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ERROR ANALYSIS OF FINITE ELEMENT METHODS WITH TRIANGLES FOR ELLIPTIC BOUNDARY VALUE PROBLEMS.

by

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1. Introduction

This paper is concerned with methods for obtaining bounds on the errors in finite element solutions to two dimensional elliptic boundary value problems defined on simply connected polygonal regions. We first mention briefly the techniques of Birkhoff, Schultz and Varga [4] for obtaining bounds when using rectangular elements, and those of Zlamal [22] and Bramble and Zlamal [5] with triangular elements. In [4] the results of Sard [12] are used, and it is through the use of the Sard kernel theorems that we obtain sharp bounds for the interpolation errors in each element, in this case a triangle. The forms of our bounds are similar to those of [22] and [5] which contain unknown constants, but we are able to compute the corresponding constants. The bounds can be used to produce bounds in the Sobolev norm for the finite element solution of the elliptic boundary value problem as in [5]. The results are then applied to problems containing boundary singularities by augmenting the spaces of trial functions with singular functions having the form of the dominant part of the singularity. In Section 6 it is seen that the approach using triangular rather than rectangular elements has the advantage that it is significantly simpler to implement computationally.

The finite element method will be discussed in the context of the homogeneous Dirichlet problem for Poisson's equation, 30 that the function u = u(x,y) satisfies

$$\left.\begin{aligned} -\Delta[u(x,y)] &= g(x,y)\,, \qquad (x,y) \in R,\\ u(x,y) &= 0\,, \qquad (x,y) \in S\,, \end{aligned}\right\} \qquad (1.1)$$

where $R \subset E^2$ is a simply connected region with closed polygonal boundary S , $g(x,y) \in L_2(R)$, and $\bar{G}=R \cup S$. Let $C_0^\infty(R)$ be the space of functions $C^\infty(R)$ satisfying the boundary conditions of (1.1) on S.

In the standard multi-index notation

$$\|u\|^2_{W_2^p} \equiv \sum_{|\alpha| \leq p} \|D^\alpha u\|^2_{L_2(R)} \qquad (1.2)$$

where p is a non-negative integer, $\|u\|^2_{L_2(R)} = \iint_R \{u(x,y)\}^2 \, dxdy$,

and the derivatives are <u>generalized derivatives</u>, (see e.g. Smirnov Vol.V, [13],P.321), defines a norm on $C_0^\infty(R)$. The completion of $C_0^\infty(R)$ in this norm defines the Sobolev space $\overset{o}{W}{}_2^P(R)$, with $W_2^p(R)$ completion of $C^\infty(R)$.

However, the pseudonorm

$$\|u\|^2_{W_2^P} = \sum_{|\alpha| = p} \|D^\alpha u\|^2_{L_2(R)}$$

will in fact be used in place of (1.2). This is permissible

because the Sobolev embedding theorem is applicable to polygonal

regions R, and it implies that

$$\left\{ \sum_{|\alpha| = p-1} \|D^\alpha u\|^2_{L_2(R)} \right\}^{\frac{1}{2}} \leqq B_R^P \left\{ \sum_{|\alpha| = p} \|D^\alpha u\|^2_{L_2(R)} \right\}^{\frac{1}{2}} \qquad (1.2')$$

for some constant B_R^P, the minimal such B_R^P being the norm of

the embedding operator from $\tilde{W}_2^p$ into $\tilde{W}_2^{p-1}$. For definitions of

$\tilde{W}_2^p$ and a proof of the above see [13], pp 339-355- We define the

bilinear form D(u,v) to be

$$D(u, v) = \iint_R (-\Delta u) v \, dxdy, \qquad u, v \in \overset{o}{W}{}_2^p(R),$$

so that

$$D(u, v) = \iint \left(\frac{\partial u}{\partial x}\frac{\partial v}{\partial x} + \frac{\partial u}{\partial y}\frac{\partial v}{\partial y} \right) dx \quad dy \quad .$$

The function $u(x, y) \in \overset{o}{W}{}_2^P(R)$ is the generalized solution of (1.1)

if for all $v \in \overset{o}{W}{}_2^p(R)$

$$D(u, v) = \iint_R g \ v \ dx \quad dy \quad . \tag{1.3}$$

As the finite element solutions of subsequent Sections will be approximations to the generalized solution u of (1.3), and as it is the classical solution of (1.1) that is desired in physical applications, the relation between these two solutions is clearly important. It can be shown that if the classical solution exists it is identical to the generalized solution, which is unique. There is also the question of why it is necessary to consider the problem (1.1) in the context of Sobolev space. In the finite element context there are two immediate reasons. Firstly Sobolev [14] proves that the solution of the non-homogeneous Dirichlet problem for Laplace's equation is the solution of a variational problem. This development can be reformulated for (1.1) so that

$$I[v] = D(v, v) - 2\iint g \ v \quad dxdy \tag{1.4}$$

is the functional to be minimized, and the relevant function space is $\overset{o}{W}{}_2^1(R)$. The solution of the variational problem is the solution of the generalized problem (1.3). In the finite element method we approximate u by U , where U is the function from some finite dimensional subspace of $\overset{o}{W}{}_2^P(R)$, which minimizes I(v) over the subspace. For this finite dimensional subspace we use the notation S^q , where q indicates the form of the approximation. In this paper S^q is a space of piecewise polynomials defined on subrectangles or subtriangles of a polygonal region. The Sobolev imbedding theorems are used in the derivation of bounds on the norms of the error in these finite element solutions, and this is the second reason for working in Sobolev space. The general method is that the solution u of (1.3) is interpolated by some $\tilde{u} \in S^q$, and a bound is found for the interpolation error $u - \tilde{u}$. From Zlamal [22] we have the following result

$$\|u - U\|^2_{\overset{o}{W}{}^1_2(R)} = D(u - U) = I[U] - I[u] = \min_{v \in S^q} I[v] - I[u]$$

$$= \min_{v \in S^q} D(v - u)$$

$$\leqq D(u - \tilde{u}) = \|u - \tilde{u}\|^2_{\overset{o}{W}{}^1_2(R)} , \qquad (1.5)$$

so that the bound on the interpolation error is a bound on the error u - U. It is here that multivariate interpolation theory is used in finite element analysis. We note that

$$\|u - U\|_{\overset{o}{W}{}^1_2(R)} \equiv \|\nabla(u - U)\|_{L_2(R)}$$

This is equivalent to (1.2) without the use of (1.2') because of the following; the linear functional $Lu = \int_S u$ is bounded in the pseudonorm $\|\nabla u\| L_2(R)$. $L(1)\neq 0$, and the u of interest are identically zero on S. The general equivalence result is given in Smirnov, p.342.

2. Error Analysis With Rectangular Elements

The region R is here a rectangular polygon, and is divided into subrectangles. The space S^q is defined on the rectangular partition, and

$$\|u - U\|_{\overset{o}{W}{}_2^1(R)} \leqq \|u - \tilde{u}\|_{\overset{o}{W}{}_2^1(R)} \quad . \tag{2.1}$$

Considering only rectangular polygons R and rectangular elements as above Birkhoff, Schultz and Varga (BSV) take as spaces of interpolants $S^{2m-1,\ 2m-1}$ which in each element have the form

$$u(x,y) = P_{2m-1,2m-1}(x,y) = \sum_{j=0}^{2m-1} \sum_{i=0}^{2m-1} a_{ij} x^i y^j \tag{2.2}$$

Their approach is to use the tensor product of univariate piecewise Hermite interpolation to the values of a function and its derivatives up to and including order 2m-1, so that on each subrectangle the interpolation conditions at each of the four corners (x_i, y_j) are

$$D^{(p,q)}\ u(x_i,y_j) = D^{(p,q)}\ [p_{2m-1,2m-1}(x_i,y_j)] \quad ,$$

$$0 \leq p,q \leq m - 1.$$

BSV consider the weak solution u of the homogeneous Dirichlet problem with a $2\ell^{th}$ order elliptic operator [4, equations (8.8),(8.9)]. With the $W_2^1(R)$ norm in (2.1) replaced by the $W_2^\ell(R)$ norm they show that

$$\|u - P_{2m-1,2m-1}\|_{\overset{o}{W}{}_2^\ell} \leqq M_{m,\ell}\, h^{2m-\ell} \tag{2.3}$$

where $0 \leqq \ell \leqq m$, h is the length of the longest side of all the rectangles of the partition, and $M_{m,\ell}$ is independent of h but involves the L_2 norm of all the $2m^{th}$ order derivatives of u. Use of (2.1), (2.2) and (2.3) with $\ell=1$ yields, where again u is the solution of (1.3),

$$\|u - U\|_{\overset{o}{W}{}^1_2(R)} < M_{m,1} \; h^{2m-1} \tag{2.4}$$

As an example we consider the space $S^{1,1}$ of bilinear trial functions, so that

$$\|u - U\|_{\overset{o}{W}{}^1_2(R)} \leqq M_{1,1} \; h. \tag{2.5}$$

In addition we find that

$$\|u - U\|_{\overset{o}{L}_2(R)} \leqq M_{1,0} \; h^2 \tag{2.6}$$

3. <u>Error Analysis With Triangular Elements.</u>

The region R is now a general polygon, and is subdivided into triangular elements. The interpolation spaces $S^{4m+\mu}$ are of trial functions $P_{4m+\mu}(x,y)$, m = 0,1, ..., μ = 1,2,3,4, where the degree of $P_{4m+\mu}$ is not greater than $4m+\mu$. We note that $S^{4m+\mu} \subset W_2^{m+1}(R)$, and it is assumed that $u \in W_2^k(R)$, $2m+2 \leqq k \leqq 4n+2$. Bounds for the interpolation error are again found, and with (1.4) these are used to bound the error in the finite element solution to (1.3). Zlamal [22] using first quadratic and then cubic trial functions shows that

$$\|u - U\|_{\overset{o}{W}{}^1_2(R)} \leqq \frac{c_1 M_{2+i}}{\sin\theta} h^{2+i}, \tag{3.1}$$

where for $i = 0,1$ respectively $M_{3+1} = \sup_{\substack{(x,y)\in R \\ \text{all partials} \\ \text{of order } 3+i}} \left|D^{s+i} u(x, y)\right|$,

and h is the largest side and $ the smallest angle in the triangulation. For (3.1) to be meaningful, M_3, and M_4 must of course be finite.

Bramble and Zlamal for a typical triangle T show that, if $P_{4m+1}(x,y)$ interpolates $u(x,y)$, and it is assumed that $u \in W_2^k$ (T), $2m+2 \leqq k \leqq 4m+2$, then for $0 \leqq n \leqq k$; $n \leqq m+1$,

$$\left\| u - P_{4m+2} \right\|_{W_2^n(T)} \leq \frac{k_2 h^{k-n}}{(\sin\theta)^{m+n}} \left\{ \sum_{|i|=k} \left\| D^i u \right\|^2_{L_2(T)} \right\}^{\frac{1}{2}}, \qquad (3.2)$$

where k_1 is a constant independent of the function u and the triangle T, and h is the length of the largest side of T. Bramble and Zlamal consider the weak solution u of the homogeneous boundary value problem with $2n^{th}$ order $\overset{o}{W}{}_2^n$- elliptic operator [5, (*2.k)*] with the $\overset{o}{W}{}_2^1$ norm in (1.5) replaced by the $\overset{o}{W}{}_2^n(R)$ norm. Extension of the result (3.2) to the whole region R under the same conditions as held for (3.2) with $u \in \overset{o}{W}{}_2^k$ (R) gives

$$\left\| u - U \right\|_{\overset{o}{W}{}_2^n(R)} \leq k_2 h^{k-n} \left\{ \sum_{|i|=k} \left\| D^i u \right\|^2_{L_2(R)} \right\}^{\frac{1}{2}}. \qquad (3.3)$$

In (3.3) the constant K_2 does not depend on u, or, when it is assumed that all the angles of all the elements are bounded away from zero, on the triangulation. Now h is the length of the largest side of the triangulation, and $n \leqq m+1$.

The calculation of the constants in the bounds of Sections 2 and 3 has always been a problem. In the next Section we give a method based on the Sard kernel theorem which at least makes the calculation possible.

4. General Scheme for the Error Bound Using Sard Kernels.

The region R is now partitioned so that all the elements are right-triangles with short sides of length h. In each triangle T the function $p_{4m+\mu}$ (x,y) interpolates u(x,y) at a set of nodal points; e.g. for $4_{m+\mu} = 1$, $p_1(x)$ a linear interpolant, the nodes are the vertices of the triangle T. We follow here the notation of Sard [12], and denote the derivatives of any function v(x,y) by subscripts;

thus $$\frac{\partial^{i+j} v}{\partial x^i \partial y^j} = v_{i,j} .$$

Our aim is to determine sharp upper bounds on the error u(x,y) - $P_{4m+\mu}(x,y)$ on the triangle T by means of the Sard kernel theorems. For the one dimensional Peano theorem case of these see Davis [7]

The Sard kernel theorem for the space $\underline{\underline{B}}_{p,q}$ enables an <u>admissible</u> functional to be written in terms of its partial derivatives of order n = p+q. $\underline{\underline{B}}_{p,q}$ is the space of functions v(x,y) that have the following Taylor expansion at the point (x,y) about the point (a,b):

$$v(x,y)=\sum_{i+j<n}(x-a)^{(i)}(y-b)^{(j)}v_{i,j}(a,b)+$$

$$+\sum_{j<q}(y-b)^{(j)}\int_a^x(x-\tilde{x})^{(n-j-i)}v_{n-j,j}(\tilde{x},b)\,d\tilde{x}+$$

$$+\sum_{i<p}(x-a)^{(i)}\int_b^y(y-\tilde{y})^{(n-i-1)}v_{i,n-i}(a,\tilde{y})\,d\tilde{y}+$$

$$+\int_a^x(x-\tilde{x})^{(p-1)}\int_b^y(y-\tilde{y})^{(q-1)}v_{p,q}(\tilde{x},\tilde{y})\,d\tilde{y}\,d\tilde{x} \tag{4.1}$$

where $(x\text{-}a)^{(i)} = \frac{(x-a)^i}{i!}$, etc. The full definition of $\underline{\underline{B}}_{p,q}$ is given in Sard p. 172, and from this we see that all the partials in (4.1) must be integrable. In fact (4.1) holds if the derivatives of v are generalized derivatives; see Barnhill and Gregory [1]. This means that the Sard spaces are compatible with the Sobolev spaces in that the same kind of derivatives are used in both. Consider a linear functional F; then for F to be an admissible functional for the Sard kernel theorem (4. 3), which follows, it must be of the form

$$Fv=\sum_{\substack{i<p\\ j<p}}\int_o^h\int_o^h v_{i,j}(x,y)\,d\mu^{i,j}(x,y)+$$

$$+\sum_{\substack{i+j<n\\ i\geqq p}}\int_o^h v_{i,j}(x,b)\,d\mu^{i,j}(x)+\sum_{\substack{i+j<n\\ j\geqq q}}\int_o^h v_{i,j}(a,y)\,d\mu^{i,j}(y), \tag{4.2}$$

where the $\mu^{i,j}$ are of bounded variation with respect to their arguments. In Section 5 we define the $\mu^{i,j}$ appropriate to a particular case. The problem of restricting F so that only values *of* partials of v on the triangle are used is solved in the examples given later. The triangle T is specified to have vertices (0,0), (h,0) and (0,h) in the following. We now state the kernel theorem for admissible func tionals on $\underline{\underline{B}}_{p,q}$,

$$F[v(x, y)] = \sum_{i+j<n} o^{i,j} v_{i,j}(a, b) + \sum_{j<q} \int_o^n v_{n-j,i}(\tilde{x}, b)\, k^{n-j,i}(x, y; \tilde{x})\, d\tilde{x} +$$

$$+ \sum_{i<p} \int_o^h v_{i,n-i}(a, \tilde{y}) k^{i,n-i}(x, y; \tilde{y}) d\tilde{y} + \int_o^h \int_o^h v_{p,q}(\tilde{x}, \tilde{y}) k^{p,q}(x, y; \tilde{x}, \tilde{y}) d\tilde{x}\, d\tilde{y}. \tag{4.3}$$

In (4,3) the $c^{i,j} = F_{(x,y)}\left[(x-a)^{(i)}(y-b)^{(j)}\right]$, where the notation

$F_{(x,y)}$ means that the functional is applied to functions of the variables x and y, and the kernels are

$$K^{n-j,j}(x, y; \tilde{x}) = F_{(x,y)}\left[(x-\tilde{x})^{(n-j-1)} \psi(a, \tilde{x}, x)(y-b)^{(j)}\right], \ j < q; \ \tilde{x} \notin Jx, \tag{4.4}$$

$$K^{i,n-i}(x, y; \tilde{y}) = F_{(x,y)}\left[(x-a)^{(i)} \psi(b, \tilde{y}, y)(y-\tilde{y})^{(n-i-1)}\right], \ i < p; \ \tilde{y} \notin Jy, \tag{4.5}$$

$$K^{p,q}(x, y; \tilde{x}, \tilde{y}) = F_{(x,y)}\left[(x-\tilde{x})^{(p-1)} \psi(a, \tilde{x}, x)(y-\tilde{y})^{(q-1)} \psi(b, \tilde{y}, y)\right], \quad \tilde{x} \notin Jx, \ \tilde{y} \notin Jy. \tag{4.6}$$

Jx and Jy are the <u>jump sets</u> of the functions $\mu^{i,j}$ in (4.2), and are defined in Sard p.172. Essentially these jump sets are the points of discontinuity of the $\mu^{i,j}$, and will be stated explicitly in the given examples. Finally

$$\psi\left(a, \tilde{x}, x\right) \equiv \begin{cases} 1 & \text{if} \quad a \leqq \tilde{x} < x, \\ -1 & \text{if} \quad x \leqq \tilde{x} < a, \\ 0 & \text{otherwise,} \end{cases}$$

and $\psi(b, \tilde{y}, y)$ is dual.

Our general schema for error bounds is as follows. Let $F[v(x,y)]$ be $R[v(x,y)]$ in (4.3) and choose $\underline{\underline{B}}_{p,q}$ so that the remainder

$$R[u(x,y)] = u(x,y) - P_{4m+\mu}(x,y)$$

of the interpolation has polynomial precision of at least $(p + q - 1)$; i.e. $4_{m+\mu}$ $p+q - 1$. Thus in (4.3) all the $c^{i,j} = 0$ and application of the triangle inequality gives:

$$\begin{aligned} |R[u(x, y)]| \leqq & \left| \sum_{j<q} \int_o^h u_{n-j,\,j}(\tilde{x}, b) K^{n-j,\,j}(x, y; \tilde{x}) d\tilde{x} \right| \\ & + \left| \sum_{i<0} \int_o^h u_{i,\,n-i}(a, \tilde{y}) K^{i,\,n-i}(x, y, \tilde{y}) d\tilde{y} \right| \\ & + \left| \int_o^h \int_o^h u_{p,\,q}(\tilde{x}, \tilde{y}) K^{p,\,q}(x, y; \tilde{x}, \tilde{y})\, d\tilde{x}\, d\tilde{y} \right| \end{aligned} \tag{4.7}$$

so that from Hölders inequality with $1/p + 1/p' = 1$

$$| R[u(x,y)]| \leqq \sum_{j<q} \|u_{n-j,j}(\tilde{x},b)\|_{L_{p'}(\tilde{x})} \|K^{n-j,j}(x,y;\tilde{x})\|_{L_p(\tilde{x})} +$$

$$+ \sum_{i<p} \|u_{i,n-i}(a,\tilde{y})\|_{L_{p'}(\tilde{y})} \|K^{i,n-i}(x,y;\tilde{y})\|_{L_p(\tilde{y})} +$$

$$+ \|u_{p,q}(\tilde{x},\tilde{y})\|_{L_{p'}(T)(\tilde{x},\tilde{y})} \|K^{p,q}(x,y;\tilde{x},\tilde{y})\|_{L_p(T)(\tilde{x},\tilde{y})} \tag{4.8}$$

The notation $L_{P'}(T)$ $(\tilde{x},\tilde{y})$ means the $L_{P'}$ norm over the triangle T with respect to the variables $(\tilde{x},\tilde{y})$. Both sides of (4.8) are functions of x and y, and the right hand side will now be denoted by G(x,y). Thus taking an L_q norm of (4.8) over the triangle T, where q is independent of p and p', we have

$$\|R[u(x,y)]\|_{L_q(T)} < \left\{ \iint_T |G(x,y)|^q \, dxdy \right\}^{1/q} .$$

The triangle inequality for L is used for **each summand** of G, and thus

$$\| R\, u(x,y)\|\, L_q(T) \leqq$$

$$\sum_{j<q} \|u_{n-j,j}(\tilde{x},b)\|_{L_{p'}(\tilde{x})} \left\| \, \|K^{n-j,j}(x,y;\tilde{x})\|_{L_p(\tilde{x})} \right\|_{L_q(T)(x,y)} +$$

$$+ \sum_{i<p} \|u_{i,n-i}(a,\tilde{y})\|_{L_{p'}(\tilde{y})} \left\| \, \|K^{i-n-i}(x,y;\tilde{y})\|_{L_p(\tilde{y})} \right\|_{L_q(T)(x,y)} +$$

$$+ \|u_{p,q}(\tilde{x},\tilde{y})\|_{L_{p'}(\tilde{x},\tilde{y})} \left\| \, \|K^{p,q}(x,y;\tilde{x},\tilde{y})\|_{L_p(\tilde{x},\tilde{y})} \right\|_{L_q(T)(x,y)}, \tag{4.9}$$

where the x norm is over [0,h], $\tilde{y}$ dually, and where the $(\tilde{x},\tilde{y})$ norm is over the triangle T. The result (4.9) is of a similar form to (3.2).

In order to calculate error bounds of the form as in (3.2) we need the $W_2^n(T)$ norm of u - P_{4m+1}; i.e. we the norms of $R_{h,k}[u(x,y)] = \partial^h/\partial x^h \left\{\partial^k/dy^k \{R[u(x,y)]\}\right\}$, Because of the form of the Sard kernel theorem, the kernels for the functional $R_{h,k}$ are $\left\{\partial^h/\partial x^h \quad \partial^k/\partial y^k\right\}$ of the corresponding kernels of R, provided that these are meaningful. Extensions of the method to produce results similar to (3.3) can be found in Barahill and Whiteman [2].

5. Implementation in $\underline{\underline{B}}_{1,1}$

For illustration we restrict ourselves to the space $\underline{\underline{B}}_{1,1}$, so that the kernel theorem (4.7) with (a,b) taken as (0,0) becomes

$$R[u(x,y)] = \int_o^h u_{2,0}(\tilde{x},0)k^{2,0}(x,y;\tilde{x})d\tilde{x} + \int_o^h u_{0,2}(0,\tilde{y})k^{0,2}(x,y;\tilde{y})d\tilde{y} + \\ + \int_o^h\int_o^h u_{1,1}(\tilde{x},\tilde{y})k^{1,1}(x,y;\tilde{x},\tilde{y})\,d\tilde{x}\,d\tilde{y} \qquad (5.1)$$

The Taylor expansion is taken about (0,0), as this makes the kernels $K^{2,0}$ and $K^{0,2}$ corresponding to R. symmetric with respect to y=x and $\tilde{y} = \tilde{x}$. For the space of linear trial functions $P_1(x,y) = a + bx + cy$ we consider the $W_2^0 \equiv L_2$ and W_2^1 norms.

Case 1 : W_2^0 norm

In the triangle T let P1.(x,y) interpolate the values u(0,0), u(0,h), u(h,0); then

$$R\,[u(x,\ y)] = u(x,\ y) - \left\{\left[1 - \frac{(x+y)}{h}\right] u(0,0) + \frac{x}{h}\, u(h,0) + \frac{y}{h}\, u(0,\ h)\right\}.$$

We now define the functions $\mu^{i,j}$ in (4.2) for the functional R on the apace $\underline{\underline{B}}_{1,1.}$ $\mu^{1,0}(X) \equiv 0 \equiv \mu^{0,1}(y)$. Denote $\mu^{0,0}$,(x,y) by μ(x,y). Initially (x,y) is considered as a fixed point of interpolation in the triangle T. (We vary (x,y) later on.) Then $\mu = \mu\ (\tilde{x},\tilde{y})$, $(\tilde{x},\tilde{y})$ varying over T, and μ is to have "jumps" as follows :

At the point $(\tilde{x},\tilde{y}) =$	(x,y) the jump is	1 ,
	$(0,0)$	$-\left\{1 - \left(\frac{x+y}{h}\right)\right\},$
	(h,0)	- x/h,
	(0,h)	- y/h •

In fact

$$\mu(\tilde{x},\tilde{y}) = v\,(x,\ y)(\tilde{x},\tilde{y}) - \left\{\left[1 - \left(\frac{x+y}{h}\right)\right] v\,(0,0)(\tilde{x},\tilde{y}) + \right.$$

$$\left. + \frac{x}{h}\, v\,(h,0)(\tilde{x},\tilde{y}) + \frac{y}{h}\, v\,(0,\ h)(\tilde{x},\tilde{y})\right\}$$

where, see Fig.1

$$v\,(x,\ y)(\tilde{x},\tilde{y}) = \begin{cases} 1\ ,\ \tilde{x} > x \text{ and } \tilde{y} > y\ , \\ 0\ ,\ \text{elsewhere} \quad \text{in}\ \ T. \end{cases}$$

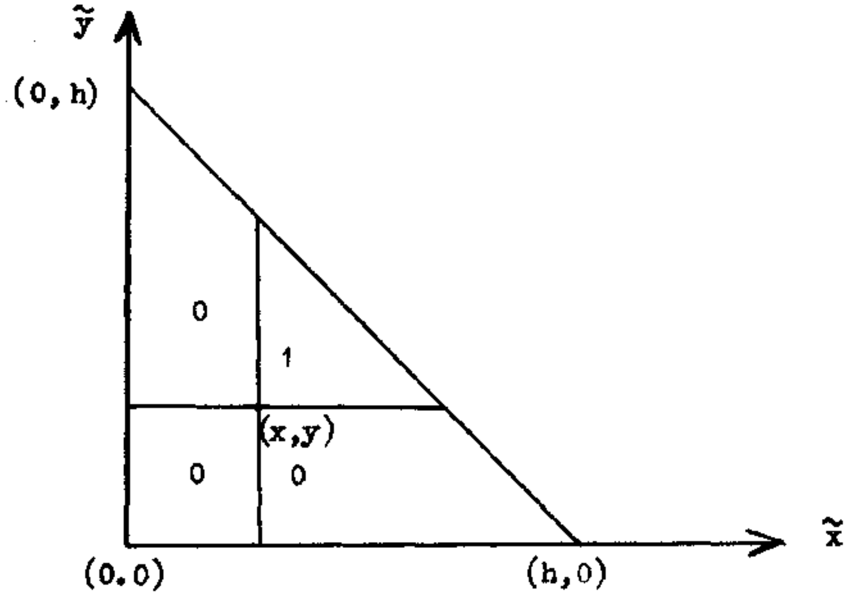


Fig 1.

$$v_{(0,0)}(\tilde{x},\tilde{y}) = v_{(x,y)}(\tilde{x},\tilde{y}) \quad \text{with } (x,y) = (0,0) \quad ,$$

$$v_{(h,0)}(\tilde{x},\tilde{y}) = \begin{cases} 1 & \text{at } (\tilde{x},\tilde{y}) = (h,0) \quad , \\ 0 & \text{elsewhere in } T, \end{cases}$$

$$v_{(0,h)}(\tilde{x},\tilde{y}) = \begin{cases} 1 & \text{at } (\tilde{x},\tilde{y}) = (0,h) \\ 0 & \text{elsewhere in } T. \end{cases}$$

For a discussion of bounded variation functions of two variables, see Sard, p.534.

The jump sets are

$$J_x = \{0, x, h\} \quad ,$$

$$J_y = \{0, y, h\} \quad ,$$

so that from (4.4), (4.5) and (4.6)

$$K^{2,0}(x,y;\tilde{x})\ R_{(x,y)}\ [(x-\tilde{x})_+^{(1)}] \quad , \qquad \tilde{x} \notin Jx,$$

$$K^{0,2}(x,y;\tilde{y}) = R_{(x,y)}\ [(y-\tilde{y})_+^{(1)}] \quad , \qquad \tilde{y} \notin Jy,$$

$$K^{1,1}(x,y;\tilde{x},\tilde{y}) = R_{(x,y)}[(x-\tilde{x})_+^{(0)} \quad (y-\tilde{y})^{(0)}\ ], \quad \tilde{x} \notin Jx,\ \tilde{y} \notin Jy.$$

where $(x-\tilde{x})_+^{(i)} \equiv (x-\tilde{x})^{(i)}$ for $x > \tilde{x}$ and zero otherwise. Hence

$$k^{2,0}(x,\ y;\ \tilde{x}) = \begin{cases} -\dfrac{\tilde{x}}{h}(h-x), & 0 \leqq \tilde{x} \leqq x, \\ \dfrac{x}{h}(h-\tilde{x}), & x \leqq \tilde{x} \leqq h, \end{cases}$$

and so $$\| k^{2,0}(x,\ y;\ \tilde{x}) \|_{L_p}(\tilde{x}) = \frac{(h-x)x}{h}\left(\frac{h}{1+p}\right)^{1/p} \qquad (5.2)$$

From the symmetry it follows that

$$\| K^{0,2}(x,\ y;\ \tilde{y}) \|_{L_p}(\tilde{y}) = \frac{(h-y)y}{h}\left(\frac{h}{1+p}\right)^{1/p}, \qquad (5.3)$$

and finally for the third kernel we have that

$$\| K^{1,1}(x,\ y;\ \tilde{x},\tilde{y}) \|_{L_p(T)}(\tilde{x},\tilde{y}) = (xy)^{1/p}. \qquad (5.4)$$

Substitution with (5.2), (5.3) and (5.4) now enables the L_p norms of the kernels in (4.8) to be replaced. Following the scheme of Section 4 we now require $\| R[u(x,y)] | L_q(T)$.

As this is cumbersome to calculate, we use the triangle inequality property of norms, and obtain the sum of the L_q norms of the three individual terms on the right hand side of the relevant form of (4.8). Thus

$$\| R[u(x,y)] \|_{L_q(T)} \leqq \left[\| u_{2,0}(\tilde{x},0) \|_{L_{p'}(\tilde{x})} + \| u_{0,2}(0,\tilde{y}) \|_{L_{p'}(\tilde{y})} \right] \cdot$$

$$\left\| \frac{(h-x)x}{h} \left(\frac{h}{1+p} \right)^{1/p} \right\|_{L_q(T)} + \| u_{1,1}(\tilde{x},\tilde{y}) \|_{L_{p'}(x,\tilde{y})} \| (xy)^{1/p} \|_{L_q(T)}.$$

(5.5)

But

$$\left\| \frac{(h-x)x}{h} \left(\frac{h}{1+p} \right)^{1/p} \right\|_{L_q(T)} = \frac{h^{1+\frac{1}{p}+\frac{2}{q}}}{(1+p)^{1/p}} \left\{ \frac{(q!)^2}{2(2q+1)!} \right\}^{\frac{1}{q}}, \qquad (5.6)$$

and, assuming $q/p \equiv r$ an integer (as will be the case in Sobolev space where $p = q = 2$ so that $r = 1$),

$$\| (xy)^{1/p} \|_{L_q(T)} = h^{\left(\frac{2}{p}+\frac{2}{q}\right)} \left\{ \frac{(r!)^2}{2(2r+1)} \right\}^{1/q}, \qquad (5.7)$$

so that substitution of (5.6) and (5.7) in (5.5) gives a bound on the L norm of $(u(x,y) - p_1(x,y))$ over T in terms of the L_p norms of the second derivatives of u. Note that two of these $L_{p'}$ norms involve univariate functions whilst the third involves bivariate functions. We are of course particularly interested in the Sobolev space $W^0_2(T)$, so that we must consider the special case $p = p' = q = 2$. In this case (5.5), (5.6) and (5.7) give

$$\| u(x,y) - p_2(x,y) \|_{W^0_2(T)} \leqq \left[\| u_{2,0}(\tilde{x},0) \|_{L_2(\tilde{x})} + \| u_{0,2}(0,\tilde{y}) \|_{L_2(\tilde{y})} \right] \frac{h^{5/2}}{6\sqrt{5}}$$

$$+ \| u_{1,1}(\tilde{x},\tilde{y}) \|_{L_2(\tilde{x},\tilde{y})} \frac{h^2}{2\sqrt{3}}. \qquad (5.8)$$

For (5.8) to be useful the norms on the right hand side must be finite . This will certainly be the case if $u \in \overset{o}{W}{}_2^2$ (R). The bound (5.8) is equivalent to that in (3.2) for the case n=m=0, k=2. which is

$$\| u - p_1 \|_{W_2^0(T)} \leqq K_1 h^2 \left\{ \sum_{|i|=2} \| D^i u \|_{L_2(T)}^2 \right\}^{\frac{1}{2}} . \tag{5.9}$$

In order to explain the apparent discrepancy in the order of h, we note that in (5.8) the norms on $u_{2,0}$ and $u_{0,2}$ involve one dimensional integrals, whilst those in (5.9) are over the triangle T and hence involve two dimensional integrals. That the respective norms produce equal orders of h in the error can be indicated as follows.

Consider the two terms $\left\{ \int_o^h \left| u_{2,0} (\tilde{x},0) \right|^2 d\tilde{x} \right\}^{\frac{1}{2}}$

and $\left\{ \int_o^h \int_o^{h-\tilde{x}} \left| u_{2,0} (\tilde{x},\tilde{y}) \right|^2 d\tilde{y}\, d\tilde{x} \right\}^{\frac{1}{2}}$. If $u_{2,0} (\tilde{x},\tilde{y}) \equiv y$

for some constant γ , then these two terms are respectively $\gamma h^{\frac{1}{2}}$ and $\gamma h/\sqrt{2}$. This implies that the corresponding terms in (5.8) and (5.9) are $\gamma h^3/6\sqrt{5}$ and $\tilde{K}_1 \gamma h^3/\sqrt{2}$, respectively, where the right hand side of (5.9) has been replaced by

$$\tilde{k}_1 h^2 \left\{ \left\| u_{2,0} \right\|_{L_2(T)} + \left\| u_{2,0} \right\|_{L_2(T)} + \left\| u_{1,1} \right\|_{L_2(T)} \right\} .$$

The lattermost is valid because of the equivalence of the induced pseudonorms on W_2^2 where k_1 in (5.9) is replaced by some constant $\tilde{K}_1$. on account of this change.

We note in passing that by the mean value theorem for integrals,

$$\int_o^h \left|u_{2,0}(\tilde{x},0)\right|^2 d\tilde{x} = \left|u_{2,0}(\theta h,0)\right|^2 h, \qquad o \leqq \theta \leqq 1,$$

and

$$\int_o^h \int_o^{h-\tilde{x}} \left|u_{2,0}(\tilde{x},\tilde{y})\right|^2 d\tilde{y}\, d\tilde{x} = \left|u_{2,0}(x^*, h-\tilde{\theta}x^*)\right|^2 \frac{h^2}{2},$$

$$0 \leqq x^* \leqq h, \qquad 0 \leqq \tilde{\theta} \leqq 1.$$

Hence if $\left|u_{2,0}(\theta h,0)\right| = \left|u_{2,0}(x^*, h-\tilde{\theta}x^*)\right|$, then this common number can serve as the γ above.

We also note that the h of Bramble and Zlamal is $\sqrt{2}$ times the h of this paper.

<u>Case 2.</u> <u>W_2^1 norm.</u>

$$R_{1,0}[u(x,y)] \equiv \frac{\partial}{\partial x} R[u(x,y)], \text{ so that}$$

$$R_{1,0}[u(x,y)] = u_{1,0}(x,y) + \frac{u(0,0)-u(h,0)}{h}. \tag{5.10}$$

Since R is precise for linear functions, so is $R_{1,0}$, and one can attempt to let $F = R_{1,0}$ in the Sard kernel theorem (4.3) for $\underline{\underline{B}}_{1,1}$.

However, $R_{1,0}$ is not an admissible functional unless (x,y) = (a,b) (See (4.2)), and so we apply $R_{1,0}$ to the Taylor expansion (4.1). Thus we have the following :

$$R_{1,0}[u(x,y)] = R_{1,0\,(x,y)}\left[\int_a^x (x-\tilde{x})u_{2,0}(\tilde{x},b)d\tilde{x}\right] +$$

$$+ R_{1,0\,(x,y)}\left[\int_b^y (y-\tilde{y})u_{2,0}(\tilde{x},b)d\tilde{x}\right] +$$

$$+ R_{1,0\,(x,y)}\left[\int_a^x\int_b^y u_{1,1}(\tilde{x},\tilde{y})d\tilde{y}\,d\tilde{x}\right]. \tag{5.11}$$

Again we let (a,b) = (0,0). This choice has some consequences for $R_{1,0}$. that it did not have for R. These stem from the fact that $R_{1,0}$. as defined in (5.10) is not an admissible linear functional unless the point of interpolation (x,y) is the same as the point of Taylor expansion (a,b).

The three summands in (5.11) are each of a different kind. The first can be evaluated in a manner similar to the summands of R[u(x,y)]. The second term will be shown to be identically zero. Finally, the third term cannot be evaluated by means of the Sard kernel theorem and the Taylor expansion is used instead.

Calculation of the first term in (5.11).

$$R_{1,0}(x,y)\left[\int_o^x (x-\tilde{x})u_{2,0}(\tilde{x},0)d\tilde{x}\right]$$

$$= \int_o^x u_{2,0}(\tilde{x},0)d\tilde{x} - \frac{1}{h}\int_o^h (h-\tilde{x})u_{2,0}(\tilde{x},0)d\tilde{x}$$

$$= \int_o^h \left[(x-\tilde{x})_+^0 - \left(\frac{h-\tilde{x}}{h}\right)\right]u_{2,0}(\tilde{x},0)d\tilde{x} \tag{5.12}$$

We apply Holder's inequality with respect to $\tilde{x}$ to (5.12), with the L , norm of $u_{2,0}(\tilde{x},0)$ and the L_p norm of the rest of the integrand, where $1/p + 1/p' = 1$. The result of the latter is the following:

$$\left[\int_o^h \left|(x-\tilde{x})_+^0 - \left(1-\frac{\tilde{x}}{h}\right)\right|^p d\tilde{x}\right]^{1/p} =$$

$$\left(\frac{1}{p+1}\right)^{1/p}\left[\frac{x^{p+1}}{h^p} + h\left(1-\frac{x}{h}\right)^{p+1}\right]^{1/p} \tag{5.13}$$

We notice that there are no jump sets involved in this result. As outlined previously for R, for $R_{1,0}u$ in (5.11) we use the triangle inequality on the right hand side and then the triangle inequality on each of the three integrals, followed by Holder's inequality on each integral. Then we take the L_q norm of the resulting inequality and use the triangle inequality for L_q on the right hand side. Thus we need the $L_q(T)(x,y)$ norm of (5.12) and we make the simplifying assumption that q=p. Then, eventually,

$$\| \, \| (x - \tilde{x})_+^0 - \left(1 - \frac{\tilde{x}}{h}\right) \|_{L_p [0,h](\tilde{x})} \|_{L_p (T)(x,y)}$$

$$= \left[\frac{1}{(p+1)(p+2)} \right]^{1/p} h^{3/p} \qquad (5.14)$$

<u>Calculation of the second term in (5.11)</u>

$$R_{1,0}(x,y)\left[\int_o^y (y - \tilde{y}) u_{0,2}(0,\tilde{y}) d\tilde{y} \right]$$

$$= \frac{\partial}{\partial x}\left[\int_o^y (y - \tilde{y}) u_{0,2}(0,\tilde{y}) d\tilde{y} \right]$$

$$+ \frac{1}{h}\left[\int_o^y (y - \tilde{y}) u_{0,2}(0,\tilde{y}) d\tilde{y} \right] \Bigg|_{(x,y)=(0,0)}^{(x,y)=(h,0)}$$

$$\equiv 0 . \qquad (5.15)$$

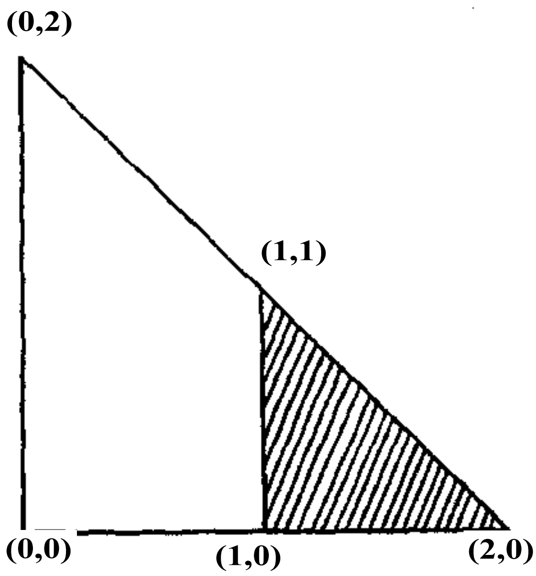


Fig. 2

Consider the symbolic "Sard triangle" of partial derivative indices for $\underline{\underline{B}}_{1,1}$, Fig.2, Equation (5.15) implies that the (1,0) partial derivative of the remainder R depends only on the partials (1,1) and (2,0) and not on the partial (0,2) which is also a part of the full core of u in $\underline{\underline{B}}_{1,1}$, (Sard,p.167) . In Barnhill and Gregory [1] is the following :

<u>Theorem.</u> If f(x,y) in $\underline{\underline{B}}_{p,q}$ is of the form f (x,y) = $p_i(x)$ h (y) where $p_i(x)$ is a polynomial in x of degree $i <$ h, and if the interpolation functional P has the property that

$$P\,[p_i\,(x)h\,(y)] = p_i(x)\,H\,(y) \tag{5.16}$$

for some function of y, H (y) , then the Sard kernels for the functional $D^{(h,k)}$ R have the property that

$$K^{i,\,p+q-i}\,(x,y;\tilde{y}) \;\equiv\; 0\;, \qquad 0 \leqq i\,,\;\; h \leqq \;\; p.$$

Dually, if f(x,y) = g (x) $q_j(y)$, where $q_{j\cdot}(y)$ is a polynomial in y of degree $j < k$ and

$$P[g(x)\;\; q_{\cdot j}(y)] \;=\; G(x)\;\; q_j(y) \tag{5.17}$$

for some function of x, G(x), then

$$k^{p+q-j,\,j}(x,\ y;\ \tilde{x}) \equiv 0, \qquad 0 \leqq j < k \leqq q.$$

The Sard kernel theorem for a remainder functional R is usually applied to a space $\underline{\underline{B}}_{p,q}$ where R. has polynomial precision of at least p + q - 1. (A frequent error is to assume that this polynomial precision must be exactly p + q - 1). In this case, assumptions (5.16) and (5.17) are frequently fulfilled; e.g. for linear interpolation on $\underline{\underline{B}}_{1,1}$,

$$P[v(x,y)] = \left[1 - \left(\frac{x-y}{h}\right)\right] v(0,0) + \left(\frac{x}{h}\right) v(h,0) + \left(\frac{y}{h}\right) v(0,h). \tag{5.18}$$

Thus if v (x,y) = v (y), a function of y alone, then

$$P[v(y)] = \left(1 \frac{y}{h}\right) v(0) + \left(\frac{y}{h}\right) v(h),$$

and (5.16) holds with i = 0. Therefore, $K^{0,2}(x, y; \tilde{y}) \equiv 0$,

which is seen to be equivalent to (5.15) if we recall that the kernel $K^{0,2}$ is obtained as follows:

$$\int_0^y (y-\tilde{y})\, u_{0,2}(0,\tilde{y})\, d\tilde{y} = \int_0^h \psi(0,\tilde{y},y)(y-\tilde{y})\, u_{0,2}(0,\tilde{y})\, d\tilde{y} \tag{5.19}$$

and

$$R_{1,0}(x,y)\left[\int_0^h \psi(0,\tilde{y},y)(y-\tilde{y})\, u_{0,2}(0,\tilde{y})\, d\tilde{y}\right]$$

$$= \int_0^h R_{1,0}(x,y)\left[\psi(0,\tilde{y},y)(y-\tilde{y})\right] u_{0,2}(0,\tilde{y})\, d\tilde{y}$$

$$\equiv \int_0^h K^{0,2}(x, y; \tilde{y})\, u_{0,2}(0,\tilde{y})\, d\tilde{y} \tag{5.20}$$

Tensor product schemes with polynomial precision of at least p+q-1 in each of the variables x and y satisfy (5.16) and (5.17) and there follows the corollary:

Corollary Tensor product schemes of precision at least $p + q - 1$ in each of the variables x and y have the property that their Sard kernels for the functional $D^{(h,k)}R$, $0 \leq h < p$, $0 \leq k < q$, are identically zero outside the shaded Sard subtriangle shown in Fig 3.

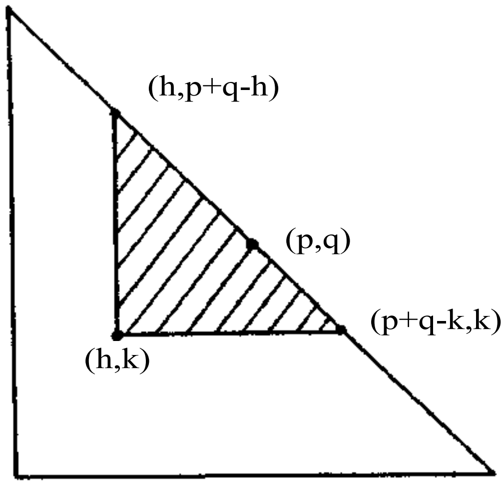


Fig. 3

We remark that, as the Corollary implies, for given h and k, p and q should be chosen so that $h < p$ and $k < q$.
The kernel $K^{0,2}$ in BSV (p.242, (4.14)) was said to be dual to $K^{2,0}$. That is incorrect since $K^{0,2} \equiv 0$ and $K^{2,0} \not\equiv 0$, More important, the above Corollary implies that there are no negative exponents in equation (4.20) of BSV, and 30 no mesh restriction of the form (4.23) of BSV is required. The fact that there need be HO mesh restriction has also been observed by Hall and Kennedy [9], who use an entirely different approach

from the above.

<u>Calculation of the third term in (5.11).</u>

$$R_{1,0}(x,y)\left[\int_o^x\int_o^y u_{1,1}(\tilde{x},\tilde{y})\,d\tilde{y}\,d\tilde{x}\right]$$
$$=\frac{\partial}{\partial x}\left[\int_o^x\int_o^y u_{1,1}(\tilde{x},\tilde{y})\,d\tilde{y}\,d\tilde{x}\right]$$
$$=\int_o^y u_{1,1}(x,\tilde{y})\,d\tilde{y}. \tag{5.21}$$

If we followed the general scheme above, we would use Holder's inequality

$$\left|\int_o^y u_{1,1}(x,\tilde{y})\,d\tilde{y}\right| \leqq y^{1/p}\,\|u_{1,1}(x,\tilde{y})\|_{L_p,[0,y]}(\tilde{y}),$$

and then take the L_q (T) (x,y) norm, along with the above assumption that q = p, to obtain

$$\left\{\int_o^h\int_o^{h-y}\left|\int_o^y u_{1,1}(x.\tilde{y})\,d\tilde{y}\right|^q dx\;dy\right\}^{1/q} \leqq$$

$$\int_o^h\int_o^{h-y} y\left\{\int_o^y |u_{1,1}(x,\tilde{y})|^{p'}\,d\tilde{y}\right\}^{q/p} dx\;dy\;\Big\}^{1/q}.$$

In order to proceed, we let q = p', which implies that q = p = p' = 2. Then, noting the shaded rectangle shown in Fig.4 we have that

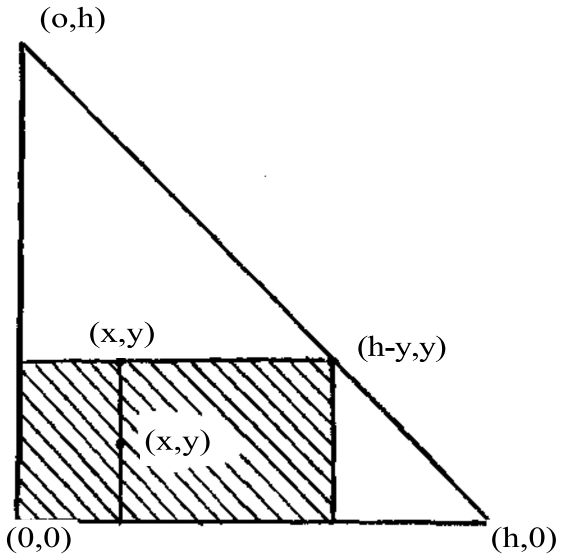


Fig. 4

$$\int_0^{h-y}\int_0^{y}|u_{1,1}(x,\tilde{y})|^2\, d\tilde{y}\, dx \le \iint_T |u_{1,1}(x,\tilde{y})|^2\, d\tilde{y}\, dx.$$

Therfore,

$$|R_{1,0}(x,y)\left[\int_0^x\int_0^y u_{1,1}(\tilde{x},\tilde{y})\,d\tilde{y}\,d\tilde{x}\right]| \le$$

$$\frac{h}{\sqrt{2}}\| u_{1,1}(x,\tilde{y})\|_{L_2(T)}(x,\tilde{y}). \tag{5.22}$$

A more general L_q bound can be found as follows:

$$\left\{\int_0^h\int_0^{h-y}\left|\int_0^y u_{1,1}(x,\tilde{y})\,d\tilde{y}\right|^q dxdy\right\}^{1/q} \le$$

$$\left\{\int_0^h\int_0^{h-y}\int_0^y|u_{1,1}(x,\tilde{y})|^q\, d\tilde{y}\quad y^{q/q'}\, dxdy\right\}^{1/q} \tag{5.23}$$

where $1/q + 1/q' = 1$. Using the above observation about the shaded rectangle in Fig. 4 we obtain that the right hand side of (5.23) is bounded above by the following:

$$\| u_{1,1}(x, \tilde{y}) \|_{L_q(T)(x, \tilde{y})} \left(\frac{h}{(q)^{1/q}} \right),$$

which is (5.22) if q = 2. We have now obtained the following :

$$\| R_{1,0}\, u(x, y) \|_{L_q(T)(x, y)} \leqq$$

$$\| u_{2,0}(\tilde{x},0) \|_{L_{p'}[0, h](\tilde{x})} \left[\frac{1}{(p+1)(p+2)} \right]^{1/p} h^{3/p} +$$

$$+ \| u_{1,1}(x, \tilde{y}) \|_{L_q(T)(x, \tilde{y})} \left(\frac{h}{(q)^{1/q}} \right) \qquad (5.24)$$

For the Sobolev space case of q = p = p' = 2, we have

$$\| R_{1,0}\, u(x, y) \|_{L_2(T)(x, y)} \leq$$

$$\| u_{2,0}(\tilde{x},0) \|_{L_2[0, h](\tilde{x})} \frac{h^{3/2}}{2\sqrt{3}} +$$

$$+ \| u_{1,1}(x, \tilde{y}) \|_{L_2(T)(x, \tilde{y})} \frac{h}{\sqrt{2}}. \qquad (5.25)$$

$R_{0,1}$ [u(x,y)] $= \frac{\partial}{\partial y}$ [Ru (x,y)] is dual to $R_{1.0}$ [u(x,y)] .This is due to the symmetry in the kernel form of R about the lines y = x and $\tilde{y} = \tilde{x}$ coming from (a,b) = (0,0) being on the line y = x, and the symmetry of the triangle T with respect to $y = x$. Therefore,

$$\| R_{0,1} [u(x, y)] \|_{L_q(T)(x,y)} \leqq$$

$$\| u_{0,2}(0, \tilde{y}) \|_{L_p,[0,h](\tilde{y})} \left[\frac{1}{(p+1)(p+2)} \right]^{1/p} h^{3/p} +$$

$$+ \| u_{1,1}(\tilde{x}, y) \|_{L_q(T)(\tilde{x},y)} \left(\frac{h}{(q)^{1/q}} \right), \tag{5.26}$$

with the obvious expression dual to (5.25). Hence

$$\|R_{1,0} [u(x,y)]\|_{L_q(T)(x,y)} + \|R_{0,1} [u(x,y)]\|_{L_q(T)(x,y)}$$

$$\leqq h^{3/p} \left[\frac{1}{(p+1)(p+2)} \right]^{1/p} \Big\{ \|u_{2,0}(\tilde{x},0)\|_{L_p,[0,h](\tilde{x})} +$$

$$+ \|u_{0,2}(0,\tilde{y})\|_{L_p,[0,h](\tilde{y})} \Big\} +$$

$$+ h \frac{2}{(q)^{1/q}} \|u_{1,1}(x',y')\|_{L_q(T)(x',y')}. \tag{5.27}$$

Finally, since $\alpha, \beta > 0$ imply $(\alpha^2 + \beta^2)^{\frac{1}{2}} \leqq \alpha + \beta$ the Sobolev space case is the following :

$$\left\{ \left(\|R_{1,0}\, u\|_{L_2(T)} \right)^2 + \left(R_{0,1}\, u\|_{L_2(T)} \right)^2 \right\}^{1/2}$$

$$\leqq \frac{h^{3/2}}{2\sqrt{3}} \left\{ \|u_{2,0}(\tilde{x},0)\|_{L_2[0,h]} + \|u_{0,2}(0,\tilde{y})\|_{L_2[0,h]} \right\}$$

$$+ \sqrt{2}\, h \|u_{1,1}(x',y')\|_{L_2(T)} \tag{5.28}$$

From the earlier notation, we have

$$\| \nabla \left(u - \tilde{u} \right) \|_{L_2(T)} = \left\{ \left(\| R_{1,0}\, u \|_{L_2(T)} \right)^2 + \left(\| R_{0,1}\, u \|_{L_2(T)} \right)^2 \right\}^{1/2} .$$

We remark that the approach of BSV does not yield the desired results for triangles. They make the clever observation that if the point (a,b) ((x_0, y_0) in BSV) of the Taylor expansion is taken as the point of interpolation (x,y), then the Sard kernel theorems can be used. Otherwise, undefined expressions such as $\frac{\partial}{\partial x} \{ \psi (a, \tilde{x}, x) \}$ are involved. However, the application of their idea to triangles with, e.g., linear interpolation, implies that the kernels $K^{2,0}$ and $K^{0,2}$ corresponding to $R_{1,0}$ and $R_{0,1}$ respectively, are not identically zero outside the triangle T. This means that values of the partials $u_{2,0}$ and $u_{0,2}$ outside the triangle must be used in the kernel theorems. For subtriangles interior to a polygonal domain, this is not a real difficulty, but for subtriangles at the boundary this involves at least an implioit extension of $u_{2,0}$ and $u_{0,2}$ outside their original domain of definition.

6. Boundary Singularities

All the preceding error bounds have involved some norm of the function u and certain of its derivatives. For the bounds to be meaningful, and in particular for them to imply convergence with decreasing mesh size of the finite element solution to the solution u(x,y) of the boundary value problem, it is necessary for the function and derivatives to be bounded in R. When the boundary S is sufficiently smooth, this condition is satisfied. However, if the boundary contains a corner at which the internal angle $\phi = k\pi/l$ is such that either $k/l < 1$ and the number l/k is non integer, or $k/l > 1$ in which case the comer is re-entrant, then u will have derivatives which are unbounded at the corner. This is illustrated by use of a local asymptotic expansion due to Lehman [11] of the solution u in the neighbourhood of the corner. In terms of local polar co - ordinates (r,θ) with origin at the corner and zero angle along one of the aims of the corner the asymptotic form of u is

$$u(r,\ \theta) = P\left(z,\ z^{1/k},\ z^{1}\log z,\ \bar{z},\ \bar{z}^{1/k},\ z^{1}\log\bar{z}\right) \qquad (6.1)$$

where $z = r\ e^{i\theta}$, $\bar{z} = r\ e^{-i\theta}$,and P is a power series in its arguments. Noting that u is the solution of (1.1), and thus has zero value on the arms of the cornr, we rewrite (6.1) as

$$u\left(r, \theta\right) = \sum_i a_i \Phi_i \left(r, \theta\right). \qquad (6.2)$$

where the Φ_i, also satisfy the boundary conditions on the arms of the corner.

For the cases of k/l above, $u \in W_2^{[1/k]+1} - W_2^{[1/k]+2}$ where [l/k] is the greatest integer ≦ l/k. Interesting cases occur when $\Phi > \pi$, and two examples of (6.2) are

(i) $\phi = 2\pi$,

$$u(r,\ \theta) = a_1 r^{\frac{1}{2}} \sin\ \theta/2 + a_2 r \sin\theta + a_3 r^{3/2} \sin 3\ \theta in + \ldots, \qquad (6.3)$$

(ii) $\phi = 3\pi\ /\ 2.$

$$u\left(r, \theta\right) = a_1 r^{2/3} \sin\ 2\theta\ /\ 3 + r^{4/3}\left[a_2 \sin\ 4\theta\theta/ + a_3\left(1-\cos\ 4\theta\theta/\right)\right] +$$
$$+\ r^{5/3}\left[a_4\left(\cos\ 5\theta\theta/ - \cos\ \theta/3 + \ldots\ .\right)\right] \qquad (6.4)$$

In both (6.3) and (6.4) it is clear that $\partial u/\partial r$ is unbounded at $r = 0$. Thus the boundary problem contains a singularity at the corner, and because of this the finite element solutions are inaccurate in the neighbourhood of this type of corner. Further the error analysis of the previous sections is not applicable, as, although $u \in W_2^1$, the assumption that $u \in W_2^k(R)$, $k \leq 2$ is violated.

In an effort to improve accuracy and to make the error bounds applicable we try to subtract off at least the dominant part of the singularity in u near each corner. Thus, we consider a region with one re-entrant corner. In the neighbourhood $N(r_i) \subset G$ of the corner, where

$$N(r_i) \equiv \{(r,\theta): \ 0 \leqq r < r_1, \ 0 \leqq \theta \leqq \Phi\},$$

for *some* fixed $r_1 > 0$, and $r_0 = r_1/2$, we form the functions

$$w_i(r, \theta) = \begin{cases} \phi_i(r, \theta), & 0 \leqq r \leqq r_0, \\ g_i(r) h_i(\theta) & r_0 \leqq r \leqq r_1, \\ 0 & r_1 < r, \end{cases} \tag{6.5}$$

1. 1,.2, ... N_i where N_i is discussed below. The $g_i(r)$ are Hermite polynomials

equation so that each function $w_i(r,\theta)$ is in $W_2^n(R)$, i.e. $g_i(r) \in S^{2n-1}[r_0, r_1]$ The $h_i(\theta)$ are *appropriat e* the function (e.g. in (6.3) $h_i(\theta) = \sin\theta/2$) so that the w_i all satisfy the homogeneous boundary conditions on the arms of the corner Using(6.5) we form the function

$$w = u - \sum_{i=3}^{N_i} c_i w_i (r, \theta), \tag{6.6}$$

and choose N_i so that w would be in $W_2^k(R)$ if the c_i were known exactly. However, the c_i are constants to be found. It is the function w that is approximated throughout R by the finite element solution U, and clearly if the were known exactly, making $W \in W_2^k(R)$, $k \geqq 2$ the error bounds (2.4) and (3.3) would then apply.

Consider the special case *of* $\varphi = 2\pi$, and the expansion of $u(r,\theta)$ in (6.3) Suppose that we want w in *(6.6)* to be in W_2^2. Then,from (*6.3),* the minimal N_i is 1 , so that only the function $w_1(r,\theta) = r^{\frac{1}{2}} \sin\theta/2$ need be considered. (We note in passing that $r \sin\theta = y$, and so this term is already included in polynomial trial functions of positive degree.) The functions $g_i(r)$ must be so chosen that smoothness of $U + \Sigma c_i w_i$ is not lost because of them; i.e. $w_i(r,\theta)$ considered as a function of r alone, $w_i(r)$, is such that

$w_i(r) \in W_2^n\ [r_o - \epsilon,\ r_1 + \delta]$ for all positive ϵ and δ such that

$[(r,\theta);\ 0 < r_o - \epsilon, \le r \le r_1 + \delta,\ 0 \leqq \theta \leqq \phi\} \subset G$. The choice of the trial functions affects only the left hand side of (3.3), that is $\|w-u\|_{\overset{o}{W}{}_2^n(R)}$,

$n \leqq 1$, so that in particular for linear, quadratic, cubic and quartic trial functions n = 1 suffices, and we only need the trial functions to he in W_2^1.

Nothing has yet been said about the choice of r_1.. In particular if we wish to consider the convergence with decreasing mesh size h of U to u, we must decide what to do about $N(r_j)$. Suppose there is a boundary singularity at the point 0, Fig. 5. If we consider a point $P = (r',\theta') \in R - N(r_i)$ with $r' > r_j$ (small) from *(6.6)* with N_i singular functions it follows that at P

$$w = u - \sum_i c_i w_i = u.$$

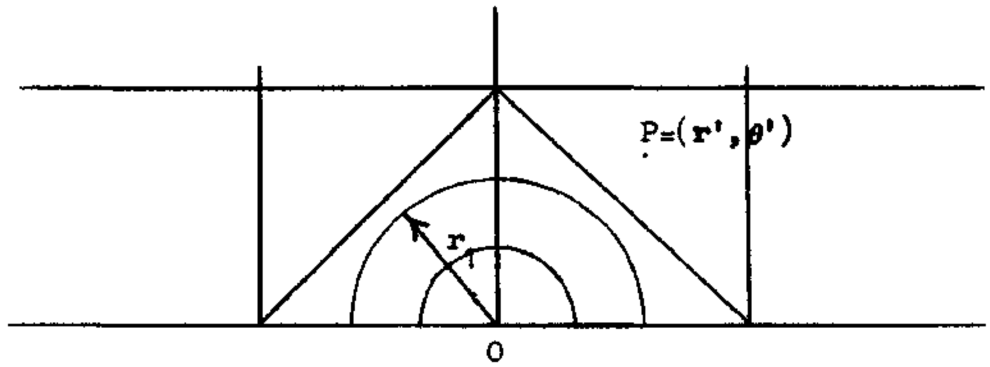


Fig. 5

Thus in fact no singular terms have been subtracted off at P, so that, if $r \rightarrow 0$ with h, at a fixed point of R nothing will have been subtracted off and $w \notin W_2^2$. In order that w remain in W_2^2 the radius r_1 must be kept fixed so that near the singular point the w_i are not zero. As the grid size is decreased, the mesh must therefore be refined <u>inside</u> $N(r_j)$. When more terms in (6.6) are retained, w is put into a higher continuity class W_2^k, $k \geqq 2$. This increasing of the smoothness of w leads to higher accuracy in $N(r_i)$, and from the manner of coupling between nodes in the calculation of the finite element solution this permeates $R - N(r_i)$.

However, the c_i can unfortunately not be calculated exactly. This can be seen from the following implementation of the finite element procedure.

The method is that of augmentation of the trial function spaces with singular functions, and was first suggested by Fix [8]. In each element of R the trial functions are taken as

$$P_m\left(x, y\right) + \sum_i c_i w_i\left(r, \theta\right),$$

so that by (6.5) these are the usual trial functions for elements in R - $N(r_1)$. Extra equations are added to the linear system which when solved gives the finite element solution, and so in practice only approximations c_i to the c_i in(6.6) are obtained from the same numerical calculation as that which gives the values of U at the nodal points. Thus although we would like $u-\Sigma c_i w_i$ to be in W_2^k, k ≧ 2, we actually have $u-\Sigma \bar{c}_i w_i \in w_2^{[1/k]+1} - w_2^{[1/k]+2}$, where the k's in the last expression arise from the angle $\phi = k\pi/l$ and are not the same as the k in W_2^k. Thus instead of having

$$\|\left(u-\Sigma c_i w_i\right)-u\|_{\overset{o}{w}{}_2^n} = \|w-u\|_{\overset{o}{w}{}_2^n} \leqq kh^{k-n}\|w\|_{\overset{o}{w}{}_2^k},$$

we have on the left hand side $\|u-\left(\Sigma \bar{c}_i w_i + u\right)\|_{\overset{o}{w}{}_2^n}$.

Hence the error bounds again do not apply since $w = u - \Sigma \bar{c}_i w_i$ is in the same space as u. However, in a qualitative way by calculating good approximations to the c_i we are able to subtract off most of the singularity, and hence w is almost in w_2^k. In fact the approximation $\left(U + \Sigma \bar{c}_i w_i\right)$ is a best approximation to u in the W_2^1 norm ; are Barnhill and Whiteman [2],

Fix uses rectangular elements and augments the spaces of trial functions defined on these. We use triangular elements, with N internal nodes in R, and demonstrate the computational advantages of doing this. Linear trial functions of the form

$$P_1(x,y) = a + bx + cy, \tag{6.7}$$

are taken in each element e, and these interpolate to the three nodal values U_i^e, U_j^e, U_k^e so that

$$\begin{bmatrix} a \\ b \\ c \end{bmatrix} = \begin{bmatrix} f_{1i} & f_{1j} & f_{1k} \\ f_{2i} & f_{2j} & f_{2k} \\ f_{3i} & f_{3j} & f_{3k} \end{bmatrix} \cdot \begin{bmatrix} U_i^e \\ U_j^e \\ U_k^e \end{bmatrix}$$

where the $f_{p,q} = 1,2,3$, depend only on the nodal co-ordinates. Thus,if $U = p_1$,

$$\frac{\partial U}{\partial x} = f_{2i} U_i + f_{2i} U_j + f_{2k} U_k ,$$

with $\partial U/\partial y$ dually. Substitution in I[v], (1.4), with summation over all the elements followed by differentiation with respect to U_n , n = 1,2, ..., N, leads to the linear system

$$\frac{\partial I[U]}{\partial U_n} = \sum_e \iint_{T_e} F\left(U_i^e, U_j^e, U_k^e\right) dx\ dy - \sum_e \iint_{T_e} g(x,y) G(x,y)\, dx\ dy = 0, \tag{6.8}$$

In (6.8)the F and Gare linear functions of their arguments, and we note that the first integral is just the area of the element, whilst the second can be difficult to compute. The above is explained in greater detail in [18].

When the trial function space is augmented by the addition of just one singular function, so that

$$P_1(x,y) = a + bx + cy + c_1 w_1(r, \theta), \tag{6.9}$$

in the elements for which $w_1 \notin 0$ there is immediately the problem of the combination of cartesian and polar co-ordinates. Thus in these elements (6.8) will be of the form

$$\frac{\partial I[U^e]}{\partial U_n} = \iint \left\{ F\left(U_i^e, U_j^e, U_k^e, r, \theta\right) + H(r,\theta) \right\} dr\, d\theta, \tag{6.10}$$

when cartesians have been changed into polar co-ordinates. The function F now involves many terms of the form $\left(U_i^e r^\alpha \sin^\beta \theta \cos^\gamma \theta\right)$, and the integrations are complicated. The necessary extra equation is formed by the inclusion of an extra node in the relevant elements. Inclusion of more singular terms correspondingly makes everything more complicated. Three ways of calculating the integrals in (6.10) are : analytically, numerically and symbolically. To date we have used only the first of these.

When the elements are rectangular, bilinear trial functions of the form

$$P_{1,1}(x, y) = a + bx + cy + dxy \qquad (6.11)$$

replace (6.7). All the subsequent analysis and computation is now correspondingly altered because of the xy term. In particular, when the singular function is incorporated as in (6.9), the integrals in (6.10) now become much more complicated on account of the interaction between the xy term and the singular term.

7. Model Problem

The above discussion has concerned only problems of type (1.1) with homogeneous Dirichlet boundary conditions. However, it is well known that with a slight modification of the functional (1.4.) the variational technique is applicable to boundary problems for Poisson's equation with non-homogeneous Dirichlet and natural boundary conditions. A much studied problem of this type, see [15]and [16 - 20], is a model harmonic mixed boundary value problem in which the function u(x,y) satisfies

$$-\Delta\,[u(x,y)] = 0,$$

in the square $-\pi/2 = x,y = \pi/2$ with the slit $y = 0,\ 0 \leqq x \leqq \pi/2$, and the boundary conditions

$$\begin{aligned}
\frac{\partial u}{\partial y}(x,\pm\pi/2) &= 0, && -\pi/2 < x < \pi/2,\\
u(\pi/2, y) &= \begin{cases} 1000, \\ 0, \end{cases} && 0 \leqq y \leqq \pi/2,\\
\frac{\partial u}{\partial x}(-\pi/2, y) &= 0, && -\pi/2 \leqq y \leqq 0,\\
\frac{\partial u}{\partial y}(x, 0+) &= 0, && 0 < x < \pi/2.
\end{aligned}$$

There is thus a re-entrant angle $(\phi = 2\pi)$ at the origin, and the asymptotic form of u near the origin is

$$u(r, \theta) = a_0 + a_1 r^{\frac{1}{2}} \cos \theta/2 + a_2 r\cos \theta + a_3 r^{3/2} \cos 3\theta/2 + \dots \qquad (7.1)$$

From the antisymmetry of the problem it suffices to consider only the upper region $G \equiv \{(x,y) \leq |x| \leqq \pi/2, 0, \leqq y \leqq \pi/2\}$, and to add the boundary condition $u(x,0) = 500$, $-\Pi/2 \leqq x \leqq 0$. An accurate (to six. significant digits) approximation to the solution $u(x,y)$ in R is derived in [19]. This is used to produce the surface of Fig.6, from which it can be seen that $\partial u/\partial r$ is unbounded at $r= 0$. Wait and Mitchell [15] use the Fix approach with rectangular elements and bilinear trial functions (6.11), and augment first with two and then with three singular functions. These are respectively the terms involving a_1 ,a_2 and a_1 a_2 a_3 of (7.1) . As no exact solution for this model problem is known, they use the results of [16] for comparison and it is clear that the introduction of the singular functions does improve the finite element solution. We note that in [15] the mesh is refined outside $N(r_1)$. We have repeated this approach, but with right triangular elements as in Fig. 7 , and using no mesh refinement. The trial functions are linear as in (6.7), and the first singular function from (7.1) is included. The results are shown in Fig. 8 together with those calculated with the standard finite element procedure. There is clearly an improvement due to the inclusion of the singular terms, and in this case comparison is made with the results of [19].

As mentioned in Zienkiewicz [21] the problem of boundary singularities is difficult. The above approach, although not completely satisfactory, goes some way towards a solution. Another interesting approach is due to Byskov [6] who uses <u>cracked elements</u>. Boundary singularities occur frequently in stress problems; e.g. in plates with cracks, see Bernal and Whiteman [3]. As the simplest governing equation here is biharmonic, the trial functions are of higher order making the calculations more complicated than those described here. We feel that a technique such as ours which reduces the amount of computation, without loss of accuracy compared to other methods, is valuable, and is likely to be more so for higher order problems.

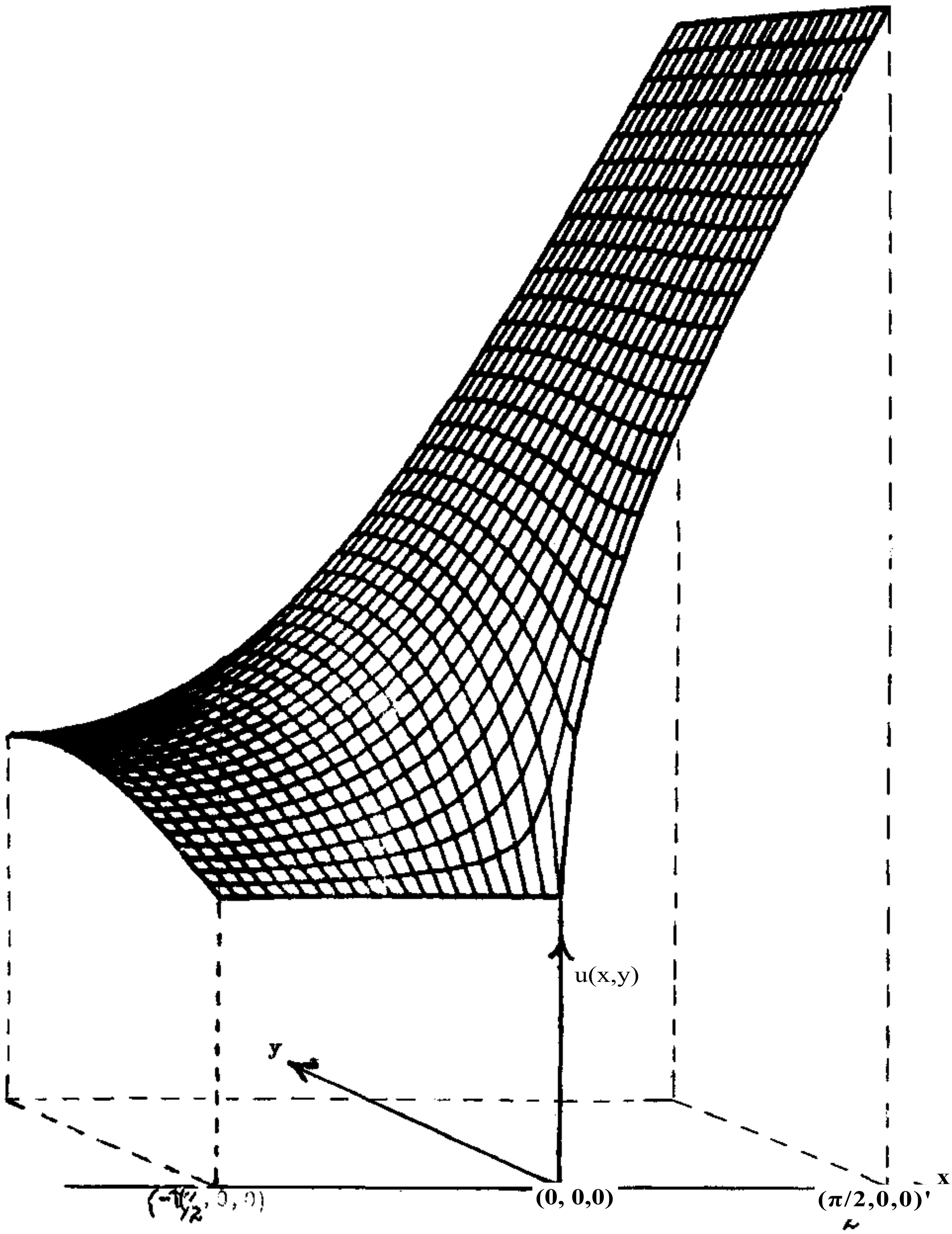


Fig. 6

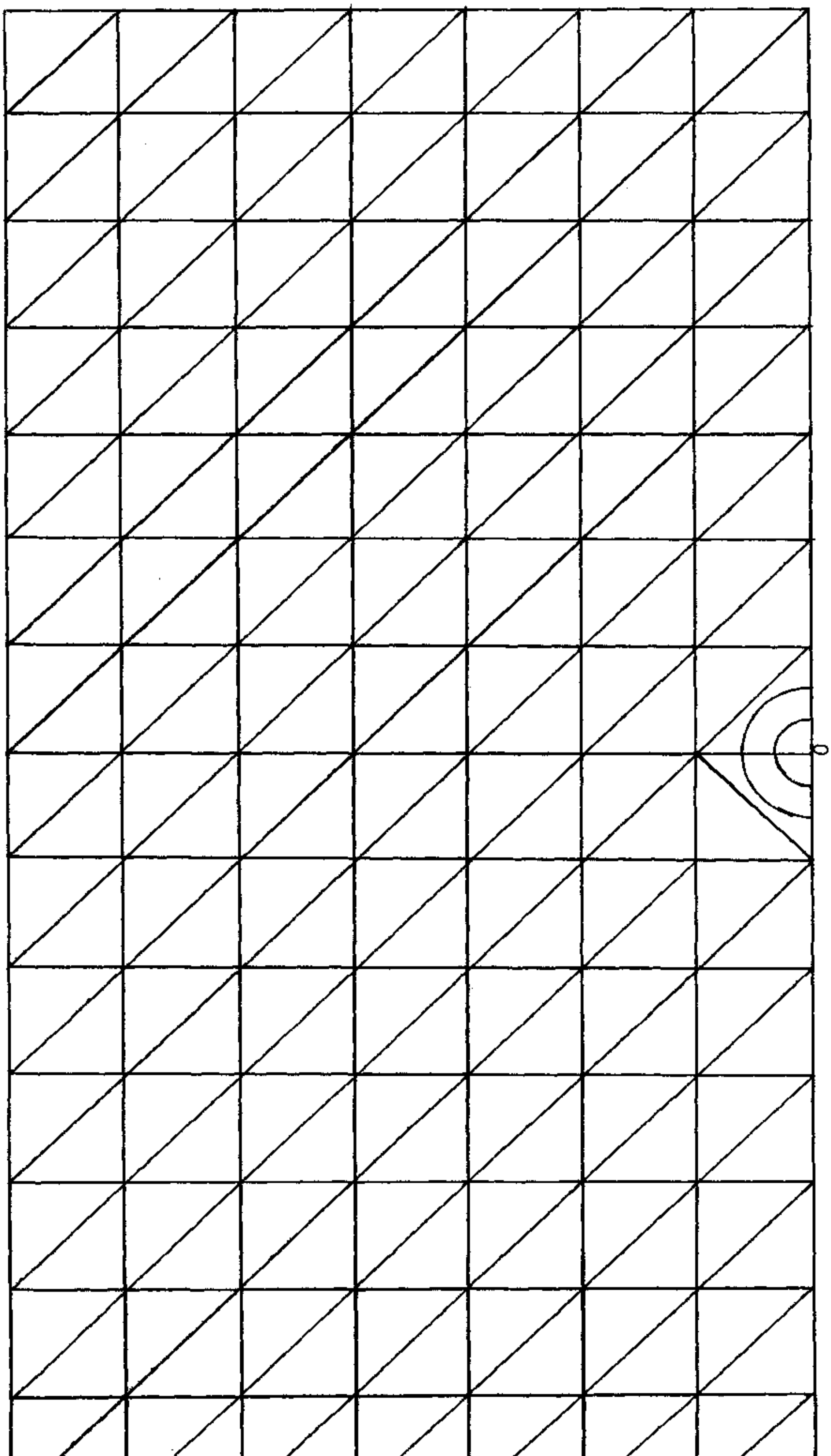
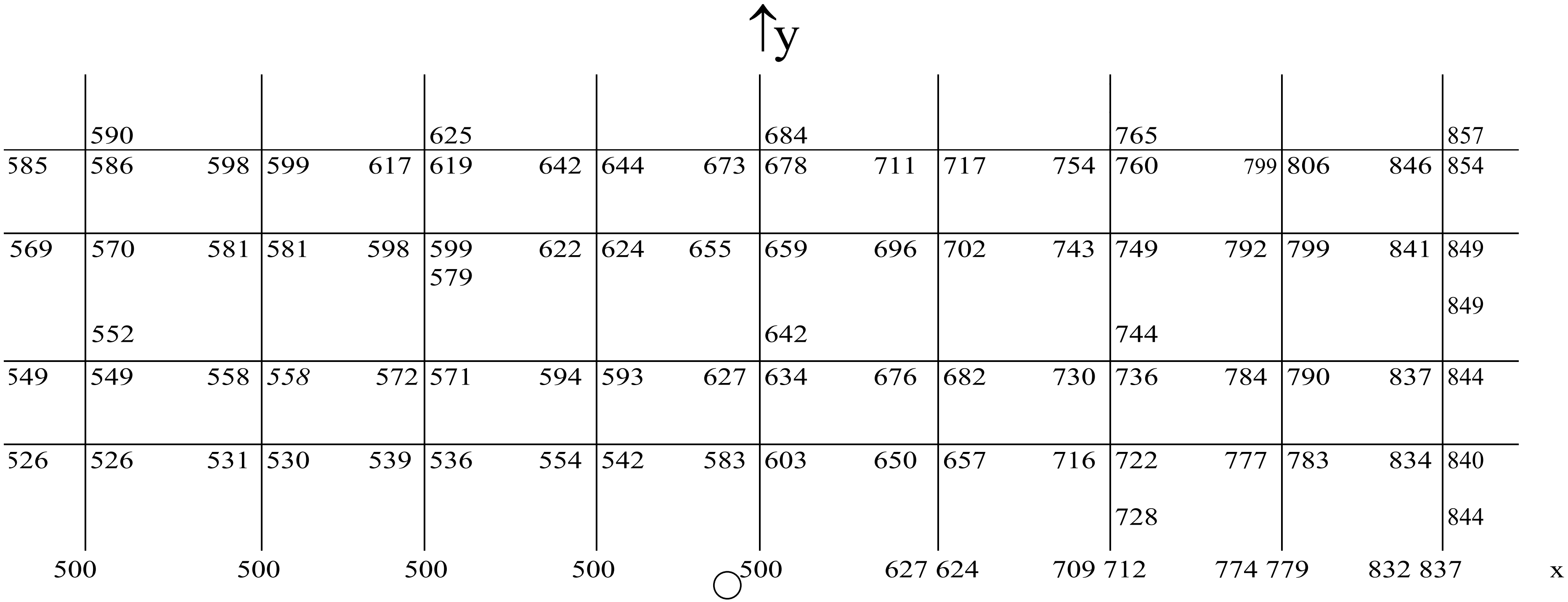


Fig. 7



Numbers due to M.Lavender [10]
(subregion - $2\pi/7 \leqq x \leqq 2\pi/7$, $0 \leqq y \leqq 2\pi/7$.)
Mesh length h = /14.

At any point P the numbers have the significance:-

	CTM[19]
F.E.Method No Singular Term	F.E.Method, $w_1(r,\theta)$ term included

Fig.8

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