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ERRATUM

Erratum to: A note on deformations of regular embeddings

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The main result of the paper Proposition 1.3 is wrongly stated. Nevertheless, the proof of Proposition 1.3 and Proposition 1.5 provides a complete description of $\mathrm{Def}_\nu(\mathbf{k}[\epsilon])$ and the paper needs only the corrections below.

Corrections

- The statement of Proposition 1.3 has to be replaced by the following, which is exactly what is proved.

Proposition 1.3 *Let* $\nu : X \hookrightarrow Y$ *be a regular closed embedding of reduced algebraic schemes and let* $\mathrm{Def}_{X/\nu/Y}$ *be the deformation functor of* ν *preserving* X *and* Y (cf. [3, §3.4.1]). *Then,*

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there exists a surjective morphism Φ *from* $\mathrm{Def}_\nu(\mathbf{k}[\epsilon])$ *to the fiber product*

$$\begin{array}{ccc}
\mathrm{Def}_\nu(\mathbf{k}[\epsilon]) & & \\
\downarrow{\scriptstyle \Phi} & & \\
\mathrm{Ext}^1_{\mathcal{O}_X}(\Omega^1_X, \mathcal{O}_X) \times_{\mathrm{Ext}^1_{\mathcal{O}_X}(\Omega^1_Y|_X, \mathcal{O}_X)} \mathrm{Ext}^1_{\mathcal{O}_Y}(\Omega^1_Y, \mathcal{O}_Y) & \xrightarrow{p_Y} & \mathrm{Ext}^1_{\mathcal{O}_Y}(\Omega^1_Y, \mathcal{O}_Y) \\
\downarrow{\scriptstyle p_X} & & \downarrow{\scriptstyle \mu} \\
\mathrm{Ext}^1_{\mathcal{O}_X}(\Omega^1_X, \mathcal{O}_X) & \xrightarrow{\quad\lambda\quad} & \mathrm{Ext}^1_{\mathcal{O}_X}(\Omega^1_Y|_X, \mathcal{O}_X)
\end{array} \tag{4}$$

whose kernel is the image of the natural map $\Delta : \mathrm{Def}_{X/\nu/Y}(\mathbf{k}[\epsilon]) \longrightarrow \mathrm{Def}_\nu(\mathbf{k}[\epsilon])$.

Recalling that

$$\mathrm{Def}_{X/\nu/Y}(\mathbf{k}[\epsilon]) \simeq \mathrm{Hom}_{\mathcal{O}_X}\left(\nu^*\Omega^1_Y, \mathcal{O}_X\right) = \mathrm{Hom}_{\mathcal{O}_X}\left(\Omega^1_Y|_X, \mathcal{O}_X\right), \tag{†}$$

by Proposition 1.5, we obtain the following result describing $\mathrm{Def}_\nu(\mathbf{k}[\epsilon])$, which is now to be considered the main result of the paper. (In the statement, the map $\beta : \Omega^1_Y|_X \longrightarrow \Omega^1_X$ is the one in the conormal sequence.)

Theorem *Let* $\nu : X \hookrightarrow Y$ *be a regular closed embedding of reduced algebraic schemes. Then, there exists a long exact sequence*

$$0 \longrightarrow \mathrm{Hom}_{\mathcal{O}_X}(\Omega^1_X, \mathcal{O}_X) \times_{\mathrm{Hom}_{\mathcal{O}_X}(\Omega^1_Y|_X, \mathcal{O}_X)} \mathrm{Hom}_{\mathcal{O}_Y}(\Omega^1_Y, \mathcal{O}_Y) \longrightarrow \mathrm{Hom}_{\mathcal{O}_X}(\Omega^1_X, \mathcal{O}_X) \times \mathrm{Hom}_{\mathcal{O}_Y}(\Omega^1_Y, \mathcal{O}_Y) \xrightarrow{\Theta}$$

$$\xrightarrow{\Theta} \mathrm{Hom}_{\mathcal{O}_X}(\Omega^1_Y|_X, \mathcal{O}_X) \xrightarrow{\Delta} \mathrm{Def}_\nu(\mathbf{k}[\epsilon]) \xrightarrow{\Phi} \mathrm{Ext}^1_{\mathcal{O}_X}(\Omega^1_X, \mathcal{O}_X) \times_{\mathrm{Ext}^1_{\mathcal{O}_X}(\Omega^1_Y|_X, \mathcal{O}_X)} \mathrm{Ext}^1_{\mathcal{O}_Y}(\Omega^1_Y, \mathcal{O}_Y) \longrightarrow 0$$

where the map Θ *is given by* $\Theta(\xi, \eta) = \xi \circ \beta - \eta|_X$.

Proof The second row of the above exact sequence follows from (the above version of) Proposition 1.3 and (†).

By the definition of $\mathrm{Hom}_{\mathcal{O}_X}(\nu^*\Omega^1_Y, \mathcal{O}_X)$ and $\mathrm{Def}_\nu(\mathbf{k}[\epsilon])$ (cf. [3, p. 158 and p. 177]), an element mapped to zero by Δ corresponds to a first order deformation

$$\tilde{\nu} : X \times \mathrm{Spec}(\mathbf{k}[\epsilon]) \to Y \times \mathrm{Spec}(\mathbf{k}[\epsilon])$$

of ν that is trivializable. More precisely, denoting by $H_X \subset \mathrm{Aut}(X \times \mathrm{Spec}(\mathbf{k}[\epsilon]))$ the space of automorphisms restricting to the identity on the closed fiber and similarly for $H_Y \subset \mathrm{Aut}(Y \times \mathrm{Spec}(\mathbf{k}[\epsilon]))$, there exist $\alpha \in H_X$ and $\beta \in H_Y$, such that

$$\alpha \circ (\nu \times \mathrm{Id}_{\mathrm{Spec}(\mathbf{k}[\epsilon])}) \circ \beta = \tilde{\nu}.$$

Then, one obtains a natural map $H_X \times H_Y \to \mathrm{Def}_{X/\nu/Y}(\mathbf{k}[\epsilon])$ whose image is the kernel of Δ. By (†) and the well-known isomorphisms $H_X \simeq \mathrm{Hom}_{\mathcal{O}_X}(\Omega^1_X, \mathcal{O}_X)$ and $H_Y \simeq \mathrm{Hom}_{\mathcal{O}_Y}(\Omega^1_Y, \mathcal{O}_Y)$ (cf. [3, Lemma 1.2.6]), this map may be identified with Θ. The kernel of Θ is by definition as in the statement. □

- In the first column of diagram (11), the vector space $\mathrm{Def}_\nu(\mathbf{k}[\epsilon])$ must be replaced by the quotient $\mathrm{Def}_\nu(\mathbf{k}[\epsilon])/\mathrm{Im}(\Delta)$.

- The paragraph "We remark that $\mathrm{Ext}^1(\delta_1, \delta_0)\ldots$ not isomorphic to it." in §1.4 has to be replaced by the following:

"We remark that $\mathrm{Ext}^1(\delta_1, \delta_0)$ coincides with $\mathrm{Def}_\nu(\mathbf{k}[\epsilon])$ in the case when $f : X \to Y$ is a regular embedding. By (11), with $\mathrm{Def}_\nu(\mathbf{k}[\epsilon])$ replaced by $\mathrm{Def}_\nu(\mathbf{k}[\epsilon])/\mathrm{Im}(\Delta)$, one has $\varphi_1 = \lambda - \mu$. Therefore,

$$\mathrm{Ext}^1_{\mathcal{O}_X}(\Omega^1_X, \mathcal{O}_X) \times_{\mathrm{Ext}^1_{\mathcal{O}_X}(\Omega^1_Y|_X, \mathcal{O}_X)} \mathrm{Ext}^1_{\mathcal{O}_Y}(\Omega^1_Y, \mathcal{O}_Y) \simeq \mathrm{Ker}(\varphi_1),$$

Δ coincides with ∂ and Θ with φ_0. Example 1.7 below gives an instance where $\partial = \Delta$ is nonzero."

- In the proof of Lemma 2.1, the exact sequence (13) is not exact on the left, but this does not affect the proof.
- Replace the statement of Corollary 2.2 by the following:

Corollary 2.2 *There is a natural surjective map*

$$\tau : \mathrm{T}_{(S,C)}\mathcal{V}_{m,\delta} \longrightarrow \mathrm{Def}_\phi(\mathbf{k}[\epsilon]) \simeq \mathrm{Def}_\nu(\mathbf{k}[\epsilon]).$$

Moreover, if X is stable, then the differential of the moduli map of $\psi_{m,\delta}$ at (S, C) factors as

$$d_{(S,C)}\psi_{m,\delta} : \mathrm{T}_{(S,C)}\mathcal{V}_{m,\delta} \xrightarrow{\tau} \mathrm{Def}_\nu(\mathbf{k}[\epsilon]) \longrightarrow \mathrm{Def}_\nu(\mathbf{k}[\epsilon])/\mathrm{Im}(\Delta) \xrightarrow{p_X} \mathrm{Ext}^1_{\mathcal{O}_X}(\Omega_X, \mathcal{O}_X) \simeq T_{[X]}\overline{\mathcal{M}}_g,$$

where p_X is the map appearing in the correct version of (11).

In particular, if $\mathrm{Ext}^2_{\mathcal{O}_Y}(\Omega^1_Y(X), \mathcal{O}_Y) = 0$, then $d_{(S,C)}\psi_{m,\delta}$ is surjective; if

$$\mathrm{Ext}^1_{\mathcal{O}_Y}\left(\Omega^1_Y(X), \mathcal{O}_Y\right) = \mathrm{Hom}_{\mathcal{O}_X}\left(\Omega^1_Y|_X, \mathcal{O}_X\right) = \mathrm{Hom}_{\mathcal{O}_Y}\left(\Omega^1_Y, \mathcal{O}_Y\right) = 0$$

then $d_{(S,C)}\psi_{m,\delta}$ is injective.

- At the end of Remark 2.3, add "In this case, using the above notation, one has $\mathrm{Hom}_{\mathcal{O}_Y}(\Omega^1_Y, \mathcal{O}_Y) = H^0(Y, T_Y) = 0$, and moreover, by [2, (4) in proof of Prop. 1.2], $\mathrm{Hom}_{\mathcal{O}_X}(\Omega^1_Y|_X, \mathcal{O}_X) = H^0(X, T_{Y|X}) = 0$."

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