Ergodicity and Fluctuations of a Fluid Particle Driven by Diffusions with Jumps

Guodong Pang

The Harold and Inge Marcus Department of Industrial and Manufacturing Engineering
College of Engineering, Pennsylvania State University, University Park PA 16802, USA
Email: gup3@psu.edu

Nikola Sandrić

Institut für Mathematische Stochastik
Fachrichtung Mathematik, Technische Universität Dresden, 01062 Dresden, Germany
and
Department of Mathematics
Faculty of Civil Engineering, University of Zagreb, 10000 Zagreb, Croatia
Email: nsandric@grad.hr

February 16, 2015

Abstract

In this paper, we study the long-time behavior of a fluid particle immersed in a turbulent fluid driven by a diffusion with jumps, that is, a Feller process associated with a non-local operator. We derive the law of large numbers and central limit theorem for the evolution process of the tracked fluid particle in the cases when the driving process: (i) has periodic coefficients, (ii) is ergodic or (iii) is a class of Lévy processes. The presented results generalize the classical and well-known results for fluid flows driven by elliptic diffusion processes.

AMS 2010 Mathematics Subject Classification: 60F17, 60G17, 60J75

Keywords and phrases: diffusion with jumps, ergodicity, Feller process, Lévy process, semimartingale characteristics, symbol

1 Introduction

Turbulence is one of the most important phenomena in nature and engineering. It is a flow regime characterized by the presence of irregular eddying motions, that is, motions with high level (high Reynolds number) of vorticity. The key problem is to describe the chaotic motion of a turbulent fluid. In practice this is done by tracking the evolution of a specially marked physical entity (particle) which is immersed in the fluid. Clearly, such a particle must be light and small enough (noninertial) so that its presence does not affect the flow pattern. In this way, the motion of the fluid may be visualized through the evolution of this passively advected particle which follows

1

the streamlines of the fluid. In experimental sciences such a particle is often called a *fluid particle* or *passive tracer*. The evolution of a fluid particle is described by the following transport equation

$$\dot{\mathbf{x}}(t) = \mathbf{v}(t, \mathbf{x}(t)), \quad \mathbf{x}(0) = x_0, \tag{1.1}$$

where $\mathbf{v}(t, x) \in \mathbb{R}^d$, $d \geq 1$, is the turbulent velocity vector field which describes the movement of the fluid at point $x \in \mathbb{R}^d$ in space at time $t \geq 0$ and $\mathbf{x}(t) \in \mathbb{R}^d$ is the position of the particle at time $t \geq 0$. However, as we mentioned, turbulence is a chaotic process. More precisely, the velocity vector field of any fluid flow which advects the fluid particle should be a solution to the Navier-Stokes equation. But, solutions to the Navier-Stokes equations for very turbulent fluids are unstable, that is, they have sensitive dependence on the initial conditions that makes the fluid flow irregular both in space and time. In other words, the velocity field $\mathbf{v}(t, x)$ at a fixed point varies with time in a nearly random manner. Similarly, $\mathbf{v}(t, x)$ at a fixed time varies with position in a nearly random manner. Due to this, a probabilistic approach to this problem might be adequate and it might bring a substantial understanding of turbulence. Accordingly, our aim is to study certain statistical properties of the turbulence through simplified random velocity field models which possess some empirical properties of turbulent fluid flows. Based on the symmetries of the Navier-Stokes equation, it is well known that random velocity fields of very turbulent flows (with high Reynolds numbers), among other properties, are time stationary, space homogeneous and isotropic. We refer the reader to [Cho94] and [Fri95] for an extensive overview on turbulent flows.

Now, instead of the transport equation (1.1) we consider the following Itô's stochastic differential equation

$$dX_t = V(t, X_t)dt + dB_t, \quad X_0 = x_0. \tag{1.2}$$

Here, $\{V(t, x)\}_{t \geq 0, \, x \in \mathbb{R}^d}$ is a d-dimensional, $d \geq 1$, random velocity vector field defined on a probability space $(\Omega_V, \mathfrak{F}_V, \mathbb{P}_V)$ which describes the movement of a turbulent fluid and $\{B_t\}_{t \geq 0}$ is a d-dimensional zero-drift Brownian motion defined on a probability space $(\Omega_B, \mathfrak{F}_B, \mathbb{P}_B)$ and given by a covariance matrix $\Sigma = (\sigma_{ij})_{i,j=1,\dots,d}$ describing the molecular diffusivity of the fluid. Recall that if for any $T > 0$ there exist random constants C_T and D_T such that

$$\sup_{0 \leq t \leq T} |V(t, x, \omega_V) - V(t, y, \omega_V)| < C_T(\omega_V)|x - y|, \quad x, y \in \mathbb{R}^d, \ \mathbb{P}_V\text{-a.s,}$$

and

$$\sup_{0 \leq t \leq T} |V(t, x, \omega_V)| < D_T(\omega_V)(1 + |x|), \quad x \in \mathbb{R}^d, \ \mathbb{P}_V\text{-a.s,}$$

then (1.2) has a unique solution (see [Øks03, Theorem 5.2.1]) defined on $(\Omega_V \times \Omega_B, \mathfrak{F}_V \times \mathfrak{F}_B, \mathbb{P}_V \times \mathbb{P}_B)$ which can be seen as an elliptic diffusion process in a turbulent random environment.

The main goal is to describe certain statistical properties of the fluid flow (that is, of $\{X_t\}_{t \geq 0}$) through the statistical properties of the velocity field $\{V(t, x)\}_{t \geq 0, \, x \in \mathbb{R}^d}$. In particular, we are interested in the long-time behavior of $\{X_t\}_{t \geq 0}$ (clearly, for small times $X_t \approx x_0$). More precisely, we investigate whether

$$\frac{X_{nt}}{n} \xrightarrow{\mathbb{P}_V \times \mathbb{P}_B \text{ - a.s.}} \bar{V}t \tag{1.3}$$

as $n \longrightarrow \infty$, for some $\bar{V} \in \mathbb{R}^d$, and, if this is the case, we analyze fluctuations of $\{X_t\}_{t \geq 0}$ around $\bar{V}$, that is, we investigate whether

$$\left\{ n^{\frac{1}{2}} \left(\frac{X_{nt}}{n} - \bar{V}t \right) \right\}_{t \geq 0} \xrightarrow{\text{d}} \{W_t\}_{t \geq 0} \tag{1.4}$$

2

as $n \longrightarrow \infty$. Here, $\xrightarrow{\text{d}}$ denotes the convergence in distribution, in the space of cádlág functions endowed with the Skorohod J_1 topology (see [Bil68] or [JS03] for details), and $\{W_t\}_{t \geq 0}$ is a d-dimensional (possibly degenerate) zero-drift Brownian motion.

Long-time behavior of $\{X_t\}_{t \geq 0}$ of type (1.3) and (1.4) has been very extensively studied in the literature. In particular, ergodicity of $\{X_t\}_{t \geq 0}$, under the assumption that $\{V(t,x)\}_{t \geq 0, \, x \in \mathbb{R}^d}$ is regular enough and has only finite dependency in time or space, has been deduced in [KK04a] and [KK04b]. Regarding the analysis of fluctuations of $\{X_t\}_{t \geq 0}$, yet in 1923 G. I. Taylor [Tay22] noticed that if the velocity field $\{V(t,x)\}_{t \geq 0, \, x \in \mathbb{R}^d}$ decorrelates sufficiently fast in time or space, then the limit in (1.4) should have a diffusive character. A rigorous mathematical analysis and proofs of this fact have occupied many authors. By assuming certain additional structural and statistical properties of the velocity field $\{V(t,x)\}_{t \geq 0, \, x \in \mathbb{R}^d}$ (time or space independence, Markovian or Gaussian nature, strong mixing property in time or space), Taylor's observation has been confirmed (see [FK97], [FK99], [FK01], [FK02], [FRP98], [FP96], [KP79], [KO01], [MK99], [PSV77], [PV81] and the reference therein). Also, let us remark that the lack of long-range (temporal or spatial) decorrelations of $\{V(t,x)\}_{t \geq 0, \, x \in \mathbb{R}^d}$ may lead to memory effects, that is, an "anomalous" (non-Markovian) diffusive behavior (fractional Brownian motion, local times of certain Markov processes, subordinated Brownian motion, exponential random variable) may appear as a limit in (1.4) (see [Fan00, Fan01], [FK00a], [FK00b], [FK03], [KNR14], [NX13], [PSV77] and the references therein).

In this paper, we consider a model in which the velocity field $\{V(t,x)\}_{t \geq 0, \, x \in \mathbb{R}^d}$ is space independent and its time dependence and randomness are governed by a diffusion with jumps. More precisely,

$$V(t,x,\omega_V) := v(F_t(\omega_V)), \quad t \geq 0, \; \omega_V \in \Omega_V,$$

where $v : \mathbb{R}^{\bar{d}} \longrightarrow \mathbb{R}^d$, $\bar{d} \geq 1$, is a certain function (specified below) and $\{F_t\}_{t \geq 0}$ is an $\mathbb{R}^{\bar{d}}$-valued diffusion with jumps (Feller process) determined by an integro-differential operator (infinitesimal generator) $(\mathcal{A}, \mathcal{D}_\mathcal{A})$ of the form

$$\mathcal{A}f(x) = \langle b(x), \nabla f(x) \rangle + \frac{1}{2} \mathrm{div}\, c(x) \nabla f(x)$$
$$+ \int_{\mathbb{R}^{\bar{d}}} \left(f(y+x) - f(x) - \langle y, \nabla f(x) \rangle 1_{\{z : |z| \leq 1\}}(y) \right) \nu(x, dy), \quad f \in \mathcal{D}_\mathcal{A}. \qquad (1.5)$$

Our work is highly motivated by the works of A. Bensoussan, J-L. Lions and G. C. Papanicolaou [BLP78], R. N. Bhattacharya [Bha82] and G. C. Papanicolaou, D. Stroock and S. R. S. Varadhan [PSV77] in which they consider a model with $\{F_t\}_{t \geq 0}$ being a diffusion process determined by a second-order elliptic operator $(\mathcal{A}, \mathcal{D}_\mathcal{A})$ of the form

$$\mathcal{A}f(x) = \langle b(x), \nabla f(x) \rangle + \frac{1}{2} \mathrm{div}\, c(x) \nabla f(x), \quad f \in \mathcal{D}_\mathcal{A},$$

and, under the assumptions that either $\{F_t\}_{t \geq 0}$ is a diffusion on the $\bar{d}$-dimensional torus $\mathbb{R}^{\bar{d}}/\mathbb{Z}^{\bar{d}}$ and $v(x) = (\mathcal{A}w_1(x), \ldots, \mathcal{A}w_d(x))$, for $w_1, \ldots, w_d \in C^2(\mathbb{R}^{\bar{d}}/\mathbb{Z}^{\bar{d}})$, or $\{F_t\}_{t \geq 0}$ is ergodic and $v(x) = (\mathcal{A}w_1(x), \ldots, \mathcal{A}w_d(x))$, for $w_1, \ldots, w_d \in C_c^\infty(\mathbb{R}^{\bar{d}})$, they derive the Brownian limit in (1.4). Here we extend their results by investigating the long-time behaviors in (1.3) and (1.4) of $\{X_t\}_{t \geq 0}$ driven by a diffusion with jumps.

We have identified three sets of conditions for the driving diffusion with jumps $\{F_t\}_{t \geq 0}$ under which the law of large numbers (LLN) in (1.3) and the central limit theorem (CLT) in (1.4) hold for the process $\{X_t\}_{t \geq 0}$. In the first case, in Theorem 3.1, the driving process $\{F_t\}_{t \geq 0}$ has periodic coefficients $(b(x), c(x), \nu(x, dy))$. Here we also assume the existence, continuity (in space variables)

and strict positivity of a transition density function $p(t, x, y)$ of $\{F_t\}_{t \geq 0}$; see discussions on the assumption in Remark 3.3. These assumptions imply implicitly that the projection of the driving diffusion with jumps on the torus $\mathbb{R}^{\bar{d}} / \mathbb{Z}^{\bar{d}}$ is ergodic. In Theorem 3.4, we simply assume that the driving diffusion with jumps is ergodic, and establish the limiting properties in (1.3) and (1.4). Since any Lévy process has constant coefficients (Lévy triplet), in Theorem 3.5 we establish the limiting properties in (1.3) and (1.4) for a class of Lévy processes with certain coefficients properties which relax the conditions from Theorem 3.1. Note that (non-trivial) Lévy processes are never ergodic, hence, in the Lévy process case, the results in Theorem 3.4 do not apply. We also discuss the cases when the driving diffusions with jumps are not necessarily ergodic in Section 6.

The main techniques used in [BLP78], [Bha82] and [PSV77] are based on proving the convergence of finite-dimensional distributions of the underlying processes, functional central limit theorems for stationary ergodic sequences and solving martingale problems, respectively. On the other hand, our approach in proving the main results, Theorems 3.1, 3.4 and 3.5, is through the characteristics of a semimartingale (note that the process $\{X_t\}_{t \geq 0}$ in our setting is a semimartingale). More precisely, by using the facts that $\{F_t\}_{t \geq 0}$ and $\{B_t\}_{t \geq 0}$ are independent and the regularity assumptions imposed upon $\{F_t\}_{t \geq 0}$, and by the classical Birkhoff ergodic theorem, we can conclude the limiting behavior in (1.3). To obtain the Brownian limit in (1.4), we reduce the problem to the convergence of the corresponding semimartingale characteristics. Namely, since $\{F_t\}_{t \geq 0}$ is a semimartingale whose characteristics is given in terms of its Lévy triplet (see [Sch98b, Lemma 3.1 and Theorem 3.5]), we explicitly compute the characteristics of $\{X_t\}_{t \geq 0}$ and show that it converges (in probability) to the characteristics of the Brownian motion $\{W_t\}_{t \geq 0}$, which, according to [JS03, Theorem VIII.2.17], proves the desired results.

The sequel of this paper is organized as follows. In Section 2, we give some preliminaries on diffusions with jumps. In Section 3, we state the main results of the paper, Theorems 3.1, 3.4 and 3.5. In Section 4, we prove Theorems 3.1 and 3.4, and in Section 5, we prove Theorem 3.5. Finally, in Section 6, we present some discussions on the ergodicity property of general diffusions with jumps and the limiting behaviors in (1.3) and (1.4) when the velocity field $\{V(t, x)\}_{t \geq 0, x \in \mathbb{R}^{\bar{d}}}$ is governed by general, not necessarily ergodic, diffusions with jumps.

2 Preliminaries on Diffusions with Jumps

Let $(\Omega, \mathcal{F}, \{\mathbb{P}^x\}_{x \in \mathbb{R}^{\bar{d}}}, \{\mathcal{F}_t\}_{t \geq 0}, \{\theta_t\}_{t \geq 0}, \{M_t\}_{t \geq 0})$, denoted by $\{M_t\}_{t \geq 0}$ in the sequel, be a Markov process with state space $(\mathbb{R}^{\bar{d}}, \mathcal{B}(\mathbb{R}^{\bar{d}}))$, where $\bar{d} \geq 1$ and $\mathcal{B}(\mathbb{R}^{\bar{d}})$ denotes the Borel σ-algebra on $\mathbb{R}^{\bar{d}}$. A family of linear operators $\{P_t\}_{t \geq 0}$ on $B_b(\mathbb{R}^{\bar{d}})$ (the space of bounded and Borel measurable functions), defined by

$$P_t f(x) := \mathbb{E}^x[f(M_t)], \quad t \geq 0, \ x \in \mathbb{R}^{\bar{d}}, \ f \in B_b(\mathbb{R}^{\bar{d}}),$$

is associated with the process $\{M_t\}_{t \geq 0}$. Since $\{M_t\}_{t \geq 0}$ is a Markov process, the family $\{P_t\}_{t \geq 0}$ forms a *semigroup* of linear operators on the Banach space $(B_b(\mathbb{R}^{\bar{d}}), ||\cdot||_\infty)$, that is, $P_s \circ P_t = P_{s+t}$ and $P_0 = I$ for all $s, t \geq 0$. Here, $||\cdot||_\infty$ denotes the supremum norm on the space $B_b(\mathbb{R}^{\bar{d}})$. Moreover, the semigroup $\{P_t\}_{t \geq 0}$ is *contractive*, that is, $||P_t f||_\infty \leq ||f||_\infty$ for all $t \geq 0$ and all $f \in B_b(\mathbb{R}^{\bar{d}})$, and *positivity preserving*, that is, $P_t f \geq 0$ for all $t \geq 0$ and all $f \in B_b(\mathbb{R}^{\bar{d}})$ satisfying $f \geq 0$. The *infinitesimal generator* $(\mathcal{A}^b, \mathcal{D}_{\mathcal{A}^b})$ of the semigroup $\{P_t\}_{t \geq 0}$ (or of the process $\{M_t\}_{t \geq 0}$) is a linear operator $\mathcal{A}^b : \mathcal{D}_{\mathcal{A}^b} \longrightarrow B_b(\mathbb{R}^{\bar{d}})$ defined by

$$\mathcal{A}^b f := \lim_{t \longrightarrow 0} \frac{P_t f - f}{t}, \quad f \in \mathcal{D}_{\mathcal{A}^b} := \left\{ f \in B_b(\mathbb{R}^{\bar{d}}) : \lim_{t \longrightarrow 0} \frac{P_t f - f}{t} \text{ exists in } ||\cdot||_\infty \right\}.$$

4

We call $(\mathcal{A}^b, \mathcal{D}_{\mathcal{A}^b})$ the B_b-*generator* for short.

A Markov process $\{M_t\}_{t\geq 0}$ is said to be a *Feller process* if its corresponding semigroup $\{P_t\}_{t\geq 0}$ forms a *Feller semigroup*. This means that the family $\{P_t\}_{t\geq 0}$ is a semigroup of linear operators on the Banach space $(C_\infty(\mathbb{R}^d), ||\cdot||_\infty)$ and it is *strongly continuous*, that is,

$$\lim_{t\longrightarrow 0} ||P_t f - f||_\infty = 0, \quad f \in C_\infty(\mathbb{R}^{\bar{d}}).$$

Here, $C_\infty(\mathbb{R}^{\bar{d}})$ denotes the space of continuous functions vanishing at infinity. Every Feller semigroup $\{P_t\}_{t\geq 0}$ can be uniquely extended to $B_b(\mathbb{R}^{\bar{d}})$ (see [Sch98a, Section 3]). For notational simplicity, we denote this extension again by $\{P_t\}_{t\geq 0}$. Also, let us remark that every Feller process possesses the strong Markov property and has càdlàg sample paths (see [Jac05, Theorems 3.4.19 and 3.5.14]). This entails that $\{M_t\}_{t\geq 0}$ is *progressively measurable*, that is, for each $t > 0$ the function $(s, \omega) \longmapsto M_s(\omega)$ on $[0, t] \times \mathbb{R}^{\bar{d}}$ is measurable with respect to the σ-algebra $\mathcal{B}([0, t]) \times \mathcal{F}_t$, where $\mathcal{B}([0, t])$ is the Borel σ-algebra on $[0, t]$ (see [Jac05, Proposition 3.6.2]). In particular, under an appropriate choice of the velocity function $v(x)$, the process $\{X_t\}_{t\geq 0}$ in (1.2) is well defined. Further, in the case of Feller processes, we call $(\mathcal{A}^\infty, \mathcal{D}_{\mathcal{A}^\infty}) := (\mathcal{A}^b, \mathcal{D}_{\mathcal{A}^b} \cap C_\infty(\mathbb{R}^{\bar{d}}))$ the *Feller generator* for short. Note that, in this case, $\mathcal{D}_{\mathcal{A}^\infty} \subseteq C_\infty(\mathbb{R}^{\bar{d}})$ and $\mathcal{A}^\infty(\mathcal{D}_{\mathcal{A}^\infty}) \subseteq C_\infty(\mathbb{R}^{\bar{d}})$. If the set of smooth functions with compact support $C_c^\infty(\mathbb{R}^{\bar{d}})$ is contained in $\mathcal{D}_{\mathcal{A}^\infty}$, that is, if the Feller generator $(\mathcal{A}^\infty, \mathcal{D}_{\mathcal{A}^\infty})$ of the Feller process $\{M_t\}_{t\geq 0}$ satisfies

(C1) $C_c^\infty(\mathbb{R}^{\bar{d}}) \subseteq \mathcal{D}_{\mathcal{A}^\infty}$,

then, according to [Cou66, Theorem 3.4], $\mathcal{A}^\infty|_{C_c^\infty(\mathbb{R}^{\bar{d}})}$ is a *pseudo-differential operator*, that is, it can be written in the form

$$\mathcal{A}^\infty|_{C_c^\infty(\mathbb{R}^{\bar{d}})} f(x) = -\int_{\mathbb{R}^{\bar{d}}} q(x, \xi) e^{i\langle \xi, x\rangle} \hat{f}(\xi) d\xi, \tag{2.1}$$

where $\hat{f}(\xi) := (2\pi)^{-\bar{d}} \int_{\mathbb{R}^{\bar{d}}} e^{-i\langle \xi, x\rangle} f(x) dx$ denotes the Fourier transform of the function $f(x)$. The function $q : \mathbb{R}^{\bar{d}} \times \mathbb{R}^{\bar{d}} \longrightarrow \mathbb{C}$ is called the *symbol* of the pseudo-differential operator. It is measurable and locally bounded in (x, ξ) and continuous and negative definite as a function of ξ. Hence, by [Jac01, Theorem 3.7.7], the function $\xi \longmapsto q(x, \xi)$ has, for each $x \in \mathbb{R}^{\bar{d}}$, the following Lévy-Khintchine representation

$$q(x, \xi) = a(x) - i\langle \xi, b(x)\rangle + \frac{1}{2}\langle \xi, c(x)\xi\rangle - \int_{\mathbb{R}^{\bar{d}}} \left(e^{i\langle \xi, y\rangle} - 1 - i\langle \xi, y\rangle 1_{\{z : |z| \leq 1\}}(y) \right) \nu(x, dy), \tag{2.2}$$

where $a(x)$ is a nonnegative Borel measurable function, $b(x)$ is an $\mathbb{R}^{\bar{d}}$-valued Borel measurable function, $c(x) := (c_{ij}(x))_{1 \leq i, j \leq \bar{d}}$ is a symmetric nonnegative definite $\bar{d} \times \bar{d}$ matrix-valued Borel measurable function and $\nu(x, dy)$ is a Borel kernel on $\mathbb{R}^{\bar{d}} \times \mathcal{B}(\mathbb{R}^{\bar{d}})$, called the *Lévy measure*, satisfying

$$\nu(x, \{0\}) = 0 \quad \text{and} \quad \int_{\mathbb{R}^{\bar{d}}} \min\{1, |y|^2\} \nu(x, dy) < \infty, \quad x \in \mathbb{R}^{\bar{d}}.$$

The quadruple $(a(x), b(x), c(x), \nu(x, dy))$ is called the *Lévy quadruple* of the pseudo-differential operator $\mathcal{A}^\infty|_{C_c^\infty(\mathbb{R}^{\bar{d}})}$ (or of the symbol $q(x, \xi)$). In the sequel, we assume the following conditions on the symbol $q(x, \xi)$:

(C2) $||q(\cdot, \xi)||_\infty \leq c(1 + |\xi|^2)$ for some $c \geq 0$ and all $\xi \in \mathbb{R}^{\bar{d}}$;

(C3) $q(x, 0) = a(x) = 0$ for all $x \in \mathbb{R}^{\bar{d}}$.

Let us remark that, according to [Sch98b, Lemma 2.1], condition (**C2**) is equivalent with the boundedness of the coefficients of the symbol $q(x, \xi)$, that is,

$$||a||_\infty + ||b||_\infty + ||c||_\infty + \left|\left|\int_{\mathbb{R}^{\bar{d}}} \min\{1, y^2\}\nu(\cdot, dy)\right|\right|_\infty < \infty,$$

and, according to [Sch98a, Theorem 5.2], condition (**C3**) (together with condition (**C2**)) is equivalent with the conservativeness property of the process $\{M_t\}_{t\geq 0}$, that is, $\mathbb{P}^x(M_t \in \mathbb{R}^{\bar{d}}) = 1$ for all $t \geq 0$ and all $x \in \mathbb{R}^{\bar{d}}$. Also, by combining (2.1), (2.2) and (**C3**), it is easy to see that $\mathcal{A}^\infty$, on $C_c^\infty(\mathbb{R}^{\bar{d}})$, has a representation as an integro-differential operator given in (1.5).

Throughout this paper, $\{F_t\}_{t\geq 0}$ denotes a Feller process satisfying conditions (**C1**), (**C2**) and (**C3**). Such a process is called a *diffusion with jumps.* If $\nu(x, dy) = 0$ for all $x \in \mathbb{R}^{\bar{d}}$, then $\{F_t\}_{t\geq 0}$ is just called a *diffusion.* Note that this definition agrees with the standard definition of elliptic diffusion processes (see [RW00]).

Further, note that in the case when the symbol $q(x, \xi)$ does not depend on the variable $x \in \mathbb{R}^{\bar{d}}$, $\{M_t\}_{t\geq 0}$ becomes a *Lévy process*, that is, a stochastic process with stationary and independent increments and càdlàg sample paths. Moreover, unlike Feller processes, every Lévy process is uniquely and completely characterized through its corresponding symbol (see [Sat99, Theorems 7.10 and 8.1]). According to this, it is not hard to check that every Lévy process satisfies conditions (**C1**), (**C2**) and (**C3**) (see [Sat99, Theorem 31.5]). Thus, the class of processes we consider in this paper contains the class of Lévy processes. Also, a Lévy process is denoted by $\{L_t\}_{t\geq 0}$. For more on diffusions with jumps we refer the readers to the monograph [BSW13].

Finally, we recall relevant definitions of ergodicity of Makov processes. Let $(\Omega, \mathcal{F}, \{\mathbb{P}^x\}_{x\in S}, \{\mathcal{F}_t\}_{t\geq 0}, \{\theta_t\}_{t\geq 0}, \{M_t\}_{t\geq 0})$, denoted by $\{M_t\}_{t\geq 0}$ in the sequel, be a Markov process on a state space $(S, \mathcal{S})$. Here, S is a nonempty set and $\mathcal{S}$ is a σ-algebra of subsets of S. A probability measure $\pi(dx)$ on $\mathcal{S}$ is called *invariant* for $\{M_t\}_{t\geq 0}$ if

$$\int_S \mathbb{P}^x(M_t \in B)\pi(dx) = \pi(B), \quad t \geq 0, \ B \in \mathcal{S}.$$

A set $B \in \mathcal{F}$ is said to be *shift-invariant* if $\theta_t^{-1}B = B$ for all $t \geq 0$. The *shift-invariant* σ-algebra $\mathcal{I}$ is a collection of all such shift-invariant sets. The process $\{M_t\}_{t\geq 0}$ is said to be *ergodic* if it possesses an invariant probability measure $\pi(dx)$ and if $\mathcal{I}$ is trivial with respect to $\mathbb{P}^\pi(d\omega)$, that is, $\mathbb{P}^\pi(B) = 0$ or 1 for every $B \in \mathcal{I}$. Here, for a probability measure $\mu(dx)$ on $\mathcal{S}$, $\mathbb{P}^\mu(d\omega)$ is defined as

$$\mathbb{P}^\mu(d\omega) := \int_S \mathbb{P}^x(d\omega)\mu(dx).$$

Equivalently, $\{M_t\}_{t\geq 0}$ is ergodic if it possesses an invariant probability measure $\pi(dx)$ and if all bounded harmonic functions are constant π-a.s. (see [MT09]). Recall, a bounded measurable function $f(x)$ is called *harmonic* (with respect to $\{M_t\}_{t\geq 0}$) if

$$\int_S \mathbb{P}^x(M_t \in dy)f(y) = f(x), \quad x \in S, \ t \geq 0.$$

The process $\{M_t\}_{t\geq 0}$ is said to be *strongly ergodic* if it possesses an invariant probability measure $\pi(dx)$ and if

$$\lim_{t\longrightarrow\infty} ||\mathbb{P}^x(M_t \in \cdot) - \pi(\cdot)||_{TV} = 0, \quad x \in S,$$

where $|| \cdot ||_{TV}$ denotes the total variation norm on the space of signed measures on $\mathcal{S}$. Recall that strong ergodicity implies ergodicity (see [Bha82, Proposition 2.5]). On the other hand, ergodicity does not necessarily imply strong ergodicity (for example, see Remark 3.7).

3 Main Results

Before stating the main results of this paper, we introduce some notation we need. Let $\tau := (\tau_1, \ldots, \tau_{\bar{d}}) \in (0, \infty)^{\bar{d}}$ be fixed and let $\tau\mathbb{Z}^{\bar{d}} := \tau_1\mathbb{Z} \times \ldots \times \tau_{\bar{d}}\mathbb{Z}$. For $x \in \mathbb{R}^{\bar{d}}$, we define

$$x_\tau := \{y \in \mathbb{R}^{\bar{d}} : x - y \in \tau\mathbb{Z}^{\bar{d}}\} \quad \text{and} \quad \mathbb{R}^{\bar{d}}/\tau\mathbb{Z}^{\bar{d}} := \{x_\tau : x \in \mathbb{R}^{\bar{d}}\}.$$

Clearly, $\mathbb{R}^{\bar{d}}/\tau\mathbb{Z}^{\bar{d}}$ is obtained by identifying the opposite faces of $[0, \tau] := [0, \tau_1] \times \ldots \times [0, \tau_{\bar{d}}]$. Next, let $\Pi_\tau : \mathbb{R}^{\bar{d}} \longrightarrow [0, \tau]$, $\Pi_\tau(x) := x_\tau$, be the covering map. A function $f : \mathbb{R}^{\bar{d}} \longrightarrow \mathbb{R}$ is called τ-periodic if $f \circ \Pi_\tau(x) = f(x)$ for all $x \in \mathbb{R}^{\bar{d}}$. For an arbitrary τ-periodic function $f : \mathbb{R}^{\bar{d}} \longrightarrow \mathbb{R}$, by $f_\tau(x_\tau)$ we denote the restriction of $f(x)$ to $[0, \tau]$.

We now state the main results of this paper, the proofs of which are given in Sections 4 and 5.

Theorem 3.1. *Let $\{F_t\}_{t\geq 0}$ be a $\bar{d}$-dimensional diffusion with jumps with symbol $q(x, \xi)$ which, in addition, satisfies:*

(C4) *the function $x \longmapsto q(x, \xi)$ is τ-periodic for all $\xi \in \mathbb{R}^{\bar{d}}$, or, equivalently, the corresponding Lévy triplet $(b(x), c(x), \nu(x, dy))$ is τ-periodic;*

(C5) *$\{F_t\}_{t\geq 0}$ possesses a transition density function $p(t, x, y)$, that is,*

$$P_t f(x) = \int_{\mathbb{R}^{\bar{d}}} f(y) p(t, x, y) dy, \quad t > 0, \ x \in \mathbb{R}^{\bar{d}}, \ f \in B_b(\mathbb{R}^{\bar{d}}),$$

such that $(x, y) \longmapsto p(t, x, y)$ is continuous and $p(t, x, y) > 0$ for all $t > 0$ and all $x, y \in \mathbb{R}^{\bar{d}}$.

Then, for any probability measure $\varrho(dx)$ on $\mathcal{B}(\mathbb{R}^d)$ having finite first moment, any initial distribution $\rho(dx)$ of $\{F_t\}_{t\geq 0}$ and any τ-periodic $w^1, \ldots, w^d \in C_b^2(\mathbb{R}^{\bar{d}})$,

$$\frac{X_{nt}}{n} \xrightarrow{\mathbb{P}_V^\rho \times \mathbb{P}_B^\varrho \text{ - a.s.}} \bar{V}t, \tag{3.1}$$

as $n \longrightarrow \infty$, and, under $\mathbb{P}_V^\rho \times \mathbb{P}_B^\varrho(d\omega_V, d\omega_B)$,

$$\left\{ n^{\frac{1}{2}} \left(\frac{X_{nt}}{n} - \bar{V}t \right) \right\}_{t\geq 0} \xrightarrow{d} \{W_t\}_{t\geq 0}, \tag{3.2}$$

as $n \longrightarrow \infty$, where

$$\bar{V} := \left(\int_{\mathbb{R}^d} x_1 \varrho(dx), \ldots, \int_{\mathbb{R}^d} x_d \varrho(dx) \right). \tag{3.3}$$

Here, $C_b^k(\mathbb{R}^{\bar{d}})$, $k \geq 0$, denotes the space of k times differentiable functions such that all derivatives up to order k are bounded, $V(t, x, \omega_V) = (\mathcal{A}w^1(F_t(\omega_V)), \ldots, \mathcal{A}w^d(F_t(\omega_V)))$, where the operator $\mathcal{A}$ is defined by (1.5), and $\{W_t\}_{t\geq 0}$ is a d-dimensional zero-drift Brownian motion determined by a covariance matrix of the form

$$C := \left(\sigma_{ij} + \int_{[0,\tau]} \left[\langle \nabla w^i(x_\tau), c(x_\tau) \nabla w^j(x_\tau) \rangle \right. \right.$$
$$\left. \left. + \int_{\mathbb{R}^{\bar{d}}} \left(w^i(y + x_\tau) - w^i(x_\tau) \right) \left(w^j(y + x_\tau) - w^j(x_\tau) \right) \nu(x_\tau, dy) \right] \pi_\tau(dx_\tau) \right)_{1 \leq i,j \leq d}, \tag{3.4}$$

where $\pi_\tau(dx_\tau)$ is an invariant measure associated with the projection of $\{F_t\}_{t\geq 0}$, with respect to $\Pi_\tau(x)$, on $[0, \tau]$.

Remark 3.2. Two nontrivial examples of diffusions with jumps satisfying the conditions in (**C5**) can be found in the classes of diffusions and stable-like processes. If $\{F_t\}_{t\geq 0}$ is a diffusion with Lévy triplet $(b(x), c(x), 0)$, such that $\inf_{x\in\mathbb{R}^{\bar{d}}}\langle \xi, c(x)\xi\rangle \geq c|\xi|^2$, for some $c > 0$ and all $\xi \in \mathbb{R}^{\bar{d}}$, and $b(x)$ and $c(x)$ are Hölder continuous with the index $0 < \beta \leq 1$, then, in [She91, Theorem A], it has been proven that $\{F_t\}_{t\geq 0}$ possesses a continuous (in space variables) and strictly positive transition density function. Note that, because of the Feller property, $\mathcal{A}^\infty(C_c^\infty(\mathbb{R}^{\bar{d}})) \subseteq C_\infty(\mathbb{R}^{\bar{d}})$, hence $b(x)$ and $c(x)$ are always continuous functions.

Let $\alpha : \mathbb{R}^{\bar{d}} \longrightarrow (0,2)$ and $\gamma : \mathbb{R}^{\bar{d}} \longrightarrow (0,\infty)$ be arbitrarily bounded and continuously differentiable functions with bounded derivatives, such that

$$0 < \inf_{x\in\mathbb{R}^{\bar{d}}} \alpha(x) \leq \sup_{x\in\mathbb{R}^{\bar{d}}} \alpha(x) < 2 \quad \text{and} \quad \inf_{x\in\mathbb{R}^{\bar{d}}} \gamma(x) > 0.$$

Under these assumptions, in [Bas88], [Kol00, Theorem 5.1] and [SW13, Theorem 3.3.] it has been shown that there exists a unique diffusion with jumps, called a *stable-like process*, determined by a symbol of the form

$$q(x,\xi) = \gamma(x)|\xi|^{\alpha(x)}$$

which satisfies condition (**C5**). Note that when $\alpha(x)$ and $\gamma(x)$ are constant functions, we deal with a rotationally invariant stable Lévy process. $\qquad\square$

Remark 3.3. In (**C5**) we assume the existence, continuity (in space variables) and strict positivity of a transition density function $p(t, x, y)$ of $\{F_t\}_{t\geq 0}$. According to [San14b, Theorem 2.6], the existence of $p(t, x, y)$ also follows from

$$\int_{\mathbb{R}^{\bar{d}}} \exp\left[-t \inf_{x\in\mathbb{R}^{\bar{d}}} \text{Re}\, q(x,\xi)\right] d\xi < \infty, \quad t > 0, \ x \in \mathbb{R}^{\bar{d}}, \tag{3.5}$$

under

$$\sup_{x\in\mathbb{R}^{\bar{d}}} |\text{Im}\, q(x,\xi)| \leq c \inf_{x\in\mathbb{R}^{\bar{d}}} \text{Re}\, q(x,\xi) \tag{3.6}$$

for some $0 \leq c < 1$ and all $\xi \in \mathbb{R}^{\bar{d}}$. According to [Fri64] and [Sat99, Theorems 7.10 and 8.1], in the Lévy process and diffusion cases, in order to ensure the existence of a transition density function, (3.6) is not necessary. Further, note that (3.5) and (3.6) also imply the continuity of $(x, y) \longmapsto p(t, x, y)$ for all $t > 0$. Indeed, according to [SW13, Theorem 2.7], we have

$$\sup_{x\in\mathbb{R}^{\bar{d}}} \left|\mathbb{E}^x\left[e^{i\xi(F_t - x)}\right]\right| \leq \exp\left[-\frac{t}{16} \inf_{x\in\mathbb{R}^{\bar{d}}} \text{Re}\, q(x, 2\xi)\right], \quad t > 0, \ \xi \in \mathbb{R}^{\bar{d}}. \tag{3.7}$$

Thus, from (3.5) and [Sat99, Proposition 2.5], we have

$$p(t,x,y) = (2\pi)^{-\bar{d}} \int_{\mathbb{R}^{\bar{d}}} e^{-i\xi(y-x)} \mathbb{E}^x\left[e^{i\xi(F_t - x)}\right] d\xi, \quad t > 0, \ x, y \in \mathbb{R}^{\bar{d}}. \tag{3.8}$$

Next, by [Sch98a, Theorem 3.2], the function $x \longmapsto \mathbb{E}^x\left[e^{i\xi(F_t - x)}\right]$ is continuous for all $\xi \in \mathbb{R}^{\bar{d}}$. Finally, the continuity of $(x, y) \longmapsto p(t, x, y)$, $t > 0$, follows directly from (3.5), (3.7), (3.8) and the dominated convergence theorem. On the other hand, the strict positivity of the transition density function $p(t, x, y)$ is a more complex problem. In the Lévy process and diffusion case this problem has been considered in [BH80], [BRŻ96], [Fri64], [Sha69] and [She91]. In the general case, the best we were able to prove is given in Proposition 6.1 in Section 6. $\qquad\square$

8

In Theorem 3.1 the strong ergodicity is hidden in assumptions (**C4**) and (**C5**) (see Section 4). In Theorem 3.4, we assume (strong) ergodicity directly and show the LLN and CLT hold. From the physical point of view, ergodicity is a natural property of turbulent flows. Namely, a system is ergodic if the underlying process visits every region of the state space. On the other hand, very turbulent flows (with high Reynolds numbers) are characterized by a low momentum diffusion and high momentum advection. In other words, a fluid particle in a very turbulent fluid has a tendency to visit all regions of the state space.

Theorem 3.4. *Let $\{F_t\}_{t\geq 0}$ be a $\bar{d}$-dimensional diffusion with jumps and let $w^1, \ldots, w^d \in C_b^2(\mathbb{R}^{\bar{d}})$ be arbitrary. If $\{F_t\}_{t\geq 0}$ is ergodic with an invariant probability measure $\pi(dx)$, then there exists a $\pi(dx)$ measure zero set $B \in \mathcal{B}(\mathbb{R}^{\bar{d}})$, such that for any probability measure $\varrho(dx)$ on $\mathcal{B}(\mathbb{R}^d)$ having finite first moment and any initial distributions $\rho(dx)$ of $\{F_t\}_{t\geq 0}$, satisfying $\rho(B) = 0$, we have*

$$\frac{X_{nt}}{n} \xrightarrow{\mathbb{P}_V^\rho \times \mathbb{P}_B^\varrho \text{ - a.s.}} \bar{V}t, \tag{3.9}$$

as $n \longrightarrow \infty$, and, under $\mathbb{P}_V^\rho \times \mathbb{P}_B^\varrho(d\omega_V, d\omega_B)$,

$$\left\{ n^{\frac{1}{2}} \left(\frac{X_{nt}}{n} - \bar{V}t \right) \right\}_{t\geq 0} \xrightarrow{d} \{W_t\}_{t\geq 0}, \tag{3.10}$$

as $n \longrightarrow \infty$. Here, $\bar{V}$ is given in (3.3), $V(t, x, \omega_V) = (\mathcal{A}w^1(F_t(\omega_V)), \ldots, \mathcal{A}w^d(F_t(\omega_V)))$, where the operator $\mathcal{A}$ is defined by (1.5), and $\{W_t\}_{t\geq 0}$ is a d-dimensional zero-drift Brownian motion determined by a covariance matrix of the form (3.4), with $\pi(dx)$ instead of $\pi_\tau(dx_\tau)$. In addition, if $\{F_t\}_{t\geq 0}$ is strongly ergodic, then the above convergences hold for any initial distribution of $\{F_t\}_{t\geq 0}$.

Note that diffusions satisfy the assumptions in (**C5**) (see [RW00] and [She91, Theorem A]). Hence, Theorems 3.1 and 3.4 generalize the results related to diffusions, presented in [Bha82] and [PSV77]. Also, note that Theorem 3.4 is not applicable to Lévy processes, since a (non-trivial) Lévy process is never ergodic. On the other hand, in the Lévy process case, we can relax the assumptions in (**C5**), that is, in order to derive the limiting behaviors in (1.3) and (1.4) the strong ergodicity will not be crucial anymore. Because of space homogeneity of Lévy processes, the assumption in (**C4**) is automatically satisfied. First, recall that for a τ-periodic locally integrable function $f(x)$ its Fourier coefficients are defined by

$$\hat{f}(k) := \frac{1}{|\tau|} \int_{[0,\tau]} e^{-i\frac{2\pi\langle k, x\rangle}{|\tau|}} f(x)dx, \quad k \in \mathbb{Z}^{\bar{d}},$$

where $|\tau| := \tau_1\tau_2 \cdots \tau_{\bar{d}}$. Under the assumption that

$$\sum_{k\in\mathbb{Z}^{\bar{d}}} |\hat{f}(k)| < \infty, \tag{3.11}$$

$f(x) = \sum_{k\in\mathbb{Z}^{\bar{d}}} \hat{f}(k)e^{i\frac{2\pi\langle k, x\rangle}{|\tau|}}$. For example, (3.11) is satisfied if $f \in C_b^1(\mathbb{R}^{\bar{d}})$ (see [Gra08, Theorems 3.2.9 and 3.2.16]). In general, $\sum_{k\in\mathbb{Z}^{\bar{d}}} |\hat{f}(k)||k|^n < \infty$, $n \geq 0$, if $f \in C_b^{n+1}(\mathbb{R}^{\bar{d}})$. Recall that we use notation $\{L_t\}_{t\geq 0}$ instead of $\{F_t\}_{t\geq 0}$ for Lévy processes as the driving process of $\{X_t\}_{t\geq 0}$.

Theorem 3.5. *Let $\{L_t\}_{t\geq 0}$ be a $\bar{d}$-dimensional Lévy process with symbol $q(\xi)$ satisfying*

$$\operatorname{Re} q\left(\frac{2\pi k}{|\tau|}\right) > 0, \quad k \in \mathbb{Z}^{\bar{d}} \setminus \{0\}, \tag{3.12}$$

and let $w^1, \ldots, w^d \in C_b^2(\mathbb{R}^{\bar{d}})$ be τ-periodic. Then, there exists a Lebesgue measure zero set $B \in \mathcal{B}(\mathbb{R}^{\bar{d}})$, such that for any probability measure $\varrho(dx)$ on $\mathcal{B}(\mathbb{R}^d)$ having finite first moment and any initial distribution $\rho(dx)$ of $\{L_t\}_{t \geq 0}$, satisfying $\rho(B) = 0$, we have

$$\frac{X_{nt}}{n} \xrightarrow{\mathbb{P}_V^\rho \times \mathbb{P}_B^\varrho \text{ - a.s.}} \bar{V}t, \tag{3.13}$$

as $n \longrightarrow \infty$, and, under $\mathbb{P}_V^\rho \times \mathbb{P}_B^\varrho(d\omega_V, d\omega_B)$,

$$\left\{ n^{\frac{1}{2}} \left(\frac{X_{nt}}{n} - \bar{V}t \right) \right\}_{t \geq 0} \xrightarrow{d} \{W_t\}_{t \geq 0}, \tag{3.14}$$

as $n \longrightarrow \infty$. Here, $\bar{V}$ is given in (3.3), $V(t, x, \omega_V) = (\mathcal{A}w^1(L_t(\omega_V)), \ldots, \mathcal{A}w^d(L_t(\omega_V)))$, where the operator $\mathcal{A}$ is defined by (1.5), and $\{W_t\}_{t \geq 0}$ is a d-dimensional zero-drift Brownian motion determined by the covariance matrix given in (3.4) with $\pi_\tau(dx_\tau) = dx_\tau / |\tau|$. In addition, if

$$\sum_{k \in \mathbb{Z}^{\bar{d}} \setminus \{0\}} |k|^2 |\hat{w}^i(k)| \left(1 + \left(\operatorname{Re} q \left(\frac{2\pi k}{|\tau|} \right) \right)^{-2} \right) < \infty, \quad i = 1, \ldots, d, \tag{3.15}$$

then the convergence in (3.14) holds for any initial distribution of $\{L_t\}_{t \geq 0}$.

Remark 3.6. Note that when

$$\operatorname{Re} q \left(\frac{2\pi k}{|\tau|} \right) > 0, \ k \in \mathbb{Z}^{\bar{d}} \setminus \{0\}, \quad \text{and} \quad \liminf_{|k| \longrightarrow \infty} \operatorname{Re} q \left(\frac{2\pi k}{|\tau|} \right) > 0,$$

the condition in (3.15) reduces to

$$\sum_{k \in \mathbb{Z}^{\bar{d}} \setminus \{0\}} |k|^2 |\hat{w}^i(k)| < \infty, \quad i = 1, \ldots, d.$$

For example, this is the case when the function $\xi \longmapsto \operatorname{Re} q(\xi)$ is radial and the function $|\xi| \longmapsto \operatorname{Re} q(\xi)$ is nondecreasing.

Remark 3.7. A simple example where Theorem 3.1 is not applicable, while Theorem 3.5 gives an answer is as follows. Let $w(x) = \sin x$ and let $\{L_t\}_{t \geq 0}$ be a one-dimensional Lévy process given by Lévy triplet of the form $(0, 0, \delta_{-1}(dy) + \delta_1(dy))$. Then, clearly,

$$\hat{w}(k) = \begin{cases} \frac{1}{2}, & k = -1, 1, \\ 0, & \text{otherwise,} \end{cases} \quad \text{and} \quad q(\xi) = 2(1 - \cos \xi).$$

Further, note that $q(k) \neq 0$ for all $k \in \mathbb{Z} \setminus \{0\}$. Thus, the condition in (3.15) (and (3.12)) holds true and consequently for any initial distribution of $\{L_t\}_{t \geq 0}$,

$$\left\{ n^{-\frac{1}{2}} \int_0^{nt} \mathcal{A}^b w(L_s) ds \right\}_{t \geq 0} \xrightarrow{d} \{W_t\}_{t \geq 0},$$

where $\{W_t\}_{t \geq 0}$ is a zero-drift Brownian motion with the variance parameter $C = 2(1 - \cos 1)$. Also, note that, according to Proposition 5.2 below, $\{L_t^{2\pi}\}_{t \geq 0}$ is ergodic but obviously it is not strongly ergodic (with respect to $dx_{2\pi}/2\pi$). $\qquad \square$

Remark 3.8. In Theorem 3.1 we implicitly assume (through conditions (**C4**) and (**C5**)) that the underlying process $\{F_t^\tau\}_{t\geq 0}$ is strongly ergodic and conclude the limiting behaviors in (3.1) and (3.2) for any initial distribution of $\{F_t\}_{t\geq 0}$. In Theorem 3.5 we implicitly assume (through (3.12)) only the ergodicity of $\{L_t^\tau\}_{t\geq 0}$ and the best we can conclude is that the limiting behaviors in (3.13) and (3.14) hold for any initial distribution of $\{L_t\}_{t\geq 0}$ whose overall mass is contained in the complement of a certain Lebesgue measure zero set. (See more discussions on the condition 3.12 in Section 5.1.) If, in addition, we assume that $\{L_t\}_{t\geq 0}$ satisfies (**C5**), then $\{L_t^\tau\}_{t\geq 0}$ becomes strongly ergodic. Conditions that certainly ensure this are the integrability of $e^{-tq(\xi)}$, $t > 0$, and that either the Brownian or jumping component is nondegenerate and possesses a strictly positive transition density function (see [Sat99, Theorem 19.2 and Lemma 27.1]). Note that for the jumping part to possess a transition density function it is necessary that $\nu(\mathbb{R}^{\bar{d}}) = \infty$. Very recently in [SW11, Theorem 4.1] it has been shown that $\{L_t^\tau\}_{t\geq 0}$ is strongly ergodic if there exists some $t_0 > 0$ such that for every $t \geq t_0$, the transition function $p(t, x, dy)$ of $\{L_t\}_{t\geq 0}$ has (with respect to the Lebesgue measure) an absolutely continuous component. According to [SW11, Theorem 4.3], a sufficient condition that guarantees the existence of an absolutely continuous component of $p(t, x, dy)$, $t > 0$, is that there exists $\varepsilon > 0$, such that for

$$\nu_\varepsilon(B) := \begin{cases} \nu(B), & \nu(\mathbb{R}^{\bar{d}}) < \infty, \\ \nu(\{x \in B : |x| \geq \varepsilon\}), & \nu(\mathbb{R}^{\bar{d}}) = \infty, \end{cases}$$

either the k-fold convolution $\nu_\varepsilon^{*k}(dy)$, $k \geq 1$, has an absolutely continuous component or there exist $\eta > 0$ and $k \geq 1$, such that

$$\inf_{x \in \mathbb{R}^{\bar{d}}, |x| \leq \eta} \nu_\varepsilon^{*k} \wedge (\delta_x * \nu_\varepsilon^{*k})(\mathbb{R}^{\bar{d}}) > 0. \tag{3.16}$$

Here, for two probability measures $\rho(dx)$ and $\varrho(dx)$, $(\rho \wedge \varrho)(dx) := \rho(dx) - (\rho - \varrho)^+(dx)$, where $(\rho - \varrho)^\pm(dx)$ is the Hahn-Jordan decomposition of the signed measure $(\rho - \varrho)(dx)$. Intuitively, condition (3.16) ensures enough jump activity of the underlying pure jump Lévy process. $\qquad\square$

4 Proofs of Theorems 3.1 and 3.4

4.1 Preliminaries on Periodic Diffusions with Jumps

We start this subsection with the following observation. Let $\{M_t\}_{t\geq 0}$ be an $\mathbb{R}^{\bar{d}}$-valued, $\bar{d} \geq 1$, Markov process with semigroup $\{P_t\}_{t\geq 0}$ and let $\Pi_\tau : \mathbb{R}^{\bar{d}} \longrightarrow [0, \tau]$ be the covering map, defined in the previous section. Recall that $\tau := (\tau_1, \ldots, \tau_{\bar{d}}) \in (0, \infty)^{\bar{d}}$ and $[0, \tau] := [0, \tau_1] \times \cdots [0, \tau_{\bar{d}}]$. Next, denote by $\{M_t^\tau\}_{t\geq 0}$ the process on $[0, \tau]$ obtained by the projection of the process $\{M_t\}_{t\geq 0}$ with respect to $\Pi_\tau(x)$, that is, $M_t^\tau := \Pi_\tau(M_t)$, $t \geq 0$. Then, if $\{M_t\}_{t\geq 0}$ is "τ-periodic", $\{M_t^\tau\}_{t\geq 0}$ is a Markov process. More precisely, by assuming that

(A1) $\{P_t\}_{t\geq 0}$ preserves the class of all τ-periodic functions in $B_b(\mathbb{R}^{\bar{d}})$, that is, $x \longmapsto P_t f(x)$ is τ-periodic for all $t \geq 0$ and all τ-periodic $f \in B_b(\mathbb{R}^{\bar{d}})$,

by [Kol11, Proposition 3.8.3], the process $\{M_t^\tau\}_{t\geq 0}$ is a Markov process on $([0, \tau], \mathcal{B}([0, \tau])$ with positivity preserving contraction semigroup $\{P_t^\tau\}_{t\geq 0}$ on the space $(B_b([0, \tau]), ||\cdot||_\infty)$ given by

$$P_t^\tau f_\tau(x_\tau) := \mathbb{E}_\tau^{x_\tau}[f_\tau(M_t^\tau)] = \int_{[0,\tau]} f_\tau(y_\tau)\mathbb{P}_\tau^{x_\tau}(M_t^\tau \in dy_\tau), \quad t \geq 0, \ x_\tau \in [0, \tau], \ f_\tau \in B_b([0, \tau]),$$

where

$$\mathbb{P}_\tau^{x_\tau}(M_t^\tau \in dy_\tau) := \sum_{k \in \tau\mathbb{Z}^{\bar{d}}} \mathbb{P}^x(M_t \in dy + k), \quad t > 0, \ x_\tau, y_\tau \in [0, \tau], \tag{4.1}$$

and x and y are arbitrary points in $\Pi_\tau^{-1}(\{x_\tau\})$ and $\Pi_\tau^{-1}(\{y_\tau\})$, respectively. Note that $\mathcal{B}([0,\tau])$ can be identified with the sub σ-algebra of "τ-periodic" sets in $\mathcal{B}(\mathbb{R}^{\bar{d}})$ (that is, the sets whose characteristic function is τ-periodic) by the relation

$$B = \bigcup_{k \in \tau\mathbb{Z}^{\bar{d}}} B_\tau + k,$$

where $B_\tau \in \mathcal{B}([0,\tau])$ and $B \in \mathcal{B}(\mathbb{R}^{\bar{d}})$ is "τ-periodic". Further, since $[0,\tau]$ is compact, it is reasonable to expect that $\{M_t^\tau\}_{t \geq 0}$ is (strongly) ergodic. By assuming, in addition, that

(A2) $\{M_t\}_{t \geq 0}$ possesses a transition density function $p(t, x, y)$, that is,

$$P_t f(x) = \int_{\mathbb{R}^{\bar{d}}} f(y) p(t, x, y) dy, \quad t > 0, \ x \in \mathbb{R}^{\bar{d}}, \ f \in B_b(\mathbb{R}^{\bar{d}}),$$

(A3) $(x, y) \longmapsto p(t, x, y)$ is continuous and $p(t, x, y) > 0$ for all $t > 0$ and all $x, y \in \mathbb{R}^{\bar{d}}$,

then, clearly, $\{M_t^\tau\}_{t \geq 0}$ has a transition density function $p_\tau(t, x_\tau, y_\tau)$, which is, according to (4.1), given by

$$p_\tau(t, x_\tau, y_\tau) = \sum_{k \in \tau\mathbb{Z}^{\bar{d}}} p(t, x_\tau, y_\tau + k), \quad t > 0, \ x_\tau, y_\tau \in [0, \tau],$$

and, by **(A3)**, it satisfies

$$\inf_{x_\tau, y_\tau \in [0,\tau]} p_\tau(t, x_\tau, y_\tau) > 0, \quad t > 0.$$

Thus, by [BLP78, the proof of Theorem III.3.1], the process $\{M_t^\tau\}_{t \geq 0}$ possesses a unique invariant probability measure $\pi_\tau(dx_\tau)$, such that

$$\sup \left\{ |P_t^\tau 1_{B_\tau}(x_\tau) - \pi_\tau(B_\tau)| : x_\tau \in [0, \tau], \ B_\tau \in \mathcal{B}([0,\tau]) \right\} \leq \Lambda e^{-\lambda t} \tag{4.2}$$

for all $t \geq 0$ and some universal constants $\lambda > 0$ and $\Lambda > 0$. In particular, $\{M_t^\tau\}_{t \geq 0}$ is strongly ergodic. Let us remark that, under **(A2)**, **(A1)** holds true if the function $x \longmapsto p(t, x, y + x)$ is τ-periodic for all $t > 0$ and all $y \in \mathbb{R}^{\bar{d}}$.

Now, based on the above observations, we prove that a diffusion with jumps which satisfies **(C4)** and **(C5)** also satisfies the conditions in **(A1)**, **(A2)** and **(A3)**.

Proposition 4.1. *Let $\{F_t\}_{t \geq 0}$ be a diffusion with jumps with Lévy triplet $(b(x), c(x), \nu(x, dy))$ and transition density function $p(t, x, y)$. Then, $\{F_t\}_{t \geq 0}$ satisfies the condition in (**C4**) if, and only if, the function $x \longmapsto p(t, x, x + y)$ is τ-periodic for all $t > 0$ and all $y \in \mathbb{R}^{\bar{d}}$.*

Proof. The sufficiency follows directly from [Jac01, the proof of Theorem 4.5.21]. To prove the necessity, first recall that there exists a suitable enlargement of the stochastic basis $(\Omega, \mathcal{F}, \{\mathbb{P}^x\}_{x \in \mathbb{R}^{\bar{d}}}, \{\mathcal{F}_t\}_{t \geq 0}, \{\theta_t\}_{t \geq 0})$, say $(\widetilde{\Omega}, \widetilde{\mathcal{F}}, \{\widetilde{\mathbb{P}}^x\}_{x \in \mathbb{R}^{\bar{d}}}, \{\widetilde{\mathcal{F}}_t\}_{t \geq 0}, \{\widetilde{\theta}_t\}_{t \geq 0})$, on which $\{F_t\}_{t \geq 0}$ is the solution to the following stochastic differential equation

12

$$F_t = x + \int_0^t b(F_{s-})ds + \int_0^t c(F_{s-})d\widetilde{W}_s$$

$$+ \int_0^t \int_{\mathbb{R}\backslash\{0\}} k(F_{s-},z)1_{\{u:|k(F_{s-},u)|\leq 1\}}(z)\left(\widetilde{\mu}(\cdot,ds,dz) - ds\widetilde{N}(dz)\right)$$

$$+ \int_0^t \int_{\mathbb{R}\backslash\{0\}} k(F_{s-},z)1_{\{u:|k(F_{s-},u)|>1\}}(z)\,\widetilde{\mu}(\cdot,ds,dz), \tag{4.3}$$

where $\{\widetilde{W}_t\}_{t\geq 0}$ is a $\bar{d}$-dimensional Brownian motion, $\widetilde{\mu}(\omega,ds,dz)$ is a Poisson random measure with compensator (dual predictable projection) $ds\widetilde{N}(dz)$ and $k : \mathbb{R}^{\bar{d}} \times \mathbb{R}\backslash\{0\} \longrightarrow \mathbb{R}^{\bar{d}}$ is a Borel measurable function satisfying

$$\widetilde{\mu}(\omega,ds,k(F_{s-}(\omega),\cdot)\in dy) = \sum_{s:\Delta F_s(\omega)\neq 0}\delta_{(s,\Delta F_s(\omega))}(ds,dy),$$

$$ds\widetilde{N}(k(F_{s-}(\omega),\cdot)\in dy) = ds\,\nu(F_{s-}(\omega),dy)$$

(see [Sch98b, Theorem 3.5] and [ÇJ81, Theorem 3.33]). Further, $\{F_t\}_{t\geq 0}$ has the same transition function on the starting and enlarged stochastic basis. Thus, because of the τ-periodicity of $(b(x),c(x),\nu(x,dy))$, directly from (4.3) we read that $\mathbb{P}^{x+\tau}(F_t \in dy) = \mathbb{P}^x(F_t + \tau \in dy)$ for all $t \geq 0$ and all $x \in \mathbb{R}^{\bar{d}}$, which proves the assertion. $\qquad\square$

Since we mainly deal with τ-periodic functions, we need to extend the operator $(\mathcal{A}^\infty|_{C_c^\infty(\mathbb{R}^{\bar{d}})}, C_c^\infty(\mathbb{R}^{\bar{d}}))$ on a larger domain which contains a certain class of τ-periodic functions. Recall that every Feller semigroup $\{P_t\}_{t\geq 0}$ can be uniquely extended to $B_b(\mathbb{R}^{\bar{d}})$. We denote this extension again by $\{P_t\}_{t\geq 0}$.

Proposition 4.2. *Let $\{F_t\}_{t\geq 0}$ be a diffusion with jumps with B_b-generator $(\mathcal{A}^b, \mathcal{D}_{\mathcal{A}^b})$ which satisfies the condition in $(\mathbf{C4})$. Then,*

$$\{f \in C_b^2(\mathbb{R}^{\bar{d}}) : f(x) \text{ is } \tau\text{-periodic}\} \subseteq \mathcal{D}_{\mathcal{A}^b}$$

and, on this class of functions, $\mathcal{A}^b$ has the representation in (1.5).

Proof. Let $\mathcal{L} : C_b^2(\mathbb{R}^{\bar{d}}) \longrightarrow B_b(\mathbb{R}^{\bar{d}})$ be defined by the relation in (1.5). Observe that actually $\mathcal{L} : C_b^2(\mathbb{R}^{\bar{d}}) \longrightarrow C_b(\mathbb{R}^{\bar{d}})$ (see [Sch98a, Remark 4.5]). Next, by [Sch98b, Corollary 3.6], we have

$$\mathbb{E}^x\left[f(F_t) - \int_0^t \mathcal{L}f(F_s)ds\right] = f(x), \quad x \in \mathbb{R}^{\bar{d}},\ f \in C_b^2(\mathbb{R}^{\bar{d}}).$$

Now, let $f \in C_b^2(\mathbb{R}^{\bar{d}})$ be τ-periodic. Then, since $x \longmapsto \mathcal{L}f(x)$ is also τ-periodic, we have

$$\lim_{t\longrightarrow 0}\left\|\frac{P_t f - f}{t} - \mathcal{L}f\right\|_\infty = \lim_{t\longrightarrow 0}\left\|\frac{1}{t}\int_0^t (P_s\mathcal{L}f - \mathcal{L}f)ds\right\|_\infty$$

$$\leq \lim_{t\longrightarrow 0}\frac{1}{t}\int_0^t \sup_{x\in[0,\tau]}|P_s\mathcal{L}f(x) - \mathcal{L}f(x)|ds$$

$$= 0,$$

where in the final step we applied [Jac01, Lemma 4.8.7]. $\qquad\square$

13

In the following proposition we derive a connection between the B_b-generators $(\mathcal{A}^b, \mathcal{D}_{\mathcal{A}^b})$ and $(\mathcal{A}_\tau^b, \mathcal{D}_{\mathcal{A}_\tau^b})$ of $\{F_t\}_{t\geq 0}$ and $\{F_t^\tau\}_{t\geq 0}$, respectively. Recall that for a τ-periodic function $f(x)$, $f_\tau(x_\tau)$ denotes its restriction to $[0, \tau]$.

Proposition 4.3. *Let $\{F_t\}_{t\geq 0}$ be a diffusion with jumps satisfying the condition in ($\boldsymbol{C4}$) and let $\{F_t^\tau\}_{t\geq 0}$ be the projection of $\{F_t\}_{t\geq 0}$ on $[0, \tau]$ with respect to $\Pi_\tau(x)$. Further, let $(\mathcal{A}^b, \mathcal{D}_{\mathcal{A}^b})$ and $(\mathcal{A}_\tau^b, \mathcal{D}_{\mathcal{A}_\tau^b})$ be the B_b-generators of $\{F_t\}_{t\geq 0}$ and $\{F_t^\tau\}_{t\geq 0}$, respectively. Then, we have*

$$\{f_\tau : f \in C_b^2(\mathbb{R}^{\bar{d}}) \text{ and } f(x) \text{ is } \tau\text{-periodic}\} \subseteq \mathcal{D}_{\mathcal{A}_\tau^b},$$

and, on this set, $\mathcal{A}_\tau^b f_\tau = \left(\mathcal{A}^b f\right)_\tau.$

Proof. First, according to Proposition 4.2, $\{f \in C_b^2(\mathbb{R}^{\bar{d}}) : f(x) \text{ is } \tau\text{-periodic}\} \subseteq \mathcal{D}_{\mathcal{A}^b}$. This and τ-periodicity automatically yield that for any τ-periodic $f \in C_b^2(\mathbb{R}^{\bar{d}})$, we have

$$\lim_{t \longrightarrow 0} \left\|\frac{P_t^\tau f_\tau - f_\tau}{t} - \left(\mathcal{A}^b f\right)_\tau\right\|_\infty = \lim_{t \longrightarrow 0} \left\|\frac{P_t f - f}{t} - \mathcal{A}^b f\right\|_\infty = 0,$$

which proves the desired result. $\qquad\square$

4.2 Proof of Theorem 3.1

Before proving the main result of this subsection (Theorem 3.1), let us recall the notion of characteristics of a semimartingale (see [JS03]). Let $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\geq 0}, \mathbb{P}, \{S_t\}_{t\geq 0})$, denoted by $\{S_t\}_{t\geq 0}$ in the sequel, be a d-dimensional semimatingale and let $h : \mathbb{R}^d \longrightarrow \mathbb{R}^d$ be a truncation function (that is, a continuous bounded function such that $h(x) = x$ in a neighborhood of the origin). We define two processes

$$\check{S}(h)_t := \sum_{s\leq t}(\Delta S_s - h(\Delta S_s)) \quad \text{and} \quad S(h)_t := S_t - \check{S}(h)_t,$$

where the process $\{\Delta S_t\}_{t\geq 0}$ is defined by $\Delta S_t := S_t - S_{t-}$ and $\Delta S_0 := S_0$. The process $\{S(h)_t\}_{t\geq 0}$ is a *special semimartingale*, that is, it admits a unique decomposition

$$S(h)_t = S_0 + M(h)_t + B(h)_t, \tag{4.4}$$

where $\{M(h)_t\}_{t\geq 0}$ is a local martingale and $\{B(h)_t\}_{t\geq 0}$ is a predictable process of bounded variation.

Definition 4.4. *Let $\{S_t\}_{t\geq 0}$ be a semimartingale and let $h : \mathbb{R}^d \longrightarrow \mathbb{R}^d$ be a truncation function. Furthermore, let $\{B(h)_t\}_{t\geq 0}$ be the predictable process of bounded variation appearing in (4.4), let $N(\omega, ds, dy)$ be the compensator of the jump measure*

$$\mu(\omega, ds, dy) := \sum_{s:\Delta S_s(\omega)\neq 0} \delta_{(s, \Delta S_s(\omega))}(ds, dy)$$

of the process $\{S_t\}_{t\geq 0}$ and let $\{C_t\}_{t\geq 0} = \{(C_t^{ij})_{1\leq i,j\leq d}\}_{t\geq 0}$ be the quadratic co-variation process for $\{S_t^c\}_{t\geq 0}$ (continuous martingale part of $\{S_t\}_{t\geq 0}$), that is,

$$C_t^{ij} = \langle S_t^{i,c}, S_t^{j,c}\rangle.$$

Then (B, C, N) is called the characteristics *of the semimartingale $\{S_t\}_{t\geq 0}$ (relative to $h(x)$). If we put $\tilde{C}(h)_t^{ij} := \langle M(h)_t^i, M(h)_t^j\rangle$, $i, j = 1, \ldots, d$, where $\{M(h)_t\}_{t\geq 0}$ is the local martingale appearing in (4.4), then $(B, \tilde{C}, N)$ is called the* modified characteristics *of the semimartingale $\{S_t\}_{t\geq 0}$ (relative to $h(x)$).*

Now, we prove Theorem 3.1.

Proof of Theorem 3.1. The proof proceeds in three steps.

Step 1. In the first step, we explain our strategy of the proof. First, note that, because of the independence of $\{F_t\}_{t\geq 0}$ and $\{B_t\}_{t\geq 0}$, [Sat99, Theorem 36.5] and Proposition 4.2, in order to prove the relation in (3.1), it suffices to prove that

$$n^{-1}\int_0^{nt}\mathcal{A}^b w^i(F_s)ds \xrightarrow{\mathbb{P}^\rho\text{ - a.s.}} 0 \tag{4.5}$$

for all $t \geq 0$, all $i = 1,\ldots,d$ and all initial distributions $\rho(dx)$ of $\{F_t\}_{t\geq 0}$. Recall that $w^1,\ldots,w^d \in C_b^2(\mathbb{R}^{\bar{d}})$ are τ-periodic. Next, due to the τ-periodicity of the Lévy triplet of $\{F_t\}_{t\geq 0}$ (which implies that $\mathcal{A}^b f(x)$ is τ-periodic for any τ-periodic $f \in C_b^2(\mathbb{R}^{\bar{d}})$) and by noting that for any τ-periodic $f : \mathbb{R}^{\bar{d}} \longrightarrow \mathbb{R}$, $f(F_t) = f(F_t^\tau)$, $t \geq 0$, we observe that we can replace $\{F_t\}_{t\geq 0}$ by $\{F_t^\tau\}_{t\geq 0}$ in (4.5), which is, by (4.2), strongly ergodic. Hence, the limiting behavior in (4.5) will simply follow by employing Proposition 4.3 and the Birkhoff ergodic theorem.

Similarly as above, because of the independence of $\{F_t\}_{t\geq 0}$ and $\{B_t\}_{t\geq 0}$ and the scaling property of $\{B_t\}_{t\geq 0}$ (that is, $\{B_t\}_{t\geq 0} \stackrel{d}{=} \{c^{-1/2}B_{ct}\}_{t\geq 0}$ for all $c > 0$), we conclude that in order to prove the limiting behavior in (3.2), it suffices to prove that for any initial distribution $\rho(dx)$ of $\{F_t\}_{t\geq 0}$,

$$\left\{n^{-\frac{1}{2}}\int_0^{nt}v(F_s)ds\right\}_{t\geq 0} \xrightarrow{\text{d}} \{\tilde{W}_t\}_{t\geq 0} \tag{4.6}$$

under $\mathbb{P}^\rho(d\omega_V)$, where $v(x) = (\mathcal{A}^b w^1(x),\ldots,\mathcal{A}^b w^d(x))$ and $\{\tilde{W}_t\}_{t\geq 0}$ is a zero-drift Brownian motion determined by a covariance matrix of the form $\tilde{C} := C - \Sigma$, where the matrices C and Σ are given in (3.4) and (1.2), respectively. Now, according to [JS03, Theorem VIII.2.17], (4.6) will follow if we prove the convergence (in probability) of the modified characteristics of $\left\{n^{-1/2}\int_0^{nt}v(F_s)ds\right\}_{t\geq 0}$ to the modified characteristics of $\{\tilde{W}_t\}_{t\geq 0}$. Accordingly, we explicitly compute the modified characteristics of $\left\{n^{-1/2}\int_0^{nt}v(F_s)ds\right\}_{t\geq 0}$ (in terms of the Lévy triplet of $\{F_t\}_{t\geq 0}$) and, again, because of the τ-periodicity of the Lévy triplet of $\{F_t\}_{t\geq 0}$, we switch from $\{F_t\}_{t\geq 0}$ to $\{F_t^\tau\}_{t\geq 0}$ and apply the Birkhoff ergodic theorem, which concludes the proof of Theorem 3.1.

Step 2. In the second step, we prove the limiting behavior in (4.5). First, observe that, by Proposition 4.3, we have

$$\mathcal{A}^b w^i(F_t) = \mathcal{A}^b w^i(F_t^\tau) = \left(\mathcal{A}^b w^i\right)_\tau (F_t^\tau) = \mathcal{A}_\tau^b w_\tau^i(F_t^\tau), \quad t \geq 0, \ i = 1,\ldots,d.$$

Using this fact, (4.2) and [Bha82, Proposition 2.5] (which states that the Birkhoff ergodic theorem for strongly ergodic Markov processes holds for any initial distribution) we conclude that for any initial distribution $\rho(dx)$ of $\{F_t\}_{t\geq 0}$, we have

$$n^{-1}\int_0^{nt}\mathcal{A}^b w^i(F_s)ds \xrightarrow{\mathbb{P}^\rho\text{ - a.s.}} t\int_{[0,\tau]}\mathcal{A}_\tau^b w_\tau^i(x_\tau)\pi_\tau(dx_\tau), \quad i = 1,\ldots,d.$$

Here, $\pi_\tau(dx_\tau)$ denotes the unique invariant probability measure of $\{F_t^\tau\}_{t\geq 0}$. Finally, we have

$$\left|\int_{[0,\tau]}\mathcal{A}_\tau^b w_\tau^i(x_\tau)\pi_\tau(dx_\tau)\right| = \lim_{t\longrightarrow 0}\left|\int_{[0,\tau]}\mathcal{A}_\tau^b w_\tau^i(x_\tau)\pi_\tau(dx_\tau) - \int_{[0,\tau]}\left(\frac{P_t^\tau w_\tau^i - w_\tau^i}{t}\right)(x_\tau)\pi_\tau(dx_\tau)\right|$$

15

$$\leq \lim_{t \to 0} \left\| \mathcal{A}_\tau^b w_\tau^i - \frac{P_t^\tau w_\tau^i - w_\tau^i}{t} \right\|_\infty$$
$$= 0,$$

where in the first equality we used the stationarity property of $\pi_\tau(dx_\tau)$.

Step 3. In the third step, we prove the limiting behavior in (4.6). Let $\rho(dx)$ be an arbitrary initial distribution of $\{F_t\}_{t \geq 0}$. According to [EK86, Proposition 4.1.7], the processes

$$S_t^{i,n} := n^{-\frac{1}{2}} \int_0^{nt} \mathcal{A}w^i(F_s)ds - n^{-\frac{1}{2}} w^i(F_{nt}) + n^{-\frac{1}{2}} w^i(F_0), \quad i = 1, \dots, d,$$

are $\mathbb{P}^\rho$-martingales (with respect to the natural filtration). Further, let $h : \mathbb{R}^d \longrightarrow \mathbb{R}^d$ be an arbitrary truncation function such that $h(x) = x$ for all $|x| \leq 2\max_{i \in \{1, \dots, d\}} \|w^i\|_\infty$. Then, $S_t^n = S^n(h)_t$ for all $t \geq 0$ and all $n \geq 1$, that is, $\{S_t^n\}_{t \geq 0}$ is a special semimartingale with $S_0^n = 0$ for all $n \geq 1$. In particular, $B_t^n = 0$ for all $t \geq 0$ and all $n \geq 1$. Now, by applying Itô's formula to S_t^n, directly from [JS03, Theorem II.2.34] and [Sch98b, Theorem 3.5], one easily obtains that

$$C_t^{i,j,n} = n^{-1} \int_0^{nt} \langle \nabla w^i(F_{s-}), c(F_{s-}) \nabla w^j(F_{s-}) \rangle ds, \quad i,j = 1, \dots, d. \tag{4.7}$$

Since $w^i w^j \in \mathcal{D}_{\mathcal{A}^b}$, $i,j = 1, \dots d$, again by [EK86, Proposition 4.1.7], the processes

$$\bar{S}_t^{i,j,n} := n^{-1} \int_0^{nt} \mathcal{A}^b(w^i w^j)(F_s)ds - n^{-1} w^i(F_{nt})w^j(F_{nt}) + n^{-1} w^i(F_0)w^j(F_0), \quad i,j = 1, \dots, d,$$

are also $\mathbb{P}^\rho$-martingales. According to (a straightforward adaption of) [EK86, Problem 2.19], this yields

$$\begin{aligned}
\tilde{C}_t^{i,j,n} &= \langle S_t^{i,n}, S_t^{j,n} \rangle \\
&= n^{-1} \int_0^{nt} \left(\mathcal{A}^b(w^i w^j)(F_{s-}) - w^i(F_{s-})\mathcal{A}^b w^j(F_{s-}) - w^j(F_{s-})\mathcal{A}^b w^i(F_{s-}) \right) ds \\
&= n^{-1} \int_0^{nt} \int_{\mathbb{R}^{\bar{d}}} \left(w^i(y + F_{s-}) - w^i(F_{s-}) \right) \left(w^j(y + F_{s-}) - w^j(F_{s-}) \right) \nu(F_{s-}, dy)ds \\
&\quad + n^{-1} \int_0^{nt} \langle \nabla w^i(F_{s-}), c(F_{s-}) \nabla w^j(F_{s-}) \rangle ds, \quad i,j = 1, \dots, d,
\end{aligned}$$

and, by (4.7), [JS03, Proposition II.2.17] and [Sch98b, Theorem 3.5],

$$N^n(\omega, ds, B) = \int_{\mathbb{R}^{\bar{d}}} 1_B \left(n^{-\frac{1}{2}} w(y + F_{s-}(\omega)) - n^{-\frac{1}{2}} w(F_{s-}(\omega)) \right) \nu(F_{s-}(\omega), dy) \, ds, \quad B \in \mathcal{B}(\mathbb{R}^{\bar{d}}).$$

Here, $w(x) := (w^1(x), \dots, w^d(x))$.

Now, we show that under $\mathbb{P}^\rho(d\omega_V)$,

$$\{S_t^n\}_{t \geq 0} \xrightarrow{\text{d}} \{\tilde{W}_t\}_{t \geq 0}.$$

To prove this, according to [JS03, Theorem VIII.2.17], it suffices to prove that

$$\int_0^{nt} \int_{\mathbb{R}^d} |g(y)| N^n(\omega, ds, dy) \xrightarrow{\mathbb{P}^\rho\text{-a.s.}} 0 \tag{4.8}$$

16

for all $t \geq 0$ and all $g \in C_b(\mathbb{R}^d)$ vanishing in a neighborhood around the origin, and

$$\tilde{C}_t^n \xrightarrow{\mathbb{P}^\rho\text{-a.s.}} t\tilde{C} \tag{4.9}$$

for all $t \geq 0$. The relation in (4.8) easily follows from the fact that the function $w(x)$ is bounded and $g(x)$ vanishes in a neighborhood around the origin. To prove the relation in (4.9), first note that because of τ-periodicity of all components,

$$\tilde{C}_t^{i,j,n} = n^{-1} \int_0^{nt} \langle \nabla w^i(F_{s-}^\tau), c(F_{s-}^\tau) \nabla w^j(F_{s-}^\tau) \rangle ds$$
$$+ n^{-1} \int_0^{nt} \int_{\mathbb{R}^{\bar{d}}} \left(w^i(y + F_{s-}^\tau) - w^i(F_{s-}^\tau) \right) \left(w^j(y + F_{s-}^\tau) - w^j(F_{s-}^\tau) \right) \nu(F_{s-}^\tau, dy) ds$$

for all $i, j = 1, \ldots, d$. Now, the desired result again follows by employing (4.2) and [Bha82, Proposition 2.5]. Finally, since $w(x)$ is bounded, [JS03, Lemma VI.3.31] shows the convergence in (4.6), and thus, (3.2). $\qquad\square$

Finally, note that $\pi_\tau(dx_\tau)$ and dx_τ are mutually absolutely continuous. Thus, due to [Bha82, Proposition 2.4], $C = \Sigma$ (that is, $\tilde{C} = 0$) if, and only if, $\mathcal{A}^b w^i(x) = 0$ for all $i = 1, \ldots, d$.

4.3 Proof of Theorem 3.4

We now prove Theorem 3.4. The main ingredients in the proof of Theorem 3.1 were the τ-periodicity of the driving diffusion with jumps $\{F_t\}_{t \geq 0}$ and velocity function $v(x)$ and the fact that all τ-periodic $f \in C_b^2(\mathbb{R}^{\bar{d}})$ are contained in the domain of the B_b-generator $(\mathcal{A}^b, \mathcal{D}_{\mathcal{A}^b})$ of $\{F_t\}_{t \geq 0}$ and, on this class of functions, $\mathcal{A}^b$ has the representation in (1.5) (Proposition 4.2). By having these facts, and assuming (**C5**), we were able to switch to the strongly ergodic process $\{F_t^\tau\}_{t \geq 0}$ and apply the Birkhoff ergodic theorem. On the other hand, in Theorem 3.4 we simply assume the (strong) ergodicity of a driving diffusion with jumps $\{F_t\}_{t \geq 0}$. Now, one might conclude that the assertion of Theorem 3.4 automatically follows by employing completely the same arguments as in the proof of Theorem 3.1. However, note that in this situation it is not clear that $C_b^2(\mathbb{R}^{\bar{d}})$ is contained in $\mathcal{D}_{\mathcal{A}^b}$ or that $\mathcal{A}^b$ can be uniquely extended to $C_b^2(\mathbb{R}^{\bar{d}})$. In order to resolve this problem, according to [BSW13, Theorem 2.37], we employ the following facts: (i) $C_\infty^2(\mathbb{R}^{\bar{d}})$ is contained in the domain of the Feller generator $(\mathcal{A}^\infty, \mathcal{D}_{\mathcal{A}^\infty})$ of $\{F_t\}_{t \geq 0}$, (ii) for any $f \in C_b^2(\mathbb{R}^{\bar{d}})$ there exists a sequence $\{f_n\}_{n \geq 1} \subseteq C_\infty^2(\mathbb{R}^{\bar{d}})$, such that $\mathcal{A}^\infty f_n$ converges (pointwise) to $\mathcal{A}f$, where the operator $\mathcal{A}$ is given by (1.5), and (iii) for any $f \in C_b^2(\mathbb{R}^{\bar{d}})$ and any initial distribution $\rho(dx)$ of $\{F_t\}_{t \geq 0}$, $\{\int_0^t \mathcal{A}f(F_s)ds - f(F_t) + f(F_0)\}_{t \geq 0}$ is a $\mathbb{P}^\rho$-martingale.

Proof of Theorem 3.4. Let $(\mathcal{A}^\infty, \mathcal{D}_{\mathcal{A}^\infty})$ be the Feller generator of $\{F_t\}_{t \geq 0}$. As we commented above, due to [BSW13, Theorem 2.37], $C_\infty^2(\mathbb{R}^{\bar{d}}) \subseteq \mathcal{D}_{\mathcal{A}^\infty}$ and, on this set, $\mathcal{A}^\infty$ has again the representation (1.5). Furthermore, according to the same reference, $(\mathcal{A}^\infty, \mathcal{D}_{\mathcal{A}^\infty})$ has a unique extension to $C_b^2(\mathbb{R}^{\bar{d}})$, denoted by $(\mathcal{A}, C_b^2(\mathbb{R}^{\bar{d}}))$, satisfying

$$\mathcal{A}f(x) = \lim_{n \to \infty} \mathcal{A}^\infty(f\phi_n)(x), \quad x \in \mathbb{R}^{\bar{d}}, \ f \in C_b^2(\mathbb{R}^{\bar{d}}),$$

for any sequence $\{\phi_n\}_{n \geq 1} \subseteq C_c^\infty(\mathbb{R}^{\bar{d}})$ with $\mathbb{1}_{\{y \in \mathbb{R}^{\bar{d}}: |y| \leq n\}}(x) \leq \phi_n(x) \leq 1$ for all $x \in \mathbb{R}^{\bar{d}}$ and all $n \geq 1$. Moreover, $\mathcal{A}$ has the representation (1.5). Note that $\mathcal{A}f \in B_b(\mathbb{R}^{\bar{d}})$. Now, by the Birkhoff ergodic theorem and dominated convergence theorem, for any $f \in C_b^2(\mathbb{R}^{\bar{d}})$ we have

$$\lim_{n \to \infty} n^{-1} \int_0^{nt} \mathcal{A}f(F_s)ds = t \int_{\mathbb{R}^{\bar{d}}} \mathcal{A}f(x)\pi(dx) = \lim_{n \to \infty} t \int_{\mathbb{R}^{\bar{d}}} \mathcal{A}^\infty(f\phi_n)(x)\pi(dx) = 0, \quad \mathbb{P}^\pi\text{-a.s.},$$

17

where $\{\phi_n\}_{n\geq 1} \subseteq C_c^\infty(\mathbb{R}^{\bar{d}})$ is as above and in the final step we used the stationarity property of $\pi(dx)$. Thus, for any $w^1, \ldots, w^d \in C_b^2(\mathbb{R}^{\bar{d}})$,

$$\mathbb{P}^\pi\left(\lim_{n\longrightarrow\infty} n^{-1}\int_0^{nt}\mathcal{A}w^i(F_s)ds = 0\right) = \int_{\mathbb{R}^{\bar{d}}}\mathbb{P}^x\left(\lim_{n\longrightarrow\infty} n^{-1}\int_0^{nt}\mathcal{A}w^i(F_s)ds = 0\right)\pi(dx) = 1$$

for all $i = 1, \ldots, d$. Therefore, there exists a $\pi(dx)$ measure zero set $B \in \mathcal{B}(\mathbb{R}^{\bar{d}})$ such that

$$\mathbb{P}^x\left(\lim_{n\longrightarrow\infty} n^{-1}\int_0^{nt}\mathcal{A}w^i(F_s)ds = 0\right) = 1, \quad x \in B^c, \; i = 1, \ldots, d,$$

which proves the desired result.

To prove the limiting behavior in (3.10), again by [BSW13, Theorem 2.37], for any $f \in C_b^2(\mathbb{R}^{\bar{d}})$ and any initial distribution $\rho(dx)$ of $\{F_t\}_{t\geq 0}$, the process

$$\left\{\int_0^t\mathcal{A}f(F_s)ds - f(F_t) + f(F_0)\right\}_{t\geq 0}$$

is a $\mathbb{P}^\rho$-martingale with respect to the natural filtration. Thus, by completely the same approach as in the proof of Theorem 3.1, the desired result follows. $\qquad\square$

5 Proof of Theorem 3.5

In this section, we prove Theorem 3.5. We start with two auxiliary results we need in the sequel. First, observe that $dx_\tau/|\tau|$ is always an invariant probability measure for $\{L_t^\tau\}_{t\geq 0}$. Indeed, let $t \geq 0$ and $B_\tau \in \mathcal{B}([0,\tau])$ be arbitrary. Then, by (4.1) and the space homogeneity property of Lévy processes, we have

$$\int_{[0,\tau]}\mathbb{P}_\tau^{x_\tau}(L_t^\tau \in B_\tau)dx_\tau = \int_{[0,\tau]}\sum_{k\in\tau\mathbb{Z}^{\bar{d}}}\int_{B_\tau + k - x_\tau}p(t,0,y)dydx_\tau$$

$$= \int_{[0,\tau]}\sum_{k\in\tau\mathbb{Z}^{\bar{d}}}\int_{\mathbb{R}^{\bar{d}}}1_{\{B_\tau + k - x_\tau\}}(y)p(t,0,y)dydx_\tau$$

$$= \int_{\mathbb{R}^{\bar{d}}}p(t,0,y)dy\int_{B_\tau}dx_\tau$$

$$= \int_{B_\tau}dx_\tau.$$

In general, $dx_\tau/|\tau|$ is not necessarily the unique invariant probability measure for $\{L_t^\tau\}_{t\geq 0}$. But, if $\{L_t\}_{t\geq 0}$ is symmetric, that is, $b = 0$ and $\nu(dy)$ is a symmetric measure, and possesses a transition density function (not necessary strictly positive), then $dx_\tau/|\tau|$ is unique (see [Yin94]). Having this fact, in the Lévy process case, the covariance matrix C (given by (3.4)) can be computed in an alternative way. Recall that $\hat{f}(k)$ denotes the k-th, $k \in \mathbb{Z}^{\bar{d}}$, Fourier coefficient of a τ-periodic locally integrable function $f(x)$.

Proposition 5.1. *Let $\{L_t\}_{t\geq 0}$ be a $\bar{d}$-dimensional Lévy process with symbol $q(\xi)$ and B_b-generator $(\mathcal{A}^b, \mathcal{D}_{\mathcal{A}^b})$. Further, let $w^1, \ldots, w^d \in C_b^2(\mathbb{R}^{\bar{d}})$ be τ-periodic, such that*

$$\sum_{k\in\mathbb{Z}^{\bar{d}}}|\hat{w}^i(k)||k|^2 < \infty, \quad i = 1, \ldots, d, \tag{5.1}$$

18

and

$$\sum_{k\in\mathbb{Z}^{\bar{d}}\setminus\{0\}} \frac{1+\operatorname{Re} q\left(\frac{2\pi k}{|\tau|}\right)}{\left|q\left(\frac{2\pi k}{|\tau|}\right)\right|^2} |\widehat{\mathcal{A}^b w^i}(k)||\widehat{\mathcal{A}^b w^j}(-k)| < \infty, \quad i,j = 1,\dots,d. \tag{5.2}$$

Then, for all $i,j = 1,\dots,d$, we have

$$\begin{aligned}
\tilde{C}_{ij} &= \lim_{n\longrightarrow\infty} \frac{1}{n|\tau|t} \int_{[0,\tau]} \mathbb{E}^{x_\tau}\left[\int_0^{nt} \mathcal{A}^b w^i(L_s)ds \int_0^{nt} \mathcal{A}^b w^j(L_r)dr\right] dx_\tau \\
&= \frac{1}{|\tau|} \int_{[0,\tau]} w^i(x_\tau)\mathcal{A}^b w^j(x_\tau)dx_\tau \\
&= 2 \sum_{k\in\mathbb{Z}^{\bar{d}}\setminus\{0\}} \frac{\operatorname{Re} q\left(\frac{2\pi k}{|\tau|}\right)}{\left|q\left(\frac{2\pi k}{|\tau|}\right)\right|^2} \widehat{\mathcal{A}^b w^i}(k)\widehat{\mathcal{A}^b w^j}(-k) \\
&= 2 \sum_{k\in\mathbb{Z}^{\bar{d}}} \operatorname{Re} q\left(\frac{2\pi k}{|\tau|}\right) \hat{w}^i(k)\hat{w}^j(-k).
\end{aligned}$$

Proof. First, we prove that

$$\sum_{k\in\mathbb{Z}^{\bar{d}}\setminus\{0\}} \frac{\operatorname{Re} q\left(\frac{2\pi k}{|\tau|}\right)}{\left|q\left(\frac{2\pi k}{|\tau|}\right)\right|^2} \widehat{\mathcal{A}^b w^i}(k)\widehat{\mathcal{A}^b w^j}(-k) = \sum_{k\in\mathbb{Z}^{\bar{d}}}^{\infty} \operatorname{Re} q\left(\frac{2\pi k}{|\tau|}\right) \hat{w}^i(k)\hat{w}^j(-k).$$

Because of τ-periodicity, from Proposition 4.2, we have

$$\begin{aligned}
\mathcal{A}^b w^i(x) =& \langle b, \nabla w^i(x)\rangle + \frac{1}{2}\operatorname{div} c\nabla w^i(x) \\
&+ \int_{\mathbb{R}^{\bar{d}}} \left(w^i(y+x) - w^i(x) - \langle y, \nabla w^i(x)\rangle 1_{\{z:|z|\le 1\}}(y)\right) \nu(dy).
\end{aligned}$$

Now, by using the assumption (5.1) and the facts that

$$\frac{\partial w^i}{\partial x_p}(x) = \frac{2\pi i}{|\tau|} \sum_{k\in\mathbb{Z}^{\bar{d}}} k_p \hat{w}^i(k)e^{i\frac{2\pi\langle k,x\rangle}{|\tau|}} \quad \text{and} \quad \frac{\partial^2 w^i}{\partial x_p \partial x_q}(x) = -\frac{4\pi^2}{|\tau|^2} \sum_{k\in\mathbb{Z}^{\bar{d}}} k_p k_q \hat{w}^i(k)e^{i\frac{2\pi\langle k,x\rangle}{|\tau|}}, \tag{5.3}$$

for $i = 1,\dots d$ and $p,q = 1,\dots \bar{d}$, we easily find

$$\widehat{\mathcal{A}^b w^i}(k) = -q\left(\frac{2\pi k}{|\tau|}\right) \hat{w}^i(k), \quad k\in\mathbb{Z}^{\bar{d}}, \ i = 1,\dots,d, \tag{5.4}$$

which proves the claim. Note that

$$\sum_{k\in\mathbb{Z}^{\bar{d}}} \operatorname{Re} q\left(\frac{2\pi k}{|\tau|}\right) |\hat{w}^i(k)||\hat{w}^j(-k)| < \infty$$

follows from (**C2**) and (5.1).

19

Next, we prove

$$\lim_{n \to \infty} \frac{1}{n|\tau|t} \int_{[0,\tau]} \mathbb{E}^{x_\tau} \left[\int_0^{nt} \mathcal{A}^b w^i(L_s) ds \int_0^{nt} \mathcal{A}^b w^j(L_r) dr \right] dx_\tau$$

$$= 2 \sum_{k \in \mathbb{Z}^{\bar{d}} \setminus \{0\}} \frac{\operatorname{Re} q\left(\frac{2\pi k}{|\tau|}\right)}{\left|q\left(\frac{2\pi k}{|\tau|}\right)\right|^2} \widehat{\mathcal{A}^b w^i}(k) \widehat{\mathcal{A}^b w^j}(-k).$$

We have

$$\frac{1}{n|\tau|t} \int_{[0,\tau]} \mathbb{E}^{x_\tau} \left[\int_0^{nt} \mathcal{A}^b w^i(L_r) dr \int_0^{nt} \mathcal{A}^b w^j(L_s) ds \right] dx_\tau$$

$$= \frac{1}{n|\tau|t} \mathbb{E}^0 \left[\int_{[0,\tau]} \int_0^{nt} \int_0^{nt} \mathcal{A}^b w^i(L_r + x_\tau) \mathcal{A}^b w^j(L_s + x_\tau) dr ds dx_\tau \right]$$

$$= \frac{1}{n|\tau|t} \mathbb{E}^0 \left[\int_{[0,\tau]} \int_0^{nt} \int_0^{nt} \sum_{k,l \in \mathbb{Z}^{\bar{d}}} \widehat{\mathcal{A}^b w^i}(k) \widehat{\mathcal{A}^b w^j}(l) e^{i\frac{2\pi \langle k, (x_\tau + L_r) \rangle}{|\tau|}} e^{i\frac{2\pi \langle l, (x_\tau + L_s) \rangle}{|\tau|}} dr ds dx_\tau \right]$$

$$= \frac{1}{nt} \sum_{k \in \mathbb{Z}^{\bar{d}}} \widehat{\mathcal{A}^b w^i}(k) \widehat{\mathcal{A}^b w^j}(-k) \int_0^{nt} \int_0^{nt} \mathbb{E}^0 \left[e^{i\frac{2\pi \langle k, (L_r - L_s) \rangle}{|\tau|}} \right] dr ds$$

$$= \frac{1}{nt} \sum_{k \in \mathbb{Z}^{\bar{d}}} \widehat{\mathcal{A}^b w^i}(k) \widehat{\mathcal{A}^b w^j}(-k) \left(\int_0^{nt} \int_0^s \mathbb{E}^0 \left[e^{-i\frac{2\pi \langle k, L_{s-r} \rangle}{|\tau|}} \right] dr ds + \int_0^{nt} \int_s^{nt} \mathbb{E}^0 \left[e^{i\frac{2\pi \langle k, L_{r-s} \rangle}{|\tau|}} \right] dr ds \right)$$

$$= \frac{1}{nt} \sum_{k \in \mathbb{Z}^{\bar{d}}} \widehat{\mathcal{A}^b w^i}(k) \widehat{\mathcal{A}^b w^j}(-k) \left(\int_0^{nt} \int_0^s e^{-(s-r)q\left(-\frac{2\pi k}{|\tau|}\right)} dr ds + \int_0^{nt} \int_s^{nt} e^{-(r-s)q\left(\frac{2\pi k}{|\tau|}\right)} dr ds \right)$$

$$= \sum_{k \in \mathbb{Z}^{\bar{d}} \setminus \{0\}} \widehat{\mathcal{A}^b w^i}(k) \widehat{\mathcal{A}^b w^j}(-k) \left(\frac{1}{q\left(-\frac{2\pi k}{|\tau|}\right)} + \frac{e^{-nt\, q\left(-\frac{2\pi k}{|\tau|}\right)} - 1}{nt\, q^2\left(-\frac{2\pi k}{|\tau|}\right)} + \frac{1}{q\left(\frac{2\pi k}{|\tau|}\right)} + \frac{e^{-nt\, q\left(\frac{2\pi k}{|\tau|}\right)} - 1}{nt\, q^2\left(\frac{2\pi k}{|\tau|}\right)} \right),$$

where in the final step we used the fact that $\widehat{\mathcal{A}^b w^i}(0) = 0$, $i = 1, \ldots, d$, that is, $\int_{[0,\tau]} \mathcal{A}^b w^i dx = 0$, $i = 1, \ldots, d$, (see the proof of Theorem 3.1). Note that the change of orders of sums and integrals is justified by (**C2**), (5.1) and (5.4). Now, the desired result follows from (5.2) and the dominated convergence theorem.

Finally, the fact that

$$\tilde{C}_{ij} = \frac{1}{|\tau|} \int_{[0,\tau]} w^i(x_\tau) \mathcal{A}^b w^j(x_\tau) dx_\tau = 2 \sum_{k \in \mathbb{Z}^{\bar{d}}} \operatorname{Re} q\left(\frac{2\pi k}{|\tau|}\right) \hat{w}^i(k) \hat{w}^j(-k)$$

follows from a straightforward computation by using (5.1), (5.3) and (5.4). □

Proposition 5.2. *Let $\{L_t\}_{t \geq 0}$ be a $\bar{d}$-dimensional Lévy process with symbol $q(\xi)$ satisfying the condition in (3.12). Then, $\{L_t^\tau\}_{t \geq 0}$ is ergodic (with respect to $dx_\tau/|\tau|$).*

Proof. First, recall that $\{L_t^\tau\}_{t \geq 0}$ is ergodic if, and only if, the only bounded measurable functions satisfying

$$\int_{[0,\tau]} p_\tau(t, x_\tau, dy_\tau) f_\tau(y_\tau) = f_\tau(x_\tau), \quad x_\tau \in [0,\tau], \ t \geq 0,$$

20

are constant dx_τ-a.s. Now, by comparing the Fourier coefficients of the left and right hand side in the above relation, we easily see that (3.12) implies that the above relation can be satisfied only for constant dx_τ-a.s. functions. $\qquad\square$

Now, we prove Theorem 3.5.

Proof of Theorem 3.5. The proof proceeds in four steps.

Step 1. In the first step, we explain our strategy of the proof. The idea of the proof is similar as in the proof of Theorem 3.1. Namely, again because of the independence of $\{L_t\}_{t\geq 0}$ and $\{B_t\}_{t\geq 0}$, [Sat99, Theorem 36.5] and Proposition 4.2, in order to prove the relation in (3.13), it suffices to prove that there exists a Lebesgue measure zero set $B \in \mathcal{B}(\mathbb{R}^{\bar d})$, such that for any initial distribution $\rho(dx)$ of $\{L_t\}_{t\geq 0}$, satisfying $\rho(B) = 0$, we have

$$n^{-1} \int_0^{nt} \mathcal{A}^b w^i(L_s)ds \xrightarrow{\mathbb{P}^\rho\text{-a.s.}} 0, \quad t \geq 0, \; i = 1,\ldots,d. \tag{5.5}$$

Recall that $w^1, \ldots, w^d \in C_b^2(\mathbb{R}^{\bar d})$ are τ-periodic. Further, since the driving diffusion with jumps is a Lévy process (hence, it has constant coefficients), we again conclude that for any τ-periodic $f : \mathbb{R}^{\bar d} \longrightarrow \mathbb{R}$, $f(L_t) = f(L_t^\tau)$, $t \geq 0$, and that $\mathcal{A}^b f(x)$ is τ-periodic for any τ-periodic $f \in C_b^2(\mathbb{R}^{\bar d})$. Thus, we can again switch from $\{L_t\}_{t\geq 0}$ to $\{L_t^\tau\}_{t\geq 0}$, which is, by (3.12), ergodic (with respect to $dx_\tau/|\tau|$). Now, the limiting behavior in (5.5) will follow by employing Proposition 4.3 and the Birkhoff ergodic theorem.

In order to prove the limiting behavior in (3.14), again because of the independence of $\{F_t\}_{t\geq 0}$ and $\{B_t\}_{t\geq 0}$ and the scaling property of $\{B_t\}_{t\geq 0}$, we conclude that it suffices to prove that there exists a Lebesgue measure zero set $B \in \mathcal{B}(\mathbb{R}^{\bar d})$, such that for any initial distribution $\rho(dx)$ of $\{L_t\}_{t\geq 0}$, satisfying $\rho(B) = 0$,

$$\left\{ n^{-\frac{1}{2}} \int_0^{nt} v(L_s)ds \right\}_{t\geq 0} \xrightarrow{\mathrm{d}} \{\tilde W_t\}_{t\geq 0} \tag{5.6}$$

under $\mathbb{P}^\rho(d\omega_V)$, where $v(x) = (\mathcal{A}^b w^1(x), \ldots, \mathcal{A}^b w^d(x))$ and $\{\tilde W_t\}_{t\geq 0}$ is a zero-drift Brownian motion determined by the covariance matrix $\tilde C := C - \Sigma$ defined in (3.4). Now, we again employ [JS03, Theorem VIII.2.17], which states that that the desired convergence is reduced to the convergence (in probability) of the modified characteristics of $\left\{ n^{-1/2} \int_0^{nt} v(L_s)ds \right\}_{t\geq 0}$ to the modified characteristics of $\{\tilde W_t\}_{t\geq 0}$. Hence, we again explicitly compute the modified characteristics of $\left\{ n^{-1/2} \int_0^{nt} v(L_s)ds \right\}_{t\geq 0}$ (in terms of the Lévy triplet of $\{L_t\}_{t\geq 0}$) and, because of the τ-periodicity of the Lévy triplet of $\{L_t\}_{t\geq 0}$, we switch from $\{L_t\}_{t\geq 0}$ to $\{L_t^\tau\}_{t\geq 0}$ and apply the Birkhoff ergodic theorem. Finally, to prove that under (3.15) the limit in (5.6) holds for any initial distribution of $\{L_t\}_{t\geq 0}$, we consider the L^2-convergence of the modified characteristics of $\left\{ n^{-1/2} \int_0^{nt} v(L_s)ds \right\}_{t\geq 0}$ to the modified characteristics of $\{\tilde W_t\}_{t\geq 0}$.

Step 2. In the second step, we prove the limiting behavior in (5.5). First, according to Proposition 4.3, we have

$$\mathcal{A}^b w^i(L_t) = \mathcal{A}^b w^i(L_t^\tau) = \left(\mathcal{A}^b w^i\right)_\tau (L_t^\tau) = \mathcal{A}_\tau^b w_\tau^i(L_t^\tau), \quad t \geq 0, \; i = 1,\ldots,d,$$

which yields

$$n^{-1} \int_0^{nt} \mathcal{A}^b w^i(L_s)ds = n^{-1} \int_0^{nt} \mathcal{A}_\tau^b w_\tau^i(L_s^\tau)ds, \quad t \geq 0, \; i = 1,\ldots,d.$$

21

Further, according to Proposition 5.2, the process $\{L_t^\tau\}_{t\geq 0}$ is ergodic (with respect to $dx_\tau/|\tau|$). Thus, the Birkhoff ergodic theorem entails

$$\mathbb{P}^{dx_\tau/|\tau|}\left(\lim_{n\longrightarrow\infty} n^{-1}\int_0^{nt}\mathcal{A}^b w^i(L_s)ds = |\tau|^{-1}t\int_{[0,\tau]}\mathcal{A}_\tau^b w_\tau^i(x_\tau)dx_\tau\right)=1, \quad i=1,\ldots,d.$$

Analogously as in the proof of Theorem 3.1, we conclude that

$$|\tau|^{-1}\int_{[0,\tau]}\mathcal{A}_\tau^b w_\tau^i(x_\tau)dx_\tau = 0, \quad i=1,\ldots,d,$$

that is,

$$|\tau|^{-1}\int_{[0,\tau]}\mathbb{P}^{x_\tau}\left(\lim_{n\longrightarrow\infty}n^{-1}\int_0^{nt}\mathcal{A}^b w^i(L_s)ds = 0\right)dx_\tau = 1, \quad i=1,\ldots,d.$$

Therefore, there exists a Lebesgue measure zero set $B\in\mathcal{B}(\mathbb{R}^{\bar d})$ such that

$$\mathbb{P}^x\left(\lim_{n\longrightarrow\infty}n^{-1}\int_0^{nt}\mathcal{A}^b w^i(L_s)ds = 0\right)=1, \quad x\in B^c,\ i=1,\ldots,d,$$

which proves the desired result.

Step 3. In the third step, we prove the limiting behavior in (5.6). We proceed similarly as in the proof of Theorem 3.1. Let $\rho(dx)$ be an arbitrary initial distribution of $\{L_t\}_{t\geq 0}$. Then, again by [EK86, Proposition 4.1.7], the processes

$$S_t^{i,n} := n^{-\frac{1}{2}}\int_0^{nt}\mathcal{A}w^i(L_s)ds - n^{-\frac{1}{2}}w^i(L_{nt}) + n^{-\frac{1}{2}}w^i(L_0), \quad i=1,\ldots,d,$$

are $\mathbb{P}^\rho$-martingales. Now, by completely the same arguments as in the proof of Theorem 3.1 we deduce that the semimartingale (modified) characteristics of $\{S_t^n\}_{t\geq 0}$ are given by

$$B_t^n = 0,$$
$$C_t^{i,j,n} = n^{-1}\int_0^{nt}\langle\nabla w^i(L_{s-}), c\nabla w^j(L_{s-})\rangle ds, \quad i,j=1,\ldots,d,$$
$$\tilde C_t^{i,j,n} = n^{-1}\int_0^{nt}\int_{\mathbb{R}^{\bar d}}\left(w^i(y+L_{s-})-w^i(L_{s-})\right)\left(w^j(y+L_{s-})-w^j(L_{s-})\right)\nu(dy)ds$$
$$+ n^{-1}\int_0^{nt}\langle\nabla w^i(L_{s-}), c\nabla w^j(L_{s-})\rangle ds, \quad i,j=1,\ldots,d,$$
$$N^n(\omega,ds,B) = \int_{\mathbb{R}^{\bar d}}1_B\left(n^{-\frac{1}{2}}w(y+L_{s-}(\omega))-n^{-\frac{1}{2}}w(L_{s-}(\omega))\right)\nu(dy)ds, \quad B\in\mathcal{B}(\mathbb{R}^{\bar d}),$$

where $w(x)=(w^1(x),\ldots,w^d(x))$. Recall that for the truncation function we again use an arbitrary $h:\mathbb{R}^d\longrightarrow\mathbb{R}^d$, such that $h(x)=x$ for all $|x|\leq 2\max_{i\in\{1,\ldots,d\}}||w^i||_\infty$.

Now, according to [JS03, Theorem VIII.2.17], in order to prove that

$$\{S_t^n\}_{t\geq 0}\overset{\mathrm{d}}{\longrightarrow}\{\tilde W_t\}_{t\geq 0},$$

under $\mathbb{P}^\rho(d\omega_V)$, it suffices to show that

$$\int_0^{nt}\int_{\mathbb{R}^d}|g(y)|N^n(\omega,ds,dy)\xrightarrow{\mathbb{P}^\rho\text{-a.s.}}0 \tag{5.7}$$

22

for all $t \geq 0$ and all $g \in C_b(\mathbb{R}^d)$ vanishing in a neighborhood around the origin, and

$$\tilde{C}_t^n \xrightarrow{\mathbb{P}^\rho\text{-a.s.}} t\tilde{C} \tag{5.8}$$

for all $t \geq 0$. The relation in (5.7) easily follows from the fact that the function $w(x)$ is bounded and $g(x)$ vanishes in a neighborhood around the origin. Also, note that (5.7) holds for any initial distribution $\rho(dx)$ of $\{L_t\}_{t \geq 0}$. Now, we prove the relation in (5.8). Similarly as in the proof of Theorem 3.1, because of τ-periodicity of all components,

$$\tilde{C}_t^{i,j,n} = n^{-1} \int_0^{nt} \langle \nabla w^i(L_{s-}^\tau), c \nabla w^j(L_{s-}^\tau) \rangle ds$$
$$+ n^{-1} \int_0^{nt} \int_{\mathbb{R}^{\bar{d}}} \left(w^i(y + L_{s-}^\tau) - w^i(L_{s-}^\tau) \right) \left(w^j(y + L_{s-}^\tau) - w^j(L_{s-}^\tau) \right) \nu(dy) ds$$

for all $i, j = 1, \ldots, d$. Now, by similar arguments as in the first step, Proposition 5.2 implies that $\{L_t^\tau\}_{t \geq 0}$ is ergodic (with respect to $dx_\tau/|\tau|$), hence the Birkhoff ergodic theorem entails that

$$\mathbb{P}^{dx_\tau/|\tau|} \left(\lim_{n \longrightarrow \infty} \tilde{C}_t^{i,j,n} = t\tilde{C}_{ij} \right) = |\tau|^{-1} \int_{[0,\tau]} \mathbb{P}^{x_\tau} \left(\lim_{n \longrightarrow \infty} \tilde{C}_t^{i,j,n} = t\tilde{C}_{ij} \right) dx_\tau = 1, \quad i, j = 1, \ldots, d.$$

Therefore, there exists a Lebesgue measure zero set $B \in \mathcal{B}(\mathbb{R}^{\bar{d}})$ such that

$$\mathbb{P}^x \left(\lim_{n \longrightarrow \infty} \tilde{C}_t^n = t\tilde{C} \right) = 1, \quad x \in B^c, \ i = 1, \ldots, d,$$

which together with [JS03, Lemma VI.3.31] proves (5.6), and thus, (3.14).

Step 4. In the fourth step, we prove that under (3.15) the limit in (5.6) holds for any initial distribution of $\{L_t\}_{t \geq 0}$. We again employ [JS03, Theorem VIII.2.17]. In the third step we derived the semimartingale (modified) characteristics $(B^n, C^n, \tilde{C}^n, N^n)$ of the semimartingales $\{S_t^n\}_{t \geq 0}$, $n \geq 1$, and proved that for any initial distribution $\rho(dx)$ of $\{L_t\}_{t \geq 0}$,

$$\int_0^{nt} \int_{\mathbb{R}^d} |g(y)| N^n(\omega, ds, dy) \xrightarrow{\mathbb{P}^\rho\text{-a.s.}} 0$$

for all $t \geq 0$ and all $g \in C_b(\mathbb{R}^d)$ vanishing in a neighborhood around the origin. Therefore, the desired result will be proven if we show that for any initial distribution $\rho(dx)$ of $\{L_t\}_{t \geq 0}$,

$$\tilde{C}_t^{i,j,n} \xrightarrow{L^2(\mathbb{P}^\rho, \Omega)} t\tilde{C}_{ij}$$

for all $i, j = 1, \ldots, d$ and all $t \geq 0$. We have

$$\mathbb{E}^\rho \left[\left(\tilde{C}_t^{i,j,n} - t\tilde{C}_{ij} \right)^2 \right] = \mathbb{E}^\rho \left[\left(\tilde{C}_t^{i,j,n} \right)^2 \right] - 2t\tilde{C}_{ij} \mathbb{E}^\rho \left[\tilde{C}_t^{i,j,n} \right] + t^2 \tilde{C}_{ij}^2.$$

First, we show that $\lim_{n \longrightarrow \infty} \mathbb{E}^\rho \left[\left(\tilde{C}_t^{i,j,n} \right)^2 \right] = t^2 \tilde{C}_{ij}^2$. We have

$$\mathbb{E}^\rho \left[\left(\tilde{C}_t^{i,j,n} \right)^2 \right] = I_1^n + I_2^n + I_3^n, \tag{5.9}$$

where

$$I_1^n := n^{-2} \sum_{k,l,p,q=1}^{\bar{d}} c_{kl} c_{pq} \int_0^{nt} \int_0^{nt} \mathbb{E}^\rho \left[\frac{\partial w^i(L_{s-})}{\partial x_k} \frac{\partial w^j(L_{s-})}{\partial x_l} \frac{\partial w^i(L_{u-})}{\partial x_p} \frac{\partial w^j(L_{u-})}{\partial x_q} \right] ds du,$$

$$I_2^n := 2n^{-2} \sum_{k,l=1}^{\bar{d}} c_{kl} \int_0^{nt} \int_0^{nt} \mathbb{E}^\rho \left[\frac{\partial w^i(L_{s-})}{\partial x_k} \frac{\partial w^j(L_{s-})}{\partial x_l} \int_{\mathbb{R}^{\bar{d}}} \left(w^i(y + L_{u-}) - w^i(L_{u-}) \right) \right.$$
$$\left. \left(w^j(y + L_{u-}) - w^j(L_{u-}) \right) \nu(dy) \right] ds du,$$

$$I_3^n := n^{-2} \int_0^{nt} \int_0^{nt} \int_{\mathbb{R}^{\bar{d}}} \int_{\mathbb{R}^{\bar{d}}} \mathbb{E}^\rho \left[\left(w^i(y + L_{s-}) - w^i(L_{s-}) \right) \left(w^j(y + L_{s-}) - w^j(L_{s-}) \right) \right.$$
$$\left. \left(w^i(z + L_{u-}) - w^i(L_{u-}) \right) \left(w^j(z + L_{u-}) - w^j(L_{u-}) \right) \right] \nu(dy)\nu(dz) ds du.$$

Now, by the same approach as in Proposition 5.1, we get

$$I_1^n = 2n^{-2} \sum_{k,l,p,q=1}^{\bar{d}} c_{kl} c_{pq} \int_0^{nt} \int_0^u \mathbb{E}^\rho \left[\frac{\partial w^i(L_{s-})}{\partial x_k} \frac{\partial w^j(L_{s-})}{\partial x_l} \frac{\partial w^i(L_{u-})}{\partial x_p} \frac{\partial w^j(L_{u-})}{\partial x_q} \right] ds du$$

$$= \frac{2^5 \pi^4}{n^2 |\tau|^4} \sum_{k,l,p,q=1}^{\bar{d}} c_{kl} c_{pq} \int_0^{nt} \int_0^u \sum_{a,b,c,d \in \mathbb{Z}^{\bar{d}}} a_k b_l c_p d_q \hat{w}^i(a) \hat{w}^j(b) \hat{w}^i(c) \hat{w}^j(d)$$
$$\mathbb{E}^\rho \left[e^{i \frac{2\pi \langle a+b, L_s \rangle}{|\tau|}} e^{i \frac{2\pi \langle c+d, L_u \rangle}{|\tau|}} \right] ds du$$

$$= \frac{2^5 \pi^4}{n^2 |\tau|^4} \sum_{k,l,p,q=1}^{\bar{d}} c_{kl} c_{pq} \int_0^{nt} \int_0^u \sum_{a,b,c,d \in \mathbb{Z}^{\bar{d}}} a_k b_l c_p d_q \hat{w}^i(a) \hat{w}^j(b) \hat{w}^i(c) \hat{w}^j(d) \hat{\rho}\left(\frac{2\pi(a+b+c+d)}{|\tau|} \right)$$
$$e^{-(u-s)q\left(\frac{2\pi(c+d)}{|\tau|} \right)} e^{-sq\left(\frac{2\pi(a+b+c+d)}{|\tau|} \right)} ds du$$

$$= \frac{2^4 \pi^4 t^2}{|\tau|^4} \sum_{k,l,p,q=1}^{\bar{d}} c_{kl} c_{pq} \sum_{a,c \in \mathbb{Z}^{\bar{d}}} a_k a_l c_p c_q \hat{w}^i(a) \hat{w}^j(-a) \hat{w}^i(c) \hat{w}^j(-c)$$

$$+ \frac{2^5 \pi^4}{n^2 |\tau|^4} \sum_{k,l,p,q=1}^{\bar{d}} c_{kl} c_{pq} \int_0^{nt} \int_0^u \sum_{\substack{a,b,c,d \in \mathbb{Z}^{\bar{d}} \\ a+b \neq 0 \text{ or } c+d \neq 0}} a_k b_l c_p d_q \hat{w}^i(a) \hat{w}^j(b) \hat{w}^i(c) \hat{w}^j(d) \hat{\rho}\left(\frac{2\pi(a+b+c+d)}{|\tau|} \right)$$
$$e^{-(u-s)q\left(\frac{2\pi(c+d)}{|\tau|} \right)} e^{-sq\left(\frac{2\pi(a+b+c+d)}{|\tau|} \right)} ds du,$$

where $\hat{\rho}(\xi)$ denotes the characteristic function of the probability measure $\rho(dx)$. Note that the change of orders of integrations and summations is justified by (3.15). Finally, again by applying (3.15), it is easy to see that

$$\lim_{n \longrightarrow \infty} I_1^n = \frac{2^4 \pi^4 t^2}{|\tau|^4} \sum_{k,l,p,q=1}^{\bar{d}} c_{kl} c_{pq} \sum_{a,c \in \mathbb{Z}^{\bar{d}}} a_k a_l c_p c_q \hat{w}^i(a) \hat{w}^j(-a) \hat{w}^i(c) \hat{w}^j(-c). \tag{5.10}$$

Similarly, we have

$$I_2^n = 4n^{-2} \sum_{k,l=1}^{\bar{d}} c_{kl} \int_0^{nt} \int_0^u \mathbb{E}^\rho \left[\frac{\partial w^i(L_{s-})}{\partial x_k} \frac{\partial w^j(L_{s-})}{\partial x_l} \int_{\mathbb{R}^{\bar{d}}} \left(w^i(y + L_{u-}) - w^i(L_{u-}) \right) \right.$$

$$\left. \left(w^j(y + L_{u-}) - w^j(L_{u-}) \right) \nu(dy) \right] dsdu$$

$$= -\frac{2^4\pi^2}{n^2|\tau|^2} \sum_{k,l=1}^{\bar{d}} c_{kl} \int_0^{nt} \int_0^u \sum_{a,b,c,d \in \mathbb{Z}^{\bar{d}}} a_k b_l \hat{w}^i(a) \hat{w}^j(b) \hat{w}^i(c) \hat{w}^j(d) \mathbb{E}^\rho \left[e^{i\frac{2\pi\langle a+b, L_s \rangle}{|\tau|}} e^{i\frac{2\pi\langle c+d, L_u \rangle}{|\tau|}} \right]$$

$$\int_{\mathbb{R}^{\bar{d}}} \left(e^{i\frac{2\pi\langle c,y \rangle}{|\tau|}} - 1 \right) \left(e^{i\frac{2\pi\langle d,y \rangle}{|\tau|}} - 1 \right) \nu(dy) dsdu$$

$$= -\frac{2^4\pi^2}{n^2|\tau|^2} \sum_{k,l=1}^{\bar{d}} c_{kl} \int_0^{nt} \int_0^u \sum_{a,b,c,d \in \mathbb{Z}^{\bar{d}}} a_k b_l \hat{w}^i(a) \hat{w}^j(b) \hat{w}^i(c) \hat{w}^j(d) \hat{\rho} \left(\frac{2\pi(a+b+c+d)}{|\tau|} \right) e^{-(u-s)q\left(\frac{2\pi(c+d)}{|\tau|} \right)}$$

$$e^{-sq\left(\frac{2\pi(a+b+c+d)}{|\tau|} \right)} \int_{\mathbb{R}^{\bar{d}}} \left(e^{i\frac{2\pi\langle c,y \rangle}{|\tau|}} - 1 \right) \left(e^{i\frac{2\pi\langle d,y \rangle}{|\tau|}} - 1 \right) \nu(dy) dsdu$$

$$= \frac{2^4\pi^2 t^2}{|\tau|^2} \sum_{k,l=1}^{\bar{d}} c_{k,l} \sum_{a,c \in \mathbb{Z}^{\bar{d}}} a_k a_l \hat{w}^i(a) \hat{w}^j(-a) \hat{w}^i(c) \hat{w}^j(-c) \int_{\mathbb{R}^{\bar{d}}} \left(1 - \cos\left(\frac{2\pi\langle c,y \rangle}{|\tau|} \right) \right) \nu(dy)$$

$$- \frac{2^4\pi^2}{n^2|\tau|^2} \sum_{k,l=1}^{\bar{d}} c_{k,l} \int_0^{nt} \int_0^u \sum_{\substack{a,b,c,d \in \mathbb{Z}^{\bar{d}} \\ a+b \neq 0 \text{ or } c+d \neq 0}} a_k b_l \hat{w}^i(a) \hat{w}^j(b) \hat{w}^i(c) \hat{w}^j(d) \hat{\rho} \left(\frac{2\pi(a+b+c+d)}{|\tau|} \right)$$

$$e^{-(u-s)q\left(\frac{2\pi(c+d)}{|\tau|} \right)} e^{-sq\left(\frac{2\pi(a+b+c+d)}{|\tau|} \right)} \int_{\mathbb{R}^{\bar{d}}} \left(e^{i\frac{2\pi\langle c,y \rangle}{|\tau|}} - 1 \right) \left(e^{i\frac{2\pi\langle d,y \rangle}{|\tau|}} - 1 \right) \nu(dy) dsdu.$$

Again, by applying (3.15), we get

$$\lim_{n \to \infty} I_2^n = \frac{2^4\pi^2 t^2}{|\tau|^2} \sum_{k,l=1}^{\bar{d}} c_{k,l} \sum_{a,c \in \mathbb{Z}^{\bar{d}}} a_k a_l \hat{w}^i(a) \hat{w}^j(-a) \hat{w}^i(c) \hat{w}^j(-c)$$

$$\int_{\mathbb{R}^{\bar{d}}} \left(1 - \cos\left(\frac{2\pi\langle c,y \rangle}{|\tau|} \right) \right) \nu(dy). \tag{5.11}$$

Finally, we have

$$I_3^n = 2n^{-2} \int_0^{nt} \int_0^u \int_{\mathbb{R}^{\bar{d}}} \int_{\mathbb{R}^{\bar{d}}} \mathbb{E}^\rho \left[\left(w^i(y + L_{s-}) - w^i(L_{s-}) \right) \left(w^j(y + L_{s-}) - w^j(L_{s-}) \right) \right.$$

$$\left. \left(w^i(z + L_{u-}) - w^i(L_{u-}) \right) \left(w^j(z + L_{u-}) - w^j(L_{u-}) \right) \right] \nu(dy)\nu(dz) dsdu$$

$$= 2n^{-2} \int_0^{nt} \int_0^u \sum_{a,b,c,d \in \mathbb{Z}^{\bar{d}}} \hat{w}^i(a) \hat{w}^j(b) \hat{w}^i(c) \hat{w}^j(d) \mathbb{E}^\rho \left[e^{i\frac{2\pi\langle a+b, L_s \rangle}{|\tau|}} e^{i\frac{2\pi\langle c+d, L_u \rangle}{|\tau|}} \right]$$

$$\int_{\mathbb{R}^{\bar{d}}} \int_{\mathbb{R}^{\bar{d}}} \left(e^{i\frac{2\pi\langle a,y \rangle}{|\tau|}} - 1 \right) \left(e^{i\frac{2\pi\langle b,y \rangle}{|\tau|}} - 1 \right) \left(e^{i\frac{2\pi\langle c,z \rangle}{|\tau|}} - 1 \right) \left(e^{i\frac{2\pi\langle d,z \rangle}{|\tau|}} - 1 \right) \nu(dy)\nu(dz) dsdu$$

25

$$= 2n^{-2} \int_0^{nt} \int_0^u \sum_{a,b,c,d \in \mathbb{Z}^{\bar{d}}} \hat{w}^i(a)\hat{w}^j(b)\hat{w}^i(c)\hat{w}^j(d)\hat{\rho}\left(\tfrac{2\pi(a+b+c+d)}{|\tau|}\right) e^{-(u-s)q\left(\tfrac{2\pi(c+d)}{|\tau|}\right)} e^{-sq\left(\tfrac{2\pi(a+b+c+d)}{|\tau|}\right)}$$

$$\int_{\mathbb{R}^{\bar{d}}} \int_{\mathbb{R}^{\bar{d}}} \left(e^{i\tfrac{2\pi\langle a,y\rangle}{|\tau|}} - 1\right)\left(e^{i\tfrac{2\pi\langle b,y\rangle}{|\tau|}} - 1\right)\left(e^{i\tfrac{2\pi\langle c,z\rangle}{|\tau|}} - 1\right)\left(e^{i\tfrac{2\pi\langle d,z\rangle}{|\tau|}} - 1\right)\nu(dy)\nu(dz)dsdu$$

$$= 4t^2 \sum_{a,c \in \mathbb{Z}^{\bar{d}}} \hat{w}^i(a)\hat{w}^j(-a)\hat{w}^i(c)\hat{w}^j(-c) \int_{\mathbb{R}^{\bar{d}}} \int_{\mathbb{R}^{\bar{d}}} \left(1 - \cos\left(\tfrac{2\pi\langle a,y\rangle}{|\tau|}\right)\right)\left(1 - \cos\left(\tfrac{2\pi\langle c,z\rangle}{|\tau|}\right)\right)\nu(dy)\nu(dz)$$

$$+ 2n^{-2} \int_0^{nt} \int_0^u \sum_{\substack{a,b,c,d \in \mathbb{Z}^{\bar{d}} \\ a+b\neq 0 \text{ or } c+d\neq 0}} \hat{w}^i(a)\hat{w}^j(b)\hat{w}^i(c)\hat{w}^j(d)\hat{\mu}\left(\tfrac{2\pi(a+b+c+d)}{|\tau|}\right) e^{-(u-s)q\left(\tfrac{2\pi(c+d)}{|\tau|}\right)} e^{-sq\left(\tfrac{2\pi(a+b+c+d)}{|\tau|}\right)}$$

$$\int_{\mathbb{R}^{\bar{d}}} \int_{\mathbb{R}^{\bar{d}}} \left(e^{i\tfrac{2\pi\langle a,y\rangle}{|\tau|}} - 1\right)\left(e^{i\tfrac{2\pi\langle b,y\rangle}{|\tau|}} - 1\right)\left(e^{i\tfrac{2\pi\langle c,z\rangle}{|\tau|}} - 1\right)\left(e^{i\tfrac{2\pi\langle d,z\rangle}{|\tau|}} - 1\right)\nu(dy)\nu(dz)dsdu.$$

Again, (3.15) implies that

$$\lim_{n \to \infty} I_3^n = 4t^2 \sum_{a,c \in \mathbb{Z}^{\bar{d}}} \hat{w}^i(a)\hat{w}^j(-a)\hat{w}^i(c)\hat{w}^j(-c)$$

$$\int_{\mathbb{R}^{\bar{d}}} \int_{\mathbb{R}^{\bar{d}}} \left(1 - \cos\left(\tfrac{2\pi\langle a,y\rangle}{|\tau|}\right)\right)\left(1 - \cos\left(\tfrac{2\pi\langle c,z\rangle}{|\tau|}\right)\right)\nu(dy)\nu(dz). \qquad (5.12)$$

Now, by putting together (5.9), (5.10), (5.11) and (5.12), Proposition 5.1 implies

$$\lim_{n \to \infty} \mathbb{E}^\rho\left[\left(\tilde{C}_t^{i,j,n}\right)^2\right] = t^2 \tilde{C}_{ij}^2.$$

In completely the same way we get

$$\lim_{n \to \infty} \mathbb{E}^\rho\left[\tilde{C}_t^{i,j,n}\right] = t\tilde{C}_{ij}.$$

Thus,

$$\tilde{C}_t^{i,j,n} \xrightarrow{L^2(\mathbb{P}^\rho, \Omega)} t\tilde{C}_{ij}, \quad i,j = 1, \dots, d,$$

that is, for any initial distribution $\rho(dx)$ of $\{L_t\}_{t \geq 0}$,

$$\{S_t^n\}_{t \geq 0} \xrightarrow{\mathrm{d}} \{\tilde{W}_t\}_{t \geq 0},$$

under $\mathbb{P}^\rho(d\omega_V)$. Finally, since the function $w(x)$ is bounded, [JS03, Lemma VI.3.31] again implies the convergence in (5.6), and thus, in (3.14). $\qquad \square$

5.1 Comments on the Condition in (3.12)

In connection to Proposition 5.2, note that if (3.12) is not satisfied for some $k_0 \neq 0$, then we cannot automatically conclude that $\{L_t^\tau\}_{t \geq 0}$ is not ergodic. For example, take a one-dimensional Lévy process $\{L_t\}_{t \geq 0}$ with symbol of the form $q(\xi) = ib\xi$, $b \neq 0$. On the other hand, in the dimension $\bar{d} \geq 2$ or when $b = 0$ (in any dimension), $\{L_t^\tau\}_{t \geq 0}$ is not ergodic.

Proposition 5.3. *Let $\{L_t\}_{t \geq 0}$ be a $\bar{d}$-dimensional Lévy process with symbol $q(\xi)$ not satisfying the condition in (3.12). Then, $\{L_t^\tau\}_{t \geq 0}$ is not strongly ergodic (with respect to $dx_\tau/|\tau|$).*

Proof. By the assumption, there exists $k_0 \in \mathbb{Z}^{\bar{d}}$, $k_0 \neq 0$, such that $\operatorname{Re} q(2\pi k_0/|\tau|) = 0$. Hence, for this $k_0 \in \mathbb{Z}^{\bar{d}}$, we have

$$\mathbb{E}^0 \left[e^{i \left\langle \frac{2\pi k_0}{|\tau|}, L_t \right\rangle} \right] = e^{i \left\langle \frac{2\pi k_0}{|\tau|}, t x_0 \right\rangle}$$

for some $x_0 \in \mathbb{R}^{\bar{d}}$. This yields

$$\mathbb{E}^0 \left[\cos \left\langle \tfrac{2\pi k_0}{|\tau|}, L_t - t x_0 \right\rangle \right] = \int_{\mathbb{R}^{\bar{d}}} \cos \left\langle \tfrac{2\pi k_0}{|\tau|}, y - t x_0 \right\rangle p(t, 0, dy) = 1.$$

Thus, $p(t, 0, dy)$ is supported on the set $\{ y \in \mathbb{R}^{\bar{d}} : \langle k_0, y - t x_0 \rangle = l|\tau|, \ l \in \mathbb{Z} \}$, $t > 0$. In particular, $p(t, 0, dy)$ is singular with respect to dx, which proves the claim. $\qquad\square$

Further, the condition in (3.12) is also not equivalent with the strong ergodicity of $\{L_t^\tau\}_{t \geq 0}$. For example, let $\{L_t\}_{t \geq 0}$ be a one-dimensional Lévy process with symbol of the form $q(\xi) = 2(1 - \cos(\kappa \xi))$ or, equivalently, with the Lévy triplet $(0, 0, \delta_{-\kappa}(dy) + \delta_\kappa(dy))$, where $\kappa > 0$ is such that $\kappa/\tau \notin \mathbb{Q}$. However, as a direct consequence of Proposition 5.3 we get that condition (**C5**) automatically implies the relation in (3.12).

Proposition 5.4. *Let $\{L_t\}_{t \geq 0}$ be a $\bar{d}$-dimensional Lévy process with symbol $q(\xi)$ and Lévy triplet $(b, 0, \nu(dy))$.*

(i) *If $\nu(dy) = 0$, then for any $\beta > 0$, any τ-periodic $f \in C_b^2(\mathbb{R}^{\bar{d}})$ and any initial distribution $\rho(dx)$ of $\{L_t\}_{t \geq 0}$,*

$$n^{-\beta} \int_0^{nt} \mathcal{A}^b f(L_s) ds \xrightarrow{\mathbb{P}^\rho\text{-}a.s.} 0.$$

(ii) *If $\bar{d} = 1$ and if (3.12) is not satisfied for some $k_0 \neq 0$, then for any $\beta > 0$, any $\tau/|k_0|$-periodic $f \in C_b^2(\mathbb{R})$ and any initial distribution $\rho(dx)$ of $\{L_t\}_{t \geq 0}$,*

$$n^{-\beta} \int_0^{nt} \mathcal{A}^b f(L_s) ds \xrightarrow{\mathbb{P}^\rho\text{-}a.s.} 0.$$

Proof. (i) Let $f \in C_b^2(\mathbb{R}^{\bar{d}})$ and $\rho(dx)$ be an arbitrary τ-periodic function and an arbitrary initial distribution of $\{L_t\}_{t \geq 0}$, respectively. Then, for any $\beta > 0$, we have

$$\begin{aligned}
\lim_{n \to \infty} n^{-\beta} \int_0^{nt} \mathcal{A}^b f(L_s) ds &= \lim_{n \to \infty} n^{-\beta} \int_0^{nt} \langle b, \nabla f(L_0 + bs) \rangle ds \\
&= \lim_{n \to \infty} n^{-\beta} \int_0^{nt} \frac{\partial f(L_0 + bs)}{\partial s} ds \\
&= \lim_{n \to \infty} n^{-\beta} (f(L_0 + bnt) - f(L_0)) \\
&= 0, \quad \mathbb{P}^\rho\text{-a.s.}
\end{aligned}$$

(ii) First, similarly as before, we conclude that $\nu(dy)$ is supported on $S := \{\tau l / k_0 : l \in \mathbb{Z}\}$. This yields that

$$\mathcal{A}^b g(x) = b g'(x) + \int_{\mathbb{R}} (g(y + x) - g(x)) \nu(dy), \quad g \in C_b^2(\mathbb{R}),$$

$\nu(\mathbb{R}) < \infty$ and $L_t = L_0 + bt + S_{N_t}$, $t \geq 0$, where $\{S_n\}_{n \geq 0}$ is a random walk on S with the jump distribution $\mathbb{P}^0(S_1 \in dy) := \nu(dy)/\nu(\mathbb{R})$ and $\{N_t\}_{t \geq 0}$ is a Poisson process with parameter $\nu(\mathbb{R})$

27

independent of $\{S_n\}_{n\geq 0}$. Now, let $f \in C_b^2(\mathbb{R})$ be an arbitrary $\tau/|k_0|$ periodic function. Then, for any $\beta > 0$, we have

$$
\begin{aligned}
n^{-\beta} \int_0^{nt} \mathcal{A}^b f(L_s) ds =\, & n^{-\beta} \int_0^{nt} b f'(L_0 + bs + S_{N_s}) ds \\
& + n^{-\beta} \int_0^{nt} \int_{\mathbb{R}} (f(y + L_0 + bs + S_{N_s}) - f(L_0 + bs + S_{N_s})) \nu(dy) ds \\
=\, & n^{-\beta} \int_0^{nt} b f'(L_0 + bs) ds \\
=\, & n^{-\beta} \int_0^{nt} \frac{\partial f(L_0 + bs)}{\partial s} ds \\
=\, & n^{-\beta} (f(L_0 + bnt) - f(L_0)),
\end{aligned}
$$

where in the second step we used the facts that $f(x)$ is $\tau/|k_0|$-periodic and $\{S_n\}_{n\geq 0}$ and $\nu(dy)$ live on S. Now, by letting $n \longrightarrow \infty$, the desired result follows.

$\square$

6 Discussions on the Cases with General Diffusions with Jumps

In Theorems 3.1, 3.4 and 3.5, we have shown the LLN and CLT for the process $\{X_t\}_{t\geq 0}$ in (1.2) driven by the process $\{F_t\}_{t\geq 0}$, a diffusion with jumps, satisfying the sets of assumed conditions, particularly the ergodicity property. In this section, we discuss the (strong) ergodicity property of general diffusions with jumps and the limiting behaviors in (1.3) and (1.4) when the velocity field $\{V(t,x)\}_{t\geq 0,\, x\in\mathbb{R}^{\bar{d}}}$ is governed by general, not necessarily ergodic, diffusions with jumps.

In the proofs of Theorems 3.1 and 3.5, the most crucial ingredient was the τ-periodicity of a driving diffusion with jumps $\{F_t\}_{t\geq 0}$ and velocity function $v(x)$. By having this property we were able to switch to a (strongly) ergodic Markov process $\{F_t^\tau\}_{t\geq 0}$ on a compact space $[0,\tau]$, satisfying $v(F_t) = v(F_t^\tau)$, and deduced the limiting behaviors in (1.3) and (1.4). In a general situation, when $\{F_t\}_{t\geq 0}$ or $v(x)$ are not τ-periodic, we cannot perform a similar trick, and unlike in the τ-periodic case, the (strong) ergodicity strongly depends on the dimension of the state space. Let us be more precise. First, recall that a progressively measurable strong Markov process $\{M_t\}_{t\geq 0}$ on the state space $(\mathbb{R}^{\bar{d}}, \mathcal{B}(\mathbb{R}^{\bar{d}}))$, $\bar{d} \geq 1$, is called

(i) *irreducible* if there exists a σ-finite measure $\varphi(dy)$ on $\mathcal{B}(\mathbb{R}^{\bar{d}})$ such that whenever $\varphi(B) > 0$ we have $\int_0^\infty \mathbb{P}^x(M_t \in B) dt > 0$ for all $x \in \mathbb{R}^{\bar{d}}$;

(ii) *recurrent* if it is φ-irreducible and if $\varphi(B) > 0$ implies $\int_0^\infty \mathbb{P}^x(M_t \in B) dt = \infty$ for all $x \in \mathbb{R}^{\bar{d}}$;

(iii) *Harris recurrent* if it is φ-irreducible and if $\varphi(B) > 0$ implies $\mathbb{P}^x(\tau_B < \infty) = 1$ for all $x \in \mathbb{R}^{\bar{d}}$, where $\tau_B := \inf\{t \geq 0 : M_t \in B\}$;

(iv) *transient* if it is φ-irreducible and if there exists a countable covering of $\mathbb{R}^{\bar{d}}$ with sets $\{B_j\}_{j\in\mathbb{N}} \subseteq \mathcal{B}(\mathbb{R}^{\bar{d}})$, such that for each $j \in \mathbb{N}$ there is a finite constant $c_j \geq 0$ such that $\int_0^\infty \mathbb{P}^x(M_t \in B_j) dt \leq c_j$ holds for all $x \in \mathbb{R}^{\bar{d}}$.

Let us remark that if $\{M_t\}_{t\geq 0}$ is a φ-irreducible Markov process, then the irreducibility measure $\varphi(dy)$ can be maximized, that is, there exists a unique "maximal" irreducibility measure $\psi(dy)$ such that for any measure $\bar{\varphi}(dy)$, $\{M_t\}_{t\geq 0}$ is $\bar{\varphi}$-irreducible if, and only if, $\bar{\varphi} \ll \psi$ (see [Twe94,

Theorem 2.1]). According to this, from now on, when we refer to irreducibility measure we actually refer to the maximal irreducibility measure. In the sequel, we consider only the so-called open set irreducible Markov processes, that is, we consider only ψ-irreducible Markov processes whose maximal irreducibility measure $\psi(dy)$ satisfies the following *open-set irreducibility* condition

(C6) $\psi(O) > 0$ for every open set $O \subseteq \mathbb{R}^{\bar{d}}$.

Obviously, the Lebesgue measure dx satisfies condition (**C6**) and a Markov process $\{M_t\}_{t\geq 0}$ will be dx-irreducible if $\mathbb{P}^x(M_t \in B) > 0$ for all $t > 0$ and all $x \in \mathbb{R}^{\bar{d}}$ whenever $B \in \mathcal{B}(\mathbb{R}^{\bar{d}})$ has positive Lebesgue measure. In particular, the process $\{M_t\}_{t\geq 0}$ will be dx-irreducible if the transition kernel $\mathbb{P}^x(M_t \in dy)$ possesses a transition density function $p(t, x, y)$, such that $p(t, x, y) > 0$ for all $t > 0$ and all $x, y \in \mathbb{R}^{\bar{d}}$. In Remark 3.3 we have commented that the question of the strict positivity of $p(t, x, y)$ of general diffusions with jumps is a non-trivial problem. The best we were able to obtain is given in the following proposition, which, regardless the possible lack of the strict positivity of $p(t, x, y)$, shows the Lebesgue irreducibility property of a class of diffusions with jumps.

Proposition 6.1. *Let $\{F_t\}_{t\geq 0}$ be a diffusion with jumps with symbol $q(x, \xi)$ satisfying (3.5) and*

$$\mathbb{E}^x\left[e^{i\langle \xi, F_t - x\rangle}\right] = \operatorname{Re} \mathbb{E}^x\left[e^{i\langle \xi, F_t - x\rangle}\right], \quad x, \xi \in \mathbb{R}^{\bar{d}}. \tag{6.1}$$

Then, $\{F_t\}_{t\geq 0}$ possesses a transition density function $p(t, x, y)$, such that for every $t_0 > 0$ and every $n \geq 1$ there exists $\varepsilon(t_0) > 0$ such that $p(t, x, y + x) > 0$ for all $t \in [nt_0, n(t_0 + 1)]$, all $x \in \mathbb{R}^{\bar{d}}$ and all $|y| < n\varepsilon(t_0)$.

Proof. First, according to [SW13, Theorem 2.1], the condition in (6.1) implies that $q(x, \xi) = \operatorname{Re} q(x, \xi)$, that is, $b(x) = 0$ and $\nu(x, dy)$ are symmetric measures for all $x \in \mathbb{R}^{\bar{d}}$. Consequently, the condition in (3.6) trivially holds true. Thus, $\{F_t\}_{t\geq 0}$ possesses a transition density function $p(t, x, y)$ which is given by (3.8), and, for every $t > 0$ and every $x, y \in \mathbb{R}^{\bar{d}}$, we have

$$\left|p(t, x, y + x) - p(t, x, x)\right| = (2\pi)^{-\bar{d}}\left|\int_{\mathbb{R}^{\bar{d}}}\left(1 - e^{-i\langle \xi, y\rangle}\right)\mathbb{E}^x\left[e^{i\langle \xi, (F_t - x)\rangle}\right]d\xi\right|$$

$$\leq (2\pi)^{-\bar{d}}\int_{\mathbb{R}^{\bar{d}}}\left|1 - e^{-i\langle \xi, y\rangle}\right|\exp\left[-\frac{t}{16}\inf_{x\in\mathbb{R}^{\bar{d}}}q(x, 2\xi)\right]d\xi.$$

Now, by (3.5) and the dominated convergence theorem, we conclude that for every $t_0 > 0$ the continuity of the function $y \longmapsto p(t, x, y)$ at x is uniformly for all $t \geq t_0$ and all $x \in \mathbb{R}^{\bar{d}}$. Next, by applying [SW13, Theorem 2.1] (under (6.1)), R. L. Schilling and J. Wang (personal communication) obtained

$$\inf_{x\in\mathbb{R}^{\bar{d}}}\mathbb{E}^x\left[e^{i\langle \xi, (F_t - x)\rangle}\right] \geq \frac{1}{2}\exp\left[-4t \sup_{|\eta|\leq|\xi|}\sup_{x\in\mathbb{R}^{\bar{d}}}q(x, \eta)\right], \quad t > 0, \ \xi \in \mathbb{R}^{\bar{d}}.$$

Hence, for every $t_0 > 0$,

$$p(t, x, x) = (2\pi)^{-\bar{d}}\int_{\mathbb{R}^{\bar{d}}}\mathbb{E}^x\left[e^{i\langle \xi, (F_t - x)\rangle}\right]d\xi \geq \frac{1}{4\pi}\int_{\mathbb{R}^{\bar{d}}}\exp\left[-4(t_0 + 1)\sup_{|\eta|\leq|\xi|}\sup_{x\in\mathbb{R}^{\bar{d}}}q(x, \eta)\right]d\xi > 0$$

uniformly for all $t \in [t_0, t_0 + 1]$ and all $x \in \mathbb{R}^{\bar{d}}$. According to this, there exists $\varepsilon(t_0) > 0$ such that $p(t, x, y + x) > 0$ for all $t \in [t_0, t_0 + 1]$, all $x \in \mathbb{R}^{\bar{d}}$ and all $|y| < \varepsilon(t_0)$. Now, for any $n \geq 1$, by the Chapman-Kolmogorov equation, we have that $p(t, x, y + x) > 0$ for all $t \in [nt_0, n(t_0 + 1)]$, all $x \in \mathbb{R}^{\bar{d}}$ and all $|y| < n\varepsilon(t_0)$, which proves the desired result. $\qquad\square$

Further, it is well known that every ψ-irreducible Markov process is either recurrent or transient (see [Twe94, Theorem 2.3]) and, clearly, every Harris recurrent Markov process is recurrent, but in general, these two properties are not equivalent. They differ on the set of the irreducibility measure zero (see [Twe94, Theorem 2.5]). However, for a diffusion with jumps satisfying condition (**C6**) these two properties are equivalent (see [San14b, Proposition 2.1]).

Next, it is shown in [Twe94, Theorem 2.6] that if $\{M_t\}_{t\geq0}$ is a recurrent process, then there exists a unique (up to constant multiples) invariant measure $\pi(dx)$. If the invariant measure is finite, then it may be normalized to a probability measure. If $\{M_t\}_{t\geq0}$ is (Harris) recurrent with finite invariant measure $\pi(dx)$, then $\{M_t\}_{t\geq0}$ is called *positive (Harris) recurrent*; otherwise it is called *null (Harris) recurrent*. One would expect that every positive (Harris) recurrent process is strongly ergodic, but in general this is not true (see [MT93]). In the case of an open-set irreducible diffusion with jumps $\{F_t\}_{t\geq0}$, these two properties coincide. In particular, for this class of processes, ergodicity coincides with strong ergodicity. Indeed, according to [MT93, Theorem 6.1] and [SW13, Theorem 3.3] it suffices to show that if $\{F_t\}_{t\geq0}$ possesses an invariant probability measure $\pi(dx)$, then it is recurrent. Assume that $\{F_t\}_{t\geq0}$ is transient. Then there exists a countable covering of $\mathbb{R}^{\bar{d}}$ with sets $\{B_j\}_{j\in\mathbb{N}} \subseteq \mathcal{B}(\mathbb{R}^{\bar{d}})$, such that for each $j \in \mathbb{N}$ there is a finite constant $M_j \geq 0$ such that $\int_0^\infty p^t(x, B_j)dt \leq M_j$ holds for all $x \in \mathbb{R}^{\bar{d}}$. Let $t > 0$ be arbitrary. Then for each $j \in \mathbb{N}$ we have

$$t\pi(B_j) = \int_0^t \int_{\mathbb{R}^d} p^s(x, B_j)\pi(dx)ds \leq M_j.$$

By letting $t \longrightarrow \infty$ we get that $\pi(B_j) = 0$ for all $j \in \mathbb{N}$, which is impossible.

Now, we can conclude that a $\bar{d}$-dimensional, $\bar{d} \geq 3$, open-set irreducible diffusion with jumps is never ergodic. Indeed, as we commented, if an invariant probability measure of an irreducible Markov process exists, then it must be recurrent. But the recurrence and transience of open set irreducible diffusions with jumps, similarly as of Lévy processes, depends on the dimension of the state space. More precisely, according to [San14b, Theorem 2.8], every truly $\bar{d}$-dimensional, $\bar{d} \geq 3$, open set irreducible diffusions with jumps is always transient. In particular, $\bar{d}$-dimensional, $\bar{d} \geq 3$, diffusions are transient. On the other hand, one-dimensional and two-dimensional symmetric diffusions are recurrent (see [San14b, Theorem 2.9]). Also $\bar{d}$-dimensional, $\bar{d} \geq 2$, stable-like processes are always transient (see [San14b, Theorem 2.10 and Corollary 3.3]). For conditions for recurrence and transience of one-dimensional stable-like processes, see [Böt11], [Fra06, Fra07], [San13a], [San13b], [San14a], [San14b] and [San14c]. For sufficient conditions for ergodicity of diffusions, see [Bha78, Bha80], [BR82]. For sufficient conditions for strong ergodicity of one-dimensional stable-like processes and diffusion with jumps, see [San13a] and [Wan08], respectively. A necessary and sufficient condition for the existence of an invariant probability measure $\pi(dx)$ of a $\bar{d}$-dimensional diffusion with jumps $\{F_t\}_{t\geq0}$ with symbol $q(x, \xi)$, for which $C_c^\infty(\mathbb{R}^{\bar{d}})$ is an operator core of the corresponding Feller generator (that is, $\mathcal{A}^\infty$ is the only extension of $\mathcal{A}^\infty|_{C_c^\infty(\mathbb{R}^{\bar{d}})}$ on $\mathcal{D}_{\mathcal{A}^\infty}$), has been given in [SB13, Theorems 3.1 and 4.1] and it reads as follows

$$\int_{\mathbb{R}^{\bar{d}}} e^{i\langle\xi,x\rangle}q(x,\xi)\pi(dx) = 0, \quad \xi \in \mathbb{R}^{\bar{d}}.$$

Also, let us remark that a (nontrivial) Lévy process is never (strongly) ergodic since it cannot possess an invariant probability measure (see [Sat99, Exercise 29.6]).

We end this paper with the following observations. Regardless the (strong) ergodicity property we have the following limiting behaviors. Let $\{L_t\}_{t\geq0}$ be a $\bar{d}$-dimensional Lévy process with Feller generator $(\mathcal{A}^\infty, \mathcal{D}_{\mathcal{A}^\infty})$ and let $f \in C_c^\infty(\mathbb{R}^{\bar{d}})$. Note that $\mathcal{A}^\infty f \in L^1(dx, \mathbb{R}^{\bar{d}}) \cap B_b(\mathbb{R}^{\bar{d}})$. Then, since

$\hat{f} \in L^1(dx, \mathbb{R}^{\bar{d}})$ (recall that $\hat{f}(\xi)$ denotes the Fourier transform of $f(x)$), we have

$$f(x) = \int_{\mathbb{R}^d} e^{i\langle \xi, x \rangle} \hat{f}(\xi) d\xi$$

and by using an analogous approach as in the proof of Theorem 3.5, we obtain that

$$n^{-\beta} \int_0^{nt} \mathcal{A}^\infty f(L_s) ds \xrightarrow{\text{d}} 0$$

for any $\beta > 0$ and any initial distribution $\rho(dx)$ of $\{L_t\}_{t\geq 0}$. Further, if $\{F_t\}_{t\geq 0}$ is a $\bar{d}$-dimensional diffusion with jumps with Feller generator $(\mathcal{A}^\infty, \mathcal{D}_{\mathcal{A}^\infty})$ and symbol $q(x,\xi)$ satisfying

$$\int_{\{|\xi|<r\}} \frac{d\xi}{\inf_{x\in\mathbb{R}^{\bar{d}}} \operatorname{Re} q(x,\xi)} < \infty, \tag{6.2}$$

for some $r > 0$, and the conditions in (3.6) and (**C6**). Then, according to [SW13, the proof of Theorem 1.1],

$$n^{-\beta} \int_0^{nt} f(F_s) ds \xrightarrow{\mathbb{P}^\rho\text{- a.s.}} 0$$

for any $\beta > 0$, any $f \in C_c^\infty(\mathbb{R}^{\bar{d}})$ and any initial distribution $\rho(dx)$ of $\{F_t\}_{t\geq 0}$. Let us remark that the condition in (6.2), together with (3.6) and (**C6**), implies the transience of $\{F_t\}_{t\geq 0}$ (see [SW13, Theorem 1.1]). In the Lévy process case, by the definition of transience, assumptions (3.6) and (**C6**) are not needed (see [Sat99, Theorem 37.5]).

Proposition 6.2. *Let $\{F_t\}_{t\geq 0}$ be a $\bar{d}$-dimensional diffusion with jumps with symbol $q(x,\xi)$ satisfying* (3.6) *and*

$$\int_0^\infty \int_{\mathbb{R}^{\bar{d}}} \exp\left[-t \inf_{x\in\mathbb{R}^{\bar{d}}} \operatorname{Re} q(x,\xi) \right] d\xi dt < \infty. \tag{6.3}$$

Then,

$$n^{-\beta} \int_0^{nt} f(F_s) ds \xrightarrow{\mathbb{P}^\rho\text{- a.s.}} 0$$

for any $\beta > 0$, any $f \in L^1(dx, \mathbb{R}^{\bar{d}})$ and any initial distribution $\rho(dx)$ of $\{F_t\}_{t\geq 0}$. In addition, if $f \in L^1(dx, \mathbb{R}^{\bar{d}}) \cap B_b(\mathbb{R}^{\bar{d}})$, then it suffices to require that (6.3) *holds on (t_0, ∞), for some $t_0 > 0$.*

Proof. First, recall that, according to [San14b, Theorem 2.6] and [SW13, Theorem 1.1], $\{F_t\}_{t\geq 0}$ has a transition density function $p(t,x,y)$ which satisfies

$$\sup_{x,y\in\mathbb{R}^{\bar{d}}} p(t,x,y) \leq (4\pi)^{-\bar{d}} \int_{\mathbb{R}^{\bar{d}}} \exp\left[-\frac{t}{16} \inf_{x\in\mathbb{R}^{\bar{d}}} \operatorname{Re} q(x,\xi) \right] d\xi.$$

By using this fact, for any $f \in L^1(dx, \mathbb{R}^{\bar{d}})$ and any initial distribution $\rho(dx)$ of $\{F_t\}_{t\geq 0}$, we have

$$\mathbb{E}^\rho\left[\int_0^\infty |f(F_s)| ds \right] = \int_{\mathbb{R}^{\bar{d}}} \int_0^\infty \int_{\mathbb{R}^{\bar{d}}} |f(y)| p(s,x,y) dy ds \rho(dx)$$

$$\leq (4\pi)^{-\bar{d}} \int_{\mathbb{R}^{\bar{d}}} |f(y)| dy \int_0^\infty \int_{\mathbb{R}^{\bar{d}}} \exp\left[-\frac{s}{16} \inf_{x\in\mathbb{R}^{\bar{d}}} \operatorname{Re} q(x,\xi) \right] d\xi ds.$$

In particular,

$$\int_0^\infty f(F_s)ds < \infty, \quad \mathbb{P}^\rho\text{-a.s.,}$$

which proves the claim.

To prove the second statement, by completely the same reasoning as above we conclude that

$$\int_{t_0}^\infty f(F_s)ds < \infty, \quad \mathbb{P}^\rho\text{-a.s.,}$$

for some $t_0 > 0$. Finally, due to boundedness of $f(x)$,

$$\int_0^\infty f(F_s)ds < \infty, \quad \mathbb{P}^\rho\text{-a.s.,}$$

which again implies the desired result. $\qquad\square$

Finally, let $\{M_t\}_{t\geq 0}$ be a $\bar{d}$-dimensional progressively measurable Markov process possessing the *local-time* process (occupation measure), that is, a nonnegative process $\{l(t,y)\}_{t\geq 0}$, such that for any $x \in \mathbb{R}^{\bar{d}}$, any $t \geq 0$ and any nonnegative $f \in B_b(\mathbb{R}^{\bar{d}})$,

$$\int_0^t f(M_s)ds = \int_{\mathbb{R}^{\bar{d}}} f(y)l(t,y)dy, \quad \mathbb{P}^x\text{-a.s.}$$

A sufficient condition for the existence of the local time for a diffusion with jumps $\{F_t\}_{t\geq 0}$ with symbol $q(x,\xi)$ satisfying (3.6) is as follows

$$\int_{\mathbb{R}^{\bar{d}}} \frac{d\xi}{1 + \inf_{x\in\mathbb{R}^{\bar{d}}} \operatorname{Re} q(x,\xi)} < \infty$$

(see [SW13, Theorem 1.1]). Now, let $f \in L^1(dx, \mathbb{R}^{\bar{d}})$. Then, for any $\beta > 0$ and any $t > 0$, we have

$$n^{\beta\bar{d}} \int_0^t f(n^\beta M_s)ds = n^{\beta\bar{d}} \int_{\mathbb{R}^{\bar{d}}} f(n^\beta y)l(t,y)dy = \int_{\mathbb{R}^{\bar{d}}} f(y)l(t, n^{-\beta}y)dy.$$

Hence, if $\{l(t,y)\}_{t\geq 0}$ is continuous in y, $\mathbb{P}^x$-a.s. for all $x \in \mathbb{R}^{\bar{d}}$, we have

$$n^{\beta\bar{d}} \int_0^t f(n^\beta M_s)ds \xrightarrow{\mathbb{P}^\rho\text{- a.s.}} l(t,0) \int_{\mathbb{R}^{\bar{d}}} f(y)dy$$

for any $\beta > 0$, any $f \in L^1(dx, \mathbb{R}^{\bar{d}})$ and any initial distribution $\rho(dx)$ of $\{L_t\}_{t\geq 0}$. In the one-dimensional Lévy process case necessary and sufficient conditions for the existence and continuity of local times have been given in [Bar85], [Bar88] and [Ber96].

Acknowledgement

References

[Bar85]   M. T. Barlow. Continuity of local times for Lévy processes. *Z. Wahrsch. Verw. Gebiete*, 69(1):23–35, 1985.

[Bar88]   M. T. Barlow. Necessary and sufficient conditions for the continuity of local time of Lévy processes. *Ann. Probab.*, 16(4):1389–1427, 1988.

[Bas88]   R. F. Bass. Uniqueness in law for pure jump Markov processes. *Probab. Theory Related Fields*, 79(2):271–287, 1988.

[Ber96]   J. Bertoin. *Lévy processes*, volume 121 of *Cambridge Tracts in Mathematics*. Cambridge University Press, Cambridge, 1996.

[BH80]   P. L. Brockett and W. N. Hudson. Zeros of the densities of infinitely divisible measures. *Ann. Probab.*, 8(2):400–403, 1980.

[Bha78]   R. N. Bhattacharya. Criteria for recurrence and existence of invariant measures for multidimensional diffusions. *Ann. Probab.*, 6(4):541–553, 1978.

[Bha80]   R. N. Bhattacharya. Correction to: "Criteria for recurrence and existence of invariant measures for multidimensional diffusions" [Ann. Probab. **6** (1978), no. 4, 541–553; MR **58** #13375]. *Ann. Probab.*, 8(6):1194–1195, 1980.

[Bha82]   R. N. Bhattacharya. On the functional central limit theorem and the law of the iterated logarithm for Markov processes. *Z. Wahrsch. Verw. Gebiete*, 60(2):185–201, 1982.

[Bil68]   P. Billingsley. *Convergence of probability measures*. John Wiley & Sons, Inc., New York-London-Sydney, 1968.

[BLP78]   A. Bensoussan, J-L. Lions, and G. C. Papanicolaou. *Asymptotic analysis for periodic structures*. North-Holland Publishing Co., Amsterdam, 1978.

[Böt11]   B. Böttcher. An overshoot approach to recurrence and transience of Markov processes. *Stochastic Process. Appl.*, 121(9):1962–1981, 2011.

[BR82]   R. N. Bhattacharya and S. Ramasubramanian. Recurrence and ergodicity of diffusions. *J. Multivariate Anal.*, 12(1):95–122, 1982.

[BRŻ96]   T. Byczkowski, B. S. Rajput, and T. Żak. Zeros of the densities of infinitely divisible measures on $\mathbf{R}^n$. *J. Theoret. Probab.*, 9(2):533–540, 1996.

[BSW13]   B. Böttcher, R. L. Schilling, and J. Wang. *Lévy matters. III.* Springer, Cham, 2013.

[Cho94]   A. J. Chorin. *Vorticity and turbulence*. Springer-Verlag, New York, 1994.

[ÇJ81]   E. Çinlar and J. Jacod. Representation of semimartingale Markov processes in terms of Wiener processes and Poisson random measures. In *Seminar on Stochastic Processes*, pages 159–242. Birkhäuser, Boston, Mass., 1981.

[Cou66]   Ph. Courrége. Sur la forme intégro-différentielle des opérateurs de C_K^∞ dans C satisfaisant au principe du maximum. *Sém. Théorie du Potentiel*, exposé 2:38 pp., 1965-1966.

[EK86]   S. N. Ethier and T. G. Kurtz. *Markov processes.* John Wiley & Sons Inc., New York, 1986.

[Fan00]   A. C. Fannjiang. Phase diagram for turbulent transport: sampling drift, eddy diffusivity and variational principles. *Phys. D*, 136(1-2):145–174, 2000.

[Fan01]   A. C. Fannjiang. Erratum to: "Phase diagram for turbulent transport: sampling drift, eddy diffusivity and variational principles" [Phy. D **136** (2000), no. 1-2, 145–147; MR1732307 (2001i:76062)]. *Phys. D*, 157(1-2):166–168, 2001.

[FK02]   A. C. Fannjiang and T. Komorowski. Diffusive and nondiffusive limits of transport in nonmixing flows. *SIAM J. Appl. Math.*, 62(3):909–923 (electronic), 2001/02.

[FK97]   A. C. Fannjiang and T. Komorowski. A martingale approach to homogenization of unbounded random flows. *Ann. Probab.*, 25(4):1872–1894, 1997.

[FK99]   A. C. Fannjiang and T. Komorowski. Turbulent diffusion in Markovian flows. *Ann. Appl. Probab.*, 9(3):591–610, 1999.

[FK00a]   A. C. Fannjiang and T. Komorowski. Fractional Brownian motions and enhanced diffusion in a unidirectional wave-like turbulence. *J. Statist. Phys.*, 100(5-6):1071–1095, 2000.

[FK00b]   A. C. Fannjiang and T. Komorowski. Fractional Brownian motions in a limit of turbulent transport. *Ann. Appl. Probab.*, 10(4):1100–1120, 2000.

[FK01]   A. C. Fannjiang and T. Komorowski. Invariance principle for a diffusion in a Markov field. *Bull. Polish Acad. Sci. Math.*, 49(1):45–65, 2001.

[FK03]   A. C. Fannjiang and T. Komorowski. Frozen path approximation for turbulent diffusion and fractional Brownian motion in random flows. *SIAM J. Appl. Math.*, 63(6):2042–2062 (electronic), 2003.

[FP96]   A.C. Fannjiang and G. Papanicolaou. Diffusion in turbulence. *Probab. Theory Related Fields*, 105(3):279–334, 1996.

[Fra06]   B. Franke. The scaling limit behaviour of periodic stable-like processes. *Bernoulli*, 12(3):551–570, 2006.

[Fra07]   B. Franke. Correction to: "The scaling limit behaviour of periodic stable-like processes" [Bernoulli **12** (2006), no. 3, 551–570]. *Bernoulli*, 13(2):600, 2007.

[Fri64]   A. Friedman. *Partial differential equations of parabolic type.* Prentice-Hall, Inc., Englewood Cliffs, N.J., 1964.

[Fri95]   U. Frisch. *Turbulence.* Cambridge University Press, Cambridge, 1995.

[FRP98]   A. C. Fannjiang, L. Ryzhik, and G. Papanicolaou. Evolution of trajectory correlations in steady random flows. In *Recent advances in partial differential equations, Venice 1996*, volume 54, pages 105–130. Amer. Math. Soc., Providence, RI, 1998.

[Gra08]   L. Grafakos. *Classical Fourier analysis.* Springer, New York, second edition, 2008.

[Jac01]   N. Jacob. *Pseudo differential operators and Markov processes. Vol. I.* Imperial College Press, London, 2001.

[Jac05]   N. Jacob. *Pseudo differential operators and Markov processes. Vol. III.* Imperial College Press, London, 2005.

[JS03]   J. Jacod and A. N. Shiryaev. *Limit theorems for stochastic processes*, volume 288. Springer-Verlag, Berlin, second edition, 2003.

[KK04a]   T. Komorowski and G. Krupa. A note on an application of the Lasota-Yorke fixed point theorem in the turbulent transport problem. *Bull. Pol. Acad. Sci. Math.*, 52(1):101–113, 2004.

[KK04b]   T. Komorowski and G. Krupa. On stationarity of Lagrangian observations of passive tracer velocity in a compressible environment. *Ann. Appl. Probab.*, 14(4):1666–1697, 2004.

[KNR14]   T. Komorowski, A. Novikov, and L. Ryzhik. Homogenization driven by a fractional Brownian motion: the shear layer case. *Multiscale Model. Simul.*, 12(2):440–457, 2014.

[KO01]   T. Komorowski and S. Olla. On homogenization of time-dependent random flows. *Probab. Theory Related Fields*, 121(1):98–116, 2001.

[Kol00]   V. N. Kolokoltsov. Symmetric stable laws and stable-like jump-diffusions. *Proc. London Math. Soc. (3)*, 80(3):725–768, 2000.

[Kol11]   V. N. Kolokoltsov. *Markov processes, semigroups and generators*, volume 38. Walter de Gruyter & Co., Berlin, 2011.

[KP79]   H. Kesten and G. C. Papanicolaou. A limit theorem for turbulent diffusion. *Comm. Math. Phys.*, 65(2):97–128, 1979.

[MK99]   A. J. Majda and P. R. Kramer. Simplified models for turbulent diffusion: theory, numerical modelling, and physical phenomena. *Phys. Rep.*, 314(4-5):237–574, 1999.

[MT93]   S. P. Meyn and R. L. Tweedie. Stability of Markovian processes. II. Continuous-time processes and sampled chains. *Adv. in Appl. Probab.*, 25(3):487–517, 1993.

[MT09]   S. Meyn and R. L. Tweedie. *Markov chains and stochastic stability.* Cambridge University Press, Cambridge, second edition, 2009.

[NX13]   D. Nuarlart and F. Xu. Limit laws for occupation times of stable processes. *Preprint. Available on arXiv: 1305.0241*, 2013.

[Øks03]   B. Øksendal. *Stochastic differential equations.* Springer-Verlag, Berlin, sixth edition, 2003.

[PSV77]   G. C. Papanicolaou, D. Stroock, and S. R. S. Varadhan. Martingale approach to some limit theorems. In *Papers from the Duke Turbulence Conference (Duke Univ., Durham, N.C., 1976), Paper No. 6.* Duke Univ., Durham, N.C., 1977.

[PV81]   G. C. Papanicolaou and S. R. S. Varadhan. Boundary value problems with rapidly oscillating random coefficients. In *Random fields, Vol. I, II (Esztergom, 1979)*, volume 27, pages 835–873. North-Holland, Amsterdam-New York, 1981.

[RW00]   L. C. G. Rogers and D. Williams. *Diffusions, Markov processes, and martingales. Vol. 1.* Cambridge University Press, Cambridge, 2000.

[San13a] N. Sandrić. Long-time behavior of stable-like processes. *Stochastic Process. Appl.*, 123(4):1276–1300, 2013.

[San13b] N. Sandrić. Recurrence and transience property for a class of Markov chains. *Bernoulli*, 19(5B):2167–2199, 2013.

[San14a] N. Sandrić. Ergodic property of stable-like Markov chains. *To appear in: J. Theoret. Probab. Available on arXiv: 1411.7497*, 2014.

[San14b] N. Sandrić. Long-time behavior for a class of feller processes. *To appear in: Trans. Am. Math. Soc. Available on arXiv: 1401.2646*, 2014.

[San14c] N. Sandrić. Recurrence and transience criteria for two cases of stable-like Markov chains. *J. Theoret. Probab.*, 27(3):754–788, 2014.

[Sat99] K. Sato. *Lévy processes and infinitely divisible distributions.* Cambridge University Press, Cambridge, 1999.

[SB13] A. Schnurr and A. Behme. A criterion for invariant measures of it processes based on the symbol. *To appear in: Bernoulli. Available on arXiv: :1310.4333*, 2013.

[Sch98a] R. L. Schilling. Conservativeness and extensions of feller semigroups. *Positivity*, 2:239–256, 1998.

[Sch98b] R. L. Schilling. Growth and Hölder conditions for the sample paths of Feller processes. *Probab. Theory Related Fields*, 112(4):565–611, 1998.

[Sha69] M. Sharpe. Zeroes of infinitely divisible densities. *Ann. Math. Statist.*, 40:1503–1505, 1969.

[She91] S.-J. Sheu. Some estimates of the transition density of a nondegenerate diffusion Markov process. *Ann. Probab.*, 19(2):538–561, 1991.

[SW11] R. L. Schilling and J. Wang. On the coupling property of Lévy processes. *Ann. Inst. Henri Poincaré Probab. Stat.*, 47(4):1147–1159, 2011.

[SW13] R. L. Schilling and J. Wang. Some theorems on Feller processes: transience, local times and ultracontractivity. *Trans. Amer. Math. Soc.*, 365(6):3255–3286, 2013.

[Tay22] G. I. Taylor. Diffusion by Continuous Movements. *Proc. London Math. Soc.*, S2-20(1):196–212, 1922.

[Twe94] R. L. Tweedie. Topological conditions enabling use of Harris methods in discrete and continuous time. *Acta Appl. Math.*, 34(1-2):175–188, 1994.

[Wan08] J. Wang. Criteria for ergodicity of Lévy type operators in dimension one. *Stochastic Process. Appl.*, 118(10):1909–1928, 2008.

[Yin94] J. Ying. Invariant measures of symmetric Lévy processes. *Proc. Amer. Math. Soc.*, 120(1):267–273, 1994.