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Eringen's small length scale coefficient for buckling of nonlocal Timoshenko beam based on microstructured beam model

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This paper presents the determination of Eringen's small length scale coefficient e_0 for buckling of nonlocal Timoshenko beam from a microstructured beam model. The microstructured beam model is composed of discrete rigid elements (of equal length), which are connected by rotational and shear springs that model the bending and shearing behaviors in a beam. The exact solution of e_0 is given for nonlocal Timoshenko beam with small length scale term appearing in the normal stress-strain relation only. It is shown that e_0 approaches $1/\sqrt{12} \approx 0.289$ which coincides with the one calibrated for nonlocal Euler beams.
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I. INTRODUCTION

Identification of the small length scale parameter is important when dealing with the characteristics of the microstructure in materials.[1] In order to allow for the small length scale effect in nanostructures, Eringen's nonlocal elasticity[2] has been widely adopted in studies on nano-scale mechanical behavior.[3] For example, Wang[4] proposed nonlocal Euler and Timoshenko beam theories for wave propagation problem. Wang and Hu[5] found that nonlocal Timoshenko beam theory can predict good agreements of wave speeds in wave propagation with those obtained by molecular dynamics simulations on carbon nanotubes. Lu *et al.*[6] studied free vibration of nonlocal Euler beams with various boundary conditions. Wang and Varadan[7] reported the frequencies of nonlocal Euler and Timoshenko beams with simply supported ends. Their theory has been developed in terms of a double-beam theory in order to represent the free vibration of double-walled carbon nanotubes. Wang and Liew[8] studied the bending of nonlocal Euler and Timoshenko beams. Reddy[9] formulated the governing equations for bending, buckling, and free vibration of beams based on nonlocal Euler, Timoshenko, Reddy, and Levinson beam theories.

In Eringen's nonlocal elasticity, the stress at a point is defined to be dependent on the interaction of all points within the range of interactions.[2] According to Eringen,[2] the nonlocal constitutive relation is given by

$$(1 - \mu^2 \ell_e^2 \nabla^2)\boldsymbol{\sigma} = \mathbf{D} : \boldsymbol{\varepsilon}, \tag{1}$$

where $\mathbf{D}$ is the fourth-order elasticity tensor, $\boldsymbol{\sigma}$ and $\boldsymbol{\varepsilon}$ are macroscopic stress second-order tensor and strain tensors, respectively. The notation ":" represents the double contraction between a fourth-order tensor and a second-order tensor and ∇^2 is the Laplacian operator. Note that $\mu = \frac{e_0 a}{\ell_e}$ where the quantity $e_0 a$ represents an intrinsic characteristic length of a material. This intrinsic length is the size of a representative elementary volume over which the local stress is integrated. The length scale coefficient e_0 is a constant appropriate to each material and a is an internal characteristic length (e.g., lattice spacing and granular distance), and ℓ_e is an external characteristic length (e.g., crack length, wavelength). It has been found that a larger value of e_0 implies a more significant effect of the small length scale.[10,11] Nevertheless, owing to the difficulty in determining the internal characteristic length a, most researchers have adopted $e_0 a$ as a single parameter.[4] For example, a conservative estimate of the scale coefficient $e_0 a < 2.0\,\text{nm}$ for a single walled carbon nanotube (SWCNT) if the measured frequency value for the SWCNT is assessed to be greater than 10 THz.[12]

In order to identify the Eringen's small length scale coefficient e_0, a promising approach is to make use of the analytical equivalence between the discrete microstructured models and nonlocal continuum models. For example, based on this equivalence, Eringen[2] identified e_0 as 0.39 by matching the dispersion curves of plane waves from nonlocal theory to the Born-Kármán model of lattice dynamics. In a discrete lattice model, the length of a representative microstructure is interpreted as the length of a discrete element or accommodated by the inter-particle spring stiffness. The basic assumption based on the equivalence between a discrete lattice model and the nonlocal continuum model is that the concerned wavelength is much longer than the characteristic length, which could be taken as the inter-particle distance. The analytical relations between equivalent continuum models and discrete lattice models have been confirmed by recent results. It has been reported that the higher order gradient continuum theories can be derived from

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discrete lattice models.[13,14] Askes and Metrikine[15] presented that continuum models can be related to discrete models through continualisation strategies. From these continualisation strategies, not only the classical continua but also higher-order continua in one-dimension and two-dimension can be developed. The continualisation on the governing equation and on the energy furnished the identical results. Pichugin *et al.*[16] established equivalence between the discrete lattice model and higher-order continuum theories. Polyzos and Fotiadis[17] derived Mindlin's first and second strain gradient elastic theories from simple lattice and continuum models.

Based on a microstructured Euler beam model comprising rigid segments connected by rotational springs, Challamel *et al.*[18,19] showed that e_0 is 0.289 for buckling of nonlocal Euler beams with any combination of end conditions. The present study extends this work to determine the small length scale coefficient e_0 for the buckling of nonlocal Timoshenko beams that allows for both the effects of small length scale and transverse shear deformation. The earlier microstructured Euler beam model used for calibrating e_0 will be refined to include shear springs at the nodes. This new model will be referred to as microstructured Timoshenko beam model. The paper is organized as follows. The governing equation and boundary condition for the buckling of nonlocal Timoshenko beam with simply supported ends will be developed based on the method of weighted residuals. The length scale coefficients in the Eringen's nonlocal theory are then estimated by comparing the buckling load formulations with the exact expression furnished by a microstructured Timoshenko beam model.

II. NONLOCAL TIMOSHENKO BEAM THEORY AND SOLUTION FOR SIMPLY SUPPORTED BEAM

Eringen's normal and shear stress-strain relations are given by

$$\sigma_{xx} - \ell_c^2 \frac{d^2\sigma_{xx}}{dx^2} = E\varepsilon_{xx} \quad \text{and} \quad \sigma_{xz} - \alpha\ell_c^2 \frac{d^2\sigma_{xz}}{dx^2} = G\gamma_{xz}, \tag{2}$$

where σ_{xx} is the normal stress, ε_{xx} is the normal strain, σ_{xz} is the shear stress, γ_{xz} is the shear strain, E is the Young's modulus, G is the shear modulus, $\ell_c = e_0 a$ is the intrinsic characteristic length, and α is a scalar indicator which may take the value of either 0 or 1. When $\alpha = 0$, we obtain the case where the small length scale effect is neglected in the shear stress-strain constitutive relation. When $\alpha = 1$, we have the case where the small length scale effect is considered in shear stress-strain constitutive relations.

According to the Timoshenko beam theory,[20] the strain-displacement relations are given by

$$\varepsilon_{xx} = -z\frac{d\phi}{dx} \quad \text{and} \quad \gamma_{xz} = \frac{dw}{dx} - \phi, \tag{3}$$

where z is the distance away from the normal plane, ϕ is the rotation due to bending, $\frac{dw}{dx} - \phi$ is the rotation due to shearing since $\frac{dw}{dx}$ is the total slope of a deformed cross section.

Multiplying Eq. (2) by zdA and integrating over the cross sectional area A of the beam, one obtains

$$M - \ell_c^2 \frac{d^2M}{dx^2} = -EI\frac{d\phi}{dx}, \tag{4a}$$

$$Q - \alpha\ell_c^2 \frac{d^2Q}{dx^2} = \kappa GA\left(\frac{dw}{dx} - \phi\right), \tag{4b}$$

where M is the bending moment, Q is the shear force, I is the second moment of area, and κ is the shear correction factor. The equilibrium equation for a Timoshenko beam under a compressive axial load P is given by Timoshenko[20,21]

$$\frac{dM}{dx} - Q = 0, \tag{5a}$$

$$\frac{dQ}{dx} - P\frac{d^2w}{dx^2} = 0. \tag{5b}$$

By substituting Eqs. (5) into Eqs. (4), the nonlocal bending moment and nonlocal shear force can be expressed as

$$M = -EI\frac{d\phi}{dx} + \ell_c^2 P\frac{d^2w}{dx^2}, \tag{6a}$$

$$Q = \kappa GA\left(\frac{dw}{dx} - \phi\right) + \alpha\ell_c^2 P\frac{d^3w}{dx^3}. \tag{6b}$$

In view of Eqs. (6a) and (6b), the governing equations given by Eqs. (5a) and (5b) can be written as

$$\kappa GA\left(\frac{dw}{dx} - \phi\right) + EI\frac{d^2\phi}{dx^2} - (1-\alpha)\,\ell_c^2 P\frac{d^3w}{dx^3} = 0, \tag{7a}$$

$$\kappa GA\left(\frac{d^2w}{dx^2} - \frac{d\phi}{dx}\right) - P\frac{\partial^2w}{\partial x^2} + \alpha\ell_c^2 P\frac{d^4w}{dx^4} = 0. \tag{7b}$$

When $\alpha = 0$, Eqs. (7a) and (7b) reduce to the fourth order differential equations of the nonlocal Timoshenko beam as derived earlier by Wang *et al.*[22] However, when $\alpha = 1$, one has to contend with a sixth order differential equation as derived by Reddy and Pang.[23]

The method of weighted residuals[24,25] may be used on the established governing equations (7a) and (7b) to obtain the boundary conditions. One can adopt $\delta\phi$ and δw as residuals for Eqs. (7a) and (7b), respectively. Therefore, the weak formulation of Eqs. (7a) and (7b) can be written as

$$\int_0^L \left[\kappa GA\left(\frac{dw}{dx} - \phi\right) + EI\frac{d^2\phi}{dx^2} - (1-\alpha)\,\ell_c^2 P\frac{d^3w}{dx^3}\right]\delta\phi dx = 0, \tag{8a}$$

$$\int_0^L \left[\kappa GA\left(\frac{d^2w}{dx^2} - \frac{d\phi}{dx}\right) - P\frac{\partial^2w}{\partial x^2} + \alpha\ell_c^2 P\frac{d^4w}{dx^4}\right]\delta w dx = 0. \tag{8b}$$

After integrating Eqs. (8a) and (8b) by parts, one obtains

$$\begin{aligned} 0 = &\int_0^L \left[EI\frac{d\phi}{dx}\delta\frac{d\phi}{dx} - (1-\alpha)\,\ell_c^2 P\frac{d^2w}{dx^2}\delta\frac{d\phi}{dx} \right.\\ &\left. + \kappa GA\left(\frac{dw}{dx} - \phi\right)\delta\phi\right]dx \\ &+ \left[(1-\alpha)\ell_c^2 P\frac{d^2w}{dx^2}\delta\phi\right]_0^L - \left[EI\frac{d\phi}{dx}\delta\phi\right]_0^L \end{aligned} \tag{9a}$$

and

$$0=\int_0^L\left[\kappa GA\left(\frac{dw}{dx}-\phi\right)\delta\frac{dw}{dx}+P\frac{dw}{dx}\delta\frac{dw}{dx}-\alpha\ell_c^2P\frac{d^2w}{dx^2}\delta\frac{d^2w}{dx^2}\right]dx-\left[\kappa GA\left(\frac{dw}{dx}-\phi\right)\delta w\right]_0^L+\left[P\frac{dw}{dx}\delta w\right]_0^L-\left[\alpha\ell_c^2P\frac{d^3w}{dx^3}\delta w\right]_0^L+\left[\alpha\ell_c^2P\frac{d^2w}{dx^2}\delta\frac{dw}{dx}\right]_0^L. \tag{9b}$$

In view of Eqs. (9a) and (9b), the boundary conditions are given by

$$\text{Specify}\begin{Bmatrix} w \\ \dfrac{dw}{dx} \\ \phi \end{Bmatrix}\text{ or }\begin{Bmatrix} \kappa GA\left(\dfrac{dw}{dx}-\phi\right)-P\dfrac{dw}{dx}+\alpha P\ell_c^2\dfrac{d^3w}{dx^3} \\ \alpha\ell_c^2P\dfrac{d^2w}{dx^2} \\ EI\dfrac{d\phi}{dx}-(1-\alpha)\ell_c^2P\dfrac{d^2w}{dx^2} \end{Bmatrix}. \tag{10}$$

Consider a nonlocal Timoshenko beam with simply supported ends. The boundary conditions for such a supported nonlocal Timoshenko beam are

$$w=0,\quad \frac{d^2w}{dx^2}=0,\quad \frac{d\phi}{dx}=0\quad \text{when}\quad \alpha=1 \tag{11}$$

and

$$w=0,\quad EI\frac{d\phi}{dx}-\ell_c^2P\frac{d^2w}{dx^2}=0\quad\text{when}\quad\alpha=0. \tag{12}$$

A. Case when $\alpha=1$

For the case when $\alpha=1$, the buckling solution of the nonlocal Timoshenko beam is obtained by solving Eqs. (7a) and (7b) with boundary conditions given in Eq. (11) as shown below. By using the following nondimensional terms:

$$\bar{x}=\frac{x}{L},\quad \bar{w}=\frac{w}{L};\quad \mu=\frac{\ell_c}{L}=\frac{e_0a}{L},\quad \Lambda_\alpha=\frac{P_\alpha L^2}{EI},\quad \Omega=\frac{EI}{\kappa GAL^2} \tag{13}$$

and after decoupling the deflection and rotation variables, the governing equations (7a) and (7b) can be written as

$$\frac{d^6\bar{w}}{d\bar{x}^6}+A_1\frac{d^4\bar{w}}{d\bar{x}^2}+A_2\frac{d^2\bar{w}}{d\bar{x}^2}=0, \tag{14a}$$

$$\frac{d^5\phi}{d\bar{x}^5}+A_1\frac{d^3\phi}{d\bar{x}^3}+A_2\frac{d\phi}{d\bar{x}}=0, \tag{14b}$$

where

$$A_1=-\frac{1}{\mu^2}-\frac{1}{\Omega}+\frac{1}{\mu^2\Lambda_1\Omega},\qquad A_2=\frac{1}{\mu^2\Omega}\quad(\Lambda_\alpha=\Lambda_1\text{ for present case}). \tag{15}$$

The general solution to Eq. (14a) is given by

$$\bar{w}(\bar{x})=C_1\cosh(r\bar{x})+C_2\sinh(r\bar{x})+C_3\sin(s\bar{x})+C_4\cos(s\bar{x})+C_5\bar{x}+C_6, \tag{16}$$

while the solution to Eq. (14b) is given by

$$\phi(\bar{x})=D_1\sinh(r\bar{x})+D_2\cosh(r\bar{x})+D_3\cos(s\bar{x})+D_4\sin(s\bar{x})+D_5, \tag{17}$$

where

$$r=\sqrt{\frac{-A_1+\sqrt{A_1^2-4A_2}}{2}}, \tag{18a}$$

$$s=\sqrt{\frac{A_1+\sqrt{A_1^2-4A_2}}{2}}, \tag{18b}$$

and constants C_i $(i=1,2\cdots6)$ and $D_j(j=1,2\cdots5)$ are related by

$$D_1=\eta C_1, D_2=\eta C_2, D_3=\chi C_3, D_4=-\chi C_4,\quad\text{and}\quad D_5=C_5, \tag{19}$$

where $\chi=\frac{s}{1+s^2\Omega}$ and $\eta=\frac{r}{1-r^2\Omega}$. Equation (19) is obtained by substituting Eqs. (16) and (17) into Eq. (7a).

The boundary conditions for a simply supported beam are specified in Eq. (11). They can be expressed in nondimensional forms as

$$\text{at }\bar{x}=0\text{ and }\bar{x}=1\ :\ \bar{w}=0,\ \frac{d^2\bar{w}}{d\bar{x}^2}=0,\text{ and }\frac{d\phi}{d\bar{x}}=0. \tag{20}$$

By substituting Eqs. (16) and (17) into Eq. (20), one obtains the following set of homogenous equation in terms of the unknown constants C_i $(i=1,2\cdots6)$ after replacing D_j $(j=1,2\cdots5)$ using Eq. (19):

$$\begin{bmatrix} 1 & 0 & 0 & 1 & 0 & 1 \\ r^2 & 0 & 0 & -s^2 & 0 & 0 \\ r\eta & 0 & 0 & -s\chi & 0 & 0 \\ \cosh(r) & \sinh(r) & \sin(s) & \cos(s) & 1 & 1 \\ r^2\cosh(r) & r^2\sinh(r) & -s^2\sin(s) & -s^2\cos(s) & 0 & 0 \\ r\eta\cosh(r) & r\eta\sinh(r) & -s\chi\sin(s) & -s\chi\cos(s) & 0 & 0 \end{bmatrix}_{6\times6}\times\begin{Bmatrix} C_1 \\ C_2 \\ C_3 \\ C_4 \\ C_5 \\ C_6 \end{Bmatrix}_{6\times1}=0. \tag{21}$$

For a nontrivial solution, the determinant of Eq. (21) must vanish. This furnishes the following characteristic equation:

$$-r^2s^2(r\chi-s\eta)^2\sinh(r)\sin(s)=0\quad\rightarrow\quad\sin(s)=0. \tag{22}$$

Based on Eqs. (18b) and (22), one can develop a buckling load relationship between nonlocal Timoshenko beam and the classical Euler beam, i.e.,

$$P_{\alpha=1} = \frac{P_E}{1 + \frac{P_E}{\kappa GA} + \ell_c^2 \frac{P_E}{EI} + \ell_c^2 \frac{P_E}{\kappa GA}\frac{P_E}{EI}}, \quad \text{for} \quad \alpha = 1, \quad (23)$$

where P_E is the buckling load of classical Euler beam with simply supported ends and is given by

$$P_E = \frac{\pi^2 EI}{L^2}. \quad (24)$$

B. Case when α=0

Consider next the case when $\alpha = 0$, i.e., when the small length scale is ignored in the shear stress-strain relation in Eq. (2). The solution has previously being obtained by Wang *et al.*[22] and it is given by

$$P_{\alpha=0} = \frac{P_E}{1 + \frac{P_E}{\kappa GA} + \ell_c^2 \frac{P_E}{EI}}, \quad \text{for} \quad \alpha = 0. \quad (25)$$

It is interesting that one can obtain the buckling load given by Eq. (25) from Eqs. (14) by dropping the sixth-order term. This approximation has been previously adopted by Reddy and Pang.[23] The nonlocal buckling loads obtained from Eqs. (23) and (25) with respect to $\mu = \frac{\ell_c}{L} = \frac{e_0 a}{L}$ are compared in Figure 1 for $\Omega = 1/300$ and $\Omega = 1/600$. The significance of these Ω values is that they represent typical properties of macro scale materials[26] and nano scale materials,[22,23] respectively. It can be seen from Figure 1 that the differences between the buckling loads (obtained from nonlocal beam theories with $\alpha = 1$ and $\alpha = 0$) are negligibly small. Therefore, one can adopt the nonlocal Timoshenko beam with $\alpha = 0$ for calibrating the Eringen's small length scale coefficient in Sec. III.

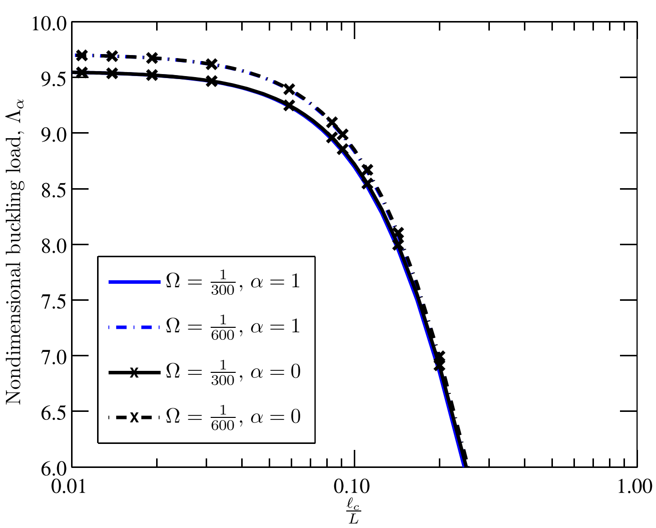

FIG. 1. Nondimensional buckling load for nonlocal Timoshenko beam with simply supported ends.

III. MICROSTRUCTURED TIMOSHENKO BEAM MODEL FOR SIMPLY SUPPORTED BEAM

Consider a microstructured Timoshenko beam model comprising n rigid segments (each segmental length $a = L/n$) that are connected by rotational and shear springs. The microstructured Timoshenko beam is subjected to a compressive axial load P. The problem at hand is to determine the buckling load of the microstructured Timoshenko beam with simply supported ends.

To illustrate the microstructured Timoshenko beam model, a five segment beam example is shown in Figure 2. There are four rotational springs and five shear springs in the five-segment microstructured Timoshenko model. At each node j, there are two degrees of freedom, representing the nodal transverse displacement w_j and nodal rotation θ_j. For a simply supported end with n segments, one will have $w_1 = w_{n+1} = \theta_{n+1} = 0$ at the ends.

The strain energy function due to deformed rotational springs is given by Ostoja-Starzewski[27] as

$$V_b = \frac{1}{2}\sum_{i=1}^{n-1} C(\theta_{j+1} - \theta_j)^2, \quad (26)$$

where $C = \frac{nEI}{L} = \frac{EI}{a}$ and $L = na$. The quantity n is the number of discrete elements. The value of C might be different for other boundary conditions instead of simply supported ends. This assumption is the same with the microstructured Euler beam model as described in the authors' previous paper.[19]

The strain energy function due to deformed shear spring is given by

$$V_s = \frac{1}{2}\sum_{j=1}^{n} S(w_{j+1} - w_j - a\theta_j)^2, \quad (27)$$

where $S = \frac{n\kappa GA}{L} = \frac{\kappa GA}{a}$.

The work done by the compressive axial load on the microstructured Timoshenko beam is given by Challamel *et al.*[19]

$$W = \frac{1}{2}\sum_{j=1}^{n} Pa\left(\frac{w_{j+1} - w_j}{a}\right)^2. \quad (28)$$

Therefore, the total potential energy function can be expressed as

$$\Pi_T = \frac{1}{2}\sum_{j=1}^{n-1} C(\theta_{j+1} - \theta_j)^2 + \frac{1}{2}\sum_{j=1}^{n} S(w_{j+1} - w_j - a\theta_j)^2 - \frac{1}{2}\sum_{j=1}^{n} Pa\left(\frac{w_{j+1} - w_j}{a}\right)^2. \quad (29)$$

Obviously this energy function in Eq. (29) for microstructured Timoshenko beam would reduce to microstructured Euler beam[18] as one sets the shear angle to zero. For example, in the five-segment beam with simply supported ends, we have

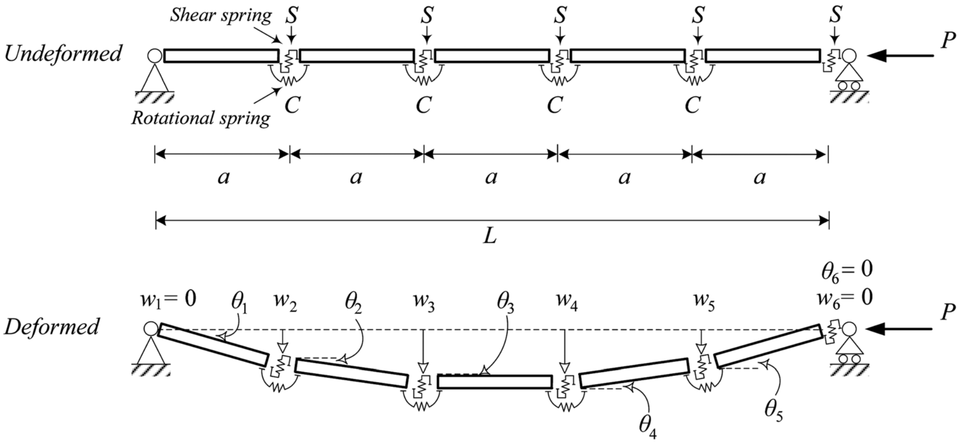

FIG. 2. Five-segment microstructured Timoshenko beam model.

$$\theta_1 = \frac{w_2}{a}, \quad \theta_2 = \frac{w_3 - w_2}{a}, \quad \theta_3 = \frac{w_4 - w_3}{a}, \quad \theta_4 = \frac{w_5 - w_4}{a}, \quad \theta_5 = -\frac{w_5}{a}. \tag{30}$$

Interestingly, by considering $\kappa GA \to \infty$ in this buckling load formulation, the buckling load parameter of the five segment Timoshenko beam reduces to 9.5492 which is the solution for the five segment Euler beam (see Table 1 of Challamel *et al.*[18]). If we keep increasing the number of elements, the buckling load converges to the buckling load of the classical Timoshenko beam.

The Euler-Lagrange equations based on the energy function in Eq. (29) are

$$aS(w_{j+1} - w_j - a\theta_j) + C(\theta_{j+1} - 2\theta_j + \theta_{j-1}) = 0, \tag{31a}$$

$$S[w_{j+1} - 2w_j + w_{j-1} - a(\theta_j - \theta_{j-1})] - \frac{P}{a}(w_{j+1} - 2w_j + w_{j-1}) = 0. \tag{31b}$$

By eliminating θ, the simplified equation can be written as

$$[w_{j+2} - 4w_{j+1} + 6w_j - 4w_{j-1} + w_{j-2}] + \frac{P'_T}{aS - P'_T}\frac{S}{C}a^2(w_{j+1} - 2w_j + w_{j-1}) = 0. \tag{32}$$

Here, we use P'_T to replace P in Eq. (31b), representing the axial force applied on the Timoshenko beam. It is noted that Eq. (32), applicable when $j \geq 3$, is a fourth-order model and rigorously corresponds to a fourth-order differential equation of the nonlocal Timoshenko beam model with $\alpha = 0$.

The microstructured Euler beam model (without the shear springs) with simply supported ends has the following total energy function:

$$\Pi_E = \frac{1}{2}\sum_{j=2}^{n} C\left[\frac{w_{j+1} - 2w_j + w_{j-1}}{a}\right]^2 - \frac{1}{2}\sum_{j=1}^{n} Pa\left(\frac{w_{j+1} - w_j}{a}\right)^2. \tag{33}$$

From Eq. (33), one can have the corresponding Euler-Lagrange equation for the microstructured Euler beam as

$$[w_{j+2} - 4w_{j+1} + 6w_j - 4w_{j-1} + w_{j-2}] + \frac{P'_E a}{C}(w_{j+1} - 2w_j + w_{j-1}) = 0, \tag{34}$$

where we use P'_E to replace P in Eq. (33). This formulation is applicable when $j \geq 3$.

By comparing Eqs. (32) and (34), their mathematical similarity[28] allows one to deduce that

$$\frac{P'_E}{C}a = \frac{P'_T}{-P'_T + aS}\frac{S}{C}a^2. \tag{35}$$

Therefore, the buckling loads of the microstructured Timoshenko beam and the microstructured Euler beam are related by

$$P'_T = \frac{P'_E}{1 + \dfrac{P'_E}{aS}}. \tag{36}$$

It has been shown[18] that the exact buckling load of the microstructured Euler beam is given by

$$\bar{p}_E = \frac{P'_E L^2}{EI} = \left[2n\sin\left(\frac{\pi}{2n}\right)\right]^2 = \pi^2\left[1 - \frac{\pi^2}{12}\frac{1}{n^2} + \frac{\pi^4}{360}\frac{1}{n^4} + \cdots\right]. \tag{37}$$

Therefore, the non-dimensional exact buckling load of the microstructured Timoshenko beam is

$$\bar{p}_T = \frac{P'_T L^2}{EI} = \frac{\bar{p}_E}{1 + \dfrac{\bar{p}_E EI}{\kappa GAL^2}}. \tag{38}$$

Note that the mathematical similarity is valid only if both the governing equation and boundary conditions match. Through the mathematical similarity between Eqs. (32) and (34), the deflections of the microstructured Timoshenko beam and the microstructured Euler beam are related by

$$w_j^T = Bw_j^E, \tag{39}$$

where B is an arbitrary constant.

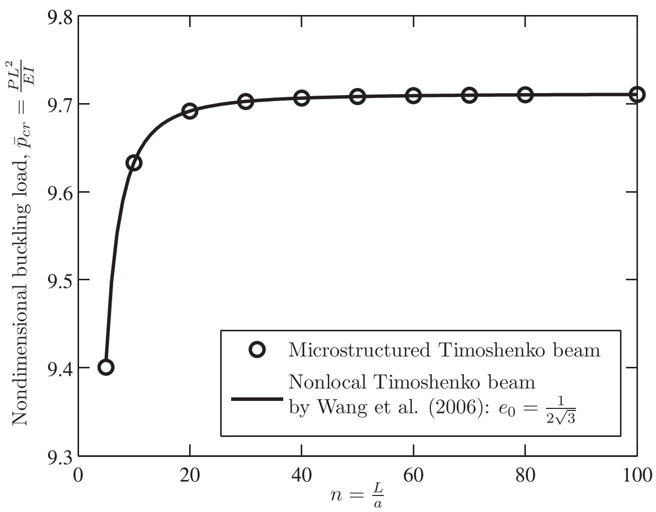

FIG. 3. Comparison of microstructured Timoshenko model and nonlocal Timoshenko model for simply supported beam ($P = P_{\alpha=0}$ for nonlocal Timoshenko beams with $\alpha = 0$ and $P = P_T'$ for microstructured Timoshenko model).

The simply supported boundary conditions in the microstructured Euler beam model are imposed by setting $w_j^E = 0$ and ignoring the rotational springs at both ends.[18,19] These boundary conditions are readily realized in the microstructured Timoshenko model by defining $w_j^T = 0$ and neglecting the rotational springs at the boundaries. Therefore, in viewing of Eq. (39), the boundary conditions for simply supported microstructured Euler and Timoshenko models have identical form as it is only necessary to define the displacement $w_j = 0$ at the boundaries.

In view of Eqs. (25) and (38), one obtains the following expression for Eringen's small length scale coefficient in the nonlocal Timoshenko beam theory (i.e. $\alpha = 0$):

$$e_0 = \frac{1}{2}\sqrt{\frac{\pi^2 - 4n^2 \sin^2\left(\frac{\pi}{2n}\right)}{\pi^2 \sin^2\left(\frac{\pi}{2n}\right)}} = \frac{1}{2\sqrt{3}}\left(1 + \frac{\pi^2}{40}\frac{1}{n^2} + \cdots\right) \quad \rightarrow \quad \lim_{n\to\infty} e_0 = \frac{1}{2\sqrt{3}}. \quad (40)$$

It is interesting to note that the value of e_0 is the same value as the one obtained for the nonlocal Euler beam. The quantity $n = \frac{L}{a}$ denotes the ratio of the external length to the internal characteristic length. When $n > 10$, the value of e_0 converges to $\frac{1}{2\sqrt{3}}$ for the present nonlocal Timoshenko beam with $\alpha = 0$.

To illustrate the formulation of e_0 in Eq. (40), we generate the numerical values for buckling load from the microstructured Timoshenko beam based on Eqs. (37) and (38). Consider the material properties given in the paper by Wang *et al.*[22] The buckling loads are also produced by nonlocal Timoshenko beam with $e_0 = \frac{1}{2\sqrt{3}}$. As shown in Figure 3, the results by nonlocal Timoshenko beam match well with that furnished by the microstructured Timoshenko model. Note that the value $e_0 = \frac{1}{2\sqrt{3}}$ has been identified for nonlocal Euler beam under buckling.[19] It is therefore found that e_0 is not affected by the kinematic assumptions of the Timoshenko theory.

IV. SUMMARY AND CONCLUSIONS

In this paper, an analytical expression has been obtained for Eringen's length scale coefficient e_0 for the buckling problem of nonlocal Timoshenko beams by using a microstructured Timoshenko beam model. The length scale coefficient is found to be asymptotic to $e_0 = \frac{1}{2\sqrt{3}}$, which coincides with the one obtained for nonlocal Euler beam. Future studies could examine the Eringen's length scale coefficient e_0 for buckling of nonlocal Timoshenko beams with other boundary conditions.

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