ERROR BOUNDS FOR NONDEGENERATE MONOTONE LINEAR COMPLEMENTARITY PROBLEMS

by

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Computer Sciences Technical Report #787

August 1988

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Technical Report #787

August 1988

Final Revision June 1989

Abstract. Error bounds and upper Lipschitz continuity results are given for monotone linear complementarity problems with a nondegenerate solution. The existence of a nondegenerate solution considerably simplifies the error bounds compared with problems for which all solutions are degenerate. Thus when a point satisfies the linear inequalities of a nondegenerate complementarity problem, the residual that bounds the distance from a solution point consists of the complementarity condition alone, whereas for degenerate problems this residual cannot bound the distance to a solution without adding the square root of the complementarity condition to it. This and other simplified results are a consequence of the polyhedral characterization of the solution set as the intersection of the feasible region $\{z|Mz+q \geq 0,\ z \geq 0\}$ with a single linear affine inequality constraint.

Key Words: Linear complementarity, error bounds, Lipschitz continuity

Abbreviated Title: Error Bounds for LCP's

[1)] This material is based on research supported by National Science Foundation Grants CCR-8723091 and DCR-8521228 and Air Force Office of Scientific Research Grant AFOSR-86-0172.

1. Introduction

We consider the classical monotone linear complementarity problem [2]

$$(1.1) \qquad Mx + q \geq 0,\ x \geq 0,\ x(Mx + q) = 0$$

where M is a given $n \times n$ real positive semidefinite matrix and q is a given vector in the n-dimensional real space R^n. For a positive semidefinite M, the map $x \to Mx+q$ is monotone. It is well known that under this assumption the linear complementarity problem (1.1) is solvable whenever it is feasible, that is whenever there is an x in R^n satisfying $Mx+q \geq 0,\ x \geq 0$. An underlying assumption of this paper (see Definition 2.1 below) is that **some** solution $\hat{x}$ of (1.1) is nondegenerate, that is

$$(1.2) \qquad \hat{x} + M\hat{x} + q > 0$$

This assumption is automatically satisfied [4, Corollary 2A] when (1.1) is feasible and M is skew-symmetric, that is $M + M^T = 0$, which is the case when the linear complementarity problem (1.1) represents a pair of feasible standard dual linear programs. The nondegeneracy assumption considerably simplifies the polyhedral characterization of the solution set of the monotone linear complementarity problem (Lemma 2.2) over previous polyhedral characterizations [1, 7]. The simplification consists in characterizing the solution set of (1.1), when it has **some** nondegenerate solution, as the solution set $\bar{S}$ (see (2.3) below) of the complementarity problem (1.1) linearized around **any** of its solution points. The first principal consequence of this simplified characterization (Theorem 2.6) is a bound on the distance between any point x in R^n and a point $\bar{x}(x)$ in the solution set of (1.1) in terms of residuals determined by only the 3 terms of (1.1) defining the complementarity problem. This is a considerably simpler residual than that of [11, Theorem 2.7] for the degenerate case. When the point x satisfies the first two inequalities of (1.1), the residual for the nondegenerate problem becomes merely the complementarity condition $x(Mx + q)$ (Corollary 2.7) instead of $x(Mx + q) + \left(x(Mx + q)\right)^{\frac{1}{2}}$ as must be the residual for the degenerate problem [11, Corollary 2.8]. This result leads to a finite perturbation formulation (2.23) of the least 2-norm solution of a nondegenerate monotone linear complementarity problem as a strongly convex quadratic program (Theorem 2.9) as well as its characterization as a problem with a weak sharp minimum (Theorem 2.11) and the finite termination of the corresponding Proximal Point Algorithm 2.12 which follows from [3]. Finally the error bound of Theorem 2.6 is used to obtain a global upper Lipschitz continuity result (Corollary 2.15) for nonnegative perturbations of q, as well as a local upper Lipschitz continuity for arbitrary local perturbations of q (Theorem 2.16).

A brief word about notation and some basic concepts employed. For a vector x in the n-dimensional real space R^n, x_+ will denote the vector in R^n with components $(x_+)_i := \max\{x_i, 0\}$, $i = 1, \ldots, n$. For a norm $\|x\|_\beta$ on R^n, $\|x\|_{\beta^*}$ will denote the dual norm [5, 12] on R^n, that is $\|x\|_{\beta^*} := \max_{\|y\|_\beta = 1} xy$, where xy denotes the scalar product $\sum_{i=1}^n x_i y_i$. The generalized Cauchy-Schwarz inequality $\|xy\| \leq \|x\|_\beta \cdot \|y\|_{\beta^*}$, for x and y in R^n, follows immediately from this definition of the dual norm. For $1 \leq p,\ q \leq \infty$, and $1/p + 1/q = 1$, the p-norm $\|x\|_p = (\sum_{i=1}^n |x_i|^p)^{1/p}$ for $1 \leq p < \infty$ and $\|x\|_p = \max_{1 \leq i \leq n} |x_i|$ for $p = \infty$, and the q-norm are dual norms on R^n [12]. For an $m \times n$ real matrix A signified by $A \in R^{m \times n}$, A_i denotes the ith row, while A^T denotes the transpose. $\|A\|_\beta$ denotes the matrix norm [12] subordinate to the vector norm $\|\cdot\|_\beta$, that is $\|A\|_\beta = \max_{\|x\|_\beta = 1} \|Ax\|_\beta$. The consistency condition $\|Ax\|_\beta \leq \|A\|_\beta \|x\|_\beta$ follows immediately from this definition of a matrix norm. A monotonic norm on R^n is any norm $\|\cdot\|$ on R^n such that for a, b in R^n, $\|a\| \leq \|b\|$ whenever $|a| \leq |b|$ or equivalently if $\|a\| = \||a|\|$, where $(|a|)_i = |a_i|$, $i = 1, \ldots, n$ [5, p. 47]. Any p-norm for $\infty \geq p \geq 1$ is monotonic [12, p. 52]. A vector of ones in any real space will be denoted by e. The identity matrix of any order will be denoted by I. The nonnegative orthant in R^n will be denoted by R^n_+.

2. Results

We begin with some simple preliminary results for the linear complementarity problem (1.1). For that purpose we make the following definitions of the feasible set

$$X := \{x | Mx + q \geq 0,\ x \geq 0\}, \tag{2.1}$$

the solution set

$$\bar{X} := \{x | Mx + q \geq 0,\ x \geq 0,\ x(Mx + q) = 0\}, \tag{2.2}$$

and the polyhedral set obtained by linearizing the linear complementarity problem (1.1) around any of its solution points

$$\bar{S} := \{x | Mx + q \geq 0,\ x \geq 0,\ x(M\bar{x} + q) + \bar{x}(Mx + q) \leq 0\} \text{ for any } \bar{x} \in \bar{X} \tag{2.3}$$

The set $\bar{S}$ will characterize the solution set $\bar{X}$ when a nondegenerate solution exists. We note immediately that, for M positive semidefinite, the definition $\bar{S}$ is independent of the specific choice of $\bar{x}$ in $\bar{X}$, because by [7, Corollary 2]

$$\bar{X} = \bar{T} := \{x | Mx + q \geq 0,\ x \geq 0,\ q(x - \bar{x}) = 0,\ (M + M^T)(x - \bar{x}) = 0\},\ \bar{x} \in \bar{X} \tag{2.4}$$

Adler and Gale [1], who gave the first polyhedral characterization of $\bar{X}$, wrote $\bar{T}$ in a slightly different but equivalent form as follows

$$\begin{aligned} \bar{T}_1 := \{x | Mx + q \geq 0,\ x \geq 0,\ x(M\bar{x} + q) + \bar{x}(Mx + q) = 0, \\ (M + M^T)(x - \bar{x}) = 0\},\ \text{for some } \bar{x} \in \bar{X} \end{aligned} \tag{2.5}$$

We also note that if we define

$$f(x) := x(Mx + q) \tag{2.6}$$

then $\bar{S}$ can be written in the equivalent form

$$\bar{S} = \{x | Mx + q \geq 0,\ x \geq 0,\ \nabla f(\bar{x})(x - \bar{x}) \leq 0\},\ \bar{x} \in \bar{X} \tag{2.7}$$

2.1 Definition If $\hat{x} + M\hat{x} + q > 0$ for some $\hat{x} \in \bar{X}$ then the linear complementarity problem (1.1) is said to be **nondegenerate**; otherwise it is called **degenerate**.

We note that in the definition (2.3) of $\bar{S}$, $\bar{x}$ need not be taken as a nondegenerate solution. We are now ready to derive our first result.

2.2 Lemma (Polyhedral characterization of $\bar{X}$ for nondegenerate problems) Let (1.1) be nondegenerate and let M be positive semidefinite. Then

$$\bar{X} = \bar{S} \tag{2.8}$$

Proof ($\bar{S} \subset \bar{X}$) Let $x \in \bar{S}$. Then since (1.1) is nondegenerate it follows by (2.3) and (2.4) that we can replace $\bar{x}$ in the definition of $\bar{S}$ by a nondegenerate solution point $\hat{x}$. Thus we have

$$Mx + q \geq 0,\ x \geq 0,\ x(M\hat{x} + q) + \hat{x}(Mx + q) = 0 \tag{2.9}$$

for some $\hat{x} \in \bar{X}$ such that $\hat{x} + M\hat{x} + q > 0$, and consequently it follows from the equality of (2.9) that $x(Mx + q) = 0$. Hence $x \in \bar{X}$.
($\bar{X} \subset \bar{S}$) Let $x \in \bar{X}$ and let $\bar{x} \in \bar{X}$. Then

$$\begin{aligned} 0 = x(Mx + q) &= \bar{x}(Mx + q) + x(M\bar{x} + q) + (x - \bar{x})M(x - \bar{x}) \\ &\geq \bar{x}(Mx + q) + x(M\bar{x} + q) \end{aligned}$$

Hence $\bar{x}(Mx + q) + x(M\bar{x} + q) \leq 0$ and $x \in \bar{S}$. ∎

2.3 Remark Note that the equality (2.8) does not hold without the nondegeneracy assumption, as evidenced by the simple example: $M = 1$, $q = 0$, for which $\bar{X} = \{0\}$ and $\bar{S} = R^1_+$. In fact for degenerate problems one must resort to the characterizations (2.4) or (2.5) [10, 11]. Thus for possibly degenerate monotone linear complementarity problems we have

$$\bar{X} = \bar{T} = \bar{T}_1 \subseteq \bar{S} \tag{2.10}$$

whereas for nondegenerate monotone linear complementarity problems we have

$$\bar{X} = \bar{T} = \bar{T}_1 = \bar{S} \tag{2.11}$$

Thus the solution set $\bar{X}$ for nondegenerate monotone problems can be characterized by the intersection of the feasible region X with the single affine linear constraint of

(2.7), $\nabla f(\bar{x})(x-\bar{x}) \leq 0$, for some $\bar{x} \in \bar{X}$. This leads to a considerable simplification of the error bounds and Lipschitz continuity results given here compared with those for the possibly degenerate case [10, 11].

2.4 Remark For a solvable monotone linear complementarity problem, relation (2.4) shows that the solution set is completely characterized by $n+1$ constants d, α defined as follows

$$\alpha := q\bar{x},\ d := (M+M^T)\bar{x}+q, \ \text{ for any } \bar{x} \in \bar{X} \tag{2.12}$$

Note that in view of (2.4), α and d are independent of the choice of $\bar{x}$ in $\bar{X}$ and are constants of the problem depending on M and q only. With the definitions (2.12) the sets $\bar{S}$, $\bar{T}$ and $\bar{T}_1$ can be rewritten as follows.

$$\bar{S} = \{x | Mx+q \geq 0,\ x \geq 0,\ dx+\alpha \leq 0\} \tag{2.13}$$

$$\bar{T} = \{x | Mx+q \geq 0,\ x \geq 0,\ qx=\alpha,\ (M+M^T)x = d-q\} \tag{2.14}$$

$$\bar{T}_1 = \{x | Mx+q \geq 0,\ x \geq 0,\ dx+\alpha = 0,\ (M+M^T)x = d-q\} \tag{2.15}$$

We are now ready to give our first error bound.

2.5 Propostion (Error bound for nondegenerate monotone LCP) Let (1.1) be a nondegenerate monotone LCP. For each x in R^n there exists an $\bar{x}(x)$ in $\bar{X}$ such that

$$\left\|x-\bar{x}(x)\right\|_\infty \leq \mu_\beta(M,q) \cdot \left\|(-Mx-q,\ -x,\ dx+\alpha)_+\right\|_\beta \tag{2.16}$$

where $\|\cdot\|_\beta$ is some norm on R^{2n+1} with dual norm $\|\cdot\|_{\beta^*}$, α and d are defined by (2.12) and

$$\mu_\beta(M,q) := \max_{(u,v,\xi)\in R_+^{2n+1}} \left\{ \left\|u,v,\xi\right\|_{\beta^*} \;\middle|\; \begin{array}{l} \left\|M^Tu+v-d\xi\right\|_1 = 1 \\ \text{Columns of } [M^T \ \ I \ \ d] \\ \text{corresponding to nonzero} \\ \text{elements of } (u,v,\xi) \\ \text{are linearly independent} \end{array} \right\} \tag{2.17}$$

Proof Just apply Theorem 2.2′ of [10] to $\bar{S}$ as defined by (2.13) and use Lemma 2.2 to conclude that $\bar{x}(x)$ is in $\bar{X}$. ∎

The condition constant $\mu_\beta(M,q)$ of (2.17), although a difficult quantity to compute, is a useful generalization of the concept of the norm of the inverse of a square matrix. For example it allows us (by Theorem 2.6 and Corollary 2.7 below) to determine which of two arbitrary points is closer to the solution set $\bar{X}$.

Note that the residual term of (2.16) contains the unknown quantity $(dx+\alpha)_+$, which precludes its practical use as an error bound. However, this term can be easily bounded as was done in [11, Lemma 2.6] by an easily computed residual as follows

$$\begin{aligned}(dx+\alpha)_+ &= \big(x(M+M^T)\bar{x} + qx + q\bar{x}\big)_+ \\ &= \big(x(Mx+q) - (x-\bar{x})M(x-\bar{x})\big)_+ \le \big(x(Mx+q)\big)_+\end{aligned}$$

We thus have the following more useful bound which involves a residual that can be calculated for any point in R^n.

2.6 Theorem (Error bound for nondegenerate monotone LCP) Let (1.1) be a nondegenerate monotone LCP. For each x in R^n there exists an $\bar{x}(x)$ in $\bar{X}$ such that

$$\|x-\bar{x}(x)\|_\infty \le \mu_\beta(M,q)\cdot \big\|\big(-Mx-q,\ -x,\ x(Mx+q)\big)_+\big\|_\beta \tag{2.18}$$

where $\mu_\beta(M,q)$ is defined by (2.17).

It is interesting to note the considerably simpler error bound (2.18) above compared with that of Theorem 2.7 of [11] for the degenerate case. When x is feasible the error bound (2.18) simplifies further as follows.

2.7 Corollary (Error bound for feasible points of nondegenerate monotone LCP) Let (1.1) be nondegenerate. For each $x \in X$ there exists an $\bar{x}(x)$ in $\bar{X}$ such that

$$\|x-\bar{x}(x)\|_\infty \le \mu_\infty(M,q)\cdot x(Mx+q) \tag{2.19}$$

where $\mu_\infty(M,q)$ is given by (2.17).

Note that (2.19) is valid for all $\mu_\beta(M,q)$ for any norm $\|\cdot\|_\beta$. We chose $\mu_\infty(M,q)$ because it is the infimum of the p-norms, for $1 \le p \le \infty$.

2.8 Remark For degenerate problems the bound (2.19) is not valid. This can be seen from the example [11, Example 2.9]

$$M = \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}, \quad q = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \quad \bar{X} = \{0\} \tag{2.20}$$

For the unique solution $\bar{x} = (0,0)$, $M\bar{x} + q = (0,1)$ and hence the problem is degenerate. For $x(\varepsilon) := (\varepsilon,\, \varepsilon^2)$, $0 \le \varepsilon \le 1$, we have

$$\frac{\left\|x(\varepsilon) - 0\right\|_\infty}{x(\varepsilon)(Mx(\varepsilon) + q)} = \frac{\varepsilon}{2\varepsilon^2 + \varepsilon^4} \to \infty \quad \text{as} \quad \varepsilon \to 0 \tag{2.21}$$

Hence the bound (2.19) fails. For such degenerate problems the bound (2.19) must be replaced by [11, Corollary 2.8]

$$\left\|x - \bar{x}(x)\right\|_\infty \le \tau_2(M,q)\left(x(Mx+q) + \left(x(Mx+q)\right)^{\frac{1}{2}}\right) \tag{2.22}$$

where $\tau_2(M,q)$, defined in [11, Equation (2.5)], is a different constant from $\mu_\beta(M,q)$.

A useful consequence of Corollary 2.7 is that the least 2-norm solution of a nondegenerate monotone linear complementarity problem can be obtained by a simple regularization of the minimization problem $\min_{x \in X} f(x)$ equivalent to (1.1). We have the following.

2.9 Theorem (Least 2-norm solution of a nondegenerate monotone linear complementarity problem) Let (1.1) be nondegenerate. There exists an $\bar{\varepsilon} > 0$ such that $\forall \varepsilon \in (0, \bar{\varepsilon}]$

$$\bar{x} = \arg\min_{x \in X} f(x) + \frac{\varepsilon}{2} xx \tag{2.23}$$

where $\bar{x}$ is the unique solution of (1.1) with least 2-norm, and f and X are as defined in (2.6) and (2.1) respectively.

Proof Since the set $\bar{S}$ as given by (2.7), which is a linearization of $\min_{x \in X} f(x)$ around $\bar{x}$, is equivalent to the solution set $\bar{X}$ by Lemma 2.2, it follows that $\bar{S}$ has the same least 2-norm element as $\bar{X}$. By [8, Theorem 5], it follows that the least 2-norm solution of (1.1) is given by (2.23). ∎

In fact Corollary 2.7 leads to a more fundamental result for the equivalent formulation $\min_{x \in X} f(x)$ of the linear complementarity problem (1.1) by showing that $\min_{x \in X} f(x)$

has a weak sharp minimum [3]. An important consequence of this is that the proximal point algorithm [14] terminates in a finite number of steps just as in the case of linear programming [13].

2.10 Definition Let $f : R^n \to R$ and $X \subset R^n$ be convex, and let the problem $\min_{x \in X} f(x)$ have a nonempty closed solution set $\bar{X}$. The problem is said to have a **weak sharp minimum** if there exists a positive constant γ such that

$$f(x) - f(\bar{x}(x)) \geq \gamma \|x - \bar{x}(x)\| \quad \forall x \in X$$

where $\bar{x}(x) \in \arg\min_{z \in \bar{X}} \|z - x\|$ and $\|\cdot\|$ is some norm on R^n.

In [9] it was shown that all solvable linear programs have weak sharp minima. We now show that nondegenerate monotone linear complementarity problems also have weak sharp minima when formulated as $\min_{x \in X} f(x)$.

2.11 Theorem (Nondegenerate monotone LCP as a weak sharp minimum) Let (1.1) be nondegenerate. Then the problem $\min_{x \in X} f(x)$, with f and X as defined in (2.6) and (2.1) respectively, has a weak sharp minimum.

Proof By Corollary 2.7 we have that for $x \in X \setminus \bar{X}$, there exists an $\bar{x}(x) \in \bar{X}$ such that

$$\mu_\infty(M, q)^{-1} \|x - \bar{x}(x)\|_\infty \leq x(Mx + q) = f(x) - f(\bar{x}(x)). \quad \blacksquare$$

A direct consequence of Theorem 2.11 is that the following proximal point algorithm for (1.1) converges in a finite number of steps [3].

2.12 Proximal Point Algorithm For a bounded sequence of positive numbers $\{\varepsilon_i\}$ and $x^0 \in R^n_+$,

$$x^{i+1} := \arg\min_{x \in X} f(x) + \frac{\varepsilon_i}{2} \left\| x - x^i \right\|_2^2 \quad i = 0, 1, \ldots$$

where f and X are as defined in (2.6) and (2.1) and M is positive semidefinite.

2.13 Finite Termination of Proximal Point Algorithm 2.12 For a nondegenerate problem (1.1) there exists $k \geq 1$ such that x^k, as determined by the Proximal Point Algorithm 2.12, solves (1.1). Furthermore for each $x^0 \in R^n_+$ there exists an $\varepsilon_0 > 0$ sufficiently small such that x^1 solves (1.1).

Proof By Theorem 9, Corollary 20 and Theorem 22 of [3]. ∎

From a computational point of view, both the proximal point subproblem and the least 2-norm problem (2.23) are strongly convex problems with unique solutions and hence should be more tractable than the original problem $\min_{x \in X} f(x)$. The least 2-norm formulation has the advantage of solving the problem only once, provided ε is chosen sufficiently small. In contrast the proximal point algorithm requires the solution of the subproblem a finite number of times. The advantage of the latter algorithm is that the sequence $\{\varepsilon_i\}$ is arbitrary, but its choice certainly affects the number of subproblems solved. For problems for which the ε of the least 2-norm problem (2.23) needs to be extremely small, the proximal point algorithm is preferable. Unfortunately this is not easy to determine a priori.

We give now some upper Lipschitz continuity results for nondegenerate monotone linear complementarity problems. For that purpose it is convenient to let LCP(q) denote the linear complementarity problem (1.1) with a fixed matrix M and a given vector q.

2.14 Proposition (Global upper Lipschitz continuity) Let M be positive semidefinite, let LCP(q^1) be nondegenerate and let LCP(q^2) be solvable. There exist solutions x^1, x^2 of LCP(q^1) and LCP(q^2) respectively such that

$$\left\|x^2 - x^1\right\|_\infty \leq \mu_\beta(M, q^1) \cdot \left\|\left(q^2 - q^1,\ 0,\ x^2(q^1 - q^2)\right)_+\right\|_\beta \tag{2.24}$$

where $\mu_\beta(M, q^1)$ is defined by (2.17) and $\|\cdot\|_\beta$ is any monotonic norm on R^{2n+1}.

Proof By Theorem 2.6 for each x^2 (solving LCP(q^2)) there exists a solution x^1 of LCP(q^1) such that

$$\begin{aligned}
\left\|x^2 - x^1\right\|_\infty &\leq \mu_\beta(M, q^1) \cdot \left\|\left(-Mx^2 - q^1,\ -x^2,\ x^2(Mx^2 + q^1)\right)_+\right\|_\beta \\
&= \mu_\beta(M, q^1) \cdot \left\|\left(-Mx^2 - q^1 + q^2 - q^2,\ 0,\ x^2(Mx^2 + q^1 + q^2 - q^2)\right)_+\right\|_\beta \\
&\leq \mu_\beta(M, q^1) \cdot \left\|\left(q^2 - q^1,\ 0,\ x^2(q^1 - q^2)\right)_+\right\|_\beta
\end{aligned}$$

where the last inequality follows from the norm-monotonicity and the fact x^2 solves LCP(q^2). ∎

2.15 Corollary (Global upper Lipschitz continuity for nonnegative perturbations of nondegenerate monotone LCP's) If the perturbation of Proposition 2.14 is nonnegative, that is

$$q^2 - q^1 \geq 0 \tag{2.25}$$

then

$$\left\|x^2 - x^1\right\|_\infty \leq \mu_\beta(M, q^1)\left\|(q^2 - q^1,\ 0,\ 0)\right\|_\beta \tag{2.26}$$

A local upper Lipschitz continuity result can be obtained for arbitrary local perturbations $q^1 - q^2$ if we assume that LCP(q^1) is nondegenerate and has a bounded solution set or if it has a nondegenerate vertex solution (which implies boundedness of the solution set).

2.16 Theorem (Local upper Lipschitz continuity for nondegenerate monotone LCP's with bounded solution sets) Let M be positive semidefinite, let $\|\cdot\|_\beta$ be a monotonic norm on R^{2n+1} and let either (i) LCP(q^1) have a bounded solution set containing some nondegenerate solution or (ii) LCP(q^1) have a nondegenerate vertex solution. Then there exist a $\gamma > 0$ and an $\varepsilon > 0$ such that for all $\left\|(q^2 - q^1,\ 0,\ 0)\right\|_\beta \leq \varepsilon$, LCP$(q^2)$ is solvable and there exist solutions x^1, x^2 of LCP(q^1) and LCP(q^2) respectively such that

$$\left\|x^2 - x^1\right\|_\infty \leq (1+\gamma)\mu_\beta(M, q^1) \cdot \left\|(q^1 - q^2,\ 0,\ 0)\right\|_\beta \tag{2.27}$$

Proof By Corollary 1 and Theorem 2 of [6], assumption (ii) of our theorem here implies assumption (i). By Theorem 2 xvi of [6], there exists an $\varepsilon > 0$ and $\gamma > 0$ such that for each $\left\|(q^2 - q^1,\ 0,\ 0)\right\|_\beta \leq \varepsilon$, LCP$(q^2)$ is solvable and each solution x^2 of LCP(q^2), satisfies $\left\|(0,\ 0,\ \left\|x^2,\ 0,\ 0\right\|_{\beta *})\right\|_\beta \leq \gamma$. By Proposition 2.14 above we have that for $\left\|(q^2 - q^1,\ 0,\ 0)\right\|_\beta \leq \varepsilon$ there exist solutions x^1, x^2 of LCP(q^1) and LCP(q^2) respectively such that

$$\begin{aligned}\left\|x^2 - x^1\right\|_\infty &\leq \mu_\beta(M, q^1) \cdot \left\|(q^2 - q^1,\ 0,\ x^2(q^1 - q^2))_+\right\|_\beta \\ &\leq \mu_\beta(M, q^1) \cdot \left[\left\|(q^2 - q^1,\ 0,\ 0)\right\|_\beta + \gamma\left\|(q^1 - q^2,\ 0,\ 0)\right\|_\beta\right] \quad \blacksquare\end{aligned}$$

Acknowledgement

I am grateful to the referees for many helpful comments.

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