ \text{ all } \ g \in N_\Phi\} \ .$$

This property is trivial for $E_I$-states , where $N_\Phi = G$ , and false in general , though not always for $E_{III}$-states .

2°) Let $\Phi$ be an ergodic state over $\{\mathcal{A},\alpha\}$ and $\bar{\Sigma}$ the set of all irreducible components of the tensor products of finite subfamilies of $\Sigma$

(see Theorem 4). If $E^{\Phi}\mathcal{H}_{\Phi}$ is separable and $\mathcal{G}_{\Phi}$ connected, the following are equivalent

(i) $\bar{\Sigma} = \hat{\mathcal{G}}_{\Phi}$ ;

(ii) $\Phi$ is an $E_{II}$-state .

If we assume $\mathcal{G}_{\Phi}$ to be connected , $E^{\Phi}\mathcal{H}_{\Phi}$ separable implies that $\{\chi | \chi \in \hat{T}, E_{\chi} \neq 0\}$ is a countable subgroup of $\hat{T}$ (see equation (19)) ; whence the equivalence of $\bar{\Sigma} = \hat{\mathcal{G}}_{\Phi}$ to the fact that $\hat{T}$ is countable ( or $n = 0$ ) .

3°) Let $\mathcal{A}$ be separable and $\Phi$ an $E_{II}$-state over $\{\mathcal{A},\alpha\}$ . Using the methods of Section 5 in [2] (where some points of rigour need to be fixed as will be done in a forthcoming paper), or alternatively of [4] , one can show the existence of an ergodic state $\varphi$ over the weakly asymptotically abelian system $\{\mathcal{A},\alpha | N_{\Phi}\}$ such that the unique decomposition of $\Phi$ into extremal $N_{\Phi}$-invariant states can be written

$$\Phi(A) = \int_{\mathcal{G}_{\Phi}} \varphi_{\xi}(A) dm(\xi)$$

where $m$ is the Haar measure on $\mathcal{G}_{\Phi}$ and $\varphi_{\xi}$ is defined by the relation

$$\varphi_{\tau(g)} = \varphi \circ \alpha_g \qquad , \qquad g \in G$$

with $\tau$ the canonical homomorphism of $G$ onto $\mathcal{G}_{\Phi}$ .

An $E_{II}$-state is thereby uniquely represented as the average of a state with "lower symmetry" .

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