Erratum

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Erratum to "Conserved vectors with conformable derivative for certain systems of partial differential equations with physical applications"

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Abstract: This erratum corrects the typing mistakes of the article "Conserved vectors with conformable derivative for certain systems of partial differential equations with physical applications," published in Open Physics 2020;18(1):164–9, https://doi.org/10.1515/phys-2020-0127.

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1 The dispersive long-wave system

In ref. [1], the classical dispersive long-wave system is given by

$$\begin{aligned}\frac{\partial u}{\partial t} &= (u^2 - u_x + 2v)_x,\\ \frac{\partial v}{\partial t} &= (2uv + v_x)_x,\end{aligned} \tag{1}$$

which is used to describe evolution of the horizontal velocity portion of water waves. We present here the correct form of (1) with conformable operator as

$$\begin{aligned}\frac{\partial^\alpha u}{\partial t^\alpha} &= 2u\frac{\partial^\alpha u}{\partial x^\alpha} - \frac{\partial^{2\alpha} u}{\partial x^{2\alpha}} + 2\frac{\partial^\alpha v}{\partial x^\alpha},\\ \frac{\partial^\alpha v}{\partial t^\alpha} &= 2u\frac{\partial^\alpha v}{\partial x^\alpha} + 2v\frac{\partial^\alpha u}{\partial x^\alpha} + \frac{\partial^{2\alpha} v}{\partial x^{2\alpha}}.\end{aligned} \tag{2}$$

We present the symmetries of the correct version of conformable dispersive long-wave system described in (2) as

$$\begin{aligned}&\xi^1 = \frac{c_1 t}{\alpha} + c_2 t^{1-\alpha}, \quad \xi^2 = \frac{c_1}{2\alpha}x + c_4 x^{1-\alpha} + \frac{c_3}{\alpha}x^{1-\alpha}t^\alpha,\\ &\eta_1 = -\frac{c_1}{2}u - \frac{c_3}{2}, \quad \eta_2 = -c_1 v.\end{aligned} \tag{3}$$

Thus, the Lie algebra of (2) is spanned by the following four generators:

$$\begin{aligned}&X_1 = \frac{t}{\alpha}\frac{\partial}{\partial t} + \frac{x}{2\alpha}\frac{\partial}{\partial x} - \frac{1}{2}u\frac{\partial}{\partial u} - v\frac{\partial}{\partial v}, \quad X_2 = t^{1-\alpha}\frac{\partial}{\partial t},\\ &X_3 = \frac{x^{1-\alpha}t^\alpha}{\alpha}\frac{\partial}{\partial x} - \frac{1}{2}\frac{\partial}{\partial u}, \quad X_4 = x^{1-\alpha}\frac{\partial}{\partial x}.\end{aligned} \tag{4}$$

Below, we will present the correct form of conserved vectors.

For the symmetry $X_1 = \frac{t}{\alpha}\frac{\partial}{\partial t} + \frac{x}{2\alpha}\frac{\partial}{\partial x} - \frac{1}{2}u\frac{\partial}{\partial u} - v\frac{\partial}{\partial v}$, we obtain

$$\begin{aligned}T_1^t &= c_1 x^{\alpha-1}\left(-\frac{1}{2}u - \frac{x}{2\alpha}u_x - \frac{t}{\alpha}u_t\right)\\ &\quad + c_2 x^{\alpha-1}\left(-v - \frac{x}{2\alpha}v_x - \frac{t}{\alpha}v_t\right),\\ T_1^x &= 2\left(-\frac{1}{2}u - \frac{x}{2\alpha}u_x - \frac{t}{\alpha}u_t\right)(c_1 u - c_2 v)t^{1-\alpha}\\ &\quad - c_1\left(\frac{1}{2}u_x + \frac{x}{2\alpha}u_{xx} + \frac{1}{2\alpha}u_x + \frac{1}{\alpha}u_{tx}\right)x^{1-\alpha}t^{\alpha-1}\\ &\quad - \left(-v - \frac{x}{2\alpha}v_x - \frac{t}{\alpha}v_t\right)(2c_1 + 2c_2 u + c_2 x^{-\alpha}\\ &\quad + c_2(1-\alpha)x^{-\alpha})t^{\alpha-1}\\ &\quad + c_2\left(v_x + \frac{x}{2\alpha}v_{xx} + \frac{1}{2\alpha}v_x + \frac{1}{\alpha}v_{tx}\right)x^{1-\alpha}t^{\alpha-1}.\end{aligned} \tag{5}$$

For the symmetry $X_2 = t^{1-\alpha}\frac{\partial}{\partial t}$, we obtain

$$\begin{aligned}T_1^t &= -c_1 x^{\alpha-1}t^{1-\alpha}u_t - c_2 x^{\alpha-1}t^{1-\alpha}v_t,\\ T_1^x &= -t^{1-\alpha}u_t(c_1 u - c_2 v)t^{1-\alpha}\\ &\quad - c_1 t^{1-\alpha}u_{tx}x^{1-\alpha}t^{\alpha-1} + t^{1-\alpha}v_t(2c_1 + 2c_2 u\\ &\quad + c_2 x^{-\alpha} + c_2(1-\alpha)x^{-\alpha})t^{\alpha-1} + c_2 v_{tx}x^{1-\alpha}.\end{aligned} \tag{6}$$

For the symmetry $X_3 = \frac{x^{1-\alpha}t^\alpha}{\alpha}\frac{\partial}{\partial x} - \frac{1}{2}\frac{\partial}{\partial u}$, we obtain

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$$\begin{aligned}
T_1^t &= c_1 x^{\alpha-1}\left(-\frac{1}{2} - \frac{x^{1-\alpha}t^{\alpha}}{\alpha}u_x\right) - c_2\frac{t^{\alpha}}{\alpha}v_x,\\
T_1^x &= \left(-\frac{1}{2} - \frac{x^{1-\alpha}t^{\alpha}}{\alpha}u_x\right)(c_1 u - c_2 v)t^{1-\alpha}\\
&\quad - c_1\left(\frac{(1-\alpha)}{\alpha}u_x + \frac{1}{\alpha}xu_{xx}\right)x^{1-2\alpha}t^{2\alpha-1}\\
&\quad + \frac{x^{1-\alpha}t^{\alpha}}{\alpha}v_x(2c_1 + 2c_2 u + c_2 x^{-\alpha}\\
&\quad + c_2(1-\alpha)x^{-\alpha})t^{\alpha-1}\\
&\quad + c_2\left(\frac{(1-\alpha)}{\alpha}v_x + \frac{1}{\alpha}xv_{xx}\right)x^{1-2\alpha}t^{2\alpha-1}.
\end{aligned} \tag{7}$$

For the symmetry $X_4 = x^{1-\alpha}\frac{\partial}{\partial x}$, we obtain

$$\begin{aligned}
T_1^t &= -c_1 u_x - c_2 v_x,\\
T_1^x &= -x^{1-\alpha}u_x(c_1 u - c_2 v)t^{1-\alpha} - c_1((1-\alpha)u_x\\
&\quad + xu_{xx})x^{1-2\alpha}t^{\alpha-1} + x^{1-\alpha}v_x(2c_1 + 2c_2 u\\
&\quad + c_2 x^{-\alpha} + c_2(1-\alpha)x^{-\alpha})t^{\alpha-1}\\
&\quad + c_2((1-\alpha)v_x + xv_{xx})x^{1-2\alpha}t^{\alpha-1}.
\end{aligned} \tag{8}$$

2 The Whitham–Broer–Kaup system

The conformable Whitham–Broer–Kaup Wilson system used in ref. [1] is given by

$$\begin{aligned}
&\frac{\partial^{\alpha}u}{\partial t^{\alpha}} + u\frac{\partial^{\alpha}u}{\partial x^{\alpha}} + \mu\frac{\partial^{2\alpha}u}{\partial x^{2\alpha}} + \frac{\partial^{\alpha}v}{\partial x^{\alpha}} = 0,\\
&\frac{\partial^{\alpha}v}{\partial t^{\alpha}} + u\frac{\partial^{\alpha}v}{\partial x^{\alpha}} + v\frac{\partial^{\alpha}u}{\partial x^{\alpha}} - \mu\frac{\partial^{2\alpha}v}{\partial x^{2\alpha}} + \beta\frac{\partial^{3\alpha}v}{\partial x^{3\alpha}} = 0.
\end{aligned} \tag{9}$$

We present the following symmetries for system (9)

$$\begin{aligned}
&\xi^1 = \frac{c_1 t}{\alpha} + c_2 t^{1-\alpha}, \quad \xi^2 = \frac{c_1}{2\alpha}x + c_4 x^{1-\alpha} + \frac{c_3}{\alpha}x^{1-\alpha}t^{\alpha},\\
&\eta_1 = -\frac{c_1}{2}u + c_3, \quad \eta_2 = -c_1 v.
\end{aligned} \tag{10}$$

Thus, the Lie algebra of (9) is spanned by the following four generators

$$\begin{aligned}
&X_1 = \frac{t}{\alpha}\frac{\partial}{\partial t} + \frac{x}{2\alpha}\frac{\partial}{\partial x} - \frac{1}{2}u\frac{\partial}{\partial u} - v\frac{\partial}{\partial v}, \quad X_2 = t^{1-\alpha}\frac{\partial}{\partial t},\\
&X_3 = \frac{x^{1-\alpha}t^{\alpha}}{\alpha}\frac{\partial}{\partial x} + \frac{\partial}{\partial u}, \quad X_4 = x^{1-\alpha}\frac{\partial}{\partial x}.
\end{aligned} \tag{11}$$

In the next step, we present the correct version of the conserved vectors for system (9).

For the symmetry $X_1 = \frac{t}{\alpha}\frac{\partial}{\partial t} + \frac{x}{2\alpha}\frac{\partial}{\partial x} - \frac{1}{2}u\frac{\partial}{\partial u} - v\frac{\partial}{\partial v}$, we obtain

$$\begin{aligned}
T_1^t &= c_1 x^{\alpha-1}\left(-\frac{1}{2}u - \frac{x}{2\alpha}u_x - \frac{t}{\alpha}u_t\right)\\
&\quad + c_2 x^{\alpha-1}\left(-v - \frac{x}{2\alpha}v_x - \frac{t}{\alpha}v_t\right),\\
T_1^x &= \left(-\frac{1}{2}u - \frac{x}{2\alpha}u_x - \frac{t}{\alpha}u_t\right)(c_1 u + c_1\mu c_2 x^{-\alpha} + c_2 v\\
&\quad + c_1\beta(1-\alpha)(1-2\alpha)x^{-2\alpha} - c_1\mu(1-\alpha)x^{-\alpha}\\
&\quad - 3c_2\beta(1-\alpha)^2x^{-\alpha} + c_2\beta(2-2\alpha)(1-\alpha)x^{-\alpha})t^{\alpha-1}\\
&\quad - \left(\frac{1}{2}u_x + \frac{x}{2\alpha}u_{xx} + \frac{1}{2\alpha}u_x + \frac{1}{\alpha}u_{tx}\right)\\
&\quad \times (c_1\mu x^{1-\alpha} + 3\beta(1-\alpha)c_2 x^{1-2\alpha} - c_2\beta x^{2-2\alpha})t^{\alpha-1}\\
&\quad - \left(\frac{1}{2}u_{xx} + \frac{1}{2\alpha}u_{xx} + \frac{x}{2\alpha}u_{xxx} + \frac{1}{2\alpha}u_{xx} + \frac{1}{\alpha}u_{txx}\right)\\
&\quad \times c_2 x^{2-2\alpha}t^{\alpha-1} + \left(-v - \frac{x}{2\alpha}v_x - \frac{t}{\alpha}v_t\right)\\
&\quad \times (c_1 + c_2 u - c_2\mu(1-\alpha)x^{1-\alpha} + c_2\mu(1-\alpha)x^{-\alpha})t^{\alpha-1}\\
&\quad - \left(v_x + \frac{x}{2\alpha}v_{xx} + \frac{1}{2\alpha}v_x + \frac{1}{\alpha}v_{tx}\right)c_2\mu x^{1-\alpha}t^{\alpha-1}.
\end{aligned} \tag{12}$$

For the symmetry $X_2 = t^{1-\alpha}\frac{\partial}{\partial t}$, we obtain

$$\begin{aligned}
T_1^t &= -c_1 x^{\alpha-1}t^{1-\alpha}u_t - c_2 x^{\alpha-1}t^{1-\alpha}v_t,\\
T_1^x &= -u_t(c_1 u + c_1\mu c_2 x^{-\alpha} + c_2 v + c_1\beta(1-\alpha)\\
&\quad \times (1-2\alpha)x^{-2\alpha} - c_1\mu(1-\alpha)x^{-\alpha}\\
&\quad - 3c_2\beta(1-\alpha)^2x^{-\alpha}\\
&\quad + c_2\beta(2-2\alpha)(1-\alpha))x^{-\alpha} - u_{tx}(c_1\mu x^{1-\alpha}\\
&\quad + 3\beta(1-\alpha)c_2 x^{1-2\alpha} - c_2\beta x^{2-2\alpha}) - u_{txx}c_2 x^{2-2\alpha}\\
&\quad - v_t(c_1 + c_2 u - c_2\mu(1-\alpha)x^{1-\alpha}\\
&\quad + c_2\mu(1-\alpha)x^{-\alpha}) - c_2\mu x^{1-\alpha}v_{tx}.
\end{aligned} \tag{13}$$

For the symmetry $X_3 = \frac{x^{1-\alpha}t^{\alpha}}{\alpha}\frac{\partial}{\partial x} + \frac{\partial}{\partial u}$, we obtain

$$\begin{aligned}
T_1^t &= c_1 x^{\alpha-1}\left(-1 - \frac{x^{1-\alpha}t^{\alpha}}{\alpha}u_x\right) - c_2\frac{t^{\alpha}}{\alpha}v_x,\\
T_1^x &= \left(-1 - \frac{x^{1-\alpha}t^{\alpha}}{\alpha}u_x\right)(c_1 u + c_1\mu c_2 x^{-\alpha} + c_2 v\\
&\quad + c_1\beta(1-\alpha)(1-2\alpha)x^{-2\alpha} - c_1\mu(1-\alpha)x^{-\alpha}\\
&\quad - 3c_2\beta(1-\alpha)^2x^{-\alpha} + c_2\beta(2-2\alpha)(1-\alpha))x^{-\alpha}t^{\alpha-1}\\
&\quad - \left(\frac{(1-\alpha)}{\alpha}u_x + \frac{1}{\alpha}xu_{xx}\right)(c_1\mu x^{1-\alpha} + 3\beta(1-\alpha)c_2 x^{1-2\alpha}\\
&\quad - c_2\beta x^{2-2\alpha})t^{2\alpha-2} - \left(\frac{(1-\alpha)}{\alpha}u_{xx} + \frac{1}{\alpha}xu_{xxx} + \frac{1}{\alpha}u_{xx}\right)\\
&\quad \times c_2 x^{2-2\alpha}t^{\alpha-1} + \frac{x^{1-\alpha}t^{\alpha}}{\alpha}v_x(2c_1 + 2c_2 u\\
&\quad + c_2 x^{-\alpha} + c_2(1-\alpha)x^{-\alpha})t^{\alpha-1}\\
&\quad + c_2\mu\left(\frac{(1-\alpha)}{\alpha}v_x + \frac{1}{\alpha}xv_{xx}\right)x^{1-2\alpha}t^{2\alpha-1}.
\end{aligned} \tag{14}$$

For the symmetry $X_4 = x^{1-\alpha}\frac{\partial}{\partial x}$, we obtain

$$
\begin{aligned}
T_1^t &= -c_1 u_x - c_2 v_x, \\
T_1^x &= -x^{1-\alpha} u_x (c_1 u + c_1 \mu c_2 x^{-\alpha} + c_2 v + c_1 \beta (1-\alpha) \\
&\quad \times (1-2\alpha) x^{-2\alpha} - c_1 \mu (1-\alpha) x^{-\alpha} - 3 c_2 \beta (1-\alpha)^2 \\
&\quad \times x^{-\alpha} + c_2 \beta (2-2\alpha)(1-\alpha) x^{-\alpha}) t^{\alpha-1} \\
&\quad - ((1-\alpha) x^{-\alpha} u_x + x^{1-\alpha} u_{xx}) \\
&\quad \times (c_1 \mu x^{1-\alpha} + 3\beta (1-\alpha) c_2 x^{1-2\alpha} - c_2 \beta x^{2-2\alpha}) t^{\alpha-1} \\
&\quad - (2(1-\alpha) x^{-\alpha} u_{xx} - \alpha (1-\alpha) x^{-\alpha-1} u_x \\
&\quad + x^{1-\alpha} u_{xxx}) c_2 x^{2-2\alpha} t^{\alpha-1} - x^{1-\alpha} v_x (c_1 + c_2 u \\
&\quad - c_2 \mu (1-\alpha) x^{1-\alpha} + c_2 \mu (1-\alpha) x^{-\alpha}) t^{\alpha-1} \\
&\quad - ((1-\alpha) x^{-\alpha} v_x + x^{1-\alpha} v_{xx}) c_2 \mu x^{1-\alpha} t^{\alpha-1}.
\end{aligned} \tag{15}
$$

References

[1] Al Qurashi MM. Conserved vectors with conformable derivative for certain systems of partial differential equations with physical applications. Open Phys. 2020;18(1):164–9.