The Annals of Applied Probability
1996, Vol. 6, No. 3, 1006–1023

ERGODIC THEOREMS FOR SOME CLASSICAL PROBLEMS IN COMBINATORIAL OPTIMIZATION[1]

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We show that the stochastic versions of some classical problems in combinatorial optimization may be imbedded in multiparameter subadditive processes having an intrinsic ergodic structure. A multiparameter generalization of Kingman's subadditive ergodic theorem is used to capture strong laws for these optimization problems, including the traveling salesman and minimal spanning tree processes. In this way we make progress on some open problems and provide alternate proofs of some well known asymptotic results.

1. Introduction. Limit theorems for random processes which arise in problems of geometric probability are relatively well understood. There are now several approaches [Steele (1981, 1988); Talagrand (1995); Rhee (1993); Redmond and Yukich (1994, 1996)] for establishing a.s. limits for the shortest path through a random sample, the length of a minimal spanning tree spanned by a random sample and the length of a minimal matching on a random sample.

In this paper we show that these and other familiar problems of combinatorial optimization may be imbedded in a subadditive multiparameter stochastic process. We do this by viewing optimization problems on $\mathbb{R}^d$ as processes indexed by d-dimensional rectangles. In this way we may use a multiparameter generalization of Kingman's subadditive ergodic theorem [Kingman (1968)] to capture strong laws for certain problems which involve minimizing sums of Euclidean distances. The role of ergodic theory has been apparently overlooked in connection with stochastic optimization problems. By drawing on their intrinsic subadditive ergodic structure, we deduce limit results for optimization problems on uniform samples, the most studied case. In this way we make progress on some open problems. The approach taken here provides a conceptual framework which yields elementary proofs of some well known results.

Our approach does not depend upon isoperimetric or other fundamental deviation inequalities [Rhee (1993); Talagrand (1995, 1996a, b)], variance bounds or the usual subadditive tools for the study of problems in geometric probability [Steele (1981)]. The only overlap with previous work involves the use of "boundary processes," a tool which simplifies both the technical

Received October 1995; revised March 1996.
[1]Research supported in part by NSA Grant MDA904-95-H-1005.
AMS 1991 *subject classifications*. 60F15, 60D05, 60C05.
Key words and phrases. Subadditive ergodic theorems, combinatorial optimization, minimal spanning tree, traveling salesman problem, boundary process.

analysis and the conceptual presentation. Unlike the seminal and classic work of Steele (1981), which in spirit resembles the theory of subadditive processes, the present work is formulated entirely in the context of subadditive multiparameter processes and subadditive ergodic theorems. This formulation makes it possible to draw upon the fundamental ergodic theorems of Akcoglu and Krengel (1981), which have been used, for example, in statistical physics to analyze the behavior of long-range random spin systems [van Enter and van Hemmen (1983)].

We focus attention on the following optimization problems.

1. *Traveling salesman problem* (TSP). Let $V := \{x_1, \dots, x_n\}$ be a set of points in $\mathbb{R}^d$, $d \geq 2$. For all $p > 0$, let $T^p(x_1, \dots, x_n)$ be the length of the shortest closed path on V with pth power weighted edges,

$$T^p\{x_1, \dots, x_n\} := \inf_T \sum_{e \in T} |e|^p,$$

where the minimum is taken over all closed paths T on V with edges e. (A closed path may pass through a vertex more than once.)
2. *Minimal spanning trees* (MST). Let $M^p(x_1, \dots, x_n)$ be the length of the shortest spanning tree on V with pth power weighted edges, namely,

$$M^p\{x_1, \dots, x_n\} := \min_T \sum_{e \in T} |e|^p,$$

where the minimum is over all connected graphs T with vertex set V.
3. *Minimal matching*. The minimal matching on V with pth power weighted edges is given by

$$S^p\{x_1, \dots, x_n\} := \min_\sigma \sum_{i=1}^{n/2} \|x_{\sigma(2i-1)} - x_{\sigma(2i)}\|^p,$$

where the minimum is over all permutations σ of the integers $1, 2, \dots, n$, and where n is assumed to have even parity.

Our main results describe the asymptotic behavior of T^p and M^p on random samples in the unit cube and also on sequences of cubes $[0, n]^d$, $n \geq 1$, of increasing edge length. Asymptotics of the latter sort resemble those considered in statistical mechanics. Throughout we let $\Pi := \Pi(1)$ denote a Poisson point process on $\mathbb{R}^d$ with intensity 1. Let $U_1, U_2, \dots$ be i.i.d. random variables with the uniform distribution on $[0, 1]^d$, $d \geq 2$. The following is our main result.

THEOREM 1.1. (a) *For all* $0 < p \leq d$ *there is a finite, positive constant* $\alpha(T^p, d)$ *such that*

$$\lim_{n \to \infty} T^p\big(\Pi \cap [0, n]^d\big)/n^d = \alpha(T^p, d) \quad a.s. \tag{1.1}$$

(b) *For all $p > 0$ there is a finite, positive constant $\alpha(M^p, d)$ such that*

$$\lim_{n\to\infty} M^p\big(\Pi \cap [0,n]^d\big)/n^d = \alpha(M^p, d) \quad a.s. \tag{1.2}$$

The next result describes the asymptotics for the power-weighted TSP and MST processes on the unit cube. This corollary is not new, but we state it for completeness and give a simple proof based on Theorem 1.1. The notation $\lim_{n\to\infty} X_n =_P Y$ means that X_n converges to Y in probability.

COROLLARY 1.2. (a) *For all $d \in \mathbb{N}^+$,*

$$\lim_{n\to\infty} T^d(U_1,\dots,U_n) =_P \alpha(T^d, d). \tag{1.1$'$}$$

(b) *For all $d \in \mathbb{N}^+$ and $p \geq d$,*

$$\lim_{n\to\infty} M^p(U_1,\dots,U_n)/n^{(p-d)/d} =_P \alpha(M^p, d). \tag{1.2$'$}$$

When $p = 1$, the asymptotics (1.1) and (1.2) are classic and go back to Beardwood, Halton and Hammersley (1959) and Steele (1988), respectively. For $1 \leq p < d$, (1.2) is essentially due to Steele (1988) and (1.1) is due to Redmond and Yukich (1996). The interest of Theorem 1.1 and Corollary 1.2 derives from the fact that they hold for the critical case $p = d$, where the usual geometric subadditivity methods break down. For the critical case $p = d$, we recall that (1.1′) and (1.2′) represent conjectures of Steele and Bland, respectively. By the uniform boundedness of T^d and M^d in dimension d, the a.s. results (1.1) and (1.2) imply similar mean versions.

When $p = d$, Aldous and Steele (1992) considered the probabilistic theory of infinite trees to obtain an L^2 version of (1.2′), thus settling Bland's conjecture. The results (1.1′) and (1.2′) were later obtained by Yukich (1995), settling Steele's conjecture. Yukich (1995) requires that the closed paths in the definition of T^p assign every vertex a degree of 2, but his methods cover the present T^p process as well.

Little is known concerning the exact value of the limiting constant $\alpha(T^p, d)$. We will identify this constant with the spatial constant of a well-defined superadditive ergodic process in the following way. Let $\mathbb{N}^d$ denote $\{1, 2, \dots\}^d$ and let $u \in \mathbb{N}^d$ have the representation $u := (u_1, u_2, \dots, u_d)$. Let $[0, u]$ denote the d-dimensional rectangle $[0, u_1] \times [0, u_2] \times \cdots \times [0, u_d]$. We will show that

$$\alpha(T^p, d) := \sup_{u\in\mathbb{N}^d} \mathbb{E} T_B^p(\Pi \cap [0,u])/u_1 \cdot u_2 \cdots u_d, \tag{1.3}$$

where T_B^p is the canonical boundary process associated with T^p; T_B^p will be defined in the sequel. An identity similar to (1.3) holds for $\alpha(M^p, d)$, adding to the more precise results of Avram and Bertsimas (1992).

For completeness, we mention another approach to Theorem 1.1 and Corollary 1.2, one which does not involve ergodic theory and which is instead inspired by the deep work of Talagrand (1995, 1996a, b) on isoperimetry and concentration inequalities. This approach, which was indicated to me by

Talagrand, has two steps: the first uses subadditivity methods to establish an L^1 version of (1.1) and (1.2). The second step converts these L^1 estimates to a.s. results by finding the appropriate concentration inequalities which describe the spread of T^p and M^p around their means. While it remains to find these concentration inequalities, they would surely be similar in spirit to the far-reaching ones developed by Talagrand. While this approach does not identify the limiting constants, it provides asymptotics for two different models of problem generation; it treats the *incrementing* model of problem generation (the existing sample points $U_1, \dots, U_n$ are incremented to get a new sample $U_1, \dots, U_{n+1}$) and also the *independent* model of problem generation (the sample $U_1, \dots, U_n$ is discarded and replaced by a completely new sample $U_1', \dots, U_{n+1}'$). The present approach is limited to the first model of problem generation.

We point out that the ergodic theoretic approach to Theorem 1.1 is not limited to the TSP and MST problems, but also applies to the minimal matching problem S^p as well as other problems; see Section 6. The methods outlined here hold potential for generalization and extension. We illustrate this in Section 6.3 by showing how to obtain strong limit theorems for optimization processes uniformly over convex domains which increase with the sample size.

2. Boundary processes. Imbedding problems of combinatorial optimization into a multiparameter subadditive process depends upon the notion of a boundary process, the key to revealing the intrinsic subadditive structure of the problems considered here. Boundary processes are defined on pairs (F, R), where $F \subset \mathbb{R}^d$ is a finite set and R is a d-dimensional rectangle. [A d-dimensional rectangle is a d-fold Cartesian product of intervals and has the form $\{(w_i): u_i \le w_i < v_i, 1 \le i \le d\}$, where (u_i) and $(v_i) \in (\mathbb{N}^+)^d$.] Loosely speaking, boundary processes on a d-dimensional rectangle R allow free travel on the boundary δR; that is, edge connections on δR incur no cost. We now provide formal definitions of the boundary TSP, MST and minimal matching processes.

2.1. *The boundary TSP process.* For all d-dimensional rectangles R, discrete sets $F \subset R$ and $p > 0$, let $T^p(F \cup a \cup b)$ represent the length of the shortest path with pth power-weighted edges through $F \cup (a, b)$ with endpoints a and b, where a and b are constrained to lie on δR. Define the boundary process T_B^p associated with T^p by

$$T_B^p(F, R) := \min\left\{T^p(F), \inf \sum_i T^p(F_i \cup a_i \cup b_i)\right\},$$

where the infimum ranges over all partitions $\{F_i\}_{i \ge 1}$ of F and all sequences of pairs of points $\{(a_i, b_i)\}_{i \ge 1}$ belonging to δR. The process $T_B^p(F, R)$ may be interpreted as the cost of the minimal closed path (with pth power weighted edges) through the set F which may repeatedly exit to the boundary of R at one point and reenter at another, incurring no cost when moving along the boundary.

2.2. *The boundary MST process.* For all d-dimensional rectangles R, discrete sets $F \subset R$ and $p > 0$, define

$$M_B^p(F, R) := \min\left\{ M^p(F), \inf \sum_i M^p(F_i \cup a_i) \right\},$$

where the infimum ranges over all partitions $(F_i)_{i \ge 1}$ of F and all sequences of points $\{a_i\}_{i \ge 1}$ on δR. The graph realizing the boundary process $M_B^p(F, R)$ may be interpreted as a collection of small trees connected via the boundary into a single large tree, where the connections along the boundary of R incur no cost.

2.3. *The boundary minimal matching process.* For all d-dimensional rectangles R, discrete sets $F \subset R$ and $p > 0$, let $\hat{S}_B^p(F, R)$ denote the length of the least Euclidean matching (with pth power weighted edges) of points in F with matching to points on δR permitted. More precisely, each point in F is paired with either a boundary point on δR or another point in F; $\hat{S}_B^p(F, R)$ minimizes the sum of the pth powers of the edge lengths over all such pairings. Define the boundary minimal matching by

$$S_B^p(F, R) := \min\left\{ S^p(F), \hat{S}_B^p(F, R) \right\}.$$

We make a simple but crucial observation concerning the geometry of boundary processes.

LEMMA 2.1. *Let L_B^p denote the boundary processes associated with either the TSP, MST or minimal matching problems. Then L_B^p is a superadditive process in the sense that for all d-dimensional rectangles R and $F \subset R$,*

$$L_B^p(F, R) \ge \sum_{i \le n} L_B^p(F \cap R_i, R_i),$$

where R_i, $i \le n$, are disjoint rectangles and have union R.

By definition, $T_B^p(F, [0,1]^d) \le T^p(F)$ for all $F \subset [0,1]^d$ and similarly for M_B^p and S_B^p. The following lemma, which is proved in the Appendix, shows an estimate in the other direction. Here and elsewhere, C denotes a universal constant whose value may change from line to line and $|F|$ denotes the cardinality of the set F.

LEMMA 2.2. *Let L denote either the TSP, MST or minimal matching problems. Then for all $p > 0$ and all $F \subset [0,1]^d$ we have*

$$L^p(F) \le L_B^p\left(F, [0,1]^d\right) + C\left[|F|^{(d-p-1)/(d-1)} \vee \log|F| \right].$$

3. Subadditive ergodic theorems. Before proceeding further we state for convenience an elegant multiparameter generalization of Kingman's deep subadditive ergodic theorem. We write $\mathscr{R} := \mathscr{R}(d)$ for the collection of d-dimensional rectangles.

THEOREM A [Akcoglu and Krengel (1981)]. *Let $L := \{L(R),\ R \in \mathscr{R}\}$ be a real-valued process defined on a probability space $(\Omega, \mathscr{A}, P)$. Suppose that L is a stationary multiparameter superadditive process. That is, satisfies the following properties.*

(i) *Stationarity: For all $k \ge 1$, $R_1, \dots, R_k \in \mathscr{R}$ and $u \in (\mathbb{R}^+)^d$, the joint distribution of $L(R_1), \dots, L(R_k)$ is the same as that of $L(R_1 + u), \dots, L(R_k + u)$.*

(ii) *Superadditivity: Given disjoint rectangles R_i, $i \le n$, with $\bigcup_{i \le n} R_i \in \mathscr{R}$, we have $L(\bigcup_{i \le n} R_i) \ge \sum_{i \le n} L(R_i)$.*

(iii) *Integrability: $L(R)$, $R \in \mathscr{R}$, are integrable.*

(iv) *Boundedness in mean: $\sup_n \mathbb{E} L([0, n]^d)/n^d < \infty$.*

Then there exists $f(L, d) \in L^1(\Omega, \mathscr{A}, P)$ such that

$$\lim_{n \to \infty} L\big([0, n]^d\big)/n^d = f(L, d) \quad a.s.$$

Moreover, $\mathbb{E} f(L, d) = \alpha(L, d) := \sup_{u \in \mathbb{N}^d} \mathbb{E} L([0, u])/u_1 \cdots u_d$.

Here $\alpha(L, d)$ is the *spatial constant* for the process L. Spatial constants are the multiparameter analogs of the "time constants" in one-dimensional subadditive theory. The identification $\mathbb{E} f(L, d) = \alpha(L, d)$ may be seen from Smythe's (1976) mean ergodic theorem.

Given an optimization problem L, the traditional approach for finding its asymptotics involves understanding how L behaves on the finite subsets of $(\mathbb{R}^+)^d$. In what follows, we change the perspective and consider L as a process defined over the parameter set $\mathscr{R}$ of d-dimensional rectangles. With this point of view the ergodic structure of L becomes transparent.

To carry out this approach *let L denote either the TSP, MST or minimal matching problem*. Recalling that $\Pi := \Pi(1)$ is a Poisson point process on $\mathbb{R}^d$ with intensity 1, we define the process L^p over the rectangles in $\mathscr{R}$ according to the convention

$$L^p(R) := L^p(\Pi \cap R), \qquad R \in \mathscr{R}.$$

Similarly, we define

$$L^p_B(R) := L^p_B(\Pi \cap R, R), \qquad R \in \mathscr{R}.$$

We obtain the multiparameter processes $L^p(R)$ and $L^p_B(R)$, $R \in \mathscr{R}$. Thus, when L is the TSP, $L^p_B(R)$ is the optimal rooted path of points in $R \cap \Pi$, with rooting to δR permitted.

We observe that the multiparameter process L^p_B satisfies the four properties of Theorem A. Indeed, property (i) is a result of the well-known translation invariance of L. Property (ii) is a result of Lemma 2.1. Property (iii) is easily verified. To prove (iv), we appeal to bounds for L_B^p based on the space-filling curve heuristic [see Steele (1990)]. These bounds tell us that when L is either the TSP, MST or minimal matching problem, then

$$\mathbb{E} L_B^p\big(\{U_1,\dots,U_k\},[0,1]^d\big) \le Ck^{(d-p)/d}$$

for U_i, $i \ge 1$, i.i.d. with the uniform distribution on $[0,1]^d$. It easily follows that if $N := N(n^d)$ is an independent Poisson random variable with parameter n^d, then by the scaling property

$$L_B^p(\alpha F, \alpha R) = \alpha^p L_B^p(F,R) \quad \text{for } F \subset R,\ \alpha > 0,$$

we have

$$\begin{aligned}\mathbb{E} L_B^p\big([0,n]^d\big) &= \mathbb{E} L_B^p\big(n(U_1,\dots,U_N),[0,n]^d\big)\\ &= n^p \mathbb{E} L_B^p\big((U_1,\dots,U_N),[0,1]^d\big)\\ &\le Cn^p \mathbb{E}(N^{(d-p)/d})\\ &\le Cn^d,\end{aligned}$$

by Jensen's inequality. The proof of (iv) is complete.

We have verified that the boundary process L_B^p is a discrete stationary multiparameter superadditive process. By Theorem A, we deduce for all $p > 0$ the existence of an L^1 function $f(L_B^p, d)$ such that

$$\lim_{n\to\infty} L_B^p\big([0,n]^d\big)/n^d = f(L_B^p,d) \quad \text{a.s.} \tag{3.1}$$

The number N of vertices in $[0,n]^d$ and the volume V of $[0,n]^d$ tend to infinity but in such a way that the particle number density N/V essentially remains finite. Borrowing a term from statistical mechanics, we say that $f(L_B^p, d)$ is the infinite-volume limit for the process L_B^p. By Theorem A, the expectation of $f(L_B^p, d)$ satisfies

$$\mathbb{E} f(L_B^p, d) = \alpha(L_B^p, d). \tag{3.2}$$

We have thus shown that Theorem A implies the following proposition.

PROPOSITION 3.1. *Let L_B denote either the boundary TSP, MST or minimal matching processes. For all $d \in \mathbb{N}^+$ and $p > 0$ there exists an L^1 function $f(L_B^p, d)$ such that*

$$L_B^p\big([0,n]^d\big)/n^d \to f(L_B^p,d) \quad a.s. \tag{3.3}$$

Before deducing Theorem 1.1 from Proposition 3.1 for general p, consider first the simple case $0 < p < d$, for which we offer a straightforward proof. Since the convergence in (3.3) is unaffected by changes in finitely many vertices, the Hewitt–Savage zero–one law implies that $f(L_B^p, d)$ reduces to its mean value $\alpha(L_B^p, d)$. Next, by Lemma 2.2, L^p is close to the boundary process L_B^p. That is, if $N := \operatorname{card}(\Pi n[0,n]^d)$, then

$$L_B^p\big([0,n]^d\big) \le L^p\big([0,n]^d\big) \le L_B^p\big([0,n]^d\big) + Cn^p\big[N^{(d-p-1)/(d-1)} \vee \log N\big].$$

When $0 < p < d$, we have $Cn^p[N^{(d-p-1)/(d-1)} \vee \log N]/n^d \to 0$ a.s. and we thus deduce

$$L^p\big([0,n]^d\big)/n^d \to \alpha\big(L_B^p, d\big) \quad \text{a.s.}$$

This is precisely (1.1).

4. Asymptotics for the TSP. In this section we prove the two main asymptotic results for the TSP, namely, Theorem 1.1(a) and Corollary 1.1(a). We use Proposition 3.1 as a starting point. With regard to Theorem 1.1(a), we consider the proof of the critical case $p = d$; this proof may be divided into two simple steps. We continue to write $T([0, n]^d)$ for $T(\Pi \cap [0, n]^d)$.

STEP 1 (Identification of the limit). Show that the random limit $f(T_B^d, d)$ in Proposition 3.1 a.s. reduces to a constant; that is, show

$$f\big(T_B^d, d\big) = \alpha\big(T_B^d, d\big) \quad \text{a.s.} \tag{4.1}$$

Step 1 gives a version of Theorem 1.1(a) for the boundary TSP process T_B^d. The next step boosts this to the standard process T^d.

STEP 2 (Asymptotics for the standard process). Show that

$$\Big|T_B^d\big([0,n]^d\big) - T^d\big([0,n]^d\big)\Big|/n^d \to 0 \quad \text{a.s.} \tag{4.2}$$

We will prove Steps 1 and 2 when $d = 2$. The proof for general dimension d follows in a similar way. Not surprisingly, the method for proving Steps 1 and 2 for the TSP process centers on high probability smoothness estimates. Throughout, the statement "E occurs with high probability" means that $P(E^c) \le Cn^{-A}$, where A is large. We will write $T_B^2(k)$ for $T_B^2(U_1, \dots, U_k)$.

LEMMA 4.1 (Smoothness for the TSP). *There is a constant C such that for all $n \ge 1$ and all $1 \le k \le n^2/2$,*

$$\big|T_B^2(n^2) - T_B^2(n^2 + k)\big| \le Ck\big(\log n/n^{3/2}\big) \tag{4.3}$$

with high probability. Moreover, for all $1 \le k \le n^2/2$,

$$\big|T_B^2(n^2) - T_B^2(n^2 - k)\big| \le Ck\big(\log n/n^{3/2}\big) \tag{4.4}$$

with high probability.

PROOF. To show (4.3) we need some notation. Given i.i.d. random variables $\{U_i\}_{i=1}^{n^2}$ and U_{n^2+j}, $j \ge 1$, with the uniform distribution on $[0,1]^2$, let F_j denote the edge of minimal Euclidean length which connects U_{n^2+j} with a point in the sample $\{U_i\}_{i=1}^{n^2}$. Notice that for all $j \ge 1$ we have with high probability,

$$|F_j| \le C(\log n)^{1/2}/n \le C \log n/n, \tag{4.5}$$

where here and elsewhere $|E|$ denotes the Euclidean length of the edge E. Given $T_B^2(n^2)$, let $E_1, \dots, E_{n^2}$ be an enumeration of the edges in the path described by $T_B^2(n^2)$. By Lemma 2.4 of Yukich (1995), the lengths $|E_j|$, $1 \le j \le n^2$, satisfy the humble but useful high probability estimate

$$|E_j| \le Cn^{-1/2}. \tag{4.6}$$

We are now ready to show (4.3). Consider the edge F_1 and, relabeling if necessary, assume without loss of generality (WLOG) that F_1 links U_{n^2+1} to U_1 and that the path $T_B^2(n^2)$ visits U_2 after U_1. By inserting the edge F_1 and replacing the edge $E_1 := (U_1, U_2)$ with the edge (U_{n^2+1}, U_2) of length at most $|F_1| + |E_1|$, we may use the tour given by $T_B^2(n^2)$ to construct a feasible tour on $U_1, \dots, U_{n^2+1}$ at an extra cost of at most $|F_1|^2 + (|F_1| + |E_1|)^2 - |E_1|^2 = 2|F_1|^2 + 2|F_1|\,|E_1|$. Thus, by (4.5) and (4.6),

$$T_B^2(n^2+1) \le T_B^2(n^2) + C \log n / n^{3/2}$$

with high probability. Iterating, we arrive at an estimate of the form

$$T_B^2(n^2+k) \le T_B^2(n^2) + Ck\left(\log n / n^{3/2}\right). \tag{4.7}$$

This holds with high probability for $1 \le k \le n^2$.

To complete the proof of (4.3), it remains to show the reverse high probability inequality,

$$T_B^2(n^2) \le T_B^2(n^2+k) + Ck\left(\log n / n^{3/2}\right), \qquad 1 \le k \le n^2/2. \tag{4.8}$$

Let $N_{n^2+1,1}$ and $N_{n^2+1,2}$ denote the two neighbors of U_{n^2+1} in the minimal tour on $U_1, \dots, U_{n^2+1}$. With high probability there is a sample point, say U_1, such that (i) U_{n^2+1} is within $C \log n / n$ of U_1 and (ii) U_1 is neither a neighbor of $N_{n^2+1,1}$ nor of $N_{n^2+1,2}$. Let E_1 denote the edge $(N_{n^2+1,1}, U_{n^2+1})$ and note that $|E_1| \le Cn^{-1/2}$ with high probability. Replace E_1 with the edge E_1' joining $N_{n^2+1,1}$ to U_1. This may be done at an extra cost of at most $(|E_1| + C \log n / n)^2 - |E_1|^2$, which is bounded by $C \log n / n^{3/2}$ with high probability. By similarly replacing the edge $E_2 := (N_{n^2+1,2}, U_{n^2+1})$ with the edge E_2' joining $N_{n^2+1,2}$ to U_1, we obtain a path on $U_1, \dots, U_{n^2}$ which is obtained at an additional cost which is bounded by $C \log n / n^{3/2}$ with high probability. Thus, with high probability,

$$T_B^2(n^2) \le T_B^2(n^2+1) + C \log n / n^{3/2}. \tag{4.9}$$

Iterating (4.9) gives (4.8) as desired.

To now establish smoothness (4.4), observe that the high probability estimate

$$T_B^2(n^2) \le T_B^2(n^2-k) + Ck\left(\log n / n^{3/2}\right), \qquad 0 \le k \le n^2/2,$$

follows from (4.7) with n^2 replaced by $n^2 - k$. The reverse inequality

$$T_B^2(n^2-k) \le T_B^2(n^2) + Ck\left(\log n / n^{3/2}\right), \qquad 0 \le k \le n^2/2,$$

follows as in (4.8) with n^2 replaced by $n^2 - k$. This concludes the proof of Lemma 4.1. □

We may now complete the two-step proof of Theorem 1.1(a).

PROOF OF THEOREM 1.1a. *Step* 1. We want to show that the convergence

$$T_B^2\big(\Pi \cap [0,n]^2, [0,n]^2\big)/n^2 \to f\big(T_B^2, 2\big) \quad \text{a.s.}$$

is unaffected by a change of finitely many vertices. To see this, notice that if Π is replaced by Π', which differs from Π in k vertices, then with high probability (i.e., probability at least $1 - n^{-A}$) we have

$$\Big|T_B^2\big(\Pi \cap [0,n]^2, [0,n]^2\big) - T_B^2\big(\Pi' \cap [0,n]^2, [0,n]^2\big)\Big| \le Ckn^2(\log n/n)^{3/2}.$$

To see this we let $N := \operatorname{card}(\Pi \cap [0,n]^2)$, we condition on N and note that with high probability N is within $Cn(\log n)^{1/2}$ of n^2. We then apply Lemma 4.1 and scale. Thus,

$$\Big|T_B^2\big(\Pi \cap [0,n]^2, [0,n]^2\big) - T_B^2\big(\Pi' \cap [0,n]^2, [0,n]^2\big)\Big|/n^2 \to 0 \quad \text{a.s.}$$

and by the triangle inequality

$$T_B^2\big(\Pi' \cap [0,n]^2, [0,n]^2\big)/n^2 \to f\big(T_B^2, 2\big) \quad \text{a.s.}$$

Thus by the Hewitt–Savage zero–one law, $f(T_B^2, 2)$ is a.s. constant, completing the proof of Step 1.

Step 2. If the cardinality of $\Pi \cap [0,n]^2$ is a fixed number, say k, then the proof of Proposition 2.3 and Lemmas 2.4–2.7 of Yukich (1995) show that

$$T^2\big(\Pi \cap [0,n]^2\big) \le T_B^2\big(\Pi \cap [0,n]^2, [0,n]^2\big) + \Delta(k),$$

where $\Delta(k) \le Cn^2 k^{-1/10}$ with probability at least $1 - k^{-A}$. Here the factor of n^2 is the appropriate scaling factor. We may use this estimate to complete the proof of Step 2. Indeed, let the cardinality of $\Pi \cap [0,n]^2$ be N, where N is a Poisson random variable with parameter n^2. Define the event

$$E := E(n) := \big\{n^2 - C(\log n)^{1/2} n \le N \le n^2 + C(\log n)^{1/2} n\big\}.$$

Then on the event E we have

$$T^2\big(\Pi \cap [0,n]^2\big) \le T_B^2\big(\Pi \cap [0,n]^2, [0,n]^2\big) + \Delta(n),$$

where $\Delta(n) \le Cn^2(n^{-1/5})$ holds with high probability, that is, with probability at least $1 - n^{-A}$. Thus, with high probability we have

$$\begin{aligned} T_B^2\big(\Pi \cap [0,n]^2, [0,n]^2\big) &\le T^2\big(\Pi \cap [0,n]^2\big) \\ &\le T_B^2\big(\Pi \cap [0,n]^2, [0,n]^2\big) + Cn^{9/5}. \end{aligned}$$

This, together with the uniform bound $T^2(\Pi \cap [0,n]^2) \le Cn^2$, yields the desired conclusion (4.2), completing Step 2 and the proof of Theorem 1.1(a). $\square$

PROOF OF COROLLARY 1.2a. We next deduce Corollary 1.2(a). Since

$$\Pi \cap [0,n]^2 =_d n(U_1, \ldots, U_N),$$

where N is a Poisson random variable with parameter n^2 and which is independent of $U_1, U_2, \dots,$ we obtain from Theorem 1.1(a) that

$$T^2(U_1, \dots, U_N) \to_P \alpha\big(T_B^2, 2\big).$$

We may de-Poissonize this convergence result as follows. Once more we consider the event $E := E(n) := \{n^2 - C(\log n)^{1/2} n \le N \le n^2 + C(\log n)^{1/2} n\}$. Notice that since T^2 is uniformly bounded and $|N - n^2|/n$ has exponential tails, we obtain

$$\big|T^2(N) - T^2(n^2)\big| \cdot 1_{E^c} \to_P 0.$$

We now show convergence to zero on the set E:

$$\big|T^2(N) - T^2(n^2)\big| \cdot 1_E \to_P 0. \tag{4.10}$$

By Lemma 4.1 we have with high probability,

$$\big|T^2(N) - T^2(n^2)\big| \cdot 1_E \le C(\log n)^2 / n^{1/2},$$

and thus (4.10) is immediate.

We have thus shown that Corollary 1.2(a) holds over samples of size n^2. We now complete the proof by showing that we have convergence for samples of all sizes. This amounts to a simple interpolation. Let $n_0 := n_0(n)$ denote the unique integer such that $n_0^2 \le n < (n_0 + 1)^2$. We wish to show

$$\big|T^2(n) - T^2\big(n_0^2\big)\big| \to_P 0. \tag{4.11}$$

By the analog of smoothness (4.3) for the standard TSP process T^2 we have with high probability,

$$\big|T^2(n) - T^2\big(n_0^2\big)\big| \le C \log n_0 / n_0^{1/2},$$

since n_0^2 and n differ by at most Cn_0. The estimate (4.11) follows. This completes the proof of Corollary 1.2(a). □

5. Asymptotics for the MST. The MST is easier to handle than the TSP process. To deduce asymptotics for the MST, we may follow the methods of Section 4. Given U_i, $i \ge 1$, i.i.d. with the uniform distribution on $[0,1]^d$, abbreviate notation by writing $M_B^p(m)$ for $M_B^p(U_1, \dots, U_m)$ and similarly for $M^p(m)$. The analog of the smoothness Lemma 4.1 for the boundary MST process M_B^p takes the following form.

LEMMA 5.1 (Smoothness for MST). *For all $p > 0$ and $d \ge 1$ there is a $C := C(p, d)$ such that for all $n \ge 1$ and $1 \le k \le n^d/2$,*

$$\big|M_B^p(n^d) - M_B^p(n^d + k)\big| \le Ck(\log n / n)^p \tag{5.1}$$

with high probability. Moreover, for all $0 \le k \le n^d/2$,

$$\big|M_B^p(n^d) - M_B^p(n^d - k)\big| \le Ck(\log n / n)^p, \tag{5.2}$$

with high probability.

PROOF. Notice that

$$M_B^p(n^d + k) \le M_B^p(n^d) + \sum_{j=n^d+1}^{n^d+k} d^p\Big(U_j, \{U_i\}_{i=1}^{n^d}\Big),$$

where $d(x, F)$ denotes the Euclidean distance between the point x and the discrete set F. For all $j \ge n^d + 1$ we have $d(U_j, \{U_i\}_{i=1}^{n^d}) \le C(\log n)^{1/d}/n \le C \log n/n$ with high probability. We obtain for all $k \le n^d/2$ the high probability estimate

$$M_B^p(n^d + k) \le M_B^p(n^d) + Ck(\log n/n)^p.$$

To complete the proof of (5.1) we need to show the reverse high probability inequality

$$M_B^p(n^d) \le M_B^p(n^d + k) + Ck(\log n/n)^p.$$

We will first show

$$M_B^p(n^d) \le M_B^p(n^d + 1) + C(\log n/n)^p \tag{5.3}$$

and then iterate. Let $\{N_j\}_{j=1}^{M(d)}$ denote the neighbors of U_{n^d+1} given by the minimal spanning tree on $U_1, \dots, U_{n^d+1}$. Since vertices in minimal spanning trees have bounded degree, $M(d)$ is finite. With high probability there is a sample point, say WLOG U_1, such that U_{n^d+1} is within $C(\log n)^{1/d}/n$ of U_1. Replace all $M(d)$ edges E_i, $1 \le i \le M(d)$, having U_{n^d+1} as a vertex with edges leading to U_1 instead. For each $1 \le i \le M(d)$, this may be achieved at a cost of at most

$$\Big(|E_i| + C(\log n)^{1/d}/n\Big)^p \le C\Big((\log n)^{1/d}/n\Big)^p \le C(\log n/n)^p$$

since $|E_i| \le C(\log n)^{1/d}/n$ with high probability. The resulting graph gives a feasible spanning tree on $U_1, \dots, U_{n^d}$ showing the high probability bound

$$M_B^p(n^d) \le M_B^p(n^d + 1) + CM(d)(\log n/n)^p,$$

which is precisely (5.3). Iterating gives (5.1). The proof of (5.2) is similar. $\square$

It is straightforward to see that the estimates (5.1) and (5.2) hold when the boundary MST process M_B^p is replaced by the standard MST M^p. Using such estimates, it is easy to verify the analog of Step 1 in the context of the MST. To prove Theorem 1.1(b), it only remains to prove the analog of Step 2, namely, we must verify

$$|M^p\big(\Pi \cap [0, n]^d\big) - M_B^p\big(\Pi \cap [0, n]^d, [0, n]^d\big)/n^d \to 0 \quad \text{a.s.} \tag{5.4}$$

We rely upon a construction which consists of adding extra edges to the components formed by $M_B^p(\Pi \cap [0, n]^d, [0, n]^d)$. We will roughly follow the methods of Yukich (1995); for the sake of completeness, we provide the details.

Let the components of $M_B^p(\Pi \cap [0, n]^d, [0, n]^d)$ be given by $T_1, \dots, T_N$, N random. Let the rooted tree T_i have endpoint B_i on $\delta([0, 1]^d)$ and let M_i

denote the unique sample point which is rooted to B_i, $1 \le i \le N$. The sum of the pth powers of the lengths of the edges rooted to the boundary is small:

$$\sum_{i=1}^{N} |M_i - B_i|^p \le Cn^{d-1}(\log n)^p \tag{5.5}$$

with high probability. To see this, consider a subcube S of edge length $n - 2$ centered within $[0, n]^d$. Without loss of generality we may assume $M_i \in S$ for all $i \ge 1$, since with high probability there are at most Cn^{d-1} points in the moat $[0, n]^d \setminus S$ and these points contribute at most Cn^{d-1} to (5.5).

To complete the proof of (5.5), let F denote a face of $[0, n]^d$. Observe that S may be covered with Cn^{d-1} rectangular solids which are perpendicular to F, have height $n - 2$ and have a base with cross-sectional diameter 1. Geometric considerations show that every such solid contains at most one edge of the graph $M_B^p(\Pi \cap [0, n]^d, [0, n]^d)$ which is rooted to F. Were there two or more such edges this would contradict optimality, as it would be more efficient to join the points rooted to F with a single edge. By considering rectangular solids which are perpendicular to the remaining faces of $[0, n]^d$ we may easily conclude that given $M_i \in S$, there is a rectangular solid R such that among all sample points in R, M_i is the one which is closest to the boundary. Thus for all M_i in S we have

$$|M_i - B_i|^p \le C(\log n)^{p/d}$$

with high probability. Since there are as many points M_i in S as there are solids, (5.5) follows.

We now add three types of edges to the trees in $M_B^p(\Pi \cap [0, n]^d, [0, n]^d)$. For all $1 \le i \le N$ insert the edge F_i joining M_i to the nearest point in the grid $G := \{G_i\}_{i=1}^{n^{d-1}}$ of regularly spaced points on $\delta[0, n]^d$. Since each B_i is within C of a grid point in G, (5.5) and the triangle inequality imply the high probability estimate

$$S_F^p := \sum_{i=1}^{N} |F_i|^p \le Cn^{d-1}(\log n)^p. \tag{5.6}$$

Next, for all $1 \le i \le n^{d-1}$, add the edge E_i joining G_i to the nearest point in $\Pi n[0, n]^d$. There are n^{d-1} such edges and since each edge satisfies the high probability bound $|E_i| \le C(\log n)^{1/d}$, it follows that

$$S_E^p := \sum_{i=1}^{n^{d-1}} |E_i|^p \le Cn^{d-1}(\log n)^p \tag{5.7}$$

with high probability.

By inserting the two types of edges F_i, $1 \le i \le N$, and E_i, $1 \le i \le n^{d-1}$, we generate a boundary rooted tree on $G \cup (\Pi n[0, n]^d)$; this tree has disjoint components, say $T_1, \dots, T_L$, $L \le N$. Say that components are neighboring if they contain neighboring grid points.

The triangle inequality implies that we may tie together any two neighboring components with a third type of edge H, which joins points in $\Pi n[0, n]^d$

and has a length which may be bounded in terms of lengths of edges of the first two types. Let H_i, $i \geq 1$, denote an enumeration of all such edges. Then the H_i, $i \geq 1$, together with the edges in $M_B^p(\Pi \cap [0,n]^d)$, form a global tree T' through $G \cup (\Pi n[0,n]^d)$ and moreover the triangle inequality, (5.6) and (5.7) imply that

$$S_H^p := \sum_{i=1} |H_i|^p \leq C\big(S_E^p + S_F^p\big) \leq Cn^{d-1}(\log n)^p \tag{5.8}$$

with high probability. Moreover, by deleting all edges in T' which involve grid points, and by adding edges of the order $C(S_E^p + S_F^p)$, we form a feasible tree T through the smaller set $\Pi n[0,n]^d$, which shows

$$M^p\big(\Pi \cap [0,n]^d\big) \leq \sum_{e \in T} |e|^p \leq M_B^p\big(\Pi \cap [0,n]^d, [0,n]^d\big) + C\big(S_E^p + S_F^p\big).$$

Now (5.4) follows from (5.8). This finishes the proof of (5.4) and completes the proof of Theorem 1.1(b).

The proof of Corollary 1.2(b) follows exactly as in the proof of Corollary 1.2(a); no new ideas are needed.

6. Extensions and open problems. By viewing classical optimization problems as processes defined on the parameter set of rectangles, we have shown that the stochastic versions of these problems enjoy an intrinsic superadditive ergodic structure. This structure yields a.s. asymptotics for the TSP, MST and minimal matching processes. The ergodic theoretic approach is not limited to these three classical processes, but applies to all optimization problems which can be viewed as a stationary superadditive process. In this way one could capture similar strong laws for the semimatching problem [Steele (1992)], the k-median problem [Hochbaum and Steele (1982)] and related problems in geometric probability and computational geometry, including the total length of a Voronoi tessellation. Here are some open problems and directions for further research.

6.1. *Minimal matchings*. Given U_i, $i \geq 1$, i.i.d. on $[0,1]^d$, write $S^p(k)$ for $S^p(U_1, \dots, U_k)$. Since $|S^p(n) - S^p(m)| \leq C|n-m|^{(d-p)/d}$ for all integers n and m, it is not difficult to verify that the proof of Theorem 1.1 may be modified to treat minimal matchings for $0 < p < d$.

PROPOSITION 6.1. *For* $0 < p < d$,

$$\lim_{n \to \infty} S^p\big(\Pi \cap [0,n]^d\big)/n^d = \alpha\big(S_B^p, d\big) \quad a.s., \tag{6.1}$$

where $\alpha(S_B^p, d)$ *is the spatial constant for the boundary minimal matching process* S_B^p.

It is likely that the methods of Section 4 could be modified to show that (6.1) also holds for powers of p in the critical region $p \geq d$, but this is not yet settled. In particular, it is not clear whether a modification of Lemma 4.1

holds for minimal matching S^p. The difficulty here centers on the construction of a feasible matching on a uniform sample of cardinality $k + 2$ given the optimal minimal matching on a uniform sample of cardinality k: one can either add one long edge or several short edges, but it is unclear whether the average combined length of additional edges is of the required small order.

6.2. *TSP*. It appears likely that a refinement of the methods in Section 4 could be used to extend Theorem 1.1(a) to TSP processes T^p, where p exceeds d, but this is also not yet resolved.

6.3. *Infinite-volume limits over general averaging sets*. By considering the approach of Akcoglu and Krengel (1981) and Krengel and Pyke (1987), it is immediate that Proposition 3.1 holds in the context of averaging sets which are more general than the cube $[0, n]^d$. Moreover, the infinite-volume limit results of Proposition 3.1 hold *uniformly* over large collections of averaging sets.

To make these ideas precise, we adopt the following notation. For measurable $A \subset \mathbb{R}^d$, $\lambda(A)$ denotes the Lebesgue measure of A and δA denotes the boundary of A. If ρ is the Euclidean distance, let $A(\delta) := \{v \in \mathbb{R}^d\colon \rho(v, \delta A) < \delta\}$ be the δ-annulus of δA. Set $nA := \{nv\colon v \in A\}$. The following uniformity result is of special interest.

THEOREM B [Krengel and Pyke (1987)]. *Suppose $\mathscr{A}$ is a collection of Borel measurable subsets of $[0,1]^d$ such that*

$$\sup\{\lambda(A(\delta))\colon A \in \mathscr{A}\} \to 0 \quad \text{as } \delta \to 0. \tag{6.2}$$

Let L be a stationary superadditive process on $\mathbb{R}^d$ defined on the Borel measurable subsets of $\mathbb{R}^d$. Then there is an $f \in L^1$ such that

$$\sup_{A \in \mathscr{A}} \{|n^{-d} L(nA) - \lambda(A) f|\} \to 0 \quad a.s.\ \text{as } n \to \infty. \tag{6.3}$$

We immediately deduce a uniformity result for the classical optimization problems. Let L_B^p denote either the boundary TSP, MST or minimal matching processes and let $\alpha(L_B^p, d)$ be the associated spatial constants (3.2). For $A \in \mathscr{A}$, let $L_B^p(A) := L_B^p(A \cap \Pi, A)$.

COROLLARY 6.2. *Let $\mathscr{A}$ be a collection of sets satisfying the regularity condition* (6.2). *Then for all $d \in \mathbb{N}^+$ and $p > 0$,*

$$\sup_{A \in \mathscr{A}} \left\{\left|n^{-d} L_B^p(nA) - \lambda(A)\,\alpha(L_B^p, d)\right|\right\} \to 0 \quad a.s.\ \text{as } n \to \infty. \tag{6.4}$$

Thus (6.4) gives, for example, the asymptotics of the boundary TSP process uniformly over the convex subsets of $[0,1]^d$. From here it is possible to deduce similar uniform results for the standard TSP; we leave this as an exercise.

APPENDIX

PROOF OF LEMMA 2.2. In order to prove Lemma 2.2 we require an additional lemma. The proof of this result, which depends upon a dyadic subdivision of the cube, sheds some light on the structure of boundary processes.

LEMMA A. *Consider the graph given by a boundary* TSP *process* $T_B^p(F)$, $|F| = n$. *Then the sum of the* pth *powers of the lengths of the edges connecting vertices in* F *with* $\delta[0,1]^d$ *is bounded by* $C[n^{(d-p-1)/(d-1)} \vee \log n]$. *Identical estimates hold for the boundary MST and boundary minimal matching processes.*

PROOF. The proof depends upon a dyadic subdivision of $[0,1]^d$ similar to that of Redmond and Yukich (1994). Let Q_0 be the cube of edge length $1/3$ and centered within $[0,1]^d$. Let Q_1 be the cube of edge length $2/3$, also centered within $[0,1]^d$. Partition $Q_1 \setminus Q_0$ into subcubes of edge length $1/6$; it is easy to verify that the number of such subcubes is bounded by $C6^{d-1}$.

Continue with the subdivision recursively, so that at the jth stage we define cube Q_j of edge length $1 - 2(3 \cdot 2^j)^{-1}$ and partition $Q_j \setminus Q_{j-1}$ into subcubes of edge length $(3 \cdot 2^j)^{-1}$. The number of such subcubes is at most $C3^{d-1}(2^j)^{d-1}$. Carry out k stages, where k is the unique integer chosen so that

$$2^{(k-1)(d-1)} \leq n < 2^{k(d-1)}.$$

This recursive subdivision partitions Q_k into at most

$$\sum_{j=0}^{k} C3^{d-1}2^{j(d-1)} = Cn$$

subcubes with the property that each subcube has an edge length which is smaller than the distance between the subcube and the boundary of $[0,1]^d$. Furthermore, by partitioning each subcube of this partition into 2^{ld} congruent subcubes, where l is the least integer satisfying $2^l > d^{1/2}$, we obtain a partition $\mathscr{Q}$ of Q_k consisting of at most Cn subcubes with the property that the diameter of each subcube is less than the distance between it and the boundary.

Observe that in an optimal rooted path on F each subcube Q in $\mathscr{Q}$ contains at most two points in F which are rooted to the boundary. Indeed, if there were three or more points in $F \cap Q$ which were rooted to the boundary, then minimality tells us that *it would be more efficient to link two of these three points with an edge*, since the diameter of the subcube is less than the distance to the boundary.

The sum of the pth powers of the lengths of the edges connecting vertices in $F \cap (Q_j \setminus Q_{j-1})$ with the boundary is thus bounded by

$$C3^{d-1}2^{j(d-1)}(3 \cdot 2^j)^{-p}.$$

Summing over all $1 \le j \le k$ gives a bound for the sum of the pth powers of the lengths of the edges connecting points in $F \cap Q_k$,

$$\text{(A.1)} \qquad \sum_{j=1}^{k} C3^{d-1}2^{j(d-1)}(3 \cdot 2^j)^{-p} \le C\left(n^{(d-p-1)/(d-1)} \vee \log n\right).$$

The sum of the pth powers of the lengths of the edges connecting vertices in $F \cap ([0,1]^d \setminus Q_k)$ with the boundary is at most the product of n and the pth power of the width of the moat $[0,1]^d \setminus Q_k$, that is, at most

$$\text{(A.2)} \qquad Cn \cdot n^{-p/(d-1)} = Cn^{(d-p-1)/(d-1)}.$$

Combining (A.1) and (A.2) gives Lemma A for the TSP.

The proof of the analogous estimates involving the MST and minimal matching processes is identical, save for the observation that there is *at most one* vertex in each subcube of $\mathscr{Q}$ which is joined to the boundary. □

To conclude the proof of Lemma 2.2 for the TSP we need to show

$$T^p(F) \le T_B^p\left(F, [0,1]^d\right) + C\left[n^{(d-p-1)/(d-1)} \vee \log n\right].$$

Consider the minimal rooted path T given by $T_B^p(F, [0,1]^d)$ and let $F' \subset F$ denote those vertices which are rooted to the boundary by T. Let $\mathfrak{M} \subset \delta[0,1]^d$ denote the set of points where the edges in T meet the boundary; let $|\mathfrak{M}| = |F'| = N$. The goal is to use T to construct a feasible path through F.

Since $\mathfrak{M}$ lies on the $d-1$-dimensional boundary it is a simple matter to find a matching S' with power-weighted edges on $\mathfrak{M}$ such that the edges in S' are contained in $\delta[0,1]^d$ and the length $l(S')$ satisfies $l(S') \le C(n^{(d-p-1)/(d-1)} \vee 1)$. The matching S' generates tours $C_1, \dots, C_R$ $(R \le N)$ on the union $F \cup \mathfrak{M}$. Given tour C_i, $1 \le i \le R$, select a point $M_i \in \mathscr{M} \cap C_i$ and set $\mathfrak{M}' := \{M_1, \dots, M_R\}$. The triangle inequality, the estimate $l(S') \le C(n^{(d-p-1)/(d-1)} \vee 1)$ and Lemma A (for the TSP) together tell us that we may add and delete edges from the tours $C_1, \dots, C_R$ to generate tours $C_1', \dots, C_R'$ on the smaller set $F \cup \mathfrak{M}'$ at an extra cost of at most $C(n^{(d-p-1)/(d-1)} \vee \log n)$. Moreover, the sum of the pth powers of the lengths of the edges with a vertex in $\mathfrak{M}'$ is bounded by $C(n^{(d-p-1)/(d-1)} \vee \log n)$.

Finally, we construct a tour through $\mathfrak{M}'$ having edges on $\delta[0,1]^d$ and a length of at most $C(n^{(d-p-1)/(d-1)} \vee 1)$.

The above construction, which is achieved at a cost of at most $T_B^p(F, [0,1]^d) + C(n^{(d-p-1)/(d-1)} \vee \log n)$, generates a connected graph G through $F \cup \mathfrak{M}'$ consisting of tours $C_1', \dots, C_R'$ through $F \cup \mathfrak{M}'$ as well as a single tour through $\mathfrak{M}'$ with length at most $C(n^{(d-p-1)/(d-1)} \vee 1)$. Since the sum of the pth powers of the lengths of the edges in G with a vertex in $\mathfrak{M}'$ is bounded by $C(n^{(d-p-1)/(d-1)} \vee \log n)$, the triangle inequality implies that we may construct a tour through F at an extra cost of at most $C(n^{(d-p-1)/(d-1)} \vee \log n)$. We have thus shown

$$T^p(F) \le T_B^p\left(F, [0,1]^d\right) + C\left(n^{(d-p-1)/(d-1)} \vee \log n\right)$$

as desired. This completes the proof of Lemma 2.2 for the TSP. The proof for the MST and minimal matching processes is essentially the same. □

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