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# ERROR ESTIMATES FOR THE NUMERICAL APPROXIMATION OF BOUNDARY SEMILINEAR ELLIPTIC CONTROL PROBLEMS

Eduardo Casas, Mariano Mateos, Fredi Tröltzsch

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**AMS subject classifications.** 49K20, 49N10, 90C46, 49M20

# ERROR ESTIMATES FOR THE NUMERICAL APPROXIMATION OF BOUNDARY SEMILINEAR ELLIPTIC CONTROL PROBLEMS *

EDUARDO CASAS†, MARIANO MATEOS‡, AND FREDI TRÖLTZSCH§

**Abstract.** We study the numerical approximation of boundary optimal control problems governed by semilinear elliptic partial differential equations with pointwise constraints on the control. The analysis of the approximate control problems is carried out. The uniform convergence of discretized controls to optimal controls is proven under natural assumptions by taking piecewise constant controls. Finally, error estimates are established.

**Key words.** Boundary control, semilinear elliptic equation, numerical approximation, error estimates

**AMS subject classifications.** 49J20, 49K20, 49M05, 65K10

**1. Introduction.** With this paper, we continue the discussion of error estimates for the numerical approximation of optimal control problems we have started for semilinear elliptic equations and distributed controls in [1]. The case of distributed control is the easiest one with respect to the mathematical analysis. In [1] it was shown that, roughly speaking, the distance between a locally optimal control $\bar{u}$ and its numerical approximation $\bar{u}_h$ has the order of the mesh size $h$ in the $L^2$-norm and in the $L^\infty$-norm. This estimate holds for a finite element approximation of the equation by standard piecewise linear elements and piecewise constant control functions.

The analysis for boundary controls is more difficult, since the regularity of the state function is lower than that for distributed controls. Moreover, the internal approximation of the domain causes problems. In the general case, we have to approximate the boundary by a polygon. This requires the comparison of the original control that is located at the boundary $\Gamma$ and the approximate control that is defined on the polygonal boundary $\Gamma_h$. Moreover, the regularity of elliptic equations in domains with corners needs special care. To simplify the analysis, we assume here that $\Omega$ is a polygonal domain of $\mathbb{R}^2$. Though this makes the things easier, the lower regularity of states in polygonal domains complicates, together with the presence of nonlinearities, the analysis.

Another novelty of our paper is the numerical confirmation of the predicted error estimates. We present two examples, where we know the exact solutions. The first one is of linear-quadratic type, while the second one is semilinear. We are able to verify our error estimates quite precisely.

Let us mention some further papers related to this subject. The case of linear-quadratic elliptic control problems by finite elements was discussed in early papers by Falk [9], Geveci [10] and Malanowski [16], and Arnautu and Neittaanmäki [2], who already proved the optimal error estimate of order $h$ in the $L^2$-norm. In [16], also the case of piecewise linear control functions is addressed.

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†Dpto. de Matemática Aplicada y Ciencias de la Computación, E.T.S.I. Industriales y de Telecomunicación, Universidad de Cantabria, 39071 Santander, Spain, e-mail: eduardo.casas@unican.es

‡Dpto. de Matemáticas, E.P.S.I. de Gijón, Universidad de Oviedo, Campus de Viesques, 33203 Gijón, Spain, e-mail: mmateos@orion.ciencias.uniovi.es

§Institut für Mathematik, Technische Universität Berlin, 10623 Berlin, Germany, e-mail: troeltzsch@mathematik.tu-berlin.de

In the recent paper [8], the case of linear-quadratic elliptic problems was investigated again from a slightly different point of view: It was assumed that only the control is approximated while considering the elliptic equation as exactly solvable. Here, all main variants of elliptic problems have been studied – distributed control, boundary control, distributed observation and boundary observation. Moreover, the case of piecewise linear control functions was studied in domains of dimension 2. Finally, we refer to [7], where error estimates were derived for elliptic problems with integral state constraints.

**2. The Control Problem.** Throughout the sequel, $\Omega$ denotes an open convex bounded polygonal set of $\mathbb{R}^2$ and $\Gamma$ is the boundary of $\Omega$. In this domain we formulate the following control problem

$$\text{(P)}\left\{\begin{array}{l} \inf \quad J(u)=\displaystyle\int_\Omega L(x,y_u(x))\,dx \;+\int_\Gamma l(x,y_u(x),u(x))\,d\sigma(x) \\ \\ \text{subject to } (y_u,u)\in H^1(\Omega)\times L^\infty(\Gamma), \\ u\in U^{ad}=\{u\in L^\infty(\Gamma)\mid \alpha\le u(x)\le\beta \;\text{ a.e. } x\in\Gamma\}, \\ (y_u,u)\;\text{ satisfying the state equation (2.1)} \end{array}\right.$$

$$\left\{\begin{array}{rcllc} -\Delta y_u(x) &=& a_0(x,y_u(x)) & \text{in} & \Omega \\ \partial_\nu y_u(x) &=& b_0(x,y_u(x))+u(x) & \text{on} & \Gamma, \end{array}\right. \tag{2.1}$$

where $-\infty<\alpha<\beta<+\infty$. Here $u$ is the control while $y_u$ is said to be the associated state. The following hypotheses are assumed about the functions involved in the control problem (P):

**(A1)** The function $L:\Omega\times\mathbb{R}\longrightarrow\mathbb{R}$ is measurable with respect to the first component, of class $C^2$ with respect to the second, $L(\cdot,0)\in L^1(\Omega)$ and for all $M>0$ there exist a function $\psi_{L,M}\in L^p(\Omega)$ $(p>2)$ and a constant $C_{L,M}>0$ such that

$$\left|\frac{\partial L}{\partial y}(x,y)\right|\le\psi_{L,M}(x),\qquad\left|\frac{\partial^2 L}{\partial y^2}(x,y)\right|\le C_{L,M},$$

$$\left|\frac{\partial^2 L}{\partial y^2}(x,y_2)-\frac{\partial^2 L}{\partial y^2}(x,y_1)\right|\le C_{L,M}|y_2-y_1|,$$

for a.e. $x,x_i\in\Omega$ and $|y|,|y_i|\le M$, $i=1,2$.

**(A2)** The function $l:\Gamma\times\mathbb{R}^2\longrightarrow\mathbb{R}$ is measurable with respect to the first component, of class $C^2$ with respect to the second and third variables, $l(x,0,0)\in L^1(\Gamma)$ and for all $M>0$ there exist a constant $C_{l,M}>0$ and a function $\psi_{l,M}\in L^p(\Gamma)$ $(p>1)$ such that

$$\left|\frac{\partial l}{\partial y}(x,y,u)\right|\le\psi_{l,M}(x),\qquad\|D^2_{(y,u)}l(x,y,u)\|\le C_{l,M},$$

$$\left|\frac{\partial l}{\partial u}(x_2,y,u)-\frac{\partial l}{\partial u}(x_1,y,u)\right|\le C_{l,M}|x_2-x_1|,$$

$$\|D^2_{(y,u)}l(x,y_2,u_2)-D^2_{(y,u)}l(x,y_1,u_1)\|\le C_{l,M}(|y_2-y_1|+|u_2-u_1|),$$

for a.e. $x, x_i \in \Gamma$ and $|y|, |y_i|, |u|, |u_i| \le M$, $i = 1, 2$, where $D^2_{(y,u)}l$ denotes the second derivative of $l$ with respect to $(y, u)$. Moreover we assume that there exists $m_l > 0$ such that

$$\frac{\partial^2 l}{\partial u^2}(x, y, u) \ge m_l, \quad \text{a.e. } x \in \Gamma \text{ and } (y, u) \in \mathbb{R}^2.$$

Let us remark that this inequality implies the strict convexity of $l$ with respect to the third variable.

**(A3)** The function $a_0 : \Omega \times \mathbb{R} \longrightarrow \mathbb{R}$ is measurable with respect to the first variable and of class $C^2$ with respect to the second,

$$a_0(\cdot, 0) \in L^p(\Omega) \ \ (p > 2), \quad \frac{\partial a_0}{\partial y}(x, y) \le 0 \ \text{ a.e. } x \in \Omega \text{ and } y \in \mathbb{R}$$

and for all $M > 0$ there exists a constant $C_{a_0,M} > 0$ such that

$$\left|\frac{\partial a_0}{\partial y}(x, y)\right| + \left|\frac{\partial^2 a_0}{\partial y^2}(x, y)\right| \le C_{a_0,M} \text{ a.e. } x \in \Omega \text{ and } |y| \le M,$$

$$\left|\frac{\partial^2 a_0}{\partial y^2}(x, y_2) - \frac{\partial^2 a_0}{\partial y^2}(x, y_1)\right| < C_{a_0,M}|y_2 - y_1| \ \text{ a.e. } x \in \Omega \text{ and } |y_1|, |y_2| \le M.$$

**(A4)** The function $b_0 : \Gamma \times \mathbb{R} \longrightarrow \mathbb{R}$ is Lipschitz with respect to the first variable and of class $C^2$ with respect to the second, $b_0(\cdot, 0) \in W^{1-1/p,p}(\Gamma)$, with $p > 2$,

$$\frac{\partial b_0}{\partial y}(x, y) \le 0$$

and for all $M > 0$ there exists a constant $C_{b_0,M} > 0$ such that

$$\left|\frac{\partial b_0}{\partial y}(x, y)\right| + \left|\frac{\partial^2 b_0}{\partial y^2}(x, y)\right| \le C_{b_0,M},$$

$$\left|\frac{\partial^2 b_0}{\partial y^2}(x, y_2) - \frac{\partial^2 b_0}{\partial y^2}(x, y_1)\right| \le C_{b_0,M}|y_2 - y_1|.$$

for all $x \in \Gamma$ and $|y|, |y_1|, |y_2| \le M$.

**(A5)** At least one of the two conditions must hold: either $(\partial a_0/\partial y)(x, y) < 0$ in $E_\Omega \times \mathbb{R}$ with $E_\Omega \subset \Omega$ of positive $n$-dimensional measure or $(\partial b_0/\partial y)(x, y) < 0$ on $E_\Gamma \times \mathbb{R}$ with $E_\Gamma \subset \Gamma$ of positive $(n-1)$-dimensional measure.

Before finishing this section let us study the state equation (2.1).

THEOREM 2.1. *For every $u \in L^2(\Gamma)$ the state equation* (2.1) *has a unique solution $y_u \in H^{3/2}(\Omega)$, that depends continuously on $u$. Moreover, there exists $p_0 > 2$ depending on the measure of the angles in $\Gamma$ such that $u \in W^{1-1/p,p}(\Gamma)$ with some $2 \le p \le p_0$ implies $y_u \in W^{2,p}(\Omega)$.*

*Proof.* Due to the Assumptions (A3)–(A5), it is classical to show the existence of a unique solution $y_u \in H^1(\Omega) \cap L^\infty(\Omega)$. From the Assumptions (A3)–(A4) we also deduce that $a_0(\cdot, y_u(\cdot)) \in L^2(\Omega)$ and $u - b_0(\cdot, y_u(\cdot)) \in L^2(\Gamma)$. In this situation Lemma 2.2 below proves that $y_u \in H^{3/2}(\Omega)$.

Let us verify the $W^{2,p}(\Omega)$ regularity. It is known that $H^{3/2}(\bar\Omega) \subset W^{1,4}(\Omega)$; see for instance Grisvard [11]. Therefore, the traces of $\bar y$ and $\bar\varphi$ belong to the space $W^{1-1/4,4}(\Gamma)$ [11, Theorem 1.5.13]. From the Lipschitz property of $b_0$ with respect to $x$ and $y$, we deduce that $b_0(\cdot, y_u(\cdot)) \in W^{1-1/4,4}(\Gamma)$ too. Now Corollary 4.4.3.8 of Grisvard [11] yields the existence of some $p_0 \in (2,4]$ depending on the measure of the angles in $\Gamma$ such that $\bar y, \bar\varphi \in W^{2,p}(\Omega)$ for any $2 \le p \le p_0$ provided that $u \in W^{1-1/p,p}(\Gamma)$. We should remind at this point that we have assumed $\Omega$ to be convex. □

LEMMA 2.2. *Let us assume that $f \in L^2(\Omega)$ and $g \in L^2(\Gamma)$ satisfy that*

$$\int_\Omega f(x)\,dx + \int_\Gamma g(x)\,d\sigma(x) = 0.$$

*Then the problem*

$$\left\{\begin{array}{rcll} -\Delta y &=& f & \text{in} \quad \Omega \\ \partial_\nu y &=& g & \text{on} \quad \Gamma \end{array}\right. \tag{2.2}$$

*has a solution $y \in H^{3/2}(\Omega)$ that is unique up to an additive constant.*

*Proof.* It is a consequence of Lax-Milgram Theorem's that (2.2) has a unique solution in $H^1(\Omega)$ up to an additive constant. Let us prove the $H^{3/2}(\Omega)$ regularity. To show this we consider the problem

$$\left\{\begin{array}{rcll} -\Delta y_1 &=& f & \text{in} \quad \Omega \\ y_1 &=& 0 & \text{on} \quad \Gamma. \end{array}\right.$$

Following Jerison and Kenig [13], this problem has a unique solution $y_1 \in H^{3/2}(\Omega)$. Moreover, from $\Delta y_1 \in L^2(\Omega)$ and $y_1 \in H^{3/2}(\Omega)$ we deduce that $\partial_\nu y_1 \in L^2(\Gamma)$; see Kenig [14].

From the equality

$$\int_\Gamma (g - \partial_\nu y_1)\,d\sigma = -\int_\Omega f\,dx - \int_\Gamma \partial_\nu y_1\,d\sigma = -\int_\Omega f\,dx - \int_\Omega \Delta y\,dx = 0$$

we deduce the existence of a unique solution $y_2 \in H^1(\Omega)$ of

$$\left\{\begin{array}{rcll} -\Delta y_2 &=& 0 & \text{in} \quad \Omega \\ \partial_\nu y_2 &=& g - \partial_\nu y_1 & \text{on} \quad \Gamma \\ \displaystyle\int_\Omega y_2\,dx &=& \displaystyle\int_\Omega (y - y_1)\,dx. & \end{array}\right.$$

Once again following Jerison and Kenig [12] we know that $y_2 \in H^{3/2}(\Omega)$. Now it is easy to check that $y = y_1 + y_2 \in H^{3/2}(\Omega)$. □

Let us note that $H^{3/2}(\Omega) \subset C(\bar\Omega)$ holds for Lipschitz domains in $\mathbb{R}^2$. As a consequence of the theorem above, we know that the functional $J$ is well defined in $L^2(\Gamma)$. Using the convexity of $l$ with respect to $u$, we can prove, as in Casas and Mateos [7], the existence of at least one global solution of (P). Let us discuss the differentiability properties of $J$.

THEOREM 2.3. *Suppose that assumptions (A3)–(A4) are satisfied. Then the mapping $G : L^\infty(\Gamma) \longrightarrow H^{3/2}(\Omega)$ defined by $G(u) = y_u$ is of class $C^2$. Moreover, for all $u, v \in L^\infty(\Gamma)$, $z_v = G'(u)v$ is the solution of*

$$\left\{ \begin{array}{rcll} -\Delta z_v & = & \dfrac{\partial a_0}{\partial y}(x, y_u) z_v & \text{in} \quad \Omega \\ \partial_\nu z_v & = & \dfrac{\partial b_0}{\partial y}(x, y_u) z_v + v & \text{on} \quad \Gamma. \end{array} \right. \tag{2.3}$$

*Finally, for every $v_1, v_2 \in L^\infty(\Omega)$, $z_{v_1 v_2} = G''(u) v_1 v_2$ is the solution of*

$$\left\{ \begin{array}{rcll} -\Delta z_{v_1 v_2} & = & \dfrac{\partial a_0}{\partial y}(x, y_u) z_{v_1 v_2} + \dfrac{\partial^2 a_0}{\partial y^2}(x, y_u) z_{v_1} z_{v_2} & \text{in} \quad \Omega \\ \partial_\nu z_{v_1 v_2} & = & \dfrac{\partial b_0}{\partial y}(x, y_u) z_{v_1 v_2} + \dfrac{\partial^2 b_0}{\partial y^2}(x, y_u) z_{v_1} z_{v_2} & \text{on} \quad \Gamma, \end{array} \right. \tag{2.4}$$

*where $z_{v_i} = G'(u) v_i$, $i = 1, 2$.*

This theorem is now standard and can be proved by using the implicit function theorem; see Casas and Mateos [6].

THEOREM 2.4. *Under the assumptions (A1)–(A4), the functional $J : L^\infty(\Gamma) \to \mathbb{R}$ is of class $C^2$. Moreover, for every $u, v, v_1, v_2 \in L^\infty(\Gamma)$*

$$J'(u)v = \int_\Gamma \left( \frac{\partial l}{\partial u}(x, y_u, u) + \varphi_u \right) v \, d\sigma \tag{2.5}$$

*and*

$$\begin{aligned} J''(u) v_1 v_2 = & \int_\Omega \left[ \frac{\partial^2 L}{\partial y^2}(x, y_u) z_{v_1} z_{v_2} + \varphi_u \frac{\partial^2 a_0}{\partial y^2}(x, y_u) z_{v_1} z_{v_2} \right] dx \\ & + \int_\Gamma \left[ \frac{\partial^2 l}{\partial y^2}(x, y_u, u) z_{v_1} z_{v_2} + \frac{\partial^2 l}{\partial y \partial u}(x, y_u, u)(z_{v_1} v_2 + z_{v_2} v_1) \right. \\ & \qquad \left. + \frac{\partial^2 l}{\partial u^2}(x, y_u, u) v_1 v_2 + \varphi_u \frac{\partial^2 b_0}{\partial y^2}(x, y_u) z_{v_1} z_{v_2} \right] d\sigma \end{aligned} \tag{2.6}$$

*where $z_{v_i} = G'(u) v_i$, $i = 1, 2$, $y_u = G(u)$, and the adjoint state $\varphi_u \in H^{3/2}(\Omega)$ is the unique solution of the problem*

$$\left\{ \begin{array}{l} -\Delta \varphi = \dfrac{\partial a_0}{\partial y}(x, y_u) \varphi + \dfrac{\partial L}{\partial y}(x, y_u) \quad \text{in } \Omega \\ \partial_\nu \varphi = \dfrac{\partial b_0}{\partial y}(x, y_u) \varphi + \dfrac{\partial l}{\partial y}(x, y_u, u) \quad \text{on } \Gamma. \end{array} \right. \tag{2.7}$$

This theorem follows from Theorem 2.3 and the chain rule.

**3. First and Second Order Optimality Conditions.** The first order optimality conditions for Problem (P) follow readily from Theorem 2.4.

THEOREM 3.1. *Assume that $\bar{u}$ is a local solution of Problem (P). Then there exist $\bar{y}, \bar{\varphi} \in H^{3/2}(\Omega)$ such that*

$$\left\{ \begin{array}{rcll} -\Delta \bar{y}(x) & = & a_0(x, \bar{y}(x)) & \text{in} \quad \Omega \\ \partial_\nu \bar{y}(x) & = & b_0(x, \bar{y}(x)) + \bar{u}(x) & \text{on} \quad \Gamma, \end{array} \right. \tag{3.1}$$

$$\text{(3.2)} \qquad \begin{cases} -\Delta\bar\varphi = \dfrac{\partial a_0}{\partial y}(x,\bar y)\bar\varphi + \dfrac{\partial L}{\partial y}(x,\bar y) & \text{in} \quad \Omega \\ \partial_\nu\bar\varphi = \dfrac{\partial b_0}{\partial y}(x,\bar y)\bar\varphi + \dfrac{\partial l}{\partial y}(x,\bar y,\bar u) & \text{on} \quad \Gamma, \end{cases}$$

$$\text{(3.3)} \qquad \int_\Gamma \left(\frac{\partial l}{\partial u}(x,\bar y,\bar u) + \bar\varphi\right)(u-\bar u)\,d\sigma \ge 0 \quad \forall u \in U^{ad}.$$

If we define

$$\bar d(x) = \frac{\partial l}{\partial u}(x,\bar y(x),\bar u(x)) + \bar\varphi(x),$$

then we deduce from (3.3) that

$$\text{(3.4)} \qquad \bar d(x) = \begin{cases} 0 & \text{for a.e. } x\in\Gamma \text{ where } \alpha < \bar u(x) < \beta, \\ \ge 0 & \text{for a.e. } x\in\Gamma \text{ where } \bar u(x) = \alpha, \\ \le 0 & \text{for a.e. } x\in\Gamma \text{ where } \bar u(x) = \beta. \end{cases}$$

In order to establish the second order optimality conditions we define the cone of critical directions

$$C_{\bar u} = \{v \in L^2(\Gamma) \text{ satisfying (3.5) and } v(x) = 0 \text{ if } |\bar d(x)| > 0\},$$

$$\text{(3.5)} \qquad v(x) = \begin{cases} \ge 0 & \text{for a.e. } x\in\Gamma \text{ where } \bar u(x) = \alpha, \\ \le 0 & \text{for a.e. } x\in\Gamma \text{ where } \bar u(x) = \beta. \end{cases}$$

Now we formulate the second order necessary and sufficient optimality conditions.

THEOREM 3.2. *If $\bar u$ is a local solution of (P), then $J''(\bar u)v^2 \ge 0$ holds for all $v \in C_{\bar u}$. Conversely, if $\bar u \in U^{ad}$ satisfies the first order optimality conditions* (3.1)–(3.3) *and the coercivity condition $J''(\bar u)v^2 > 0$ holds for all $v \in C_{\bar u} \setminus \{0\}$, then there exist $\delta > 0$ and $\varepsilon > 0$ such that*

$$\text{(3.6)} \qquad J(u) \ge J(\bar u) + \delta\|u-\bar u\|^2_{L^2(\Gamma)}$$

*is satisfied for every $u \in U^{ad}$ such that $\|u-\bar u\|_{L^\infty(\Omega)} \le \varepsilon$.*

The necessary condition provided in the theorem is quite easy to get. The sufficient conditions are proved by Casas and Mateos [6, Theorem 4.3] for distributed control problems with integral state constraints. The proof can be translated in a straightforward way to the case of boundary controls. The hypothesis $(\partial^2 l/\partial u^2) \ge m_l > 0$ introduced in Assumption **(A2)** as well as the linearity of $u$ in the state equation is essential to apply the mentioned Theorem 4.3. The same result can be proved by following the approach of Bonnans and Zidani [5].

REMARK 3.3. *By using the assumption $(\partial^2 l/\partial u^2)(x,y,u) \ge m_l > 0$, we deduce from Casas and Mateos [6, Theorem 4.4] that the following two conditions are equivalent:*

*(1) $J''(\bar u)v^2 > 0$ for every $v \in C_{\bar u} \setminus \{0\}$.*

*(2) There exist* $\delta > 0$ *and* $\tau > 0$ *such that* $J''(\bar{u})v^2 \geq \delta\|v\|^2_{L^2(\Gamma)}$ *for every* $v \in C^\tau_{\bar{u}}$, *where*

$$C^\tau_{\bar{u}} = \{v \in L^2(\Gamma) \text{ satisfying } (3.5) \text{ and } v(x) = 0 \text{ if } |\bar{d}(x)| > \tau\}.$$

*It is clear that that* $C^\tau_{\bar{u}}$ *contains strictly* $C_{\bar{u}}$, *so the condition (2) seems to be stronger than (1), but in fact they are equivalent.*

We finish this section by providing a characterization of the optimal control $\bar{u}$ and deducing from it the Lipschitz regularity of $\bar{u}$ as well as some extra regularity of $\bar{y}$ and $\bar{\varphi}$.

THEOREM 3.4. *Suppose that* $\bar{u}$ *is a local solution of (P), then for all* $x \in \Gamma$ *the equation*

$$\bar{\varphi}(x) + \frac{\partial l}{\partial u}(x, \bar{y}(x), t) = 0 \tag{3.7}$$

*has a unique solution* $\bar{t} = \bar{s}(x)$. *The mapping* $\bar{s} : \Gamma \longrightarrow \mathbb{R}$ *is Lipschitz and it is related with* $\bar{u}$ *through the formula*

$$\bar{u}(x) = Proj_{[\alpha,\beta]}(\bar{s}(x)) = \max\{\alpha, \min\{\beta, \bar{s}(x)\}\}. \tag{3.8}$$

*Moreover* $\bar{u} \in C^{0,1}(\Gamma)$ *and* $\bar{y}, \bar{\varphi} \in W^{2,p}(\Omega) \subset C^{0,1}(\bar{\Omega})$ *for some* $p > 2$.

*Proof.* Let us remind that $\bar{y}, \bar{\varphi} \in H^{3/2}(\Omega) \subset C(\bar{\Omega})$ because $n = 2$. We fix $x \in \Gamma$ and consider the real function $g : \mathbb{R} \longrightarrow \mathbb{R}$ defined by

$$g(t) = \bar{\varphi}(x) + \frac{\partial l}{\partial u}(x, \bar{y}(x), t).$$

From assumption **(A2)** we have that $g$ is $C^1$ with $g'(t) \geq m_l > 0$ for every $t \in \mathbb{R}$. Therefore, there exists a unique real number $\bar{t}$ satisfying $g(\bar{t}) = 0$. Consequently $\bar{s}$ is well defined and relation (3.8) is an immediate consequence of (3.4). Let us prove the regularity results. Invoking once again assumption **(A2)** along with (3.7) and (3.8), we get for every $x_1, x_2 \in \Gamma$

$$|\bar{u}(x_2) - \bar{u}(x_1)| \leq |\bar{s}(x_2) - \bar{s}(x_1)| \leq \frac{1}{m_l}\left|\frac{\partial l}{\partial u}(x_2, \bar{y}(x_2), \bar{s}(x_2)) - \frac{\partial l}{\partial u}(x_2, \bar{y}(x_2), \bar{s}(x_1))\right| \leq$$

$$\frac{1}{m_l}\left\{|\bar{\varphi}(x_2) - \bar{\varphi}(x_1)| + \left|\frac{\partial l}{\partial u}(x_1, \bar{y}(x_1), \bar{s}(x_1)) - \frac{\partial l}{\partial u}(x_2, \bar{y}(x_2), \bar{s}(x_1))\right|\right\} \leq$$

$$C\left\{|x_2 - x_1| + |\bar{\varphi}(x_2) - \bar{\varphi}(x_1)| + |\bar{y}(x_2) - \bar{y}(x_1)|\right\}. \tag{3.9}$$

The embedding $H^{3/2}(\bar{\Omega}) \subset W^{1,4}(\Omega)$ ensures that the traces of $\bar{y}$ and $\bar{\varphi}$ belong to the space $W^{1-1/4,4}(\Gamma)$. Exploiting that $n = 2$ and taking in this space the norm

$$\|z\|_{W^{1-1/4,4}(\Gamma)} = \left\{\|z\|^4_{L^4(\Gamma)} + \int_\Gamma\int_\Gamma \frac{|z(x_2) - z(x_1)|^4}{|x_2 - x_1|^4}\, d\sigma(x_1)d\sigma(x_2)\right\}^{1/4},$$

the regularity $\bar{u}, \bar{s} \in W^{1-1/4,4}(\Gamma) \subset W^{1-1/p,p}(\Gamma)$ $(1 \leq p \leq 4)$ follows from (3.9). Now Theorem 2.1 leads to the regularity $\bar{y} \in W^{2,p}(\Omega)$. The same is also true for $\bar{\varphi}$. Indeed, it is enough to use Corollary 4.4.3.8 of Grisvard [11] as in the proof of Theorem 2.1. Using the embedding $W^{2,p}(\Omega) \subset C^{0,1}(\bar{\Omega})$ and (3.9) we get the Lipschitz regularity of $\bar{u}$ and $\bar{s}$. ∎

**4. Approximation of (P) by Finite Elements and Piecewise Constant Controls.** Here, we define a finite-element based approximation of the optimal control problem $(P)$. To this aim, we consider a family of triangulations $\{\mathcal{T}_h\}_{h>0}$ of $\bar\Omega$: $\bar\Omega = \cup_{T\in\mathcal{T}_h} T$. This triangulation is supposed to be regular in the usual sense that we state exactly here. With each element $T \in \mathcal{T}_h$, we associate two parameters $\rho(T)$ and $\sigma(T)$, where $\rho(T)$ denotes the diameter of the set $T$ and $\sigma(T)$ is the diameter of the largest ball contained in $T$. Let us define the size of the mesh by $h = \max_{T\in\mathcal{T}_h}\rho(T)$. The following regularity assumption is assumed.

$(H)$ - There exist two positive constants $\rho$ and $\sigma$ such that

$$\frac{\rho(T)}{\sigma(T)} \le \sigma, \qquad \frac{h}{\rho(T)} \le \rho$$

hold for all $T \in \mathcal{T}_h$ and all $h > 0$.

For fixed $h > 0$, we denote by $\{T_j\}_{j=1}^{N(h)}$ the family of triangles of $\mathcal{T}_h$ with a side on the boundary of $\Gamma$. If the edges of $T_j \cap \Gamma$ are $x_\Gamma^j$ and $x_\Gamma^{j+1}$ then $[x_\Gamma^j, x_\Gamma^{j+1}] := T_j \cap \Gamma$, $1 \le j \le N(h)$, with $x_\Gamma^{N(h)+1} = x_\Gamma^1$. Associated with this triangulation we set

$$U_h = \{u \in L^\infty(\Gamma) \mid u \text{ is constant on every side } (x_\Gamma^j, x_\Gamma^{j+1}) \text{ for } 1 \le j \le N(h)\},$$

$$Y_h = \{y_h \in C(\bar\Omega) \mid y_{h|T} \in \mathcal{P}_1, \text{ for all } T \in \mathcal{T}_h\},$$

where $\mathcal{P}_1$ is the space of polynomials of degree less than or equal to 1. For each $u \in L^\infty(\Gamma)$, we denote by $y_h(u)$ the unique element of $Y_h$ that satisfies

$$a(y_h(u), z_h) = \int_\Omega a_0(x, y_h(u)) z_h\,dx + \int_\Gamma [b_0(x, y_h(u)) + u] z_h\,dx \quad \forall z_h \in Y_h, \tag{4.1}$$

where $a : Y_h \times Y_h \longrightarrow \mathbb{R}$ is the bilinear form defined by

$$a(y_h, z_h) = \int_\Omega \nabla y_h(x) \nabla z_h(x)\,dx.$$

The finite dimensional control problem is defined by

$$(\mathrm{P}_h)\begin{cases} \min J_h(u_h) = \int_\Omega L(x, y_h(u_h)(x))\,dx + \int_\Gamma l(x, y_h(u_h)(x), u_h(x))\,d\sigma(x), \\ \text{subject to } \ (y_h(u_h), u_h) \in Y_h \times U_h^{ad}, \end{cases}$$

where

$$U_h^{ad} = U_h \cap U^{ad} = \{u_h \in U_h \mid \alpha \le u_h(x) \le \beta \text{ for all } x \in \Gamma\}.$$

Since $J_h$ is a continuous function and $U_h^{ad}$ is compact, we get that $(\mathrm{P}_h)$ has at least one global solution. The first order optimality conditions can be written as follows:

THEOREM 4.1. *Assume that $\bar u_h$ is a local optimal solution of $(P_h)$. Then there exist $\bar y_h$ and $\bar\varphi_h$ in $Y_h$ satisfying*

$$a(\bar y_h, z_h) = \int_\Omega a_0(x, \bar y_h) z_h\,dx + \int_\Gamma (b_0(x, \bar y_h) + \bar u_h) z_h\,dx \quad \forall z_h \in Y_h, \tag{4.2}$$

$$a(\bar\varphi_h, z_h) = \int_\Omega \left(\frac{\partial a_0}{\partial y}(x,\bar y_h)\bar\varphi_h + \frac{\partial L}{\partial y}(x,\bar y_h)\right) z_h\,dx +$$

$$\int_\Gamma \left(\frac{\partial b_0}{\partial y}(x,\bar y_h)\bar\varphi_h + \frac{\partial l}{\partial y}(x,\bar y_h,\bar u_h)\right) z_h\,d\sigma(x) \quad \forall z_h \in Y_h, \tag{4.3}$$

$$\int_\Gamma \left(\bar\varphi_h + \frac{\partial l}{\partial u}(x,\bar y_h,\bar u_h)\right)(u_h - \bar u_h)\,d\sigma(x) \ge 0 \quad \forall u_h \in U_h^{ad}. \tag{4.4}$$

The following result is the counterpart of Theorem 3.4.

THEOREM 4.2. *Let us assume that $\bar u_h$ is a local solution of problem ($P_h$). Then for every $1 \le j \le N(h)$, the equation*

$$\int_{x_\Gamma^j}^{x_\Gamma^{j+1}} \left(\bar\varphi_h(x) + \frac{\partial l}{\partial u}(x,\bar y_h(x),t)\right) d\sigma(x) = 0 \tag{4.5}$$

*has a unique solution $\bar s_j$. The mapping $\bar s_h \in U_h$, defined by $\bar s_h(x) = \bar s_j$ on every side $(x_\Gamma^j, x_\Gamma^{j+1})$, is related to $\bar u_h$ by the formula*

$$\bar u_h(x) = Proj_{[\alpha,\beta]}(\bar s_h(x)) = \min\{\alpha, \max\{\beta, \bar s_h(x)\}\}. \tag{4.6}$$

**4.1. Convergence Results.** Our main aim is to prove the convergence of the local solutions of ($P_h$) to local solutions of (P) as well as to derive error estimates. Before doing this we need to establish the order of convergence of the solutions of the discrete equation (4.1) to the solution of the state equation (2.1). An analogous result is needed for the adjoint state equation.

THEOREM 4.3. *For any $u \in L^2(\Gamma)$ there exists a constant $C = C(\|u\|_{L^2(\Gamma)}) > 0$ independent of $h$ such that*

$$\|y_u - y_h(u)\|_{L^2(\Omega)} + \|\varphi_u - \varphi_h(u)\|_{L^2(\Omega)} \le Ch, \tag{4.7}$$

*where $y_u$ denotes the solution of* (2.1) *and $\varphi_u$ is the solution of* (3.2) *with $(\bar y, \bar u)$ being replaced by $(y, u)$. Moreover, if $u \in W^{1-1/p,p}(\Gamma)$ holds for some $p > 2$ and $u_h \in U_h$ then*

$$\|y_u - y_h(u_h)\|_{H^1(\Omega)} + \|\varphi_u - \varphi_h(u_h)\|_{H^1(\Omega)} \le C\{h + \|u - u_h\|_{L^2(\Gamma)}\}. \tag{4.8}$$

*Finally, if $u_h \to u$ weakly in $L^2(\Gamma)$, then $y_h(u_h) \to y_u$ and $\varphi_h(u_h) \to \varphi_u$ strongly in $C(\bar\Omega)$.*

*Proof.* Let us prove the theorem for the state $y$. The corresponding proof for the adjoint state $\varphi$ follows the same steps. Inequality (4.7) is proved by Casas and Mateos [7]. Let us prove (4.8). The regularity of $u$ implies that $y_u \in H^2(\Omega)$, then

$$\|y_u - y_h(u)\|_{H^1(\Omega)} \le Ch\|y_u\|_{H^2(\Omega)} = hC(\|u\|_{H^{1/2}(\Gamma)});$$

see Casas and Mateos [7].

On the other hand, from the monotonicity of $a_0$ and $b_0$ and the assumption **(A5)** it is easy to get by classical arguments

$$\|y_h(u) - y_h(u_h)\|_{H^1(\Omega)} \le C\|u - u_h\|_{L^2(\Gamma)}.$$

Combining both inequalities we achieve the desired result for the states. For the proof of the uniform convergence of the states and adjoint states the reader is also referred to [7]. □

Now we can prove the convergence of the discretizations.

THEOREM 4.4. *For every $h > 0$ let $\bar u_h$ be a global solution of problem $(P_h)$. Then there exist weakly*-converging subsequences of $\{\bar u_h\}_{h>0}$ in $L^\infty(\Gamma)$ (still indexed by $h$). If the subsequence $\{\bar u_h\}_{h>0}$ is converging weakly* to $\bar u$, then $\bar u$ is a solution of* (P) ,

$$\lim_{h\to 0} J_h(\bar u_h) = J(\bar u) = \inf(P) \quad \text{and} \quad \lim_{h\to 0} \|\bar u - \bar u_h\|_{L^\infty(\Gamma)} = 0. \tag{4.9}$$

*Proof.* Since $U_h^{ad} \subset U^{ad}$ holds for every $h > 0$ and $U^{ad}$ is bounded in $L^\infty(\Gamma)$, $\{\bar u_h\}_{h>0}$ is also bounded in $L^\infty(\Gamma)$. Therefore, there exist weakly*-converging subsequences as claimed in the statement of the theorem. Let $\bar u_h$ be the of one of these subsequences. By the definition of $U^{ad}$ it is obvious that $\bar u_h \in U^{ad}$. Let us prove that the weak* limit $\bar u$ is a solution of (P). Let $\tilde u \in U^{ad}$ be a solution of (P) and consider the operator $\Pi_h : L^1(\Gamma) \to U_h$ defined by

$$\Pi_h u \mid_{(x_\Gamma^j, x_\Gamma^{j+1})} = \frac{1}{|x_\Gamma^{j+1} - x_\Gamma^j|} \int_{x_\Gamma^j}^{x_\Gamma^{j+1}} u(x)\,d\sigma(x).$$

According to Theorem 4.3 we have that $\tilde u \in C^{0,1}(\Gamma)$ and then

$$\|\tilde u - \Pi_h \tilde u\|_{L^\infty(\Gamma)} \le Ch\|\tilde u\|_{C^{0,1}(\Gamma)}.$$

Remark that $\Pi_h \tilde u \in U_h^{ad}$ for every $h$. Now using the convexity of $l$ with respect to $u$ and the uniform convergence $\bar y_h = y_h(\bar u_h) \to \bar y = y_{\bar u}$ and $y_h(\Pi_h \tilde u) \to y_{\tilde u}$ (Theorem 4.3) along with the assumptions on $L$ and $l$ we get

$$J(\bar u) \le \liminf_{h\to 0} J_h(\bar u_h) \le \limsup_{h\to 0} J_h(\bar u_h) \le \limsup_{h\to 0} J_h(\Pi_h \tilde u) = J(\tilde u) = \inf(P).$$

This proves that $\bar u$ is a solution of (P) as well as the convergence of the optimal costs. Let us verify the uniform convergence of $\{\bar u_h\}$ to $\bar u$. From (3.8) and (4.6) we obtain

$$\|\bar u - \bar u_h\|_{L^\infty(\Gamma)} \le \|\bar s - \bar s_h\|_{L^\infty(\Gamma)},$$

therefore it is enough to prove the uniform convergence of $\{\bar s_h\}_{h>0}$ to $\bar s$. On the other hand, from (4.5) and the continuity of the integrand with respect to $x$ we deduce the existence of a point $\xi_\Gamma^j \in (x_\Gamma^j, x_\Gamma^{j+1})$ such that

$$\bar\varphi_h(\xi_\Gamma^j) + \frac{\partial l}{\partial u}(\xi_\Gamma^j, \bar y_h(\xi_\Gamma^j), \bar s_h(\xi_\Gamma^j)) = 0. \tag{4.10}$$

Given $x \in \Gamma$, let us take $1 \le j \le N(h)$ such that $x \in (x_\Gamma^j, x_\Gamma^{j+1})$. By the fact that $\bar s_h$ is constant on each of these intervals we get

$$|\bar s(x) - \bar s_h(x)| \le |\bar s(x) - \bar s(\xi_\Gamma^j)| + |\bar s(\xi_\Gamma^j) - \bar s_h(\xi_\Gamma^j)| \le$$

$$\Lambda_s|x - \xi_\Gamma^j| + |\bar s(\xi_\Gamma^j) - \bar s_h(\xi_\Gamma^j)| \le \Lambda_s h + |\bar s(\xi_\Gamma^j) - \bar s_h(\xi_\Gamma^j)|,$$

where $\Lambda_s$ is the Lipschitz constant of $\bar s$. So it remains to prove the convergence $\bar s_h(\xi_\Gamma^j) \to \bar s(\xi_\Gamma^j)$ for every $j$. For it we use the strict positivity of the second derivative of $l$ with respect to $u$ (Assumption **(A2)**) along with the equations (3.7) satisfied by $\bar{s}(x)$ and (4.10) to deduce

$$m_l|\bar{s}(\xi_\Gamma^j) - \bar{s}_h(\xi_\Gamma^j)| \le \left|\frac{\partial l}{\partial u}(\xi_\Gamma^j, \bar{y}_h(\xi_\Gamma^j), \bar{s}(\xi_\Gamma^j)) - \frac{\partial l}{\partial u}(\xi_\Gamma^j, \bar{y}_h(\xi_\Gamma^j), \bar{s}_h(\xi_\Gamma^j))\right| \le$$

$$\left|\frac{\partial l}{\partial u}(\xi_\Gamma^j, \bar{y}_h(\xi_\Gamma^j), \bar{s}(\xi_\Gamma^j)) - \frac{\partial l}{\partial u}(\xi_\Gamma^j, \bar{y}(\xi_\Gamma^j), \bar{s}(\xi_\Gamma^j))\right| +$$

$$\left|\frac{\partial l}{\partial u}(\xi_\Gamma^j, \bar{y}(\xi_\Gamma^j), \bar{s}(\xi_\Gamma^j)) - \frac{\partial l}{\partial u}(\xi_\Gamma^j, \bar{y}_h(\xi_\Gamma^j), \bar{s}_h(\xi_\Gamma^j))\right| =$$

$$\left|\frac{\partial l}{\partial u}(\xi_\Gamma^j, \bar{y}_h(\xi_\Gamma^j), \bar{s}(\xi_\Gamma^j)) - \frac{\partial l}{\partial u}(\xi_\Gamma^j, \bar{y}(\xi_\Gamma^j), \bar{s}(\xi_\Gamma^j))\right| + |\bar{\varphi}(\xi_\Gamma^j) - \bar{\varphi}_h(\xi_\Gamma^j)| \to 0$$

because of the uniform convergence of $\bar{y}_h \to \bar{y}$ and $\bar{\varphi}_h \to \bar{\varphi}$; see Theorem 4.3. ☐

The next theorem is a kind of reciprocal result of the previous one. At this point we are wondering if every local minimum $\bar{u}$ of (P) can be approximate by a local minimum of ($P_h$). The following theorem answers positively this question under the assumption that $\bar{u}$ satisfies the second order sufficient optimality conditions given in Theorem 3.2. In the sequel, $B_\rho(u)$ will denote the open ball of $L^\infty(\Gamma)$ centered at $u$ with radius $\rho$. By $\bar{B}_\rho(u)$ we denote the corresponding closed ball.

THEOREM 4.5. *Let $\bar{u}$ be a local minimum of (P) satisfying the second order sufficient optimality condition given in Theorem 3.2. Then there exist $\varepsilon > 0$ and $h_0 > 0$ such that ($P_h$) has a local minimum $\bar{u}_h \in B_\varepsilon(\bar{u})$ for every $h < h_0$. Furthermore, the convergences* (4.9) *hold.*

*Proof.* Let $\varepsilon > 0$ be given by Theorem 3.2 and consider the problems

$$(\mathrm{P}_\varepsilon) \begin{cases} \min J(u) \\ \text{subject to } \ (y_u, u) \in H^1(\Omega) \times (U^{ad} \cap \bar{B}_\varepsilon(\bar{u})) \end{cases}$$

and

$$(\mathrm{P}_{h\varepsilon}) \begin{cases} \min J_h(u_h) \\ \text{subject to } \ (y_h(u_h), u_h) \in Y_h \times (U_h^{ad} \cap \bar{B}_\varepsilon(\bar{u})). \end{cases}$$

According to Theorem 3.2, $\bar{u}$ is the unique solution of ($P_\varepsilon$). Moreover $\Pi_h\bar{u}$ is a feasible control for ($P_{h\varepsilon}$) for every $h$ small enough. Therefore $U_h^{ad} \cap \bar{B}_\varepsilon(\bar{u})$ is a non empty compact set and consequently ($P_{h\varepsilon}$) has at least one solution $\bar{u}_h$. Now we can argue as in the proof of Theorem 4.4 to deduce that $\bar{u}_h \to \bar{u}$ uniformly, hence $\bar{u}_h$ is a local solution of ($P_h$) in the open ball $B_\varepsilon(\bar{u})$ as required. ☐

**4.2. Error Estimates.** In this section we denote by $\bar{u}$ a fixed local reference solution of (P) satisfying the second order sufficient optimality conditions and by $\bar{u}_h$ the associated local solution of ($P_h$) converging uniformly to $\bar{u}$. As usual $\bar{y}$, $\bar{y}_h$ and $\bar{\varphi}$, $\bar{\varphi}_h$ are the state and adjoint states correponsding to $\bar{u}$ and $\bar{u}_h$. The goal is to estimate $\|\bar{u} - \bar{u}_h\|_{L^2(\Gamma)}$. Let us start by proving a first estimate for this term.

LEMMA 4.6. *Let $\delta > 0$ given as in Remark 3.3,(2). Then there exists $h_0 > 0$ such that*

$$\frac{\delta}{2}\|\bar{u} - \bar{u}_h\|^2_{L^2(\Gamma)} \le (J'(\bar{u}_h) - J'(\bar{u}))(\bar{u}_h - \bar{u}) \quad \forall h < h_0. \tag{4.11}$$

*Proof.* Let us set

$$\bar{d}_h(x) = \frac{\partial l}{\partial u}(x, \bar{y}_h(x), \bar{u}_h(x)) + \bar{\varphi}_h(x)$$

and take $\delta > 0$ and $\tau > 0$ as introduced in Remark 3.3-(2). We know that $\bar{d}_h \to \bar{d}$ uniformly in $\Gamma$, therefore there exists $h_\tau > 0$ such that

$$\|\bar{d} - \bar{d}_h\|_{L^\infty(\Gamma)} < \frac{\tau}{4} \quad \forall h \le h_\tau. \tag{4.12}$$

For every $1 \le j \le N(h)$ we define

$$I_j = \int_{x_\Gamma^j}^{x_\Gamma^{j+1}} \bar{d}_h(x)\, d\sigma(x).$$

From Theorem (4.1) we deduce by the classical argumentation that

$$\bar{u}_h \mid_{(x_\Gamma^j, x_\Gamma^{j+1})} = \begin{cases} \alpha & \text{if } I_j > 0 \\ \beta & \text{if } I_j < 0. \end{cases}$$

Let us take $0 < h_1 \le h_\tau$ such that

$$|\bar{d}(x_2) - \bar{d}(x_1)| < \frac{\tau}{4} \ \text{ if } \ |x_2 - x_1| < h_1.$$

This inequality along with (4.12) implies that

$$\text{if } \ \xi \in (x_\Gamma^j, x_\Gamma^{j+1}) \text{ and } \bar{d}(\xi) > \tau \Rightarrow \bar{d}_h(x) > \frac{\tau}{2} \ \ \forall x \in (x_\Gamma^j, x_\Gamma^{j+1}), \ \ \forall h < h_1,$$

which implies that $I_j > 0$, hence $\bar{u}_h \mid_{(x_\Gamma^j, x_\Gamma^{j+1})} = \alpha$, in particular $\bar{u}_h(\xi) = \alpha$ . From (3.4) we also deduce that $\bar{u}(x) = \alpha$. Therefore $(\bar{u}_h - \bar{u})(\xi) = 0$ whenever $\bar{d}(\xi) > \tau$ and $h < h_1$. Analogously we can prove that the same is true when $\bar{d}(\xi) < -\tau$. Moreover since $\alpha \le \bar{u}_h(x) \le \beta$, it is obvious that $(\bar{u}_h - \bar{u})(x) \ge 0$ if $\bar{u}(x) = \alpha$ and $(\bar{u}_h - \bar{u})(x) \le 0$ if $\bar{u}(x) = \beta$. Thus we have proved that $(\bar{u}_h - \bar{u}) \in C_{\bar{u}}^\tau$ and according to Remark 3.3-(2) we have

$$J''(\bar{u})(\bar{u}_h - \bar{u})^2 \ge \delta \|\bar{u}_h - \bar{u}\|_{L^2(\Gamma)}^2 \ \ \forall h < h_1. \tag{4.13}$$

On the other hand, by applying the mean value theorem we get for some $0 < \theta_h < 1$

$$(J'(\bar{u}_h) - J'(\bar{u}))(\bar{u}_h - \bar{u}) = J''(\bar{u} + \theta_h(\bar{u}_h - \bar{u}))(\bar{u}_h - \bar{u})^2 \ge$$

$$(J''(\bar{u} + \theta_h(\bar{u}_h - \bar{u})) - J''(\bar{u}))(\bar{u}_h - \bar{u})^2 + J''(\bar{u})(\bar{u}_h - \bar{u})^2 \ge$$

$$(\delta - \|J''(\bar{u} + \theta_h(\bar{u}_h - \bar{u})) - J''(\bar{u})\|)\, \|\bar{u}_h - \bar{u}\|_{L^2(\Gamma)}^2.$$

Finally it is enough to choose $0 < h_0 \le h_1$ such that

$$\|J''(\bar{u} + \theta_h(\bar{u}_h - \bar{u})) - J''(\bar{u})\| \le \frac{\delta}{2} \ \ \forall h < h_0$$

to prove (4.11). The last inequality can be obtained easily from the relation (2.6) thanks to the uniform convergence of $(\bar\varphi_h, \bar y_h, \bar u_h) \to (\bar\varphi, \bar y, \bar u)$ and the assumptions **(A1)-(A4)**. ☐

The next step consists of estimating the convergence of $J'_h$ to $J'$.

LEMMA 4.7. *For every $\rho > 0$ there exists $C_\rho > 0$ independent of $h$ such that*

$$|(J'_h(\bar u_h) - J'(\bar u_h))v| \le \left(C_\rho h + \rho\|\bar u_h - \bar u\|_{L^2(\Gamma)}\right)\|v\|_{L^2(\Gamma)} \quad \forall v \in L^2(\Gamma). \tag{4.14}$$

*Proof.* From the hypotheses on $l$ it is readily deduced

$$|(J'_h(\bar u_h) - J'(\bar u_h))v| \le \int_\Gamma \left(|\bar\varphi_h - \varphi_{\bar u_h}| + \left|\frac{\partial l}{\partial u}(x, \bar y_h, \bar u_h) - \frac{\partial l}{\partial u}(x, y_{\bar u_h}, \bar u_h)\right|\right) v\, d\sigma(x) \le$$

$$C\left(\|\bar\varphi_h - \varphi_{\bar u_h}\|_{L^2(\Gamma)} + \|\bar y_h - y_{\bar u_h}\|_{L^2(\Gamma)}\right)\|v\|_{L^2(\Gamma)}, \tag{4.15}$$

where $y_{\bar u_h}$ and $\varphi_{\bar u_h}$ are the solutions of (2.1) and (2.7) corresponding to $\bar u_h$.

We use the following well known property. For every $\varepsilon > 0$ there exists $C_\varepsilon > 0$ such that

$$\|z\|_{L^2(\Gamma)} \le \varepsilon\|z\|_{H^1(\Omega)} + C_\varepsilon\|z\|_{L^2(\Omega)}.$$

Thus we get with the aid of (4.7)

$$\|\bar y_h - y_{\bar u_h}\|_{L^2(\Gamma)} = \|y_h(\bar u_h) - y_{\bar u_h}\|_{L^2(\Gamma)} \le$$

$$\varepsilon\|y_h(\bar u_h) - y_{\bar u_h}\|_{H^1(\Omega)} + C_\varepsilon\|y_h(\bar u_h) - y_{\bar u_h}\|_{L^2(\Omega)} \le$$

$$\varepsilon\|y_h(\bar u_h) - y_{\bar u_h}\|_{H^1(\Omega)} + C_\varepsilon Ch = \varepsilon\|\bar y_h - y_{\bar u_h}\|_{H^1(\Omega)} + C_\varepsilon Ch.$$

Thanks to the monotonicity of $a_0$ and $b_0$ and the assumption **(A5)** we obtain from the state equation in the standard way

$$\|\bar y - y_{\bar u_h}\|_{H^1(\Omega)} \le C\|\bar u - \bar u_h\|_{L^2(\Gamma)}.$$

On the other hand, (4.8) leads to

$$\|\bar y - \bar y_h\|_{H^1(\Omega)} \le C\left(h + \|\bar u - \bar u_h\|_{L^2(\Gamma)}\right).$$

Combining the last three inequalities we deduce

$$\|\bar y_h - y_{\bar u_h}\|_{L^2(\Gamma)} \le C\left(\varepsilon\left(h + \|\bar u - \bar u_h\|_{L^2(\Gamma)}\right) + C_\varepsilon h\right).$$

The same arguments can be applied to the adjoint states, so (4.14) follows from (4.15). Inequality (4.14) is obtained by choosing $C\varepsilon = \rho$ and $C_\rho = C_\varepsilon + C\varepsilon$. ☐

One key point in the proof of error estimates is to get a discrete control $u_h \in U_h^{ad}$ that approximates $\bar u$ conveniently and satisfies $J'(\bar u)\bar u = J'(\bar u)u_h$. Let us find such a control. Let $\bar d$ be defined as in §3 and set $I_j$ for every $1 \le j \le N(h)$ as in the proof of Lemma 4.6

$$I_j = \int_{x_\Gamma^j}^{x_\Gamma^{j+1}} \bar d(x)\, d\sigma(x).$$

Now we define $u_h \in U_h$ with $u_h(x) \equiv u_h^j$ on the intervals $(x_\Gamma^j, x_\Gamma^{j+1})$ by the expression

$$
u_h^j = \begin{cases} \dfrac{1}{I_j} \displaystyle\int_{x_\Gamma^j}^{x_\Gamma^{j+1}} \bar{d}(x)\bar{u}(x)\, d\sigma(x) & \text{if } I_j \neq 0 \\[2ex] \dfrac{1}{|x_\Gamma^j - x_\Gamma^{j+1}|} \displaystyle\int_{x_\Gamma^j}^{x_\Gamma^{j+1}} \bar{u}(x)\, d\sigma(x) & \text{if } I_j = 0. \end{cases} \tag{4.16}
$$

This $u_h$ satisfies our requirements.

LEMMA 4.8. *There exists $h_0 > 0$ such that for every $0 < h < h_0$ the following properties hold:*

1. $u_h \in U_h^{ad}$.
2. $J'(\bar{u})\bar{u} = J'(\bar{u})u_h$.
3. *There exists $C > 0$ independent of $h$ such that*

$$
\|\bar{u} - u_h\|_{L^\infty(\Gamma)} \leq Ch. \tag{4.17}
$$

*Proof.* Let $L_u > 0$ be the Lipschitz constant of $\bar{u}$ and take $h_0 = (\beta - \alpha)/(2L_u)$, then

$$
|\bar{u}(\xi_2) - \bar{u}(\xi_1)| \leq L_u|\xi_2 - \xi_1| \leq L_u h < \frac{\beta - \alpha}{2} \quad \forall \xi_1, \xi_2 \in [x_\Gamma^j, x_\Gamma^{j+1}],
$$

which implies that $\bar{u}$ can not admit the values $\alpha$ and $\beta$ on one segment $[x_\Gamma^j, x_\Gamma^{j+1}]$ for all $h < h_0$. Hence the sign of $\bar{d}$ on $[x_\Gamma^j, x_\Gamma^{j+1}]$ must be constant due to (3.4). Therefore, $I_j = 0$ if and only if $\bar{d}(x) = 0$ for all $x \in [x_\Gamma^j, x_\Gamma^{j+1}]$. Moreover if $I_j \neq 0$, then $\bar{d}(x)/I_j \geq 0$ for every $x \in [x_\Gamma^j, x_\Gamma^{j+1}]$. As a first consequence of this we get that $\alpha \leq u_h^j \leq \beta$, which means that $u_h \in U_h^{ad}$. On the other hand

$$
J'(\bar{u})u_h = \sum_{j=1}^{N(h)} \int_{x_\Gamma^j}^{x_\Gamma^{j+1}} \bar{d}(x)\, d\sigma(x) u_h^j = \sum_{j=1}^{N(h)} \int_{x_\Gamma^j}^{x_\Gamma^{j+1}} \bar{d}(x)\bar{u}(x)\, d\sigma(x) = J'(\bar{u})\bar{u}.
$$

Finally let us prove (4.17). Since the sign of $\bar{d}(x)/I_j$ is always non negative and $\bar{d}$ is a continuous function, we get for any of the two possible definitions of $u_h^j$ the existence of a point $\xi^j \in [x_\Gamma^j, x_\Gamma^{j+1}]$ such that $u_h^j = \bar{u}(\xi_j)$. Therefore, for any $x \in [x_\Gamma^j, x_\Gamma^{j+1}]$

$$
|\bar{u}(x) - u_h(x)| = |\bar{u}(x) - u_h^j| = |\bar{u}(x) - \bar{u}(\xi^j)| \leq L_u|x - \xi^j| \leq L_u h,
$$

which leads to (4.17) ☐

Finally, we derive the main error estimate.

THEOREM 4.9. *There exists a constant $C > 0$ independent of $h$ such that*

$$
\|\bar{u} - \bar{u}_h\|_{L^2(\Gamma)} \leq Ch. \tag{4.18}
$$

*Proof.* Setting $u = \bar{u}_h$ in (3.3) we get

$$
J'(\bar{u})(\bar{u}_h - \bar{u}) = \int_\Gamma \left( \frac{\partial l}{\partial u}(x, \bar{y}, \bar{u}) + \bar{\varphi} \right) (\bar{u}_h - \bar{u})\, d\sigma \geq 0. \tag{4.19}
$$

From (4.4) with $u_h$ defined by (4.16) it follows

$$J_h'(\bar u_h)(u_h - \bar u_h) = \int_\Gamma \left(\bar\varphi_h + \frac{\partial l}{\partial u}(x, \bar y_h, \bar u_h)\right)(u_h - \bar u_h)\,d\sigma(x) \geq 0$$

and then

$$J_h'(\bar u_h)(\bar u - \bar u_h) + J_h'(\bar u_h)(u_h - \bar u) \geq 0. \tag{4.20}$$

By adding (4.19) and (4.20) and using Lemma 4.8-2, we derive

$$\left(J'(\bar u) - J_h'(\bar u_h)\right)(\bar u - \bar u_h) \leq J_h'(\bar u_h)(u_h - \bar u) = \left(J_h'(\bar u_h) - J'(\bar u)\right)(u_h - \bar u).$$

For $h$ small enough, this inequality and (4.11) lead to

$$\frac{\delta}{2}\|\bar u - \bar u_h\|^2_{L^2(\Gamma)} \leq \left(J'(\bar u) - J'(\bar u_h)\right)(\bar u - \bar u_h) \leq$$

$$\left(J_h'(\bar u_h) - J'(\bar u_h)\right)(\bar u - \bar u_h) + \left(J_h'(\bar u_h) - J'(\bar u)\right)(u_h - \bar u). \tag{4.21}$$

Arguing as in (4.15) and using (4.8) and (4.17) we get

$$\left|\left(J_h'(\bar u_h) - J'(\bar u)\right)(u_h - \bar u)\right| \leq C\left(\|\bar\varphi_h - \bar\varphi\|_{L^2(\Gamma)} + \|\bar y_h - \bar y\|_{L^2(\Gamma)}\right)\|u_h - \bar u\|_{L^2(\Gamma)} \leq$$

$$C\left(h + \|\bar u - \bar u_h\|_{L^2(\Gamma)}\right)\|u_h - \bar u\|_{L^2(\Gamma)} \leq C\left(h^2 + h\|\bar u - \bar u_h\|_{L^2(\Gamma)}\right). \tag{4.22}$$

On the other hand, using (4.14)

$$\left|\left(J_h'(\bar u_h) - J'(\bar u_h)\right)(\bar u - \bar u_h)\right| \leq \left(C_\rho h + \rho\|\bar u - \bar u_h\|_{L^2(\Gamma)}\right)\|\bar u - \bar u_h\|_{L^2(\Gamma)}.$$

By taking $\rho = \delta/4$, we deduce from this inequality along with (4.21) and (4.22)

$$\frac{\delta}{4}\|\bar u - \bar u_h\|^2_{L^2(\Gamma)} \leq Ch^2 + (C + C_\rho)h\|\bar u - \bar u_h\|_{L^2(\Gamma)},$$

which proves (4.18) for a convenient constant $C$ independent of $h$. ∎

**5. Numerical confirmation.** In this section we shall verify our error estimates by numerical test examples for which we know the exact solution. We report both on a linear-quadratic problem and on a semilinear problem.

**5.1. A linear-quadratic problem and primal-dual active set strategy.** Let us consider the problem

$$\text{(E1)}\begin{cases} \min J(u) = \dfrac{1}{2}\displaystyle\int_\Omega (y_u(x) - y_\Omega(x))^2 dx + \dfrac{\mu}{2}\int_\Gamma u(x)^2 d\sigma(x) + \\ + \displaystyle\int_\Gamma e_u(x)u(x)d\sigma(x) + \int_\Gamma e_y(x)y_u(x)d\sigma(x) \\ \\ \text{subject to } (y_u, u) \in H^1(\Omega) \times L^\infty(\Gamma), \\ u \in U_{ad} = \{u \in L^\infty(\Gamma) \,|\, 0 \leq u(x) \leq 1 \text{ a.e. } x \in \Gamma\}, \\ (y_u, u) \text{ satisfying the linear state equation (5.1)} \end{cases}$$

$$\text{(5.1)}\qquad \left\{\begin{array}{rcll} -\Delta y_u(x)+c(x)y_u(x) &=& e_1(x) & \text{in} \quad \Omega \\ \partial_\nu y_u(x)+y_u(x) &=& e_2(x)+u(x) & \text{on} \quad \Gamma. \end{array}\right.$$

We fix the following data: $\Omega=(0,1)^2$, $\mu=1$, $c(x_1,x_2)=1+x_1^2-x_2^2$, $e_y(x_1,x_2)=1$, $y_\Omega(x_1,x_2)=x_1^2+x_1x_2$, $e_1(x_1,x_2)=-2+(1+x_1^2-x_2^2)(1+2x_1^2+x_1x_2-x_2^2)$,

$$e_u(x_1,x_2)=\left\{\begin{array}{ll} -1-x_1^3 & \text{on } \Gamma_1 \\ -1-\min\left\{\begin{array}{l} 8(x_2-0.5)^2+0.5, \\ 1-16x_2(x_2-0.25)(x_2-0.75)(x_2-1) \end{array}\right\} & \text{on } \Gamma_2 \\ -1-x_1^2 & \text{on } \Gamma_3 \\ -1+x_2(1-x_2) & \text{on } \Gamma_4 \end{array}\right.$$

and

$$e_2(x_1,x_2)=\left\{\begin{array}{ll} 1-x_1+2x_1^2-x_1^3 & \text{on } \Gamma_1 \\ 7+2x_2-x_2^2-\min\{8(x_2-.5)^2+.5,1\} & \text{on } \Gamma_2 \\ -2+2x_1+x_1^2 & \text{on } \Gamma_3 \\ 1-x_2-x_2^2 & \text{on } \Gamma_4, \end{array}\right.$$

where $\Gamma_1$ to $\Gamma_4$ are the four sides of the square, starting at the bottom side and turning counterclockwise. This problem has the following solution $(\bar y,\bar u)$ with adjoint state $\bar\varphi$: $\bar y(x)=1+2x_1^2+x_1x_2-x_2^2$, $\bar\varphi(x_1,x_2)=1$ and

$$\bar u(x_1,x_2)=\left\{\begin{array}{ll} x_1^3 & \text{on } \Gamma_1 \\ \min\{8(x_2-.5)^2+.5,1\} & \text{on } \Gamma_2 \\ x_1^2 & \text{on } \Gamma_3 \\ 0 & \text{on } \Gamma_4. \end{array}\right.$$

It is not difficult to check that the state equation (5.1) is satisfied by $(\bar y,\bar u)$. The same refers to the adjoint equation

$$\left\{\begin{array}{rcll} -\Delta\bar\varphi(x)+c(x)\bar\varphi(x) &=& \bar y(x)-y_\Omega(x) & \text{in} \quad \Omega \\ \partial_\nu\bar\varphi(x)+\bar\varphi(x) &=& e_y & \text{on} \quad \Gamma. \end{array}\right.$$

In example (E1), the function

$$\bar d(x)=\bar\varphi(x)+e_u(x)+\bar u(x)=\left\{\begin{array}{ll} 0 & \text{on } \Gamma_1 \\ \min\{0,16x_2(x_2-0.25)(x_2-0.75)(x_2-1)\} & \text{on } \Gamma_2 \\ 0 & \text{on } \Gamma_3 \\ x_2(1-x_2) & \text{on } \Gamma_4 \end{array}\right.$$

satisfies the relations (3.4) (see figure 5.1), hence the first order necessary condition (3.3) is fulfilled. Since (E1) is a convex problem, this condition is also sufficient for $(\bar y,\bar u)$ to be global minimum.

Let us briefly describe how we have performed the optimization. We define the following operators: $S: L^2(\Gamma)\to L^2(\Omega)$, and $\tau: L^2(\Gamma)\to L^2(\Gamma)$. For $u\in L^2(\Gamma)$, $Su=y$, and $\tau u=y_{|\Gamma}$, where

$$\left\{\begin{array}{rcll} -\Delta y(x)+c(x)y(x) &=& 0 & \text{in} \quad \Omega \\ \partial_\nu y(x)+y(x) &=& u(x) & \text{on} \quad \Gamma. \end{array}\right.$$

If we define $y_0$ as the state associated to $u(x)=0$ for all $x\in\Gamma$ and set $y_d(x)=y_\Omega(x)-y_0(x)$ then minimizing $J(u)$ is equivalent to minimize

$$\tilde J(u)=\frac{1}{2}(S^*Su+u,u)_{L^2(\Gamma)}+(e_u+\tau S^*e_y-S^*y_d,u)_{L^2(\Gamma)},$$

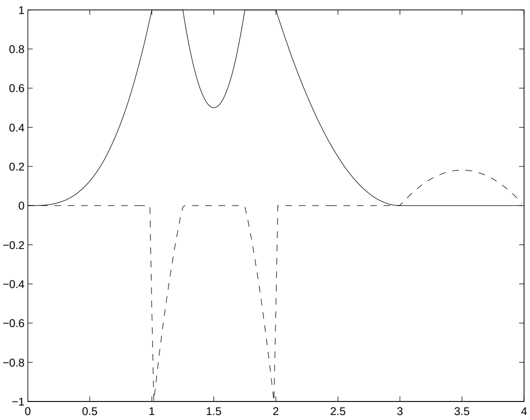



FIG. 5.1. *solid:* $\bar{u}(x_1, x_2)$*, dashed:* $\bar{d}(x_1, x_2)$

subject to $u \in U_{ad}$ where $(\cdot,\cdot)_X$ denotes the inner scalar product in the space $X$.

We perform the discretization in two steps. First we discretize the control and thereafter the state. Let us take $\{e_j\}_{j=1}^{N(h)}$ as a basis of $U_h$. If $u_h(x) = \sum_{j=1}^{N(h)} u_j e_j(x)$ for $x \in \Gamma$, we must perform the optimization over $U_h$ of

$$\tilde{J}(u_h) = \frac{1}{2}\sum_{i,j=1}^{N(h)} u_i u_j (S^*Se_i + e_i, e_j)_{L^2(\Gamma)} + \sum_{j=1}^{N(h)} u_j (e_j, e_u + \tau S^* e_y - S^* y_d)_{L^2(\Gamma)}$$

subject to $0 \le u_j \le 1$ for $j = 1, \dots, N(h)$.

If we set $A_{i,j} = (S^*Se_i + e_i, e_j)_{L^2(\Gamma)}$, $b_i = (e_i, e_u + \tau S^* e_y - S^* y_d)_{L^2(\Gamma)}$ and $\vec{u} = (u_1, \dots . u_{N(h)})^T$, then we must minimize

$$f(\vec{u}) = \frac{1}{2}\vec{u}^T A \vec{u} + \vec{b}^T \vec{u}$$

subject to $0 \le u_j \le 1$ for $j = 1, \dots, N(h)$. Calculating the matrix $A$ explicitely would require solving $2N(h)$ partial differential equations, and this is numerically too expensive. Therefore usual routines to perform quadratic constrained minimization should not be used. General optimization programs that require only an external routine providing the function and its gradient do not take advantage of the fact that we indeed have a quadratic functional. Therefore, we have implemented our own routine for a primal-dual active set strategy according to Bergounioux and Kunisch [4]; see also Kunisch and Rösch [15]. Let us briefly describe the main steps of this iterative method. At each step $n$, we solve an unconstrained problem to get $(\vec{u}_{n+1}, y_{n+1}, \varphi_{n+1})$. To get the next iterate, we fix the current active sets by

$$A^n_{h,+} = \{j \in \{1, \dots, N(h)\} \mid u^j_n - \partial_{u_j} f(\vec{u}_n) > 1\}$$

and

$$A^n_{h,-} = \{j \in \{1, \dots, N(h)\} \mid u^j_n - \partial_{u_j} f(\vec{u}_n) < 0\}.$$

Notice that $\partial_{u_j} f(\vec{u}_n) = J'(u_{h,n})e_j$. We define a vector $\vec{u}^{act}_{n+1}$ that has zeros in all its components, except those belonging to $A^n_{h,+}$, which are set to 1 and those belonging to $A^n_{h,-}$ which are set to the lower bound (which is also zero in this problem). Set $m = N(h) - |A^n_{h,+}| - |A^n_{h,-}|$. We define a matrix $K$ with $n$ rows and $m$ columns such that row $j$ is the zero vector if $j \in A^n_{h,+} \cup A^n_{h,-}$ and the submatrix formed by the rest of the rows is the identity $m \times m$ matrix. At each iteration we must minimize $f(K\vec{v} + \vec{u}^{act}_{n+1})$, where $v \in \mathbb{R}^m$. This is equivalent to minimizing

$$q(v) = \frac{1}{2}\vec{v}^T K^T A K \vec{v} + (K^T(\vec{b} + A\vec{u}^{act}_{n+1}))^T \vec{v}$$

for $\vec{v} \in \mathbb{R}^m$. Since it is not possible to compute the whole matrix $A$, we solve this problem by the conjugate gradient method. At each iteration of this method we must evaluate $A\vec{w}$ for some $\vec{w} \in \mathbb{R}^{N(h)}$. If we define $w = \sum_{j=1}^{N(h)} w_j e_j$, the component $i$ of the vector $A\vec{w}$ is given by $(e_i, \varphi + w)_{L^2(\Gamma)}$, where $\varphi$ is obtained solving the two partial differential equations

$$\left\{ \begin{array}{rl} -\Delta y(x) + c(x)y(x) = & 0 \text{ in } \Omega \\ \partial_\nu y(x) + y(x) = & w(x) \text{ on } \Gamma \end{array} \right. \quad \text{and} \quad \left\{ \begin{array}{rl} -\Delta\varphi(x) + c(x)\varphi(x) = & y(x) \text{ in } \Omega \\ \partial_\nu \varphi(x) + \varphi(x) = & 0 \text{ on } \Gamma. \end{array} \right.$$

These equations are solved by the finite element method. We have used the MATLAB PDE Toolbox just to get the mesh for $\Omega$, but we have performed the assembling of the mass and stiffness matrices and of the right hand side vector with our own routines to determine all the integrals in an exact way. We had two reasons to do this. First, we have not included the effect of integration errors in our previous research, and secondly, when making a non-exact integration, the approximate adjoint state is possibly not the adjoint state of the approximate state. This fact may negatively affect the convergence. In practice, a low order integration method slows down the convergence.

The solution is achieved if $A^n_{h,+} = A^{n+1}_{h,+}$ and $A^n_{h,-} = A^{n+1}_{h,-}$. It is shown in Kunisch and Rösch [15] that the algorithm terminates in finitely many iterations for the discretized problem.

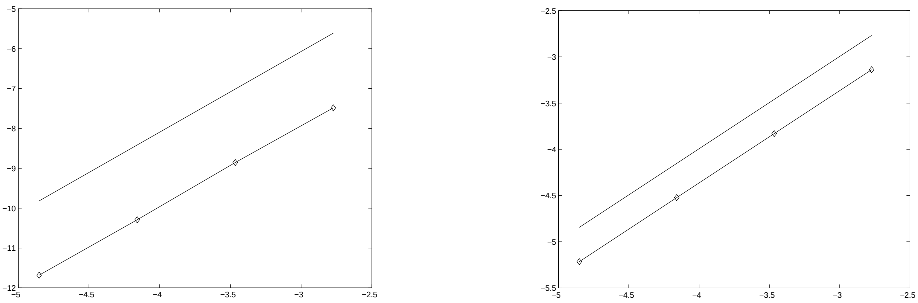
Observe that the discretization of the state can be done independently of the discretization of the controls. We have performed two tests to show that the bottleneck of the error in the control is the discretization of the controls. In the first test we have chosen the same mesh sizes both for the state and the control. In the second test we have chosen a fixed small mesh size for the state and we have varied the mesh size for the control. These are the results:

**Test 1.**

| $h$ | $\|\bar{y} - \bar{y}_h\|_{L^2(\Omega)}$ | $\|\bar{y} - \bar{y}_h\|_{H^1(\Omega)}$ | $\|\bar{u} - \bar{u}_h\|_{L^2(\Gamma)}$ | $\|\bar{u} - \bar{u}_h\|_{L^\infty(\Gamma)}$ |
|---|---|---|---|---|
| $2^{-4}$ | $5.617876e-04$ | $7.259364e-02$ | $4.330776e-02$ | $1.146090e-01$ |
| $2^{-5}$ | $1.423977e-04$ | $3.635482e-02$ | $2.170775e-02$ | $5.990258e-02$ |
| $2^{-6}$ | $3.500447e-05$ | $1.800239e-02$ | $1.086060e-02$ | $3.060061e-02$ |
| $2^{-7}$ | $8.971788e-06$ | $8.950547e-03$ | $5.431141e-03$ | $1.546116e-02$ |

The orders of convergence obtained are $h^2$ for $\|\bar{y} - \bar{y}_h\|_{L^2(\Omega)}$ and $h$ for the seminorm in $H^1(\Omega)$ the $L^2(\Gamma)$ and $L^\infty(\Gamma)$ norms. We can see this comparing a double logarithmic plot of $h$ and one of the error estimate with the plot of $p\log(h)$, where $p$ is the order of convergence obtained.

The estimates $|\bar{y} - \bar{y}_h|_{H^1(\Omega)} \le Ch$ and for $\|\bar{u} - \bar{u}_h\|_{L^2(\Gamma)} \le Ch$ are the ones expected from inequalities (4.8) and (4.18). The estimate $\|\bar{y} - \bar{y}_h\|_{L^2(\Omega)} \le Ch^2$ is indeed better than the one we can expect from inequality (4.7). This cannot only be explained by the information that $\bar{y} \in H^2(\Omega)$ ensures order $h^2$ for the FEM. Neverheless, the observed order $h^2$ can be theoretically justified. A forthcoming paper by A. Rösch studies this case.



Test 1: $\|\bar{y} - \bar{y}_h\|_{L^2(\Omega)} \leq Ch^2$ Test 1: $\|\bar{u} - \bar{u}_h\|_{L^2(\Gamma)} \leq Ch$

FIG. 5.2. *Solid line: $p \log h$. Dotted line: Data from Test 1.*

**Test 2.** We fix now the mesh size for the state to $h_y = 2^{-7}$. This ensures a fairly accurate solution of the partial differential equations.

| $h$ | $\|\bar{y} - \bar{y}_h\|_{L^2(\Omega)}$ | $\|\bar{y} - \bar{y}_h\|_{H^1(\Omega)}$ | $\|\bar{u} - \bar{u}_h\|_{L^2(\Gamma)}$ | $\|\bar{u} - \bar{u}_h\|_{L^\infty(\Gamma)}$ |
|---|---|---|---|---|
| $2^{-4}$ | $1.831053e-04$ | $9.837630e-03$ | $4.330774e-02$ | $1.145890e-01$ |
| $2^{-5}$ | $4.648617e-05$ | $9.026588e-03$ | $2.170775e-02$ | $5.989731e-02$ |
| $2^{-6}$ | $1.424508e-05$ | $8.952289e-03$ | $1.086060e-02$ | $3.059955e-02$ |
| $2^{-7}$ | $8.971788e-06$ | $8.950547e-03$ | $5.431141e-03$ | $1.546116e-02$ |

The error for the state is very small from the beginning. The order is again $h$ for the last two columns. We observe that refining the mesh for the state does not improve the approximation of the control.

**5.2. A semilinear example.** Let us next consider the problem

$$\text{(E2)} \begin{cases} \min J(u) = \dfrac{1}{2}\displaystyle\int_\Omega (y_u(x) - y_\Omega(x))^2 dx + \dfrac{\mu}{2}\int_\Gamma u(x)^2 d\sigma(x) + \\ + \displaystyle\int_\Gamma e_u(x)u(x)d\sigma(x) + \int_\Gamma e_y(x)y_u(x)d\sigma(x) \\ \text{subject to } (y_u, u) \in H^1(\Omega) \times L^\infty(\Gamma), \\ u \in U_{ad} = \{u \in L^\infty(\Gamma) \,|\, 0 \leq u(x) \leq 1 \text{ a.e. } x \in \Gamma\}, \\ (y_u, u) \text{ satisfying the semilinear state equation (5.2)} \end{cases}$$

$$\text{(5.2)} \quad \begin{cases} -\Delta y_u(x) + c(x)y_u(x) = e_1(x) & \text{in } \Omega \\ \partial_\nu y_u(x) + y_u(x) = e_2(x) + u(x) - y(x)|y(x)| & \text{on } \Gamma. \end{cases}$$

The term $y|y|$ stands for $y^2$ that does not satisfy the assumptions on monotonicity required for our current work. However, in our computations negative values of $y$ never occured so that in fact $y^2$ was used. This also assures that locally assumption **(A4)** is satisfied.

We fix: $\Omega = (0,1)^2$, $\mu = 1$, $c(x_1, x_2) = x_2^2 + x_1x_2$, $e_y(x_1, x_2) = -3 - 2x_1^2 - 2x_1x_2$, $y_\Omega(x_1, x_2) = 1 + (x_1 + x_2)^2$, $e_1(x_1, x_2) = -2 + (1 + x_1^2 + x_1x_2)(x_2^2 + x_1x_2)$,

$$e_u(x_1,x_2) = \begin{cases} 1 - x_1^3 & \text{on } \Gamma_1 \\ 1 - \min\left\{ \begin{array}{l} 8(x_2-0.5)^2+0.5, \\ 1-16x_2(x_2-0.25)(x_2-0.75)(x_2-1) \end{array} \right\} & \text{on } \Gamma_2 \\ 1 - x_1^2 & \text{on } \Gamma_3 \\ 1 + x_2(1-x_2) & \text{on } \Gamma_4 \end{cases}$$

and

$$e_2(x_1,x_2) = \begin{cases} 2 - x_1 + 3x_1^2 - x_1^3 + x_1^4 & \text{on } \Gamma_1 \\ 8 + 6x_2 + x_2^2 - \min\{8(x_2-.5)^2+.5, 1\} & \text{on } \Gamma_2 \\ 2 + 4x_1 + 3x_1^2 + 2x_1^3 + x_1^4 & \text{on } \Gamma_3 \\ 2 - x_2 & \text{on } \Gamma_4. \end{cases}$$

This problem has the following solution $(\bar{y},\bar{u})$ with adjoint state $\bar{\varphi}$: $\bar{y}(x) = 1 + 2x_1^2 + x_1x_2$, $\bar{\varphi}(x_1,x_2) = -1$ and $\bar{u}$ is the same as in example (E1). Again, it holds $\bar{d}(x) = \bar{\varphi}(x) + e_u(x) + \bar{u}(x)$, which is also the same as in example (E1) and satisfies relation (3.4) so that the first order necessary condition (3.3) is fulfilled. The second derivative of $J(\bar{u})$ is, according to (2.6),

$$J''(\bar{u})v^2 = \int_\Omega z_v(x)^2dx + \int_\Gamma v(x)^2 d\sigma(x) + \int_\Gamma (-2)\text{sign}(\bar{y}(x))\bar{\varphi}(x) z_v(x)^2 d\sigma(x),$$

where $z_v$ is given by equation (2.3). Since $\bar{\varphi}(x) \le 0$ and $\bar{y}(x) \ge 0$, clearly $J''(\bar{u})v^2 \ge \|v\|^2_{L^2(\Gamma)}$ holds. Therefore the second order sufficient conditions are fulfilled.

For the optimization, a standard SQP method was implemented. Given $w_k = (y_k, u_k, \varphi_k)$, at step $k+1$ we have to solve the following linear-quadratic problem to find $(y_{k+1}, u_{k+1})$:

$$(\text{QP})_{k+1} \begin{cases} \min J_{k+1}(u_{k+1}) = \dfrac{1}{2}\displaystyle\int_\Omega (y_{k+1}(x) - y_\Omega(x))^2 dx + \dfrac{1}{2}\int_\Gamma u_{k+1}(x)^2 d\sigma(x) + \\ + \displaystyle\int_\Gamma e_u(x) u_{k+1}(x) d\sigma(x) + \int_\Gamma e_y(x) y_{k+1}(x) d\sigma(x) - \\ - \displaystyle\int_\Gamma \text{sign}(y_k(x))\varphi_k(x)(y_{k+1}(x) - y_k(x))^2 d\sigma(x) \\ \text{subject to } (y_{k+1}, u_{k+1}) \in H^1(\Omega) \times L^\infty(\Gamma), \\ u_{k+1} \in U_{ad}, \\ (y_{k+1}, u_{k+1}) \text{ satisfying the linear state equation (5.3)} \end{cases}$$

$$(5.3) \begin{cases} -\Delta y_{k+1}(x) + c(x) y_{k+1}(x) &= e_1(x) & \text{in } \Omega \\ \partial_\nu y_{k+1}(x) + y_{k+1}(x) &= e_2(x) + u_{k+1}(x) - y_k(x)|y_k(x)| - & \\ & \quad -2|y_k(x)|(y_{k+1}(x) - y_k(x)) & \text{on } \Gamma. \end{cases}$$

The new iterate $\varphi_{k+1}$ is the solution of the associated adjoint equation. It is known that the sequence $\{w_k\}$ converges quadratically to $\bar{w} = \{(\bar{y},\bar{u},\bar{\varphi})\}$ in the $L^\infty$ norm provided that the initial guess is taken close to $\bar{w}$, where $(\bar{y},\bar{u})$ is a local solution of (E2) and $\bar{\varphi}$ is the associated adjoint state:

$$\|w_{k+1} - \bar{w}\|_{C(\bar{\Omega})\times L^\infty(\Gamma)\times C(\bar{\Omega})} \le C\|w_k - \bar{w}\|^2_{C(\bar{\Omega})\times L^\infty(\Gamma)\times C(\bar{\Omega})}.$$

To solve each of the linear-quadratic problems $(QP)_k$ we have applied the primal-dual active set strategy explained for (E1). For the semilinear example the same tests were made as for (E1). First we considered the same mesh both for control and state. Next a very fine mesh was taken for the state while refining the meshes for the control.

**Test 1.**

| $h$ | $\|\bar{y}-\bar{y}_h\|_{L^2(\Omega)}$ | $\|\bar{y}-\bar{y}_h\|_{H^1(\Omega)}$ | $\|\bar{u}-\bar{u}_h\|_{L^2(\Gamma)}$ | $\|\bar{u}-\bar{u}_h\|_{L^\infty(\Gamma)}$ |
|---|---|---|---|---|
| $2^{-4}$ | $3.178397e-04$ | $3.547400e-02$ | $4.330792e-02$ | $1.145619e-01$ |
| $2^{-5}$ | $8.094299e-05$ | $1.769994e-02$ | $2.170777e-02$ | $5.988813e-02$ |
| $2^{-6}$ | $1.983313e-05$ | $8.783231e-03$ | $1.086060e-02$ | $3.059566e-02$ |
| $2^{-7}$ | $4.938929e-06$ | $4.365300e-03$ | $5.431140e-03$ | $1.546130e-02$ |

The observed orders of convergence are again $h^2$ for $\|\bar{y}-\bar{y}_h\|_{L^2(\Omega)}$ and $h$ for the other columns.

**Test 2.** We fix now the mesh size for the state to $h_y = 2^{-7}$. This ensures a fairly accurate solution of the partial differential equations. The order of convergence for the error in the control is again $h$.

| $h$ | $\|\bar{y}-\bar{y}_h\|_{L^2(\Omega)}$ | $\|\bar{y}-\bar{y}_h\|_{H^1(\Omega)}$ | $\|\bar{u}-\bar{u}_h\|_{L^2(\Gamma)}$ | $\|\bar{u}-\bar{u}_h\|_{L^\infty(\Gamma)}$ |
|---|---|---|---|---|
| $2^{-4}$ | $1.093204e-04$ | $5.695770e-03$ | $4.330780e-02$ | $1.145649e-01$ |
| $2^{-5}$ | $2.782787e-05$ | $4.498224e-03$ | $2.170776e-02$ | $5.988683e-02$ |
| $2^{-6}$ | $8.585435e-06$ | $4.367794e-03$ | $1.086060e-02$ | $3.059585e-02$ |
| $2^{-7}$ | $4.938929e-06$ | $4.365300e-03$ | $5.431140e-03$ | $1.546130e-02$ |

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