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ERGODIC THEORY
OF DIFFERENTIABLE DYNAMICAL SYSTEMS

by DAVID RUELLE

Dedicated to the memory of Rufus Bowen

Abstract. — If f is a $C^{1+\varepsilon}$ diffeomorphism of a compact manifold M, we prove the existence of stable manifolds, almost everywhere with respect to every f-invariant probability measure on M. These stable manifolds are smooth but do not in general constitute a continuous family. The proof of this stable manifold theorem (and similar results) is through the study of random matrix products (multiplicative ergodic theorem) and perturbation of such products.

0. Introduction.

Let M be a smooth compact manifold, f a diffeomorphism, and ρ an f-invariant probability measure on M. The asymptotic behavior for large n of the tangent map $T_x f^n$ is determined for ρ-almost all x by the multiplicative ergodic theorem of Oseledec [11]. This theorem (see (1.6) below) is a sort of spectral theorem for random matrix products. It treats the ergodic theory of the diffeomorphism f so to say in *linear approximation.* The aim of the present paper is to tackle the *nonlinear theory,* and our main result is an " almost everywhere " stable manifold theorem (see Theorem (6.3)). This theorem says that for ρ-almost all x, the points y such that the distance of $f^n x$ and $f^n y$ tends to zero at a suitable exponential rate (when $n \to +\infty$) form a differentiable manifold (¹). The proof goes via a study of perturbations of the matrix products (Theorem (4.1)) occurring in the multiplicative ergodic theorem. The proof of the multiplicative ergodic theorem given by Oseledec is not appropriate for our discussion, and we use a proof due to Raghunathan [15]. A version of this proof is reproduced in Section 1.

We have included in the present paper some results of general interest, which fitted naturally, but are not needed for the proof of Theorem (6.3). The reader who only wants to get to the stable manifold theorem may thus omit Section 3 and the

(¹) That something like this should be true was suggested by Smale in [20].

Appendices B and C. We have not tried to present all our results in the greatest generality. Since the articulation of the proofs is reasonably simple, the reader should be able to obtain further results without too much work.

Our theorem (6.3) is very close to results of Pesin ([12], [13], [14]) who has a stable manifold theorem almost everywhere with respect to a smooth invariant measure, assuming that such a measure exists. Our techniques are however rather different from those of Pesin. We refer the reader to the monograph of Hirsch, Pugh and Shub [6] for the much studied case where a continuous splitting of the tangent space exists.

The present paper originated in an attempt at proving certain conjectures on the asymptotic behavior of differentiable dynamical systems. These conjectures, presented in [18], generalize results obtained for Axiom A systems (see [19], [16], [2]). The results obtained here constitute a preliminary step towards proving the conjectures of [18]. Another step is contained in [17] (see also Katok [8]). Ultimately, this work should serve to determine the measures which describe hydrodynamic turbulence, and more generally the asymptotic behavior of dissipative physical systems.

(0.1) *Note on the multiplicative ergodic theorem.*

Besides its applications to differentiable dynamical systems, the multiplicative ergodic theorem has applications to algebraic groups. The idea is due to Margulis (see Tits [21]), and involves extending the theorem to local fields. The original proof of the multiplicative ergodic theorem is due to Oseledec, and applies to flows as well as maps. In view of the applications to algebraic groups, Raghunathan [15] devised a simpler proof, based on a theorem of Furstenberg and Kesten [4]. This theorem in turn is a corollary (Corollary (1.2) below) of Kingman's subadditive ergodic theorem ([9], [10]) (see Theorem (1.1) and Appendix A). An extension of the subadditive ergodic theorem to quasi-invariant measures has been obtained by Akcoglu and Sucheston [1], and would permit a similar extension of all our results. While Raghunathan's results apply to maps, an extension to flows, following the ideas of Oseledec, is easy, and carried out in Appendix B ([1]).

(0.2) *Terminology.*

Here are a few definitions which might be helpful for what follows.

A class Σ of subsets of a space M is a σ-algebra if $\emptyset \in \Sigma$, and if Σ is stable under countable intersections and complementation $(X \mapsto M \backslash X)$.

A (finite) *measure space* (M, Σ, ρ) is a space M with a σ-algebra Σ of subsets (measurable sets) and a countably additive function $\rho : \Sigma \to \mathbf{R}_+$. The function ρ is

([1]) I am indebted to A. Connes, M. Herman, and D. Sullivan for pointing out to me the literature on the subadditive ergodic theorem, and in general for encouragement in writing the present paper. I also want to thank J. Tits who informed me of the work of Raghunathan.

a (finite positive) *measure*. We also assume completeness: if $\rho(X)=0$ and $Y \subset X$ then $Y \in \Sigma$ (and $\rho(Y)=0$). If $\rho(M)=1$, we say that (M, Σ, ρ) is a *probability space*, and ρ a *probability measure*.

Let M be a topological space; the elements of the σ-algebra generated by the open sets are called *Borel sets*. In particular, if M is compact metrizable, and ρ is a positive Radon measure on M, one can define $\rho(X)$ when X is a Borel set. A measure space (M, Σ, ρ) is then defined where the measurable sets are all the sets $X \cup N$ with $N \subset Y$, X and Y Borel, and $\rho(Y)=0$.

Let S be a topological space and M a measure space (resp. a topological space). A map $\varphi : M \to S$ is called measurable (resp. Borel) if $\varphi^{-1}\mathcal{O}$ is measurable (resp. Borel) for every open $\mathcal{O} \subset S$. These definitions extend to sections of fiber bundles, using local trivializations. As usual a map from a measure space to a measure space is measurable if the inverse image of a measurable set is measurable.

1. Some basic results.

In this section (M, Σ, ρ) is a fixed probability space, and $\tau : M \to M$ is a measurable map preserving ρ. Almost everywhere means ρ-almost everywhere.

We denote by f^+ the positive part of a function $f : f^+(x) = \max(0, f(x))$.

Theorem **(1.1)** (Subadditive ergodic theorem).

Let $(f_n)_{n>0}$ *be a sequence of measurable functions* $M \to \mathbf{R} \cup \{-\infty\}$ *satisfying the conditions*:

a) *integrability*: $f_1^+ \in L^1(M, \rho)$;

b) *subadditivity*: $f_{m+n} \leq f_m + f_n \circ \tau^m$ a.e.

Then, there exists a τ-invariant measurable function $f : M \to \mathbf{R} \cup \{-\infty\}$ *such that* $f^+ \in L^1(M, \rho)$,

$$\lim_{n\to\infty} \frac{1}{n} f_n = f \text{ a.e.,}$$

and

$$\lim_{n\to\infty} \frac{1}{n} \int f_n(x)\rho(dx) = \inf_n \frac{1}{n} \int f_n(x)\rho(dx) = \int f(x)\rho(dx).$$

This is one version of Kingman's theorem (see [10], Theorem (1.8)). In Appendix A we reduce Theorem (1.1) to another version, for which an easy proof has been given by Derriennic [3].

Corollary **(1.2)**. — *Let* $T : M \to \mathbf{M}_m$ *be a measurable function to the real* $m \times m$ *matrices such that*

$$\log^+ \| T(\cdot) \| \in L^1(M, \rho).$$

Write $T_x^n = T(\tau^{n-1}x) \cdot \ldots \cdot T(\tau x) \cdot T(x)$.

Then there exists a τ-invariant measurable function $\chi : M \to \mathbf{R}\cup\{-\infty\}$ *such that* $\chi^+ \in L^1(M, \rho)$,

$$\lim_{n\to\infty} \frac{1}{n} \log\|T_x^n\| = \chi(x)$$

for almost all x, *and*

$$\lim_{n\to\infty} \frac{1}{n}\int \log\|T_x^n\|\rho(dx) = \inf_n \frac{1}{n}\int \log\|T_x^n\|\rho(dx) = \int \chi(x)\rho(dx).$$

This is proved by taking $f_n(x) = \log\|T_x^n\|$ in Theorem (1.1).

Proposition **(1.3)** [1]. — *Let* $(T_n)_{n>0}$ *be a sequence of real* $m\times m$ *matrices such that*

$$\limsup_{n\to\infty} \frac{1}{n}\log\|T_n\| \leq 0. \tag{1.1}$$

We write:

$$T^n = T_n \cdot \ldots \cdot T_2 \cdot T_1$$

and assume that the limits:

$$\lim_{n\to\infty} \frac{1}{n}\log\|(T^n)^{\wedge q}\|$$

exist for $q = 1, \ldots, m$. *Then*:

a) $$\lim_{n\to\infty} (T^{n*}T^n)^{1/2n} = \Lambda$$

exists, where $*$ *denotes matrix transposition.*

b) *Let* $\exp\lambda^{(1)} < \ldots < \exp\lambda^{(s)}$ *be the eigenvalues of* Λ (*real* $\lambda^{(r)}$, *possibly* $\lambda^{(1)} = -\infty$), *and* $U^{(1)}, \ldots, U^{(s)}$ *the corresponding eigenspaces. Writing* $V^{(0)} = \{0\}$ *and* $V^{(r)} = U^{(1)} + \ldots + U^{(r)}$, *we have*:

$$\lim_{n\to\infty} \frac{1}{n}\log\|T^n u\| = \lambda^{(r)} \quad \text{when} \quad u \in V^{(r)}\backslash V^{(r-1)}$$

for $r = 1, \ldots, s$.

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(1) If the assumptions of the proposition are satisfied, and $\det\Lambda \neq 0$ (*i.e.* $\lambda_1 > -\infty$), (1.1) can be replaced by

$$\lim_{n\to\infty} \frac{1}{n}\log\|T_n\| = \lim_{n\to\infty} \frac{1}{n}\log\|T_n^{-1}\| = 0.$$

(In view of *a*), $\lim_{N\to\infty} \frac{1}{N}\sum_{n=1}^{N} \log|\det T_n| = \log\det\Lambda$, hence $\lim_{n\to\infty}\frac{1}{n}\log|\det T_n| = 0$, and since

$$\|T_n^{-1}\| \leq \|T_n\|^{m-1}/|\det T_n|,$$

we have $\limsup_{n\to\infty} \frac{1}{n}\log\|T_n^{-1}\| \leq 0$).

Let $t_n^{(1)} \leq \ldots \leq t_n^{(m)}$ be the eigenvalues of $(T^{n*}T^n)^{1/2}$. By assumption, the limits:

$$\lim_{n\to\infty} \frac{1}{n} \log \prod_{p=m-q+1}^{m} t_n^{(p)} = \lim_{n\to\infty} \frac{1}{n} \log \|(T^n)^{\wedge q}\|$$

exist for $q = 1, \ldots, m$, and therefore also the limits:

$$\lim_{n\to\infty} \frac{1}{n} \log t_n^{(p)} = \chi^{(p)}$$

for $p = 1, \ldots, m$. Let $\lambda^{(1)} < \ldots < \lambda^{(s)}$ be the distinct $\chi^{(p)}$, and $U_n^{(r)}$ be the space spanned by the eigenvectors of $(T^{n*}T^n)^{1/2}$ corresponding to the eigenvalues $t_n^{(p)}$ such that

$$(1.2) \qquad \lim_{n\to\infty} \frac{1}{n} \log t_n^{(p)} = \lambda^{(r)}.$$

We interrupt now the proof of Proposition (1.3) for a lemma. For simplicity we shall assume that $\lambda^{(1)} \neq -\infty$.

Lemma **(1.4)**. — *Given* $\delta > 0$, *there is* $K > 0$ *such that, for all* $k > 0$,

$$(1.3) \qquad \max\{|(u, u')| : u \in U_n^{(r)}, u' \in U_{n+k}^{(r')}, \|u\| = \|u'\| = 1\} \leq K \exp(-n(|\lambda^{(r')} - \lambda^{(r)}| - \delta)).$$

We first prove (1.3) for $r < r'$. Equivalently, it suffices to prove that, if $v_{rr'}^k$ is the orthogonal projection of $u \in \sum_{t \leq r} U_n^{(t)}$ in $\sum_{t' \geq r'} U_{n+k}^{(t')}$, then

$$(1.4) \qquad \|v_{rr'}^k\| \leq K \|u\| \exp(-n(\lambda^{(r')} - \lambda^{(r)} - \delta)).$$

It will be convenient to assume δ less than all $|\lambda^{(r')} - \lambda^{(r)}|$ for $r \neq r'$, and to write $\delta^* = \delta/s$. In view of (1.1) there is $C > 0$ such that, for all n,

$$\log \|T_{n+1}\| \leq C + n\frac{\delta^*}{4}.$$

For large n we have thus:

$$\begin{aligned} \|v_{rr'}^1\| \exp\left((n+1)\left(\lambda^{(r')} - \frac{\delta^*}{4}\right)\right) &\leq \|T^{n+1}u\| \\ &\leq \|T_{n+1}\| . \|T^n u\| \\ &\leq \exp\left(C + n\frac{\delta^*}{4}\right) . \|u\| \exp\left(n\left(\lambda^{(r)} + \frac{\delta^*}{4}\right)\right). \end{aligned}$$

If n is so large that $C - \lambda^{(r')} + \frac{\delta^*}{4} \leq n\frac{\delta^*}{4}$, this gives:

$$\|v_{rr'}^1\| \leq \|u\| \exp(-n(\lambda^{(r')} - \lambda^{(r)} - \delta^*)).$$

From this we obtain in particular:

$$\begin{aligned} \|v_{r,r+1}^k\| &\leq \sum_{j=0}^{k-1} \|u\| \exp(-(n+j)(\lambda^{(r+1)} - \lambda^{(r)} - \delta^*)) \\ &\leq K_1 \|u\| \exp(-n(\lambda^{(r+1)} - \lambda^{(r)} - \delta^*)) \end{aligned}$$

with $K_1=(1-\exp(-(\lambda^{(r+1)}-\lambda^{(r)}-\delta^*)))^{-1}$. Therefore also:

$$||v^k_{r,r+2}||\leq \sum_{j=0}^{k-1}||u||\exp(-(n+j)(\lambda^{(r+2)}-\lambda^{(r)}-\delta^*))$$
$$+\sum_{j=0}^{k-1}K_1||u||\exp(-n(\lambda^{(r+1)}-\lambda^{(r)}-\delta^*))\exp(-(n+j)(\lambda^{(r+2)}-\lambda^{(r+1)}-\delta^*))$$
$$\leq K_2||u||\exp(-n(\lambda^{(r+2)}-\lambda^{(r)}-2\delta^*)).$$

In general:

$$||v^k_{rr'}||\leq K_{r'-r}||u||\exp(-n(\lambda^{(r')}-\lambda^{(r)}-(r'-r)\delta^*).$$

Since $(r'-r)\delta^*<\delta$, this proves (1.4).

Notice that the lemma gives bounds on the elements of the $m\times m$ matrix S of scalar products between the eigenvectors of $(T^{n*}T^n)^{1/2}$ and those of $(T^{(n+k)*}T^{n+k})^{1/2}$. We have proved up to now the bounds for the elements on one side of the diagonal of S. The other bounds are readily obtained from the calculation of $S^*=S^{-1}$ by the minors of S. Allowing for change of δ and K, it suffices to use the bounds already obtained, and the fact that all matrix elements are bounded by 1 in absolute value. This conclude the proof of the lemma.

Lemma (1.4) shows that $(U_n^{(r)})_{n>0}$ is a Cauchy sequence for each r. Part a) of Proposition (1.3) follows from this and (1.2). Let $U^{(r)}=\lim_{n\to\infty}U_n^{(r)}$; (1.3) then becomes:

$$\max\{|(u,u')|: u\in U^{(r)}, u'\in U_n^{(r')}, ||u||=||u'||=1\}\leq K\exp(-n(|\lambda^{(r')}-\lambda^{(r)}|-\delta)).$$

Therefore we have, for large n, if $0\neq u\in U^{(r)}$,

$$\lambda^{(r)}-2\delta\leq\frac{1}{n}\log\frac{||T^n u||}{||u||}\leq\lambda^{(r)}+2\delta,$$

hence:

$$\lim_{n\to\infty}\frac{1}{n}\log||T^n u||=\lambda^{(r)}\qquad\text{if}\qquad u\in U^{(r)}\backslash\{0\}$$

and part b) of the proposition follows.

Corollary **(1.5)** (*of Proposition* (1.3)). — *Let* $\lambda^{(r)}<\lambda<\lambda^{(r+1)}$ (*put* $\lambda^{(r+1)}=+\infty$ *if* $r=s$). *Then:*

$$R=\{u\in\mathbf{R}^m: ||T^n u||<e^{n\lambda}\text{ for all } n\geq 0\}$$

is a bounded open neighborhood of 0 *in* $V^{(r)}$.

That $R\subset V^{(r)}$ is clear from Proposition (1.3) b). Furthermore, we have:

$$\lim_{n\to\infty}||T^n u||e^{-n\lambda}=0$$

uniformly for u in the unit ball B of $V^{(r)}$. Since $R \subset B$, there is N finite such that

$$R = \{u \in V^{(r)} : \|T^n u\| < e^{n\lambda} \text{ for } 0 \leq n \leq N\}$$

proving the Corollary.

Theorem **(1.6)** (Multiplicative ergodic theorem). — *Let* $T : M \to \mathbf{M}_m$ *be a measurable function to the real* $m \times m$ *matrices such that*

$$\log^+ \|T(\cdot)\| \in L^1(M, \rho). \tag{1.5}$$

Write $T_x^n = T(\tau^{n-1}x) \cdot \ldots \cdot T(\tau x) . T(x)$, *and use* * *to denote matrix transposition.*

There is $\Gamma \subset M$ *such that* $\tau\Gamma \subset \Gamma$, $\rho(\Gamma) = 1$, *and the following properties hold if* $x \in \Gamma$:

a) $$\lim_{n \to \infty} (T_x^{n*} T_x^n)^{1/2n} = \Lambda_x$$

exists.

b) *Let* $\exp \lambda_x^{(1)} < \ldots < \exp \lambda_x^{(s)}$ *be the eigenvalues of* Λ_x (*where* $s = s(x)$, *the* $\lambda_x^{(r)}$ *are real, and* $\lambda_x^{(1)}$ *may be* $-\infty$), *and* $U_x^{(1)}, \ldots, U_x^{(s)}$ *the corresponding eigenspaces. Let* $m_x^{(r)} = \dim U_x^{(r)}$. *The functions* $x \mapsto \lambda_x^{(r)}$, $m_x^{(r)}$ *are* τ-*invariant. Writing* $V_x^{(0)} = \{0\}$ *and* $V_x^{(r)} = U_x^{(1)} + \ldots + U_x^{(r)}$, *we have*:

$$\lim_{n \to \infty} \frac{1}{n} \log \|T_x^n u\| = \lambda_x^{(r)} \quad \textit{when} \quad u \in V_x^{(r)} \backslash V_x^{(r-1)}$$

for $r = 1, \ldots, s$.

According to (1.5) and the ergodic theorem, there is $\Gamma_1 \subset M$ such that $\tau\Gamma_1 \subset \Gamma_1$, $\rho(\Gamma_1) = 1$, and

$$\lim_{n \to \infty} \frac{1}{n} \log^+ \|T(\tau^{n-1}x)\| = 0 \quad \text{if} \quad x \in \Gamma_1.$$

By Corollary (1.2), there is also Γ_2 such that $\tau\Gamma_2 \subset \Gamma_2$, $\rho(\Gamma_2) = 1$, and, for $q = 1, \ldots, m$,

$$\lim_{n \to \infty} \frac{1}{n} \log \|(T_x^n)^{\wedge q}\|$$

exists, and is a τ-invariant function of x.

Let $\Gamma = \Gamma_1 \cap \Gamma_2$. The theorem follows from Proposition (1.3) applied to $T_n = T(\tau^{n-1}x)$ for $x \in \Gamma$.

Corollary **(1.7)**. — *Let* $x \in \Gamma$, $u \in \mathbf{R}^m$; *then:*

$$\lim_{n \to \infty} \frac{1}{n} \log \|T_x^n u\| = \chi(x, u) \tag{1.6}$$

exists, finite or $-\infty$. *If* $\lambda \in \mathbf{R}$, *the linear space*

$$V_x^\lambda = \{u \in \mathbf{R}^m : \chi(x, u) \leq \lambda\}$$

is a measurable function of $x \in \Gamma$.

This is an immediate consequence of Theorem (1.6). We have $\chi(x, u) = \lambda_x^{(r)}$ if $u \in V_x^{(r)} \backslash V_x^{(r-1)}$, and $V_x^\lambda = \bigcup \{V_x^{(r)} : \lambda_x^{(r)} \leq \lambda\}$.

Remark **(1.8)**. — (1.6) implies

$$\chi(\tau x, \mathrm{T}(x)u) = \chi(x, u).$$

In particular $\mathrm{T}(x)\mathrm{V}_x^\lambda \subset \mathrm{V}_{\tau x}^\lambda$, $\mathrm{T}(x)\mathrm{V}_x^{(r)} \subset \mathrm{V}_{\tau x}^{(r)}$. If $\lambda_x^{(1)} \neq -\infty$, $\mathrm{T}(x)$ is invertible and therefore $\mathrm{T}(x)\mathrm{V}_x^{(r)} = \mathrm{V}_{\tau x}^{(r)}$, $\mathrm{T}(x)\mathrm{V}_x^\lambda = \mathrm{V}_{\tau x}^\lambda$. On the other hand, the $\mathrm{U}_x^{(r)}$ do not transform simply under $\mathrm{T}(x)$.

2. The spectrum.

As in Section 1, $(\mathrm{M}, \Sigma, \rho)$ is a probability space, and $\tau : \mathrm{M} \to \mathrm{M}$ a measurable map preserving ρ; $\mathrm{T} : \mathrm{M} \to \mathbf{M}_m$ is a measurable function such that

$$\log^+ \| \mathrm{T}(\cdot) \| \in \mathrm{L}^1(\mathrm{M}, \rho).$$

We write $\mathrm{T}_x^n = \mathrm{T}(\tau^{n-1}x) \cdot \ldots \cdot \mathrm{T}(\tau x) \cdot \mathrm{T}(x)$. According to Corollary (1.2) and the multiplicative ergodic theorem (Theorem (1.6) and Corollary (1.7)), there is $\Gamma \subset \mathrm{M}$ with $\tau\Gamma \subset \Gamma$, $\rho(\Gamma) = 1$, such that, if $x \in \Gamma$, we can define Λ_x; $s = s(x)$; $\lambda_x^{(1)} < \ldots < \lambda_x^{(s)} = \chi(x)$; $\mathrm{U}_x^{(1)}, \ldots, \mathrm{U}_x^{(s)}$; $\{0\} = \mathrm{V}_x^{(0)} \subset \mathrm{V}_x^{(1)} \subset \ldots \subset \mathrm{V}_x^{(s)} = \mathbf{R}^m$; and the functions $u \mapsto \chi(x, u)$, $\lambda \mapsto \mathrm{V}_x^\lambda$.

Let $m_x^{(r)} = \dim \mathrm{U}_x^{(r)} = \dim \mathrm{V}_x^{(r)} - \dim \mathrm{V}_x^{(r-1)}$. The numbers $\lambda_x^{(r)}$ are called *characteristic exponents*; with the multiplicities $m_x^{(r)}$ they constitute the *spectrum* of (τ, T), or T, at x. We shall say that $\mathrm{V}_x^{(1)} \subset \ldots \subset \mathrm{V}_x^{(s)}$ is the *associated filtration* of $\mathbf{R}^m$. The spectrum is τ-invariant. If ρ is τ-ergodic, the spectrum is almost everywhere constant. In what follows we shall determine the spectrum of $(\tau, \mathrm{T}^\wedge)$, $(\tau^{-1}, \mathrm{T}^*)$ and (τ, T^{*-1}).

(2.1) *Spectrum of* $(\tau, \mathrm{T}^\wedge)$.

Let $\mathrm{T}^{\wedge p} : \mathrm{M} \mapsto \mathbf{M}_{\binom{m}{p}}$ be the p-th exterior power of T. We have:

$$\mathrm{T}^{\wedge p}(\tau^{n-1}x) \cdot \ldots \cdot \mathrm{T}^{\wedge p}(\tau x) \cdot \mathrm{T}^{\wedge p}(x) = (\mathrm{T}_x^n)^{\wedge p}$$

and

$$\lim_{n \to \infty} ((\mathrm{T}_x^n)^{\wedge p *} (\mathrm{T}_x^n)^{\wedge p})^{1/2n} = \Lambda_x^{\wedge p}.$$

This determines the spectrum of $\mathrm{T}^{\wedge p}$ and the associated filtration of $\mathbf{R}^{\binom{m}{p}}$.

Writing $\mathrm{T}^\wedge = \bigoplus_{p=0}^{m} \mathrm{T}^{\wedge p}$, we obtain in particular:

$$\lim_{n \to \infty} \frac{1}{n} \log \| (\mathrm{T}_x^n)^\wedge \| = \sum_{r : \lambda_x^{(r)} > 0} m_x^{(r)} \lambda_x^{(r)}.$$

(2.2) *Spectrum of* $(\tau^{-1}, \mathrm{T}^*)$.

Suppose that τ has a measurable inverse, we shall show that the spectrum of $(\tau^{-1}, \mathrm{T}^*)$ is almost everywhere the same as that of (τ, T). Let $\hat{\Lambda}_x = \lim_{n \to \infty} (\hat{\mathrm{T}}_x^{n*} \hat{\mathrm{T}}_x^n)^{1/2n}$ where $\hat{\mathrm{T}}_x^n = \mathrm{T}^*(\tau^{-n+1}x) \cdot \ldots \cdot \mathrm{T}^*(\tau^{-1}x) \cdot \mathrm{T}^*(x)$. Since the spectrum of $\hat{\Lambda}_x$ is τ-invariant it is also the limit almost everywhere of the spectra of the $(\check{\mathrm{T}}_x^{n*} \check{\mathrm{T}}_x^n)^{1/2n}$, where

$$\check{\mathrm{T}}_x^n = \mathrm{T}^*(x) \mathrm{T}^*(\tau x) \cdot \ldots \cdot \mathrm{T}^*(\tau^{n-1}x).$$

The spectrum of $\check{T}_x^{n*}\check{T}_x^n$ is the same as that of $\check{T}_x^n\check{T}_x^{n*}=T_x^{n*}T_x^n$. Therefore the spectrum of $\hat{\Lambda}_x$ is the same as the spectrum of Λ_x.

(2.3) *Spectrum of* (τ, T^{*-1}).

Suppose that T is almost everywhere invertible and that

$$\log^+||T^{-1}(\cdot)||\in L^1(M, \rho).$$

Define $\widetilde{\Lambda}_x=\lim_{n\to\infty}(\widetilde{T}_x^{n*}\widetilde{T}_x^n)^{1/2n}$, where $\widetilde{T}_x^n=T^{*-1}(\tau^{n-1}x)\cdot\ldots\cdot T^{*-1}(\tau x)\,.\,T^{*-1}(x)$. We have then $\widetilde{\Lambda}_x=\Lambda_x^{-1}$. Therefore the spectrum of (τ, T^{*-1}) is obtained by changing the sign of the spectrum of $(\tau, T):\widetilde{\lambda}_x^{(r)}=-\lambda_x^{(s-r+1)}$. The filtration of $\mathbf{R}^m$ associated with (τ, T^{*-1}) is the orthogonal of the filtration associated with $(\tau, T):\widetilde{V}_x^{(r)}=V_x^{(s-r)\perp}$.

3. The invertible case.

In this section, (M, Σ, ρ) is a probability space, and $\tau: M\to M$ is a measurable map with measurable inverse preserving ρ.

Theorem **(3.1)**. — *Let* $T: M\to\mathbf{GL}_m$ *be a measurable function to the invertible real* $m\times m$ *matrices, such that*

$$\log^+||T(\cdot)||,\ \log^+||T^{-1}(\cdot)||\in L^1(M, \rho).$$

Write:

$$T_x^n=T(\tau^{n-1}x)\cdot\ldots\cdot T(\tau x)\,.\,T(x)$$

$$T_x^{-n}=T^{-1}(\tau^{-n}x)\cdot\ldots\cdot T^{-1}(\tau^{-2}x)\,.\,T^{-1}(\tau^{-1}x).$$

There is then $\Delta\subset M$ *such that* $\tau\Delta=\Delta$, $\rho(\Delta)=1$, *and a measurable splitting* $x\mapsto W_x^{(1)}\oplus\ldots\oplus W_x^{(s)}$ *of* $\mathbf{R}^m$ *over* Δ *(with* $s=s(x)$*), such that*

$$\lim_{k\to\pm\infty}\frac{1}{k}\log||T_x^k u||=\lambda_x^{(r)} \quad \textit{if} \quad 0\neq u\in W_x^{(r)}.$$

Let again the numbers $\lambda_x^{(1)}<\ldots<\lambda_x^{(s)}$ with multiplicities $m_x^{(1)}, \ldots, m_x^{(s)}$ constitute the spectrum of (τ, T) at x. Let $V_x^{(1)}\subset\ldots\subset V_x^{(s)}$ be the associated filtration of $\mathbf{R}^m$. From Sections (2.2) and (2.3) we know that the spectrum of $(\tau^{-1}, T^{-1}\circ\tau^{-1})$ at x consists of the numbers $-\lambda_x^{(s)}<\ldots<-\lambda_x^{(1)}$ with multiplicities $m_x^{(s)}, \ldots, m_x^{(1)}$. Let:

$$V_x^{(-s)}\subset\ldots\subset V_x^{(-1)}$$

be the associated filtration. Suppose that we can show that

$$\textbf{(3.1)}\qquad V_x^{(r-1)}\cap V_x^{(-r)}=\{0\}$$

$$\textbf{(3.2)}\qquad V_x^{(r-1)}+V_x^{(-r)}=\mathbf{R}^m$$

for $r=2, \ldots, s$, and almost all x. Then, putting

$$W_x^{(r)}=V_x^{(r)}\cap V_x^{(-r)}$$

we obtain:

$$\mathbf{R}^m = V_x^{(-1)} \cap (V_x^{(1)} + V_x^{(-2)}) \cap (V_x^{(2)} + V_x^{(-3)}) \cap \ldots \cap V_x^{(s)}$$
$$= W_x^{(1)} \oplus W_x^{(2)} \oplus \ldots \oplus W_x^{(s)}$$

and the theorem holds. It remains thus to prove (3.1) and (3.2).

Define S as the set of those x such that (3.1) does not hold. Given $\delta > 0$ and $r \in [2, s]$, let S_n be the subset of S such that, if $x \in S_n$,

(3.3) $$||T_x^n u|| \leq ||u|| \exp n(\lambda_x^{(r-1)} + \delta) \quad \text{and}$$

(3.4) $$||T_x^{-n} u|| \leq ||u|| \exp n(-\lambda_x^{(r)} + \delta)$$

for all $u \in V_x^{(r-1)} \cap V_x^{(-r)}$. From (3.4) we get, if $x \in \tau^{-n} S_n$,

(3.5) $$||T_x^n u|| \geq ||u|| \exp n(\lambda_x^{(r)} - \delta)$$

for all $u \in V_x^{(r-1)} \cap V_x^{(-r)}$. For $x \in S_n \cap \tau^{-n} S_n$, (3.3) and (3.5) yield $\lambda_x^{(r)} - \lambda_x^{(r-1)} \leq 2\delta$. Since $\rho(S_n \cap \tau^{-n} S_n) \to \rho(S)$ we have $\lambda_x^{(r)} - \lambda_x^{(r-1)} \leq 2\delta$ for almost all $x \in S$ and, since δ is arbitrary, we get $\rho(S) = 0$. We have proved (3.1); (3.2) follows because

$$\dim V_x^{(r-1)} + \dim V_x^{(-r)} = m.$$

(3.2) *Spectrum and associated splitting.*

The characteristic exponents $\lambda_x^{(1)} < \ldots < \lambda_x^{(s)}$ with multiplicities $m_x^{(r)} = \dim W_x^{(r)}$ constitute the spectrum of (τ, T) at x. We call $W_x^{(1)} \oplus \ldots \oplus W_x^{(s)}$ the *associated splitting* of $\mathbf{R}^m$. Notice that the $\lambda_x^{(r)}$ are all finite, and that:

$$T(x) W_x^{(r)} = W_{\tau x}^{(r)} \qquad r = 1, \ldots, s.$$

(See Remark (1.8).)

The spectrum of $(\tau, T^{\wedge p})$ at x consists of the numbers $\mu = \sum_r n_r \lambda_x^{(r)}$ with $0 \leq n_r \leq m_x^{(r)}$, and $\sum_r n_r = p$. The subspace corresponding to μ in the associated splitting of $\mathbf{R}^{\binom{p}{m}}$ is generated by $u_1 \wedge \ldots \wedge u_m$ where $u_j \in W_x^{(r_j)}$ and $\sum_{j=1}^{p} \lambda^{(r_j)} = \mu$. (This follows readily from Section (2.1).)

The spectrum of $(\tau^{-1}, T^{-1} \circ \tau^{-1})$ at x consists of the numbers $-\lambda_x^{(s)} < \ldots < -\lambda_x^{(1)}$ with multiplicities $m_x^{(s)}, \ldots, m_x^{(1)}$. The associated splitting of $\mathbf{R}^m$ is $W_x^{(s)} \oplus \ldots \oplus W_x^{(1)}$.

The spectrum of (τ, T^{*-1}) at x consists of the numbers $-\lambda_x^{(s)} < \ldots < -\lambda_x^{(1)}$ with multiplicities $m_x^{(s)}, \ldots, m_x^{(1)}$. The associated splitting of $\mathbf{R}^m$ is $W_x^{(-s)} \oplus \ldots \oplus W_x^{(-1)}$ where $W_x^{(-r)}$ is the orthogonal complement of $\sum_{r' : r' \neq r} W_x^{(r')}$ in $\mathbf{R}^m$. (This follows readily from Section (2.3).)

The spectrum of $(\tau^{-1}, T^* \circ \tau^{-1})$ at x is the same as that of (τ, T). The associated splitting of $\mathbf{R}^m$ is $W_x^{(-1)} \oplus \ldots \oplus W_x^{(-s)}$. (This follows from what has been said of (τ, T^{*-1}) and $(\tau^{-1}, T^{-1} \circ \tau^{-1})$.)

Corollary **(3.3)**. — *Define*:

$$\nu_r(x) = \max\{|(u, u')| : u \in W_x^{(r)},\ u' \in \sum_{r' : r' \neq r} W^{(r')},\ ||u|| = ||u'|| = 1\}$$

(*put* $\gamma_r(x) = 0$ *if* $s(x) = 1$). *Then*:

$$\begin{aligned}\delta_r(x) &= (1 - \gamma_r(x)^2)^{1/2}\\ &= \min_{u \in W_x^{(r)} : ||u|| = 1} \max\{|(u, v)| : v \in W_x^{(-r)},\ ||v|| = 1\}\end{aligned}$$

and

$$\lim_{k \to \pm\infty} \frac{1}{k} \log \delta_r(\tau^k x) = 0.$$

Let indeed $p = m_x^{(r)}$, $q = m - p$, $0 \neq w \in (W_x^{(r)})^{\wedge p}$, $0 \neq w' \in (\sum_{r' : r' \neq r} W_x^{(r')})^{\wedge q}$, then:

$$||((T_x^k)^{\wedge p} w) \wedge ((T_x^k)^{\wedge q} w')|| \leq \delta_r(\tau^k x)\, ||(T_x^k)^{\wedge p} w||\,.\,||(T_x^k)^{\wedge q} w'||$$

and it suffices to apply what has been said on the spectrum of $T^{\wedge}$ in Section (3.2).

4. A perturbation theorem.

Theorem **(4.1)**. — *Let* $T = (T_n)_{n>0}$ *be a sequence of real* $m \times m$ *matrices such that* [1]

(4.1) $$\limsup_{n \to \infty} \frac{1}{n} \log ||T_n|| \leq 0.$$

We write $T^n = T_n . \cdots . T_2 . T_1$ *and assume the existence of*

(4.2) $$\lim_{n \to \infty} (T^{n*} T^n)^{1/2n} = \Lambda$$

with $\det \Lambda \neq 0$. *Denote by* $\lambda^{(1)} < \ldots < \lambda^{(s)}$ *the eigenvalues of* $\log \Lambda$.

Let $\eta > 0$ *be given and, for* $T' = (T'_n)_{n>0}$, *write*

$$||T' - T|| = \sup_n ||T'_n - T_n||\, e^{3n\eta}$$

and $T'^n = T'_n . \cdots . T'_2 . T'_1$. *Then there are* δ, $A > 0$ *and, given* $\varepsilon > 0$, *there are* $B_\varepsilon > 0$, $B'_\varepsilon > 1$ *with the following properties*:

If $||T' - T|| \leq \delta$,

(4.3) $$\lim_{n \to \infty} (T'^{n*} T'^n)^{1/2n} = \Lambda'$$

exists and has the same eigenvalues as Λ *(including multiplicity). Furthermore, if* $P^{(r)}(T')$ *denotes the orthogonal projection of* Λ' *corresponding to* $\exp \lambda^{(r)}$, *and* $||T'' - T|| \leq \delta$, *we have*:

(4.4) $$||P^{(r)}(T') - P^{(r)}(T'')|| \leq A\, ||T' - T''||$$

(4.5) $$B_\varepsilon \exp n(\lambda^{(r)} - \varepsilon) \leq ||T'^n P^{(r)}(T')|| \leq B'_\varepsilon \exp n(\lambda^{(r)} + \varepsilon).$$

---

(1) Instead of (4.1) one could write:

$$\lim_{n \to \infty} \frac{1}{n} \log ||T_n|| = 0.$$

See the footnote to Proposition (1.3).

If (4.1) holds, it is known (Proposition (1.3) a)) that the existence of the limit (4.2) is equivalent to the existence of the limits

$$\lim_{n\to\infty}\frac{1}{n}\log||(T^n)^{\wedge q}||$$

for $q=1, \ldots, m$. Since (4.1) and $||T'-T||<+\infty$ imply

$$\limsup_{n\to\infty}\frac{1}{n}\log||T'_n||\leq 0,$$

(4.3) will follow if we can prove the existence of

$$\lim_{n\to\infty}\frac{1}{n}\log||(T'^n)^{\wedge q}||$$

for $q=1, \ldots, m$. Furthermore these limits determine uniquely the eigenvalues of Λ'. Therefore, to prove (4.3) and the fact that Λ' has the same eigenvalues as Λ, it suffices to show that

$$\lim_{n\to\infty}\frac{1}{n}\log||(T'^n)||^{\wedge q}=\lim_{n\to\infty}\frac{1}{n}\log||(T^n)^{\wedge q}||. \tag{4.6}$$

Let $0<\eta'<\eta$ and define:

$$||T'^{\wedge q}-T^{\wedge q}||=\sup_n||T'^{\wedge q}_n-T^{\wedge q}_n||\,e^{3n\eta'}.$$

Then (4.1) implies the existence of $E_q>0$ such that

$$||T'^{\wedge q}-T^{\wedge q}||\leq E_q||T'-T|| \tag{4.7}$$

for $\delta\leq 1$. Therefore, the replacements $T_n\mapsto T^{\wedge q}_n$, $T'_n\mapsto T'^{\wedge q}_n$ reduce the proof of (4.6) to the case $q=1$, *i.e.*:

$$\lim_{n\to\infty}\frac{1}{n}\log||T'^n||=\lambda^{(s)}.$$

Equivalently, it suffices to find an open set $U\subset\mathbf{R}^m$ such that

$$\lim_{n\to\infty}\frac{1}{n}\log||T'^n u||=\lambda^{(s)}\quad\text{for}\quad u\in U.$$

To see this take $u^{(1)}, \ldots, u^{(m)}$ linearly independent in U and notice that the matrix norm $|||\,.\,|||$ defined by:

$$|||X|||=||Xu^{(1)}||+\ldots+||Xu^{(m)}||$$

is equivalent to $||\cdot||$. The existence of the limit (4.3), and the fact that Λ and Λ' have the same eigenvalues, are therefore a consequence of the following result:

Lemma **(4.2)**. — *Let* $\lambda^{(r(1))}\leq\ldots\leq\lambda^{(r(m))}=\lambda^{(s)}$ *be the eigenvalues of* $\log\Lambda$ *repeated according to multiplicity. Let* $\xi^{(0)}_1, \ldots, \xi^{(0)}_m$ *be unit vectors spanning* $\mathbf{R}^m$ *and such that*

$$\lim_{n\to\infty}\frac{1}{n}\log||T^n\xi^{(0)}_k||=\lambda^{(r(k))}. \tag{4.8}$$

There is then $\delta>0$ *such that*

$$\lim_{n\to\infty}\frac{1}{n}\log||\mathrm{T}'^{n}u||=\lambda^{(s)}$$

whenever $0<\alpha\leq 1$, $||\mathrm{T}'-\mathrm{T}||\leq\delta\alpha$, *and* $u\in\mathrm{U}$, *where*:

$$\mathrm{U}=\left\{\sum_{k=1}^{m-1}u_k\frac{\xi_k^{(0)}}{\alpha}+u_m\xi_m^{(0)}:\ \max_{k<m}|u_k|<|u_m|\right\}.$$

The existence of $\xi_1^{(0)},\ldots,\xi_m^{(0)}$ satisfying (4.8) follows from Proposition (1.3) b). The reason for not assuming the $\xi_k^{(0)}$ orthogonal will appear in Remark (4.7).

(4.3) *Proof of the lemma and further inequalities.*

By Proposition (1.3) a):

(4.9)
$$\lim_{n\to\infty}\frac{1}{n}\log||\mathrm{T}^n\xi_1^{(0)}\wedge\ldots\wedge\mathrm{T}^n\xi_m^{(0)}||=\sum_{k=1}^{m}\lambda^{(r(k))}.$$

Let $\xi_k^{(n)}$ be a unit vector proportional to $\mathrm{T}^n\xi_k^{(0)}$, and write:

(4.10)
$$\mathrm{T}_n\xi_k^{(n-1)}=t_k^{(n)}\xi_k^{(n)}.$$

Let also $\xi_{jk}^{(n)}$ be the j-th component of $\xi_k^{(n)}$. The matrix $\xi^{(n)}=(\xi_{jk}^{(n)})$ satisfies $||\xi^{(n)}||<\sqrt{m}$ and, because of (4.9),

$$\lim_{n\to\infty}\frac{1}{n}\log|\det\xi^{(n)}|=0.$$

Therefore

$$\lim_{n\to\infty}\frac{1}{n}\log||\xi^{(n)-1}||=0$$

and given $\varepsilon>0$, we have:

(4.11)
$$\mathrm{D}_\varepsilon=\sup_n e^{-n\varepsilon}||\xi^{(n)-1}||<+\infty.$$

We write $\mathrm{D}_\eta=\mathrm{D}$.

In view of proving (4.5) we shall obtain a result somewhat stronger than the lemma. We suppose that $||\mathrm{T}'-\mathrm{T}||\leq\delta\alpha$ and estimate the components $u_1^{(n)},\ldots,u_{m-1}^{(n)},u_m^{(n)}$ of $\mathrm{T}'^n u$ along $\xi_1^{(n)}/\alpha,\ldots,\xi_{m-1}^{(n)}/\alpha,\xi_m^{(n)}$, for any $u\neq 0$ in $\mathbf{R}^m$.

Let μ be the smallest integer such that

(4.12)
$$(\forall n)\quad\max_{j\leq\mu}|u_j^{(n)}|\geq\max_{k>\mu}|u_k^{(n)}|.$$

(In particular if $u\in\mathrm{U}$, we have $\mu=m$.) Because of (4.10) and (4.11) we have:

$$|u_k^{(n)}|\leq t_k^{(n)}|u_k^{(n-1)}|+\mathrm{D}\delta e^{-2n\eta}\sum_\ell|u_\ell^{(n-1)}|.$$

We weaken these inequalities if we replace the $t_k^{(n)}$ by $t_k^{(n)*}\geq t_k^{(n)}$ such that

a) $$\lim_{N\to\infty}\frac{1}{N}\sum_{n=1}^{N}\log t_k^{(n)*}=\lambda^{(r(\mu))}\quad\text{for}\quad k\leq\mu\quad\text{and}$$

b) $$t_\mu^{(n)*}=t_\mu^{(n)}.$$

In view of (4.8), (4.10), this can be achieved by multiplying the sequences $(t_k^{(n)})$, for $k<\mu$, by constants ≥ 1. Since $\eta>0$, *a)* implies the existence of $C>0$ such that, for all $\nu\geq 0$, $N>\nu$, and $k,\ell\leq\mu$,

$$\textbf{(4.13)}\qquad \prod_{n=\nu+1}^{N-1} t_\ell^{(n)*}\Big/\prod_{n=\nu+1}^{N} t_k^{(n)*}\leq Ce^{N\eta}.$$

One can also choose C independent of μ.

In view of the above we have $|u_k^{(n)}|\leq U_k^{(n)}$ for $n\geq\nu$, provided

$$U_k^{(\nu)}=U^{(\nu)}=\max_\ell |u_\ell^{(\nu)}|,$$

and $$U_k^{(n)}\geq t_k^{(n)*}U_k^{(n-1)}+D\delta e^{-2n\eta}.m.\max_{\ell\leq\mu}U_\ell^{(n-1)}.$$

Using (4.13) we see that this is satisfied by

$$\textbf{(4.14)}\qquad U_k^{(N)}=\prod_{n=\nu+1}^{N} t_k^{(n)*}.\prod_{n=\nu+1}^{N}(1+mCD\delta e^{-n\eta}).U^{(\nu)}.$$

We choose

$$\textbf{(4.15)}\qquad \delta=\frac{1}{mCD}\prod_{n=1}^{\infty}(1-e^{-n\eta})^2.$$

In this way $mCD\delta<1$, and

$$\textbf{(4.16)}\qquad C'=\frac{\prod_{n=1}^{\infty}(1+mCD\delta e^{-n\eta})}{\prod_{n=1}^{\infty}(1-e^{-n\eta})}\leq\prod_{n=1}^{\infty}\frac{1+e^{-n\eta}}{1-e^{-n\eta}}\leq\prod_{n=1}^{\infty}(1-e^{-n\eta})^{-2}=\frac{1}{mCD\delta}.$$

Therefore (4.14) gives:

$$\textbf{(4.17)}\qquad |u_k^{(N)}|\leq U_k^{(N)}\leq C'\prod_{n=\nu+1}^{N} t_k^{(n)*}.\prod_{n=\nu+1}^{N}(1-e^{-n\eta}).U^{(\nu)}.$$

In view of the definition of μ by (4.12), we may choose ν such that

$$|u_\mu^{(\nu)}|=\max_k |u_k^{(\nu)}|=U^{(\nu)}.$$

Using (4.13) and (4.17) we obtain then, for $N>\nu$:

$$\begin{aligned}|u_\mu^{(N)}|&\geq t_\mu^{(N)}|u_\mu^{(N-1)}|-D\delta e^{-2N\eta}\sum_\ell |u_\ell^{(N-1)}|\\&\geq t_\mu^{(N)}|u_\mu^{(N-1)}|-mCC'D\delta e^{-N\eta}\prod_{n=\nu+1}^{N} t_\mu^{(n)}.\prod_{n=\nu+1}^{N-1}(1-e^{-n\eta}).|u_\mu^{(\nu)}|.\end{aligned}$$

Using (4.16) gives:

$$|u_\mu^{(N)}| \geqslant t_\mu^{(N)}(|u_\mu^{(N-1)}| - e^{-N\eta} \prod_{n=\nu+1}^{N-1} t_\mu^{(n)} \cdot \prod_{n=\nu+1}^{N-1} (1-e^{-n\eta}) \cdot |u_\mu^{(\nu)}|)$$

which implies, by induction,

$$\textbf{(4.18)} \qquad |u_\mu^{(N)}| \geq \prod_{n=\nu+1}^{N} t_\mu^{(n)} \cdot \prod_{n=\nu+1}^{N} (1-e^{-n\eta}) \cdot |u_\mu^{(\nu)}|.$$

From (4.11), (4.17) and (4.18) we obtain:

$$\textbf{(4.19)} \qquad \lim_{n\to\infty} \frac{1}{N} \log ||T'^n u|| = \lambda^{(r(\mu))}.$$

In particular, if $u \in U$ we have $r(\mu) = s$, and the lemma results from (4.19).

(4.4) *Partial proof of* (4.4).

Suppose that the eigenvalue $\exp \lambda^{(s)}$ is simple, *i.e.* the corresponding projection $P^{(s)}(T)$ is one-dimensional. If $u = \sum_{k=1}^{m-1} u_k \frac{\xi_k^{(0)}}{\alpha} + u_m \xi_m'^{(0)} \notin U$, we have:

$$||P^{(s)}(T)u|| \leq |u_m| \leq \max_{k<m} |u_k| \leq \alpha ||\xi^{(0)-1}|| . ||u||.$$

Let ξ be a unit vector in the range of $P^{(s)}(T)$. Since the kernel of $P^{(s)}(T')$ cannot intersect U, we have, in view of the above estimate and triangle similarity:

$$||(1-P^{(s)}(T'))P^{(s)}(T)|| = ||(1-P^{(s)}(T'))\xi|| \leq \alpha ||\xi^{(0)-1}||,$$

hence

$$||P^{(s)}(T')P^{(s)}(T)||^2 \geq 1 - \alpha^2 ||\xi^{(0)-1}||^2.$$

We apply this result to the situation where Λ' is replaced by $\Lambda'^{\wedge p}$, p being the sum of the multiplicities of the largest eigenvalues of Λ' corresponding to the projections $P^{(r)}(T'), \ldots, P^{(s)}(T')$.

Writing α_p instead of α, $\alpha' = \alpha_p ||(\xi^{(0)-1})^{\wedge p}||$, and

$$\widetilde{P} = P^{(r)}(T) + \ldots + P^{(s)}(T)$$

$$\widetilde{P}' = P^{(r)}(T') + \ldots + P^{(s)}(T')$$

we obtain:

$$||\widetilde{P}^{\wedge p}\widetilde{P}'^{\wedge p}\widetilde{P}^{\wedge p}|| = ||\widetilde{P}'^{\wedge p}\widetilde{P}^{\wedge p}||^2 \geq 1 - \alpha'^2.$$

$\widetilde{P}\widetilde{P}'\widetilde{P}$ has at most p non zero eigenvalues, with product $\geq 1-\alpha'^2$, so each eigenvalue is $\geq 1-\alpha'^2$. Therefore $||\widetilde{P} - \widetilde{P}\widetilde{P}'\widetilde{P}|| \leq \alpha'^2$, or $||(1-\widetilde{P}')\widetilde{P}|| \leq \alpha'$. Similarly:

$$||(1-\widetilde{P})\widetilde{P}'|| \leq \alpha',$$

so that $||\tilde{P}'-\tilde{P}||\leq 2\alpha'$ provided $||T'^{\wedge p}-T^{\wedge p}||\leq\delta_p\alpha_p$ with δ_p determined by the lemma. In view of (4.7), we can take $\alpha_p=\frac{1}{\delta_p}E_p||T'-T||$. This is less than 1 because we choose δ in the theorem $<\delta_p/E_p$ for each p. Thus:

$$||\tilde{P}'-\tilde{P}||\leq 2\alpha'=\frac{2}{\delta_p}E_p||(\xi^{(0)-1})^{\wedge p}||.||T'-T||.$$

Therefore:

$$||P^{(r)}(T')-P^{(r)}(T)||\leq A||T'-T|| \tag{4.20}$$

with

$$A=\max_p\frac{4}{\delta_p}E_p||(\xi^{(0)-1})^{\wedge p}||. \tag{4.21}$$

(4.5) *Proof of* (4.5).

If u is in the range of $P^{(r)}(T')$, and $u\neq 0$, (4.19) shows that $r(\mu)=r$. In particular, we may use (4.17) with $\nu=0$ to obtain

$$||T'^n u||\leq B'_\varepsilon||u||\exp n(\lambda^{(r)}+\varepsilon)$$

which is the second half of (4.5).

If $u\in U$, then $\mu=m$, and one can take $\nu=0$ in (4.18). Therefore (4.11), (4.17), and (4.18) show that, given $\varepsilon>0$, there are C_ε, $C'_\varepsilon>0$ such that

$$C_\varepsilon||u||\exp n(\lambda^{(s)}-\varepsilon)\leq||T'^n u||\leq C'_\varepsilon||u||\exp n(\lambda^{(s)}+\varepsilon). \tag{4.22}$$

We shall now prove that δ may be decreased so that these inequalities hold for all u in the range of $P^{(s)}(T')$ when $||T'-T||\leq\delta$.

Let u be a unit vector in the range of $P^{(s)}(T)$, and u' be such that

$$||u'||\leq 1,\qquad ||u'-u||\leq(2m||\xi^{(0)-1}||)^{-1}. \tag{4.23}$$

Write

$$u=\sum_k u_k\xi_k^{(0)},\qquad u'=\sum_k u'_k\xi_k^{(0)}.$$

Then $\sum_{k:r(k)=s} u_k\xi_k^{(0)}$ has norm ≥ 1. Therefore $|u_k|>\frac{1}{m}$ for some k with $r(k)=s$ and, by renumbering the $\xi_k^{(0)}$, we may assume $|u_m|>\frac{1}{m}$. Since

$$|u'_m-u_m|\leq||\xi^{(0)-1}||.||u'-u||\leq\frac{1}{2m},$$

we have $|u'_m|>\frac{1}{2m}$. And since $||u'||\leq 1$ we have $|u'_k|\leq||\xi^{(0)-1}||$ for $k<m$. Therefore $u'\in U$ when $\alpha\leq(2m||\xi^{(0)-1}||)^{-1}$. According to (4.20), every vector in the range of

$P^{(s)}(T')$ is proportional to u' satisfying (4.23), provided $A\|T'-T\|\leq(2m\|\xi^{(0)-1}\|)^{-1}$. We also want $\|T'-T\|\leq\alpha\delta$. This is achieved by replacing δ by

(4.24) $$\min((2m\|\xi^{(0)-1}\|)^{-1}\delta,\ (2mA\|\xi^{(0)-1}\|)^{-1}).$$

With this choice (4.22) holds whenever u is in the range of $P^{(s)}(T')$ and $\|T'-T\|\leq\delta$.

Let q be the sum of the multiplicities of $\exp\lambda^{(r+1)}, \ldots, \exp\lambda^{(s)}$, and apply (4.22) with T' replaced by $T'^{\wedge q}$ and $T'^{\wedge(q+1)}$. One finds, for u in the range of $P^{(r)}(T')$, and $v\neq 0$ in the range of $(P^{(r+1)}(T')+\ldots+P^{(s)}(T'))^{\wedge q}$:

$$\|T'^n u\|\geq\frac{\|T'^n u\wedge(T'^n)^{\wedge q}v\|}{\|(T'^n)^{\wedge q}v\|}\geq\frac{C_{\varepsilon/2}^{(q+1)}}{C_{\varepsilon/2}'^{(q)}}\cdot\|u\|.\exp n(\lambda^{(r)}-\varepsilon).$$

Therefore:

$$\|T'^n P^{(r)}(T')\|\geq B_\varepsilon \exp n(\lambda^{(r)}-\varepsilon).$$

This completes the proof of (4.5).

(4.6) *Proof of* (4.4).

The earlier " partial proof of (4.4) " in Section (4.4) yields (4.20). We obtain (4.4) from (4.20) by the replacement $T\mapsto T''$ if A can be chosen independent of T''. In view of (4.21) this is achieved if we can replace T by T'' in Lemma (4.2) and get bounds on δ_p^{-1}, E_p (defined by (4.7)) and $\|\xi^{(0)-1}\|$ uniform in T''. Since $\|T_n''\|\leq\|T_n\|+\delta$ it is easy to obtain a bound on E_p. We take the vectors $\xi_1^{(0)}, \ldots, \xi_m^{(0)}$ in the lemma to be orthogonal, so that $\|\xi^{(0)-1}\|=1$.

The choice of δ made in the proof of the lemma is given by (4.15). Therefore it suffices that we find upper bounds to C and D independent of T''. Remember that C is given by (4.13), and D is given by (4.11). In view of (4.5) we can bound C by $(B'_{\eta/4}/B_{\eta/4})^2.\exp(-\lambda^{(s)})$. Applying (4.5) to $T''^{\wedge m}$ we obtain an estimate:

$$|\det\xi^{(n)}|\geq\frac{B_\varepsilon^{(m)}}{(B'_\varepsilon)^m}\exp(-n(m+1)\varepsilon).$$

Taking $\varepsilon=\eta/(m+1)$ yields the desired bound on D.

(4.7) *Complement to Theorem* (4.1).

If, instead of $(T_n)_{n>0}$, we consider the sequence $T^{(\ell)}=(T_{n+\ell})_{n>0}$, the conditions of Theorem (4.1) are again satisfied. We check here that δ^{-1}, A, and B'_ε can be chosen to increase with ℓ at most like $e^{3\ell\eta}$, $e^{2\ell\eta}$, and $e^{\ell\eta}$ respectively. This result will be used in Remark (5.2) *c)* and the proof of Theorem (6.3).

First, we replace in Lemma (4.2) the vectors $\xi_1^{(0)}, \ldots, \xi_m^{(0)}$ by $\xi_1^{(\ell)}, \ldots, \xi_m^{(\ell)}$. Then D_ε and C are multiplied at most by $e^{\ell\varepsilon}$ and $e^{\ell\eta}$. Therefore δ is multiplied by a factor not smaller than $e^{-2\ell\eta}$.

Replacement of T by $T^{\wedge p}$ replaces δ by δ_p which is multiplied by a factor not smaller than $e^{-2\ell\eta'}$. The E_p (see (4.7)) are multiplied by at most $e^{3\ell(\eta-\eta')}$, and therefore δ_p/E_p is multiplied by at least $e^{-\ell(3\eta-\eta')}$. Remember that $\min_p \delta_p/E_p$ is the choice of δ used to prove the existence of the limit (4.3), and also in Section (4.4). From (4.11) it follows that the choice of A given by (4.21) does not grow faster than $e^{\ell(3\eta-\eta'+\varepsilon)}$. The choice of δ in (4.24) therefore does not decrease faster than $e^{-\ell(3\eta-\eta'+2\varepsilon)}$, *i.e.* $e^{-3\ell\eta}$ if $\eta'=2\varepsilon$; going over to $\min \delta_p/E_p$ does not change this.

In Section (4.5), B'_ε, C'_ε, C_ε^{-1}, B_ε^{-1} do not increase faster than $e^{\ell\varepsilon'}$ for any $\varepsilon'>0$, *e.g.* $\varepsilon'=\eta$. Therefore in Section (4.6) we obtain finally that A does not increase faster than $e^{2\ell\eta}$.

5. A nonlinear ergodic theorem.

In what follows we denote by $B(\alpha)$ the open unit ball of radius α centered at the origin of $\mathbf{R}^m$, and by $\overline{B}(\alpha)$ its closure. We shall say that a map is of class $C^{r,\theta}$ if its derivatives up to order r are Hölder continuous of exponent θ; similarly for manifolds.

Theorem **(5.1)**. — *Let* (M, Σ, ρ) *be a probability space and* $\tau: M\to M$ *a measurable map preserving* ρ. *Given an integer* $r\geq 1$, *and* $\theta\in(0, 1]$, *let* $x\mapsto F_x$ *map* M *to* $C^{r,\theta}(\overline{B}(1), 0; \mathbf{R}^m, 0)$. *We write*

$$F_x^n = F_{\tau^{n-1}x}\circ\ldots\circ F_{\tau x}\circ F_x$$

and denote by $T(x)$ *the derivative of* F_x *at* 0. *We assume that* $x\mapsto T(x)$, $\|F_x\|_{r,\theta}$ *are measurable and that*

$$\int \log^+\|F_x\|_{r,\theta}\,\rho(dx)<+\infty. \tag{5.1}$$

We choose $\lambda<0$ *and assume that almost everywhere the spectrum of* T *at x contains neither* λ *nor* $-\infty$ (*the spectrum is finite, in particular* $T(x)$ *is invertible*).

There is then a measurable set $\Gamma\subset M$ *such that* $\tau\Gamma\subset\Gamma$, $\rho(\Gamma)=1$, *and there are measurable functions* $\beta>\alpha>0$, $\gamma>1$ *on* Γ *with the following properties:*

a) *If* $x\in\Gamma$ *the set*

$$\mathcal{V}_x^\lambda=\{u\in\overline{B}(\alpha(x)) : \|F_x^n u\|\leq\beta(x)e^{n\lambda} \text{ for all } n\geq 0\}$$

is a $C^{r,\theta}$ *submanifold of* $\overline{B}(\alpha(x))$, *tangent at* 0 *to* V_x^λ.

b) *If* $u, v\in\mathcal{V}_x^\lambda$, *then*

$$\|F_x^n u-F_x^n v\|\leq\gamma(x)\|u-v\|e^{n\lambda}.$$

If ρ *is ergodic, the spectrum may be assumed constant on* Γ. *If* $\lambda'<\lambda$ *and the interval* $[\lambda', \lambda]$ *is disjoint from the spectrum, there exists* γ' *measurable on* Γ *with the property:*

b') *when* $u, v\in\mathcal{V}_x^\lambda$, *then*

$$\|F_x^n u-F_x^n v\|\leq\gamma'(x)\|u-v\|e^{n\lambda'}.$$

We first study the case $r=1$; the case $r>1$ will be dealt with later.

We may take $\Gamma \subset M$ such that $\tau\Gamma \subset \Gamma$, $\rho(\Gamma)=1$, and, if $x \in \Gamma$:

(5.2) $$\lim_{n\to\infty} (T_x^{n*} T_x^n)^{1/2n} = \Lambda_x,$$

(5.3) $$\det \Lambda_x \neq 0, \quad \det(\Lambda_x - e^\lambda 1) \neq 0 \quad \text{and}$$

(5.4) $$\lim_{n\to\infty} \frac{1}{n} \log^+ \| F_{\tau^{n-1}x} \|_{1,\theta} = 0.$$

This follows from Theorem (1.6) for (5.2), by assumption for (5.3) and from (5.1) and the ergodic theorem for (5.4). Notice that (5.4) implies

$$\lim_{n\to\infty} \frac{1}{n} \log^+ \| T(\tau^{n-1}x) \| = 0.$$

Let $0 < 4\eta \leq -\lambda\theta$; we may then write, using (5.4),

(5.5) $$\begin{cases} G = \sup_n \| F_{\tau^{n-1}x} \|_{1,\theta} \exp(-n\eta - \lambda\theta) < +\infty \\ \| F_{\tau^{n-1}x} \|_{1,\theta} \exp(n(\lambda\theta + 3\eta) - \lambda\theta) \leq G. \end{cases}$$

Given $x \in \Gamma$, we write $\Lambda_x = \Lambda$ and let $\log \Lambda$ have the eigenvalues $\lambda^{(1)} < \ldots < \lambda^{(s)}$ (characteristic exponents) with the multiplicities $m^{(1)}, \ldots, m^{(s)}$. Let $V^{(1)} \subset \ldots \subset V^{(s)}$ be the associated filtration of $\mathbf{R}^m$. We assume that $\lambda^{(p)}$ is the largest characteristic exponent $< \lambda$. Therefore, with the notation of Corollary (1.7), $V_x^\lambda = V^{(p)}$. We write

(5.6) $$\varepsilon = \lambda - \lambda^{(p)}.$$

Given β, $0 < \beta \leq 1$, we shall use the definitions:

(5.7) $$S^\nu(\beta) = \{u \in \mathbf{R}^m : \| F_x^n u \| \leq \beta e^{n\lambda} \text{ for } 0 \leq n \leq \nu\},$$

(5.8) $$S(\beta) = \{u \in \mathbf{R}^m : \| F_x^n u \| \leq \beta e^{n\lambda} \text{ for all } n \geq 0\}.$$

There are δ, $A > 0$ such that Theorem (4.1) holds with η as defined above and $T_n = T(\tau^{n-1}x)$. We can make δ smaller so that

(5.9) $$A\delta < \frac{1}{\sqrt{2}}$$

and then choose $\beta = \beta(x)$ satisfying

(5.10) $$0 < \beta < 1, \quad G\beta^\theta < \delta.$$

The functions $x \mapsto \delta$, A may be assumed measurable, as follows from their (essentially) explicit construction in the proof of Theorem (4.1). Therefore also $x \mapsto \beta$ may be assumed measurable.

Take $\varkappa > 1$ such that $\varkappa\beta \leq 1$, $G(\varkappa\beta)^\theta \leq \delta$. We shall show that there is $\alpha \in (0, \beta)$ such that, for all $\nu > 0$,

(5.11) $$\overline{B}(\alpha) \cap S^\nu(\beta) \cap \{u \in \mathbf{R}^m : \| T_{f^\nu x}^{n-\nu} F_x^\nu u \| \leq \beta e^{n\lambda} \text{ for all } n > \nu\} = \overline{B}(\alpha) \cap S^\nu(\varkappa\beta) \cap (F_x^\nu)^{-1} V_{f^\nu x}^\lambda.$$

Let indeed $u \in S^\nu(\varkappa\beta) \cap (F_x^\nu)^{-1} V_{f^\nu x}^\lambda$. The $m \times m$ matrices:

$$T'_n = \int_0^1 DF_{\tau^{n-1}x}(tF_x^{n-1}u)\,dt \quad \text{if} \quad n \leq \nu,$$
$$T'_n = T_n \quad \text{if} \quad n > \nu,$$

are such that

$$T'^n u = T'_n \cdot \ldots \cdot T'_1 u = F_x^n u \quad \text{if} \quad n \leq \nu,$$

and using (5.7), (5.5), we have

$$\|T' - T\| = \sup_n \|T'_n - T_n\| e^{3n\eta} \leq \sup_{n \leq \nu} \|DF_{\tau^{n-1}x}\|_\theta (\varkappa\beta)^\theta \exp(n(\lambda\theta + 3\eta) - \lambda\theta) \leq G(\varkappa\beta)^\theta \leq \delta.$$

Therefore, Theorem (4.1) applies. In particular u is in the space $V'^{(p)} \subset \mathbf{R}^m$ spanned by the eigenvectors of $\log \Lambda'$ corresponding to the eigenvalues $\lambda^{(1)}, \ldots, \lambda^{(p)}$. Using (5.6), (4.5) gives

$$\|T'^n u\| \leq B'_\varepsilon e^{n\lambda} \|u\|$$

uniformly in ν and $u \in S^\nu(\kappa\beta) \cap (F_x^\nu)^{-1} V_{f^\nu x}^\lambda$. If $\alpha = \beta / B'_\varepsilon < \beta$, we see that the right-hand side of (5.11) is contained in the left-hand side. The converse inclusion is immediate. As for β, we can assume that $x \mapsto \alpha$ is measurable.

Let $D^\nu(\alpha)$ be the set defined by (5.11). Since the boundary of $S^\nu(\kappa\beta)$ is disjoint from $S^\nu(\beta)$, and hence from $D^\nu(\alpha)$, we conclude from (5.11) that $D^\nu(\alpha)$ *is open and closed in* $\overline{B}(\alpha) \cap (F_x^\nu)^{-1} V_{f^\nu x}^\lambda$.

Let now $u, v \in D^\nu(\alpha)$ or $u, v \in \overline{B}(\alpha) \cap S(\beta) = \upsilon_x^\lambda$ (in the latter case, write $\nu = \infty$). The $m \times m$ matrices

$$T'_n = \int_0^1 DF_{\tau^{n-1}x}(tF_x^{n-1}u + (1-t)F_x^{n-1}v)\,dt \quad \text{if} \quad n \leq \nu,$$
$$T'_n = T_n \quad \text{if} \quad n > \nu,$$

are such that

$$T'^n(u - v) = T'_n \cdot \ldots \cdot T'_1(u - v) = F_x^n u - F_x^n v \quad \text{if} \quad n \leq \nu,$$

and, using (5.7) or (5.8), and (5.10), we have

$$\|T' - T\| \leq \sup_{n \leq \nu} \|DF_{\tau^{n-1}x}\|_\theta \beta^\theta \exp(n(\lambda\theta + 3\eta) - \lambda\theta)$$
$$\leq G\beta^\theta < \delta.$$

Therefore Theorem (4.1) applies and, since $u - v$ is in the range of $P^{(p)}(T')$, (4.5) yields

(5.12) $$\|F_x^n u - F_x^n v\| \leq \gamma \|u - v\| e^{n\lambda}.$$

In this formula we have written $\gamma = B'_\varepsilon > 1$, and $x \mapsto \gamma$ may be assumed measurable. This proves part b) of the theorem. Part b') is also obtained if we take $\varepsilon = \lambda' - \lambda^{(p)}$ instead of (5.6). From (4.4) we obtain:

(5.13) $$\|(1 - P^{(p)}(T))(u - v)\| = \|(P^{(p)}(T') - P^{(p)}(T))(u - v)\| \leq A\delta \|u - v\|$$

which implies:

$$(5.14) \qquad ||(1-P^{(p)}(T))(u-v)|| \leq \frac{A\delta}{\sqrt{1-(A\delta)^2}}||P^{(p)}(T)(u-v)||.$$

Define $\Phi : (V^{(p)} \cap \overline{B}(\alpha)) \times (V^{(p)\perp} \cap \overline{B}(\alpha)) \to \overline{B}(\alpha)$ by:

$$\Phi(u_1, u_2) = \frac{u_1}{\alpha}\sqrt{\alpha^2 - ||u_2||^2} + u_2.$$

Let $\Phi(u_1, u_2), \Phi(u_1', u_2') \in D^\nu(\alpha)$ or $\overline{B}(\alpha) \cap S(\beta)$. Then (5.13) yields $||u_2||, ||u_2'|| \leq A\delta\alpha$ and, by (5.14):

$$\begin{aligned}\frac{\sqrt{1-(A\delta)^2}}{A\delta}\cdot||u_2-u_2'|| &\leq \left\|\frac{1}{\alpha}\sqrt{\alpha^2-||u_2||^2}u_1 - \frac{1}{\alpha}\sqrt{\alpha^2-||u_2'||^2}u_1'\right\| \\ &\leq \left|\sqrt{\alpha^2-||u_2||^2}-\sqrt{\alpha^2-||u_2'||^2}\right|\frac{||u_1||}{\alpha} \\ &\qquad + \frac{1}{\alpha}\sqrt{\alpha^2-||u_2'||^2}\cdot||u_1-u_1'|| \\ &\leq \frac{A\delta\alpha\,|\,||u_2||-||u_2'||\,|}{\sqrt{\alpha^2-(A\delta\alpha)^2}} + ||u_1-u_1'|| \\ &\leq \frac{A\delta}{\sqrt{1-(A\delta)^2}}||u_2-u_2'|| + ||u_1-u_1'||\end{aligned}$$

so that

$$(5.15) \qquad ||u_2-u_2'||\left(\frac{\sqrt{1-(A\delta)^2}}{A\delta} - \frac{A\delta}{\sqrt{1-(A\delta)^2}}\right) \leq ||u_1-u_1'||.$$

In view of (5.9) the expression in parenthesis is >0. Since $D^\nu(\alpha)$ is open and closed in $\overline{B}(\alpha) \cap (F_x^\nu)^{-1}V^\lambda_{f^\nu x}$ as a consequence of (5.11), we conclude from (5.15) that $D^\nu(\alpha)$ is the connected component of 0 in $\overline{B}(\alpha) \cap (F_x^\nu)^{-1}V^\lambda_{f^\nu x}$. Furthermore $\Phi^{-1}D^\nu(\alpha)$ is the graph of a C^1 function $\varphi^\nu : V^{(r)} \cap \overline{B}(\alpha) \to V^{(r)\perp} \cap \overline{B}(\alpha)$ with derivative bounded uniformly with respect to ν.

Let φ be the limit of a uniformly convergent subsequence of (φ_ν). Since $\Phi(\text{graph } \varphi^\nu) = D^\nu(\alpha) \subset \overline{B}(\alpha) \cap S^\nu(\beta)$, we have $\Phi(\text{graph } \varphi) \subset \overline{B}(\alpha) \cap S(\beta)$. The converse inclusion follows from (5.15) applied to $\overline{B}(\alpha) \cap S(\beta)$. Therefore

$$\Phi(\text{graph } \varphi) = \overline{B}(\alpha) \cap S(\beta) = \upsilon_x^\lambda,$$

and, by uniqueness of φ,

$$\lim_{\nu\to\infty} \varphi^\nu = \varphi \quad \text{uniformly.}$$

Let $u, v \in D^\nu(\alpha)$ and define $m \times m$ matrices:

$$T_n' = DF_{\tau^{n-1}x}(F_x^{n-1}u), \qquad T_n'' = DF_{\tau^{n-1}x}(F_x^{n-1}v) \quad \text{if} \quad n \leq \nu,$$

$$T_n' = T_n'' = T_n \quad \text{if} \quad n > \nu.$$

Then $\|T'-T\|$, $\|T''-T\|<\delta$. Using (5.12) we have also

$$\|T'_n-T''_n\|\leq\|DF_{\tau^{n-1}x}\|_\theta . \|F_x^{n-1}u-F_x^{n-1}v\|^\theta$$
$$\leq\|F_{\tau^{n-1}x}\|_{1,\theta}\gamma^\theta\|u-v\|^\theta\exp(n-1)\lambda\theta$$

if $n\leq\nu$, and therefore

$$\|T'-T''\|\leq G\gamma^\theta\|u-v\|^\theta.$$

By (4.4) we have then

$$\|P^{(p)}(T')-P^{(p)}(T'')\|\leq(AG\gamma^\theta)\|u-v\|^\theta,$$

where the ranges of $P^{(p)}(T')$ and $P^{(p)}(T'')$ are the tangent spaces to $D^\nu(\alpha)$ at u and v. Letting $\nu\to\infty$ we find that the tangent space to $\overline{B}(\alpha)\cap S(\beta)=\upsilon_x^\lambda$ at w also depends Hölder continuously on w, with exponent θ. This tangent space is the range of $P^{(p)}(\widetilde{T})$, where $\widetilde{T}_n=DF_{\tau^{n-1}x}(F_x^{n-1}w)$ for all n: to see this notice that we may assume $\|T'-\widetilde{T}\|\to 0$ as $\nu\to\infty$, and apply (4.4). In particular the tangent space to υ_x^λ at 0 is $P^{(p)}(T)$, *i.e.* V_x^λ. This proves part *a)* of the theorem when $r=1$.

We prove now that υ_x^λ is $C^{r,\theta}$ by induction on r for $r>1$. Let

$$\widetilde{F}_x:\ \overline{B}(1)\oplus\mathbf{R}^m\to\mathbf{R}^m\oplus\mathbf{R}^m$$

be the $C^{r-1,\theta}$ map defined by

$$\widetilde{F}_x(u,v)=(F_xu,\ DF_x(u)v).$$

We can apply the results obtained till now to $\widetilde{F}$ instead of F. In particular, let $S(\beta)$ be replaced by $\widetilde{S}(\beta)\subset\mathbf{R}^m\oplus\mathbf{R}^m$. The above identification of the tangent space to $S(\beta)$ as the range of $P^{(p)}(\widetilde{T})$ shows that $(u,v)\in\overline{B}(\alpha)\cap\widetilde{S}(\beta)$ if and only if $u\in\overline{B}(\alpha)\cap S(\beta)$ and v is tangent to $S(\beta)$ at u and sufficiently small. Since $\overline{B}(\alpha)\cap\widetilde{S}(\beta)$ is $C^{r-1,\theta}$ by induction, the dependence on u of the tangent space to $B(\alpha)\cap S(\beta)$ at u is $C^{r-1,\theta}$. Therefore $\overline{B}(\alpha')\cap S(\beta)$ is $C^{r,\theta}$ if $\alpha'<\alpha$.

Remarks **(5.2)**. — *a)* The theorem as we have stated it assumes only the measurability of $x\mapsto T(x)$, $\|F_x\|_{r,\theta}$. One could easily give an "abstract" version for a sequence of maps $F_n\in C^{r,\theta}(\overline{B}(1),0;\mathbf{R}^m,0)$ satisfying conditions corresponding to (5.2), (5.3), (5.4). On the other hand further measurability properties of $x\mapsto F_x$ would imply measurability properties of $x\mapsto\upsilon_x^\lambda$. Such properties follow from the fact that υ_x^λ is the C^r limit, as $\nu\to\infty$, of the connected component $D^\nu(\alpha)$ of 0 in $\overline{B}(\alpha)\cap(F_x^\nu)^{-1}V_{f^\nu x}^\lambda$ (with $C^{r,\theta}$ estimates uniform in ν).

b) Let $\widetilde{T}_n=DF_{\tau^{n-1}x}(F_x^{n-1}u)$. *The range of* $P^{(q)}(\widetilde{T})$, *for* $q=1,\ldots,s$, *has* $C^{r-1,\theta}$ *dependence on* $u\in\upsilon_x^\lambda$. This was shown above when $q=p$. For general q the step $r=1$ is the same; the argument used for $r>1$ has to be modified by writing

$$\widetilde{F}_x(u,v)=(F_xu,\ e^{\lambda-\lambda'}DF_x(u)v)$$

where λ' is not in the spectrum and q is the largest characteristic exponent $<\lambda'$.

c) From Section (4.7) and (5.9), it follows that we can take δ at $\tau^\ell x$ to decrease at most like $e^{-3\ell\eta}$. From (5.5) we see that G increases at most like $e^{\ell\eta}$. Therefore, by (5.10), we can take $\beta(\tau^\ell x)$ to decrease at most like $e^{-4\ell\eta/\theta}$. Since $\alpha=\beta/B'_\varepsilon$, $\alpha(\tau^\ell x)$ decreases at most like $e^{-5\ell\eta/\theta}$.

(5.3) *The* C^∞ *case.*

Theorem (5.1) has a C^∞ version as we now indicate. *Let* $x\mapsto F_x$ *map* M *to* $C^\infty(\overline{B}(1), 0; \mathbf{R}^m, 0)$. *We assume that* $x\mapsto T(x)$, $\|F_x\|_r$ *are measurable and, instead of* (5.1), *that*

$$\textbf{(5.16)} \qquad \int \log^+\|F_x\|_r\,\rho(dx) < +\infty$$

for every integer $r>0$. *Then the conclusions of Theorem* (5.1) *hold with* υ_x^λ *a* C^∞ *submanifold of* $B(\alpha(x))$.

Let Γ_r, α_r, β_r, γ_r be a choice of Γ, α, β, γ according to Theorem (5.1), for $r\geq 1$ and any $\theta\in(0,1]$, say $\theta=\frac{1}{2}$. Let:

$$\Gamma_r(n)=\{x\in\Gamma_r:\ \beta_1(x)e^{n\lambda}<\alpha_r(\tau^n x)\}.$$

We have $0<\beta_1<1$ (see (5.10)) and $0<\alpha_r<1$; therefore $\int_{M\setminus\Gamma_r(n)}\alpha_r(\tau^n x)\rho(dx)\leq e^{n\lambda}$, implying $\lim_{n\to\infty}\rho(M\setminus\Gamma_r(n))=0$. Let $\Gamma'_r=\bigcup_{n\geq 0}\Gamma_r(n)$; then $\rho(\Gamma'_r)=1$. If $x\in\Gamma'_r$, there is some $n\geq 0$ such that F_x^n maps $\upsilon_{x,1}^\lambda$ (*i.e.* υ_x^λ defined with α_1 and β_1) onto a subset of the C^r manifold $\upsilon_{\tau^n x,r}^\lambda$ (*i.e.* $\upsilon_{\tau^n x}^\lambda$ defined with α_r and β_r). Since F_x^n is C^r, and is a C^1 diffeomorphism on $\upsilon_{x,1}^\lambda$, it is also a C^r diffeomorphism, and $\upsilon_{x,1}^\lambda$ is therefore C^r. Let now $\Gamma_\infty=\bigcap_{r=1}^\infty\bigcap_{k=0}^\infty\tau^{-k}\Gamma'_r$. We have $\rho(\Gamma_\infty)=1$ and $\tau\Gamma_\infty\subset\Gamma_\infty$; let α_∞, β_∞, γ_∞ be the restrictions of α_1, β_1, γ_1 to Γ_∞. Then the desired C^∞ version of Theorem (5.1) is obtained with Γ_∞, α_∞, β_∞, γ_∞ in place of Γ, α, β, γ.

Notice that we have also shown the following: *if the conditions of Theorem* (5.1) *are satisfied, the functions* α, β, γ *can be determined by considering* $x\mapsto F_x$ *as a map from* M *to* $C^{1,\theta}(\overline{B}(1), 0; \mathbf{R}, 0)$ (*but* Γ *might depend on* r).

(5.4) *The analytic case.*

Let B(1) denote here the open unit ball centered at 0 in $\mathbf{C}^m$ and $H(\overline{B}(1), 0, \mathbf{C}^m, 0)$ be the space of maps holomorphic in B(1) and continuous on $\overline{B}(1)$. The holomorphic version of Theorem (5.1) is as follows. *Let* $x\mapsto F_x$ *map* M *to* $H(\overline{B}(1), 0; \mathbf{C}^m, 0)$. *We assume that* $x\mapsto T(x)$, $\|F_x\|_1$ *are measurable and, instead of* (5.1), *that*

$$\textbf{(5.17)} \qquad \int \log^+\|F_x\|_1\,\rho(dx) < +\infty.$$

Then the conclusions of Theorem (5.1) *hold with* υ_x^λ *a holomorphic submanifold of* $B(\alpha(x))$.

Notice that (5.17) implies:

$$\int \log^+\|F_x\|'_2\,\rho(dx) < +\infty$$

where $||\cdot||_2'$ is the C^2 norm on a ball with radius <1. Therefore a C^1 manifold is defined by Theorem (5.1). By construction, this manifold is a limit of holomorphic manifolds $D^\nu(\alpha)$, defined by (5.11), and therefore $\mathcal{V}_x^\lambda$ is holomorphic.

In Section 6, this result on holomorphic maps will be used to handle real-analytic maps.

6. Stable manifold theorem.

Let M be a compact differentiable manifold, and $f: M \to M$ a C^1 map. Applying Appendix D with $\tau = f$, $E = TM$, $T = Tf$ yields a Borel set $\Gamma \subset M$ with the following properties:

I. $f\Gamma \subset \Gamma$ and $\sigma(\Gamma) = 1$ for every f-invariant probability measure σ on M.

II. For each $x \in \Gamma$, the spectrum $\{\lambda_x^{(1)}, \ldots, \lambda_x^{(s)}\}$ of Tf and the associated filtration $V_x^{(1)} \subset \ldots \subset V_x^{(s)} = T_xM$ of T_xM are defined. We write $V_x^\lambda = \bigcup \{V_x^{(p)} : \lambda_x^{(p)} \leq \lambda\}$.

III. Γ is the union of disjoint Borel subsets Γ_ρ indexed by the f-ergodic measures, such that $\tau\Gamma_\rho \subset \Gamma_\rho$, and

$$\lim_{n \to \infty} \frac{1}{n} \sum_{k=0}^{n-1} \varphi(f^k x) = \rho(\varphi)$$

whenever $x \in \Gamma_\rho$ and $\varphi : M \to \mathbf{R}$ is continuous. The spectrum is constant on each Γ_ρ.

Theorem **(6.1)**. — *Let* M *be a compact differentiable manifold and* $f: M \to M$ *a differentiable map of class* $C^{r,\theta}$ *(r integer* ≥ 1, $\theta \in (0, 1]$). *Let d be a Riemann metric on* M *and denote by* $B(x, \alpha)$ *the open ball of (sufficiently small) radius* α *centered at x in* M. *Given* $\lambda < 0$ *there are Borel functions* $\beta > \alpha > 0$ *and* $\gamma > 1$ *on the set*

$$\Gamma^\lambda = \{x \in \Gamma : \text{the spectrum of } Tf \text{ at } x \text{ contains neither } \lambda \text{ nor } -\infty\}$$

with the following properties:

a) *If* $x \in \Gamma^\lambda$, *the set*

$$\mathcal{V}_x^\lambda(\alpha(x)) = \{y \in \overline{B}(x, \alpha(x)) : d(f^n y, f^n x) \leq \beta(x) e^{n\lambda} \text{ for all } n \geq 0\}$$

is a $C^{r,\theta}$ *submanifold of* $\overline{B}(x, \alpha(x))$, *tangent at* 0 *to* V_x^λ. *If* M *and f are* C^∞ (resp. C^ω, i.e. *real-analytic*), *then* $\mathcal{V}_x^\lambda(\alpha(x))$ *is* C^∞ (resp. C^ω).

b) *If* $y, z \in \mathcal{V}_x^\lambda(\alpha(x))$, *then*

$$d(f^n y, f^n z) \leq \gamma(x) d(y, z) e^{n\lambda}.$$

Given ρ *ergodic, if* $\lambda' < \lambda$ *and the interval* $[\lambda', \lambda]$ *is disjoint from the (constant) spectrum on* Γ_ρ, *there exists a Borel function* $x \mapsto \gamma' \geq \gamma$ *with the property*:

b') *If* $y, z \in \mathcal{V}_x^\lambda(\alpha(x))$, *then*

$$d(f^n y, f^n z) \leq \gamma'(x) d(y, z) e^{n\lambda'}.$$

We may assume that M is C^∞. There is then a C^∞ map $(x, u) \mapsto \psi_x(u)$ of TM to M such that ψ_x maps the open unit ball of T_xM diffeomorphically onto a subset of M and $\psi_x(0)=x$. With a finite Borel partition of M trivializing the tangent bundle we may associate a Borel map $(x,u)\mapsto\psi'_x(u)$ which is piecewise C^∞, is a bijection of $M\times\mathbf{R}^m$ onto TM, and is such that $\psi'_x:\mathbf{R}^m\to T_xM$ is a linear contraction. We choose δ so small that the image by $f\psi_x\psi'_x$ of the closed ball $\overline{B}(\delta)$ is contained in $\psi_{fx}\psi'_{fx}B(1)$ for all x. Define now $\Psi_x=\psi_x\psi'_x\delta:\overline{B}(1)\to M$ and $F_x=\Psi_{fx}^{-1}\circ f\circ\Psi_x$. Given $\lambda<0$, the properties (5.2), (5.3), (5.4) hold if $x\in\Gamma^\lambda$. Therefore Theorem (5.1) holds (with Γ^λ replacing Γ) and we obtain readily the $C^{r,\theta}$ version of the present theorem. The (essentially) explicit construction of $x\mapsto\alpha,\beta,\gamma$ ensures that these functions are Borel.

In the C^∞ case, Section (5.3) should be used instead of Theorem (5.1).

If M and f are C^ω, let $\hat{f}$ be a holomorphic extension of f to a neighborhood N of M in a complexification $\hat{M}$. There is then a C^∞ map $(x,u)\mapsto\psi_x(u)$ of $T_M\hat{M}$ to $\hat{M}$ such that ψ_x restricted to the open unit ball of $T_x\hat{M}$ is a holomorphic reality preserving diffeomorphism onto a subset of N, and $\psi_x(0)=x$. With a finite Borel partition of M trivializing $T_M\hat{M}$ we may associate a Borel map $(x,u)\mapsto\psi'_x(u)$ which is piecewise C^ω, is a bijection of $M\times\mathbf{C}^m$ onto $T_M\hat{M}$, and is such that $\psi'_x:\mathbf{C}^m\to T_x\hat{M}$ is a $\mathbf{C}$-linear reality preserving contraction. Choose now δ so small that the image by $\hat{f}\psi_x\psi'_x$ of the closed ball $\overline{B}(\delta)$ (in $\mathbf{C}^m$) is contained in $\psi_{fx}\psi'_{fx}B(1)$ for all x. Define $\Psi_x=\psi_x\psi'_x\delta:\overline{B}(1)\to N$ and $F_x=\Psi_{fx}^{-1}\circ\hat{f}\circ\Psi_x$. Given $\lambda<0$ we may apply section (5.4) and we obtain a family of holomorphic manifolds. Their real parts are the desired C^ω manifolds $\upsilon_x^\lambda(\alpha(x))$.

Corollary **(6.2)**. — *If ρ is ergodic and all the characteristic exponents of* Tf *are strictly negative on* Γ_ρ, *then ρ is carried by an attracting periodic orbit.*

Let the characteristic exponents be $<\lambda<0$. There is $x\in\Gamma_\rho$ such that

$$\rho(\overline{B}(x,\alpha(x)))=\varepsilon>0.$$

Since we have here $\upsilon_x^\lambda(\alpha(x))=\overline{B}(x,\alpha(x))$, we find

$$f^n\overline{B}(x,\alpha(x))\subset\overline{B}(f^nx,\beta(x)e^{n\lambda})$$

and we have

$$\rho(f^n\overline{B}(x,\alpha(x)))\geq\varepsilon.$$

Thus the set $\overline{B}(f^nx,\beta(x)e^{n\lambda})$ has measure at least ε. Using compactness, and taking a limit, we find a point with mass $\geq\varepsilon$. Its orbit carries ρ by ergodicity, and is finite. Clearly, it is also attracting.

Theorem **(6.3)**. — *Let* M *be a compact differentiable manifold, and f a diffeomorphism of class* $C^{r,\theta}$. *We have here* $f\Gamma=\Gamma$, *and the following properties hold:*

a) *Let* $\lambda_x^{(1)}<\ldots<\lambda_x^{(q)}$ *be the strictly negative characteristic exponents at* $x\in\Gamma$. *Define* $\upsilon_x^{(1)}\subset\ldots\subset\upsilon_x^{(q)}$ *by*

$$\upsilon_x^{(p)}=\left\{y\in M:\ \limsup_{n\to\infty}\frac{1}{n}\log d(f^nx,f^ny)\leq\lambda_x^{(p)}\right\}$$

for $p=1,\ldots,q$. *Then* $\upsilon_x^{(p)}$ *is the image of* $V_x^{(p)}$ *by an injective* $C^{r,\theta}$ *immersion* I_x *tangent to the identity at* x.

b) *If* $x \in \Gamma$ *and* $\lambda_x^{(p)} < 0$, *then* $\upsilon_x^{(p)} \subset \Gamma_\rho$ *for some ergodic* ρ. *The filtration* $V_y^{(1)} \subset \ldots \subset V_y^{(s)}$ *has* $C^{r-1,\theta}$ *dependence on* $I_x^{-1} y \in V_x^{(p)}$.

One may in the above replace $C^{r,\theta}$ (resp. $C^{r-1,\theta}$ *in* b)) *by* C^∞ *or* C^ω.

With $q = q(x)$ defined above, choose a Borel function ζ on Γ such that

$$0 < \zeta(x) < -\lambda_x^{(q)}.$$

We take $q+1$ numbers $\lambda_1, \ldots, \lambda_q, \eta$ such that

$$\lambda_x^{(1)} < \lambda_1 < \lambda_x^{(2)} < \lambda_2 < \ldots < \lambda_x^{(q)} < \lambda_q < -\zeta(x), \tag{6.1}$$

$$0 < \eta < \frac{1}{5} \theta \zeta(x),$$

and such that $\lambda_1, \ldots, \lambda_q, \eta$ are constant on a countable family of f-invariant Borel sets forming a partition of Γ. On each one of these sets, and for $p = 1, \ldots, q$, a function α is defined by Theorem (6.1) with respect to $\lambda = \lambda_p$. We call again α the minimum over p of these functions. This new function α defined on Γ is again Borel, and is such that whenever λ is one of the λ_p, $\upsilon_x^\lambda(\alpha(x))$ is defined and Theorem (6.1) holds. The number η is that appearing in the proof of Theorem (5.1), it satisfies $0 < 4\eta \leq -\lambda\theta$ as it should.

By reference to Appendix D one sees readily that if $x \in \Gamma_\rho$, then $\upsilon_x^\lambda(\alpha(x)) \subset \Gamma_\rho$. (The main point is to check that $\upsilon_x^\lambda(\alpha(x)) \subset \Gamma'$. This follows from the inequality $\|\widetilde{T} - T\| \leq \delta$ in the proof of Theorem (5.1), and application of Theorem (4.1).) In particular, if $\lambda = \lambda_p$, $\upsilon_x^\lambda(\alpha(x))$ is tangent to $V_y^{(p)}$ for each $y \in \upsilon_x^\lambda(\alpha(x))$. Also the filtration $V_y^{(1)} \subset \ldots \subset V_y^{(s)}$ has $C^{r-1,\theta}$ dependence on y as noted in Remark (5.2) *b)*. (In the C^∞ case, the dependence is C^∞ (cf. Section (5.3)); in the C^ω case the dependence is C^ω: use a complex extension of f and M as in the proof of Theorem (6.1).)

We come now to the proof of the theorem. For the fact that $f\Gamma = \Gamma$ see Remark D.2 *a)*.

By Remark (5.2) *c)*, we know that $\alpha(f^\ell x)$ decreases at most like $e^{-5\ell\eta/\theta}$. (This asymptotic behavior is not changed by the mappings in the proof of Theorem (6.1).) Here $5\eta/\theta < \zeta(x)$ by (6.1), so that $\alpha(f^\ell x)$ decreases less fast than $e^{-\ell\zeta(x)}$. Therefore for each $k > 0$ there are arbitrarily large integers $\ell > 0$ such that

$$k e^{-\ell\zeta(x)} \leq \alpha(f^\ell x). \tag{6.2}$$

Let $x \in \Gamma$, $\lambda_x^{(p)} < 0$, and $\lambda = \lambda_p$. If $y \in \upsilon_x^{(p)}$, there is k such that, for all $n \geq 0$,

$$d(f^n x, f^n y) \leq k e^{n\lambda}.$$

In particular there are arbitrarily large $\ell \geq 0$ such that, for $n \geq 0$,

$$d(f^{\ell+n} y, f^{\ell+n} x) \leq k e^{-\ell\zeta(x)} e^{n\lambda} \leq \alpha(f^\ell x) e^{n\lambda} \leq \beta(f^\ell x) e^{n\lambda}.$$

Therefore $f^\ell y \in \upsilon_{f^\ell x}^\lambda(\alpha(f^\ell x))$, hence

$$\upsilon_x^{(p)} = \bigcup_{\ell=0}^{\infty} f^{-\ell} \upsilon_{f^\ell x}^\lambda(\alpha(f^\ell x)).$$

The argument applied above to $y \in \mathcal{V}_x^{(p)}$ also applies uniformly to all $y \in f^{-n}\mathcal{V}^{\lambda}_{f^n x}(\alpha(f^n x))$. Thus, for each $n \geq 0$ there is $\ell > n$ such that

$$f^{-n}\mathcal{V}^{\lambda}_{f^n x}(\alpha(f^n x)) \subset f^{-\ell}\mathcal{V}^{\lambda}_{f^\ell x}(\alpha(f^\ell x)),$$

and we may assume that $f^{\ell-n}\mathcal{V}^{\lambda}_{f^n x}(\alpha(f^n x))$ is contained in the open ball $B(f^\ell x, \alpha(f^\ell x))$. In particular $\mathcal{V}_x^{(p)}$ is the union of an increasing sequence of "disks" $f^{-\ell}\mathcal{V}^{\lambda}_{f^\ell x}(\alpha(f^\ell x))$ tangent to $V_x^{(p)}$ at x. It readily follows that $\mathcal{V}_x^{(p)}$ is the image of $V_x^{(p)}$ by an injective immersion tangent to the identity at x (see Hirsch [5], Chapter 2, Section 5). This proves *a)* and *b)*.

Remarks **(6.4)**. — *a)* If f is replaced by f^{-1}, Γ will be replaced by a set Γ^-, and there is no reason to expect in general that Γ and Γ^- will coincide.

b) One sees easily that

$$\mathcal{V}_x^{(p)} = \left\{ y \in M : \lim_{n\to\infty} \frac{1}{n} \log d(f^n x, f^n y) \leq \lambda_x^{(p)} \right\}$$
$$= \left\{ y \in M : \limsup_{n\to\infty} \frac{1}{n} \log d(f^n x, f^n y) < \min(0, \lambda_x^{(p+1)}) \right\}.$$

In particular the manifolds

$$\mathcal{V}_x^{(q)} = \left\{ y \in M : \limsup_{n\to\infty} \frac{1}{n} \log d(f^n x, f^n y) < 0 \right\}$$

may be called *stable manifolds* for f. They provide a foliation (discontinuous in general) of the set Γ. Theorem (6.3) provides a variety of other foliations depending on the choice of an f-invariant function $\lambda < 0$ on Γ.

APPENDIX A. *Proof of Theorem* (1.1).

The assumptions a) and b) of the theorem imply that

$$f_n^+ \leq f_1^+ + f_1^+ \circ \tau + \ldots + f_1^+ \circ \tau^{n-1} \in L^1.$$

Therefore $I_n = \int f_n(x)\rho(dx)$ exists, finite or $-\infty$, and b) gives $I_{m+n} \leq I_m + I_n$. The sequence (I_n) being subadditive, we have

(A.1) $$\lim_{n\to\infty} \frac{1}{n} \int f_n(x)\rho(dx) = \inf_n \frac{1}{n} \int f_n(x)\rho(dx).$$

For every positive integer N, define

$$f_n^{(N)}(x) = \max\{f_n(x), -nN\}.$$

It is easy to see that the sequences $(f_n^{(N)})_{n>0}$ again satisfy the subadditivity condition b) of the theorem. *Let us assume that the conclusions of the theorem hold for these sequences.* Then $\frac{1}{n} f_n^{(N)}$ has a limit $f^{(N)}$ a.e. for each N, and there is $f: M \to \mathbf{R} \cup \{-\infty\}$ such that

$$f^{(N)}(x) = \max\{f(x), -N\}$$

for almost all x. In particular

(A.2) $$\lim_{n\to\infty}\frac{1}{n}f_n=f \quad \text{a.e.} \qquad \text{and}$$
$$\int f(x)\rho(dx)=\inf_{N}\int f^{(N)}(x)\rho(dx).$$

Since the conclusions of Theorem (1.1) hold for $(f_n^{(N)})_{n>0}$ by assumption, we have

$$\int f^{(N)}(x)\rho(dx)=\inf_{n}\int\frac{1}{n}f_n^{(N)}(x)\rho(dx).$$

Thus

(A.3) $$\int f(x)\rho(dx)=\inf_{N,n}\int\frac{1}{n}f_n^{(N)}(x)\rho(dx)$$
$$=\inf_{n}\int\frac{1}{n}f_n(x)\rho(dx).$$

In view of (A.1), (A.2), (A.3), Theorem (1.1) is a consequence of the following result.

Theorem **(A.1)**. — *Let* $(f_n)_{n>0}$ *be a sequence of real functions such that*

a) $f_n\in L^1(M,\rho)$;

b) $f_{m+n}\leq f_m+f_n\circ\tau^m$ a.e.;

c) *there is* $N>0$ *such that* $\int f_n(x)\rho(dx)\geq -nN$.

Then there is a τ*-invariant real function* $f\in L^1(M,\rho)$ *such that* $\frac{1}{n}f_n$ *tends to* f *almost everywhere. Furthermore*:

(A.4) $$\lim_{n\to\infty}\frac{1}{n}\int f_n(x)\rho(dx)=\inf_{n}\frac{1}{n}\int f_n(x)\rho(dx)$$
$$=\int f(x)\rho(dx).$$

For a proof, see Derriennic [3].

Appendix B. *Semiflows.*

In this Appendix, (M,Σ,ρ) is a fixed probability space, and $(\tau^t)_{t\geq 0}: M\to M$ is a measurable semiflow preserving ρ. (This means that $(x,t)\mapsto\tau^t x$ is measurable $M\times\mathbf{R}_+\to M$, τ^0 is the identity, $\tau^{s+t}=\tau^s\circ\tau^t$, and each τ^t preserves ρ.) Almost everywhere means ρ-almost everywhere.

Theorem **(B.1)** (subadditive ergodic theorem).

Let the map $(x,t)\mapsto f_t(x): M\times\mathbf{R}_+\to\mathbf{R}\cup\{-\infty\}$ *be measurable and satisfy the conditions*:

a) *integrability*:

$$\varphi_1=\sup_{0\leq u\leq 1}f_u^+\in L^1(M,\rho),\qquad \varphi_2=\sup_{0\leq u\leq 1}f_{1-u}^+\circ\tau^u\in L^1(M,\rho)$$

b) *subadditivity*:

$$f_{s+t} \leq f_s + f_t \circ \tau^s \quad \text{a.e.}$$

Then there exists a (τ^t)*-invariant measurable function* $f = \mathrm{M} \to \mathbf{R} \cup \{-\infty\}$ *such that*

$$f^+ \in \mathrm{L}^1(\mathrm{M}, \rho),$$

$$\lim_{t\to\infty} \frac{1}{t} f_t = f \quad \text{a.e.,} \qquad \textit{and}$$

$$\lim_{t\to\infty} \frac{1}{t} \int f_t(x) \rho(dt) = \inf_t \frac{1}{t} \int f_t(x) \rho(dx) = \int f(x) \rho(dx).$$

Let n be the integral part of t. We have then

$$f_{n+1} - \varphi_2 \circ \tau^n \leq f_t \leq f_n + \varphi_1 \circ \tau^n$$

and, since $\varphi_1, \varphi_2 \in \mathrm{L}^1$, we have

$$\lim_{n\to\infty} \frac{1}{n} \varphi_1 \circ \tau^n = \lim_{n\to\infty} \frac{1}{n} \varphi_2 \circ \tau^n = 0 \quad \text{a.e.}$$

by the ergodic theorem. The above theorem follows thus from the corresponding theorem (1.1).

(B.2) *Cocycles.*

A map $(x, t) \mapsto \mathrm{T}_x^t$ from $\mathrm{M} \times \mathbf{R}_+$ to $\mathbf{M}_m$ (the real $m \times m$ matrices) will be called a cocycle if

$$\mathrm{T}_x^{s+t} = \mathrm{T}_{\tau^s x}^t \mathrm{T}_x^s.$$

We also assume that the cocycle is measurable $\mathrm{M} \times \mathbf{R}_+ \to \mathbf{M}_m$ and that the functions φ_1, φ_2 defined by

(B.1) $$\varphi_1(x) = \sup_{0 \leq u \leq 1} \log^+ ||\mathrm{T}_x^u||$$

(B.2) $$\varphi_2(x) = \sup_{0 \leq u \leq 1} \log^+ ||\mathrm{T}_{\tau^u x}^{1-u}||$$

are in $\mathrm{L}^1(\mathrm{M}, \rho)$. From Theorem (B.1) we obtain the existence, for $q = 1, \ldots, m$, and almost all x, of

(B.3) $$\lim_{t\to\infty} \frac{1}{t} \log ||(\mathrm{T}_x^t)^{\wedge q}||.$$

We also have, for almost all x,

(B.4) $$\lim_{t\to\infty} \sup \frac{1}{t} \sup_{0 \leq u \leq 1} \log ||\mathrm{T}_{\tau^t x}^{1+u}|| \leq 0.$$

To see this, write $t=n+v$ (n integer, $0 \leq v \leq 1$) and observe that

$$||T^{1+u}_{\tau^{n+v}x}|| \leq ||T^{u+v}_{\tau^{n+1}x}|| \cdot ||T^{1-v}_{\tau^{v}(\tau^{n}x)}||$$

$$||T^{u+v}_{\tau^{n+1}x}|| \leq ||T^{u+v-1}_{\tau^{n+2}x}|| \cdot ||T^{1}_{\tau^{n+1}x}|| \qquad (\text{if } u+v>1)$$

yield

$$\log^{+}||T^{1+u}_{\tau^{t}x}|| \leq \varphi_2(\tau^{n}x)+\varphi_1(\tau^{n+1}x)+\varphi_1(\tau^{n+2}x).$$

(B.4) follows then from the ergodic theorem.

Using the existence of (B.3), and (B.4), the proof of the multiplicative ergodic theorem in Section 1 is easily adapted to flows.

Theorem **(B.3)** (Multiplicative ergodic theorem).

Let $(T^{t})_{t\geq 0}$ *be a measurable cocycle with values in* $\mathbf{M}_m$ *(the real* $m\times m$ *matrices) such that the functions* φ_1, φ_2 *defined by* (B.1), (B.2) *are in* $L^1(M, \rho)$.

There is $\Gamma \subset M$ *such that* $\tau^{t}\Gamma \subset \Gamma$ *for all* $t\geq 0$, *and the following properties hold if* $x\in\Gamma$:

a) $$\lim_{t\to\infty} (T^{t*}_{x}T^{t}_{x})^{1/2t} = \Lambda_x$$

exists.

b) *Let* $\exp\lambda^{(1)}_x < \ldots < \exp\lambda^{(s)}_x$ *be the eigenvalues of* Λ_x *(where* $s=s(x)$, *the* $\lambda^{(r)}_x$ *are real, and* $\lambda^{(1)}_x$ *may be* $-\infty$), *and* $U^{(1)}_x, \ldots, U^{(s)}_x$ *the corresponding eigenspaces. Let* $m^{(r)}_x = \dim U^{(r)}_x$. *The functions* $x\mapsto\lambda^{(r)}_x$, $m^{(r)}_x$ *are* (τ^t)*-invariant. Writing* $V^{(0)}_x=\{0\}$ *and* $V^{(r)}_x = U^{(1)}_x + \ldots + U^{(r)}_x$, *we have*

$$\lim_{t\to\infty} \frac{1}{t}\log||T^{t}_{x}u|| = \lambda^{(r)}_x \qquad \textit{when} \qquad u\in V^{(r)}_x \backslash V^{(r-1)}_x$$

for $r=1, \ldots, s$.

Appendix C. *Local fields.*

The multiplicative ergodic theorem extends to local fields (1), as noticed by Margulis.

If **R** is replaced by **C**, matrix transposition has to be replaced by Hermitean conjugation in Theorem (1.6). In general, replacing $m\times m$ complex matrices by $2m\times 2m$ real matrices reduces the complex case to the real case. $\left(\text{Let } 1\mapsto\begin{pmatrix}1 & \\ & 1\end{pmatrix},\ i\mapsto\begin{pmatrix} & 1\\ -1 & \end{pmatrix}\right)$.

We shall not discuss ultrametric local fields, for which see Raghunathan [15].

Appendix D. *Continuous maps.*

Let M be a metrizable compact space, $\tau: M\to M$ a continuous map, $\pi: E\to M$ a continuous m-dimensional vector bundle over M, and $T: E\to E$ a continuous vector bundle map over τ.

(1) For definitions, see Weil [22].

Proposition **(D.1)**. — *There is a Borel set* $\Gamma \subset \mathrm{M}$ *such that* $\tau\Gamma \subset \Gamma$, *and* $\sigma(\Gamma) = 1$ *for every* τ*-invariant probability measure* σ *on* M. *Furthermore, for each* $x \in \Gamma$, *there is a* τ*-ergodic probability measure* ρ_x *such that*

$$\textbf{(D.1)} \qquad \lim_{n\to\infty} \frac{1}{n} \sum_{k=1}^{n} \varphi(\tau^k x) = \rho_x(\varphi)$$

for all continuous $\varphi : \mathrm{M} \to \mathbf{R}$, *and*

$$\lim_{n\to\infty} \frac{1}{n} \log \|(\mathrm{T}_x^n)^{\wedge q}\| = \lim_{n\to\infty} \frac{1}{n} \int \log \|(\mathrm{T}_y^n)^{\wedge q}\| \, \rho_x(dy) = \inf_n \frac{1}{n} \int \log \|(\mathrm{T}_y^n)^{\wedge q}\| \, \rho_x(dy)$$

for $q = 1, \ldots, m$, *where we have written* $\mathrm{T}_x^n = \mathrm{T}(\tau^{n-1}x) \cdot \ldots \cdot \mathrm{T}(x)$.

Let Γ_1 consist of those $x \in \mathrm{M}$ such that

$$\mathrm{F}^q(x) = \lim_{n\to\infty} \frac{1}{n} \log \|(\mathrm{T}_x^n)^{\wedge q}\|$$

exists, and

$$\textbf{(D.2)} \qquad \mathrm{F}^q(x) = \mathrm{F}^q(\tau x)$$

for $q = 1, \ldots, m$. Then Γ_1 is a Borel set and $\sigma(\Gamma_1) = 1$ for every invariant probability measure σ by Theorem (1.1). We write $\Gamma' = \bigcap_{n \geq 0} \tau^{-n}\Gamma_1$.

Let Γ_2 be the set of all $x \in \mathrm{M}$ for which there is a τ-ergodic probability measure ρ_x such that

$$\text{vague} \lim_{n\to\infty} \frac{1}{n} \sum_{k=0}^{n-1} \delta_{\tau^k x} = \rho_x.$$

Then Γ_2 is a Borel set, $\tau\Gamma_2 \subset \Gamma_2$, and $\sigma(\Gamma_2) = 1$ for every τ-invariant probability measure σ. This follows from the Bogoliubov-Krylov theory (see Jacobs [7]). Furthermore, if σ is a τ-invariant probability measure we have, by (D.2), for σ-almost all $x \in \Gamma' \cap \Gamma_2$,

$$\textbf{(D.3)} \qquad \mathrm{F}^q(x) = \int \mathrm{F}^q(y) \rho_x(dy).$$

We define continuous functions $\mathrm{F}^q_{\ell n}$ by

$$\mathrm{F}^q_{\ell n}(x) = \max\{\log \|(\mathrm{T}_x^n)^{\wedge q}\|, -\ell\}.$$

From Theorem (1.1) we get:

$$\begin{aligned}
\int \mathrm{F}^q(y)\rho_x(dy) &= \lim_{n\to\infty} \frac{1}{n} \int \log \|(\mathrm{T}_y^n)^{\wedge q}\| \, \rho_x(dy) \\
&= \lim_{n\to\infty} \lim_{\ell\to\infty} \frac{1}{n} \int \mathrm{F}^q_{\ell n}(x)\rho_x(dy) \\
&= \lim_{n\to\infty} \lim_{\ell\to\infty} \lim_{\mathrm{N}\to\infty} \frac{1}{n} \int \frac{1}{\mathrm{N}} \sum_{k=0}^{\mathrm{N}-1} \mathrm{F}^q_{\ell n}(\tau^k x).
\end{aligned}$$

Therefore the set Γ of those $x \in \Gamma' \cap \Gamma_2$ for which (D.3) holds is Borel, $\tau\Gamma \subset \Gamma$, $\sigma(\Gamma) = 1$, and Proposition (D.1) holds.

Remarks (**D.2**). — *a)* The proof of the proposition gives $\tau\Gamma = \Gamma$ when $T(x)$ is invertible for all x.

b) Since E can be trivialized by a finite Borel partition of M, a multiplicative ergodic theorem follows from Proposition (D.1) and Proposition (1.3). The arbitrariness in the choice of norm on E is without consequence for the definition of the spectrum of (τ, T) at $x \in \Gamma$, and the associated filtration of $E(x)$.

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