Error Correction for Continuous Quantum Variables

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We propose an error correction coding algorithm for continuous quantum variables. We use this algorithm to construct a highly efficient 5-wave-packet code which can correct arbitrary single wave-packet errors. We show that this class of continuous variable codes is robust against imprecision in the error syndromes. A potential implementation of the scheme is presented. [S0031-9007(98)05865-7]

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Quantum computers hold the promise for efficiently factoring large integers [1]. However, to do this beyond a most modest scale they will require quantum error correction [2]. The theory of quantum error correction is already well studied in two-level or spin-$\frac{1}{2}$ systems (in terms of qubits or quantum bits) [2–7]. Some of these results have been generalized to higher-spin systems [8–11]. This work applies to discrete systems like the hyperfine levels in ions but is not suitable for systems with continuous spectra, such as unbound wave packets. Simultaneously with this paper, Lloyd and Slotine present the first treatment of a quantum error correction code for continuous quantum variables [12], demonstrating a 9-wave-packet code in analogy with Shor's 9-qubit coding scheme [2]. Such codes hold exciting prospects for the *complete* manipulation of quantum systems, including both discrete and continuous degrees of freedom, in the presence of inevitable noise [13].

In this Letter we consider a highly efficient and compact error correction coding algorithm for continuous quantum variables. As an example, we construct a 5-wave-packet code which can correct arbitrary single-wave-packet errors. We show that such continuous variable codes are robust against imprecision in the error syndromes and discuss potential implementation of the scheme. This paper is restricted to one-dimensional wave packets which might represent the wave function of a nonrelativistic one-dimensional particle or the state of a single polarization of a transverse mode of electromagnetic radiation. We shall henceforth refer to such descriptions by the generic term wave packets [14].

Rather than starting from scratch we shall use some of the theory that has already been given for error correction on qubits. In particular, Steane has noted that the Hadamard transform,

$$\hat{H} = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix}, \tag{1}$$

maps phase flips into bit flips and can therefore be used to form a class of quantum error correction codes that consist of a pair of classical codes, one for each type of "flip" [3]. This mapping between phase and amplitude bases is achieved with a rotation about the y axis by $\pi/2$ radians in the Bloch sphere representation of the state. In analogy, the position and momentum bases of a continuous quantum state may be transformed into each other by $\pi/2$ rotations in phase space. This transition is implemented by substituting the Hadamard rotation in the Bloch sphere by a Fourier transform between position and momentum in phase space. This suggests that we could develop the analogous quantum error correction codes for continuous systems [15].

We shall find it convenient to use a units-free notation where

$$\begin{aligned} \text{position} &= x \times (\text{scale length}), \\ \text{momentum} &= p/(\text{scale length}), \end{aligned} \tag{2}$$

where x is a scaled length, p is a scaled momentum, and we have taken $\hbar = \frac{1}{2}$. (We henceforth drop the modifier "scaled.") The position basis eigenstates $|x\rangle$ are normalized according to $\langle x'|x\rangle = \delta(x' - x)$ with the momentum basis given by

$$|x\rangle = \frac{1}{\sqrt{\pi}} \int dp\, e^{-2ixp} |p\rangle. \tag{3}$$

To avoid confusion we shall work in the position basis throughout and so define the Fourier transform as an active operation on a state by

$$\hat{\mathcal{F}}|x\rangle = \frac{1}{\sqrt{\pi}} \int dy\, e^{2ixy} |y\rangle, \tag{4}$$

where both x and y are variables in the position basis. Note that Eqs. (3) and (4) correspond to a change of representation and a physical change of the state, respectively.

In addition to the Fourier transform we shall require an analog to the bit-wise exclusive-OR (XOR) gate for continuous variables. The XOR gate has many interpretations including controlled-NOT gate, addition modulo 2, and parity associated with it. Of these interpretations the natural generalization to continuous variables is addition without a cyclic condition, which maps

$$|x, y\rangle \to |x, x + y\rangle. \tag{5}$$

By removing the cyclic structure of the XOR gate we have produced a gate which is no longer its own inverse. Thus, in addition to the Fourier transform and this generalized XOR gate, we include their inverses on our list of useful gates. This generalized XOR operation performs translations over the entire real line, which are related to the infinite additive group on $\mathbb{R}$. The characters χ of this group satisfy the multiplicative property $\chi(x + y) = \chi(x)\chi(y)$ for all $x, y \in \mathbb{R}$ and obey the sum rule

$$\frac{1}{\pi} \int_{-\infty}^{\infty} dx\, \chi(x) = \delta(x), \tag{6}$$

where $\chi(x) = e^{2ix}$. Interestingly, this sum rule has the same form as that found by Chau in higher-spin codes [10]. Once we have recognized the parallel, it is sufficient to take the code of a spin-$\frac{1}{2}$ system as a basis for our continuous-variable code.

Based on these parallel group properties, we are tempted to speculate a much more general and fundamental relation: We conjecture that n-qubit error correction codes can be paralleled with n-wave-packet codes by replacing the discrete-variable operations (Hadamard transform and XOR gate) by their continuous-variable analogs (Fourier transform, generalized XOR, and their inverses). As a last remark before embarking on the necessary substitutions (in a specific example), we point out that the substitution conjecture is only valid for qubit codes whose circuits involve only these ($\hat{H}$ and XOR) elements. We shall therefore restrict our attention to this class of codes.

An example of a suitable 5-qubit code was given by Laflamme *et al.* [16]. We show an equivalent circuit in Fig. 1 [17]. As we perform the substitutions, we must determine which qubit-XOR gates to replace with the generalized XOR and which with its inverse. To resolve this ambiguity, two conditions are imposed. First, we demand that the code retain its properties under the parity operation (on each wave packet). We conclude that either gate may be chosen for the first operation on initially zero-position eigenstates. Ambiguity remains for the last four XOR substitutions. As a second step, the necessary and sufficient condition for quantum error correction [5,6],

$$\langle x'_{\text{encode}} | \hat{\mathcal{E}}_\alpha^\dagger \hat{\mathcal{E}}_\beta | x_{\text{encode}} \rangle = \delta(x' - x)\lambda_{\alpha\beta}\,, \quad \forall\ \alpha, \beta\,, \tag{7}$$

must be met. Here $|x_{\text{encode}}\rangle$ encodes a single wave-packet's position eigenstate in a multiwave-packet state, $\hat{\mathcal{E}}_\alpha$ is a possible error that can be handled by the code, and $\lambda_{\alpha\beta}$ is a complex constant independent of the encoded states. [Condition (7) says that correctable errors do not mask the orthogonality of encoded states.]

In the case of a single-wave-packet error, for our 5-wave-packet code, it turns out that among the conditions of Eq. (7) only $\langle x'_{\text{encode}} | \hat{\mathcal{E}}_{4\alpha}^\dagger \hat{\mathcal{E}}_{5\beta} | x_{\text{encode}} \rangle$, having errors on wave packets 4 and 5, is affected by the ambiguity (see detail below). An explicit calculation of *all* the conditions shows that the circuit of Fig. 2 yields a satisfactory quantum error correction code (as do variations of this circuit due to the extra freedom with respect to the choice of operator acting on wave packets 1–3). By analogy with the results for higher-spin codes, we know that this code is optimal (though not perfect) and that no 4-wave-packet code would suffice [10]. The code thus constructed has the form

$$|x_{\text{encode}}\rangle = \frac{1}{\pi^{3/2}} \int dw\, dy\, dz\, e^{2i(wy + xz)} \times |z, y + x, w + x, w - z, y - z\rangle. \tag{8}$$

Let us demonstrate the calculation of one of the conditions specified by Eq. (7),

$$\begin{aligned}
\langle x'_{\text{encode}} | \hat{\mathcal{E}}_{4\alpha}^\dagger \hat{\mathcal{E}}_{5\beta} | x_{\text{encode}} \rangle &= \frac{1}{\pi^3} \int dw'\, dy'\, dz'\, dw\, dy\, dz\, e^{2i(wy + xz - w'y' - x'z')} \\
&\quad \times \delta(z' - z)\delta(y' - y + x' - x)\delta(w' - w + x' - x) \\
&\quad \times \langle w' - z' | \hat{\mathcal{E}}_\alpha^\dagger | w - z \rangle \langle y' - z' | \hat{\mathcal{E}}_\beta | y - z \rangle \\
&= \frac{e^{-2i(x'-x)^2}}{\pi^3} \int dw\, dy\, dz\, e^{2i(x'-x)(w+y-z)} \\
&\quad \times \langle w - x' + x - z | \hat{\mathcal{E}}_\alpha^\dagger | w - z \rangle \langle y - x' + x - z | \hat{\mathcal{E}}_\beta | y - z \rangle.
\end{aligned} \tag{9}$$

Making the replacements $w \to w + z$ and $y \to y + z$ in this last expression we obtain

$$\begin{aligned}
&= \frac{e^{-2i(x'-x)^2}}{\pi^3} \int dw\, dy\, dz\, e^{2i(x'-x)(w+y+z)} \langle w - x' + x | \hat{\mathcal{E}}_\alpha^\dagger | w \rangle \langle y - x' + x | \hat{\mathcal{E}}_\beta | y \rangle \\
&= \frac{\delta(x' - x)}{\pi^2} \int dw\, dy\, \langle w | \hat{\mathcal{E}}_\alpha^\dagger | w \rangle \langle y | \hat{\mathcal{E}}_\beta | y \rangle \equiv \delta(x' - x)\lambda_{\alpha\beta}\,.
\end{aligned} \tag{10}$$

For the other cases we find by explicit calculation, for wave packets $j \neq k$, that

$$\langle x'_{\text{encode}} | \hat{\mathcal{E}}_{j\alpha}^\dagger \hat{\mathcal{E}}_{k\beta} | x_{\text{encode}} \rangle = \delta(x' - x)\lambda_{\alpha\beta}\,. \tag{11}$$

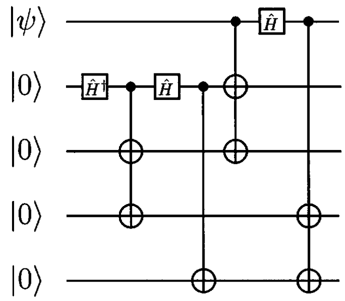

FIG. 1. Quantum error correction circuit from [17]. The qubit $|\psi\rangle$ is rotated into a 5-particle subspace by the unitary operations represented by the operations shown in this circuit. Note that the 3-qubit gates are simply pairs of XORs.

For $j = k$ this constant is found to be

$$\lambda_{\alpha\beta} = \frac{C}{\pi^2} \int dw \, \langle w | \hat{\mathcal{E}}_\alpha^\dagger \hat{\mathcal{E}}_\beta | w \rangle, \tag{12}$$

where C is formally infinite.

We shall argue that this infinity vanishes when the syndrome is read with only finite precision, which is always going to be the real situation. However, this requires us to demonstrate that our codes are robust and that for a sufficiently good precision we may correct single-wave-packet errors to any specified accuracy. In order to understand how the error syndromes are measured, let us consider a simpler code, namely, the continuous version of Shor's original 9-qubit code,

$$|x_{\text{encode}}\rangle = \frac{1}{\pi^{3/2}} \int dw \, dy \, dz \, e^{2ix(w+y+z)} \times |w, w, w, y, y, y, z, z, z\rangle, \tag{13}$$

where parity alone removes all ambiguity. (This code has been independently obtained by Lloyd and Slotine [12].) Since this 9-wave-packet code corrects position errors and momentum errors separately, it is sufficient to study the subcode

$$|x_{\text{encode}}\rangle = |x, x, x\rangle, \tag{14}$$

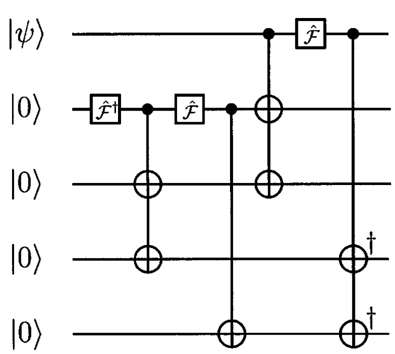

FIG. 2. This "circuit" unitarily maps a one-dimensional single-wave-packet state $|\psi\rangle$ into a 5-wave-packet error correction code. Here the auxiliary wave packets $|0\rangle$ are initially zero-position eigenstates. For degrees of freedom larger than qubits the ideal XOR is not its own inverse; here the daggers on the XOR gates represent the inverse operation.

designed to correct position errors on a single wave packet. The most general position error (on a single wave packet) is given by some function of the momentum of that system $\hat{\mathcal{E}}(\hat{p})$ and need not be unitary on the code subspace [Eq. (7)]. The action of such an error on a wave packet may be written in the position basis as

$$\hat{\mathcal{E}}(\hat{p}) |x\rangle = \frac{1}{\pi} \int dy \, dp \, e^{2ip(y-x)} \mathcal{E}(p) \, |y\rangle = \int dy \, \tilde{\mathcal{E}}(y) \, |x - y\rangle, \tag{15}$$

where $\tilde{\mathcal{E}}(x)$ is the Fourier transform of $\mathcal{E}(p)$. Thus, the most general position error looks like a convolution of the wave packet's ket with some unknown (though not completely arbitrary) function. Suppose this error occurs on wave packet 1 in the repetition code (14). Further, let us use auxiliary wave packets (so-called ancillae) and compute the syndrome as shown in Fig. 3, then the resulting state may be written as

$$\int dy \, \tilde{\mathcal{E}}(y) \, |x - y, x, x, -y, 0, y\rangle. \tag{16}$$

Everything up till now has been unitary and assumed ideal. Now measure the syndrome: Ideally, it would be $\{-y, 0, y\}$ collapsing the wave packet for a specific y. Correcting the error is now easy, because we know the location, value, and sign of the error. Shifting the first wave packet by the amount y retrieves the correctly encoded state $|x, x, x\rangle$. Note that this procedure uses only very simple wave-packet gates: The comparison stage is done *classically*, in contrast to the scheme of Lloyd and Slotine, where the comparison is performed at the amplitude level and involves significantly more complicated interactions [12].

It is now easy to see what imprecise measurements of the syndromes will do. Suppose each measured value of a syndrome s_j' is distributed randomly about the true value s_j according to the distribution $p_{\text{meas}}(s_j' - s_j)$. We find two conditions for error correction to proceed smoothly. First, $p_{\text{meas}}(x)$ must be narrow compared to any important length scales in $\tilde{\mathcal{E}}(x)$. This guarantees that the

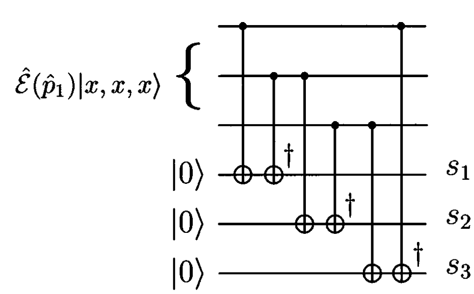

FIG. 3. Syndrome calculation and measurement: A state with a single-wave-packet position error (here on wave packet 1) enters, and the differences of each pair of positions are computed. The syndrome $\{s_1, s_2, s_3\}$ may now be directly measured in the position basis.

chance for "correcting" the wrong wave packet is negligible and reduces the position-error operator to an uninteresting prefactor. If the original unencoded state had been $\int dx\, \psi(x)\, |x\rangle$, then after error correction we would obtain the mixed state

$$\int dx'\, dx\, dz\, \psi(x)\, \psi^*(x') p_{\rm meas}(z) \times |x - z, x, x\rangle\langle x' - z, x', x'| . \qquad (17)$$

Thus, unless $p_{\rm meas}(x)$ is *also* narrow compared to any important length scales in $\psi(x)$, decoherence will appear in the off-diagonal terms for wave packet 1 of the corrected state (17). This second condition is also seen in the quantum teleportation of continuous variables due to inaccuracies caused by measurement [13]. These conditions roughly match those described by Lloyd and Slotine [12]. We note that any syndrome imprecision will degrade the encoded states, although this precision may be improved by repeated measurements of the syndromes. For our 5-wave-packet example (8), syndromes consist of sums of two or more wave-packet positions or momenta and are measured similarly.

It should be noted that Chau's higher-spin code [10] could have been immediately taken over into a quantum error correction code for continuous quantum variables in accordance with our substitution procedure. However, we have produced an equivalent code with a more efficient circuit prescription: Whereas Chau gives a procedure for constructing his higher-spin code using nine generalized XOR operations, the circuit in Fig. 2 requires only seven such gates or their inverses. In fact, we could run this substitution backwards to obtain a cleaner 5-particle higher-spin code based on Eq. (8).

In order to consider potential implementations of the above code let us restrict our attention to a situation where the wave packets are sitting in background harmonic-oscillator potentials. By the virial theorem the form of a wave packet in such a potential is preserved up to a trivial rotation in phase space with time. The two operations required may be implemented simply as follows: The rotation in phase space, Eq. (4), may be obtained by delaying the phase of one wave packet relative to the others, and the XOR operation, Eq. (5), should be implemented via a quantum nondemolition coupling. There exists extensive experimental literature on these operations both for optical fields and for trapped ions [13,18–21].

The conjecture put forth in this Letter leads to a simple, two-step design of error correction codes for continuous quantum variables. According to this conjecture, any qubit code, whose circuit operations include only a specific Hadamard transformation, its inverse, and the ideal XOR, may be translated to a continuous quantum-variable code, by substituting these operators with their continuous analogs and then imposing two criteria—parity invariance and the error-correction condition—which remove any ambiguities in the choice of operators. We demonstrate the success of this coding procedure in two examples (one based on Shor's 9-qubit code [2], and a second based on a variation of the Laflamme *et al.* 5-qubit code [16,17]). The 5-wave-packet code presented here is the optimal continuous encoding of a single one-dimensional wave packet that protects against arbitrary single-wave-packet errors. We show that this code (and, in fact, the entire class of codes derived in this manner) is robust against imprecision in the error syndromes. The potential implementation of the proposed class of circuits in optical-field and ion-trap setups is an additional incentive for further investigation of the robust manipulation of continuous quantum variables.

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