Error Rate Performance of Coded Free-Space Optical Links Over Strong Turbulence Channels

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***Abstract*—Error control coding can be used over free-space optical (FSO) links to mitigate turbulence-induced fading. In this letter, we present error rate performance bounds for coded FSO communication systems operating over atmospheric turbulence channels, which are modeled as correlated K distribution under strong turbulence conditions. We derive an upper bound on the pairwise error probability (PEP) and then apply the union-bound technique in conjunction with the derived PEP to obtain upper bounds on the bit error rate. Simulation results are further demonstrated to verify the analytical results.**

***Index Terms*—Atmospheric turbulence channel, free-space optical communication (FSO), pairwise error probability (PEP).**

I. Introduction

Wireless optical communications, also known as free-space optical (FSO) communications, is a cost-effective and high bandwidth access technique, which is receiving growing attention with recent commercialization successes [1], [2]. One major impairment over FSO links is the atmospheric turbulence, which results in fluctuations at the received signal, severely degrading the link performance. Error control coding, as well as diversity techniques, can be used over FSO links to improve the error rate performance [2]–[4]. In [4], Zhu and Kahn studied the performance of coded FSO links assuming a log-normal channel model for atmospheric turbulence. Specifically, they derived an approximate upper bound on the pairwise error probability (PEP) for a coded FSO communication system with intensity modulation/direct direction (IM/DD) and provided upper bounds on the bit error rate (BER) using the transfer function technique. Although lognormal distribution is the most widely used model for the probability density function (pdf) of the irradiance due to its simplicity, this pdf model is only applicable to weak turbulence conditions [2]. As the strength of turbulence increases, multiple scattering effects must be taken into account and log-normal statistics exhibits large deviations compared to experimental data. One of the widely accepted models under strong turbulence regime is the K *distribution* [2]. It was shown in [5], [6] that this channel model provides good agreement with experimental data in a variety of experiments involving radiation scattered by strong turbulent media. It should be further noted that K distribution was also proposed as a good approximation to Rayleigh-lognormal channels in the wireless radio frequency (RF) communication literature [7] and used in the error rate performance analysis [8]. However, one should be careful of the different underlying detection techniques in wireless optical and wireless RF systems: In a typical IM/DD FSO system the received current from the optical detector is proportional to the square of the electromagnetic field and thus statistical models for atmospheric-induced turbulence (i.e. intensity fading) correspond to those applied to *power* in the coherent RF problem where the received current is proportional to the field. Therefore, the results in [8] can not be applied to performance analysis of FSO links in a straightforward manner.

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II. The Correlated K Channel Model

In the K channel model for atmospheric turbulence channels [5], [6], the irradiance I can be considered as a product of two independent random variables $I = yz$, where y and z follow exponential distribution and gamma distribution, respectively

$$f_y(y) = \exp(-y), \qquad y > 0 \tag{1}$$

$$f_z(z) = \frac{\alpha^{\alpha} z^{\alpha-1}\exp(-\alpha z)}{\Gamma(\alpha)}, \qquad z > 0. \tag{2}$$

Here $\Gamma(.)$ stands for the gamma function and α is a channel parameter related to the effective number of discrete scatterers. By first fixing y and writing $z = I/y$, we obtain the conditional pdf

$$f_{I|z}(I|z) = \left(\frac{1}{z}\right) f_y\left(\frac{I}{z}\right) = \left(\frac{1}{z}\right)\exp\left(\frac{-I}{z}\right) \tag{3}$$

in which z is the (conditional) mean value of I. The distribution for the irradiance is then found as [2]

$$f_I(I) = \frac{2}{\Gamma(\alpha)}\alpha^{\frac{(\alpha+1)}{2}} I^{\frac{(\alpha-1)}{2}} K_{\alpha-1}\left(2\sqrt{\alpha I}\right), \qquad I > 0 \tag{4}$$

where $K_a(.)$ is the modified Bessel function of the second kind of order a. For most of the practical FSO applications, the intensity fading is temporally correlated. Unfortunately, a correlation model for K-distributed atmospheric turbulence channels is not available in the wireless optical literature. However, following the rich literature in modeling correlated K clutters for radar communications, we assume an exponential time correlation model [9] where the fading coefficients follow the joint pdf

$$f_{I_1,\ldots I_M}(I_1,\ldots I_M)=\int_0^\infty\cdots\int_0^\infty f_y\left(\frac{I_1}{z_1}\right)\cdots f_y\left(\frac{I_M}{z_M}\right)\times\frac{f_{z_1,\ldots z_M}(z_1,\ldots z_M)}{z_1\times\ldots\times z_M}dI_1\ldots dI_M \quad (5)$$

with

$$\begin{aligned} f_{z_1,\ldots z_M}&(z_1,\ldots z_M)\\ &= f_{z_M|z_{M-1}}(z_M|z_{M-1})\ldots f_{z_2|z_1}(z_2|z_1)f_{z_1}(z_1)\\ f_{z_m|z_{m-1}}&(z_m|z_{m-1})\\ &=\frac{\alpha}{\rho^{\alpha-1}(1-\rho^2)}\left(\frac{z_m}{z_{m-1}}\right)^{\frac{\alpha-1}{2}}\exp\left(-\frac{\alpha\left(z_m+\rho^2 z_{m-1}\right)}{1-\rho^2}\right)\\ &\times I_{\alpha-1}\left[\frac{2\alpha\rho}{1-\rho^2}\sqrt{z_m z_{m-1}}\right]. \end{aligned} \quad (6)$$

In (5), the gamma random variable z_m can be written as $z_m = (1/2\alpha)\sum_{i=1}^{2\alpha}(g_i^m)^2$ in terms of channel parameter α and zero-mean, unit variance Gaussian random variables g_i^m. The correlation among g_i^m and $g_i^{m'}$ is given as $\rho_g = \rho^{|m-m'|}$ following the exponential model [9], where $0 \le \rho < 1$ is the correlation coefficient. It can be easily verified that the correlation between fading coefficients I_m and $I_{m'}$ is then obtained as $\rho_I = \rho_g/(2+\alpha)$.

III. Derivation of Upper Bounds on PEP and BER

The PEP represents the probability of choosing the coded bit sequence $\hat{\boldsymbol{X}} = (\hat{x}_1,\hat{x}_2,\ldots,\hat{x}_M)$ when indeed $\boldsymbol{X} = (x_1,x_2,\ldots,x_M)$ was transmitted. We consider an IM/DD link using on-off keying (OOK). Following [4], we assume that the receiver signal-to-noise ratio (SNR) is limited by shot noise caused by ambient light which is much stronger than the desired signal and/or by thermal noise. In this case, the noise can be modeled as additive white Gaussian noise (AWGN) with zero mean and variance $N_0/2$, independent of the on/off state of the received bit. Under the assumption of maximum-likelihood soft decoding with perfect channel state information (CSI), the conditional PEP with respect to fading coefficients $\boldsymbol{I} = (I_1, I_2,\ldots,I_M)$ is given as [4]

$$P(\boldsymbol{X},\hat{\boldsymbol{X}}|\boldsymbol{I}) = Q\left(\sqrt{\frac{\varepsilon(\boldsymbol{X},\hat{\boldsymbol{X}})}{2N_0}}\right) \quad (7)$$

where $Q(.)$ is the Gaussian-Q function and $\varepsilon(\boldsymbol{X},\hat{\boldsymbol{X}})$ is the energy difference between two codewords. Since OOK is used, the receiver would only receive signal light subject to fading during the on-state transmission. Therefore, (7) is given as

$$P(\boldsymbol{X},\hat{\boldsymbol{X}}|\boldsymbol{I}) = Q\left(\sqrt{\frac{E_s}{2N_0}\sum_{k\in\Omega}I_k^2}\right) = Q\left(\sqrt{\frac{E_s}{2N_0}\sum_{k\in\Omega}y_k^2 z_k^2}\right) \quad (8)$$

where E_s is the total transmitted energy and Ω is the set of bit locations where $\boldsymbol{X}$ and $\hat{\boldsymbol{X}}$ differ from each other. Defining the SNR as $\tau = E_s/N_0$, using the alternative form for Gaussian-Q function, $Q(x) = (1/2\pi)\int_0^{\pi/2}\exp(-x^2/2\sin^2\theta)$ [10], we obtain the unconditional PEP as

$$P(\boldsymbol{X},\hat{\boldsymbol{X}}) = \frac{1}{\pi}\int_0^{\frac{\pi}{2}}\prod_{k\in\Omega}\underset{y_k,z_k}{\mathrm{E}}\left[\exp\left(-\frac{\tau}{4}\frac{y_k^2 z_k^2}{\sin^2\theta}\right)\right]d\theta \quad (9)$$

where the expectation operation in (9) should be performed over y_k and z_k, $k\in\Omega$. Here, $|\Omega|$ is the cardinality of Ω, which also corresponds to the length of error event. For further simplifications, we rewrite (9), exploiting the fact that the underlying distribution is a *conditional* negative exponential distribution with its mean following gamma distribution, i.e.,

$$P(\boldsymbol{X},\hat{\boldsymbol{X}}) = \frac{1}{\pi}\int_0^{\frac{\pi}{2}}\underset{z_k}{E}\left\{\prod_{k\in\Omega}\underset{y_k}{E}\left[\exp\left(-\frac{\tau}{4}\frac{y_k^2 z_k^2}{\sin^2\theta}\right)\right]\right\}d\theta. \quad (10)$$

The inner expectation in (10) yields

$$\underset{y_k}{E}\left[\exp\left(-\frac{\tau}{4}\frac{y_k^2 z_k^2}{\sin^2\theta}\right)\right] = \sqrt{\frac{4\pi\sin^2\theta}{\tau y_k^2}}\times\exp\left(\frac{\sin^2\theta}{\tau y_k^2}\right)Q\left(\sqrt{\frac{2\sin^2\theta}{\tau y_k^2}}\right) \quad (11)$$

where the integration in (11) is solved following [11, p. 113, Eq. 2.33]. Replacing $Q(\sqrt{t}) \le 0.5\exp(-t/2)$ in (11) and inserting the resulting expression in (10), we obtain

$$P(\boldsymbol{X},\hat{\boldsymbol{X}}) \le \frac{1}{\pi}\int_0^{\frac{\pi}{2}}\underset{z_k}{E}\left[\prod_{k\in\Omega}z_k^{-1}\sqrt{\frac{\pi\sin^2\theta}{\tau}}\right]d\theta \quad (12)$$

where a closed form expression can only be found for $|\Omega| = 2$ as

$$P(\boldsymbol{X},\hat{\boldsymbol{X}}) \le \frac{\alpha^2}{\tau}\frac{(1-\rho^2)^{\alpha-2}}{\Gamma(\alpha)}\sum_{i=0}^{\infty}\rho^{2i}\frac{\Gamma^2(\alpha+i-1)}{\Gamma(i+1)\Gamma(\alpha+i)}. \quad (13)$$

For the general case, i.e. $|\Omega| > 2$, the multidimensional integrals in (12) can be carried out using NAG C Library routines [12].

The derived PEP expression can be used to obtain upper bounds on the error probability for a coded system. For example, consider a rate $R = k/n$ convolutional code with the all-zero codeword C_0 and nonzero codewords C_j, $j > 0$. The average BER can be upper bounded by [13]

$$P_b \le \frac{1}{k}\sum_{j=1}^{\infty}B_j P(C_j, C_0) \quad (14)$$

where B_j is the Hamming weight of the information sequence and $P(C_j, C_0)$ is the PEP expression given by (12). Alternatively, (14) can be written as

$$P_b \le \frac{1}{k}\sum_{w=w_{\text{free}}}^{\infty}\sum_{\substack{l=1\\ C_{j_l}\in S_w}}^{n_w}B_{j_l}P(C_{j_l}, C_0) \quad (15)$$

where the pairwise error events are sorted by their Hamming distances. Here, w_{free} is the free distance of the code, S_w is

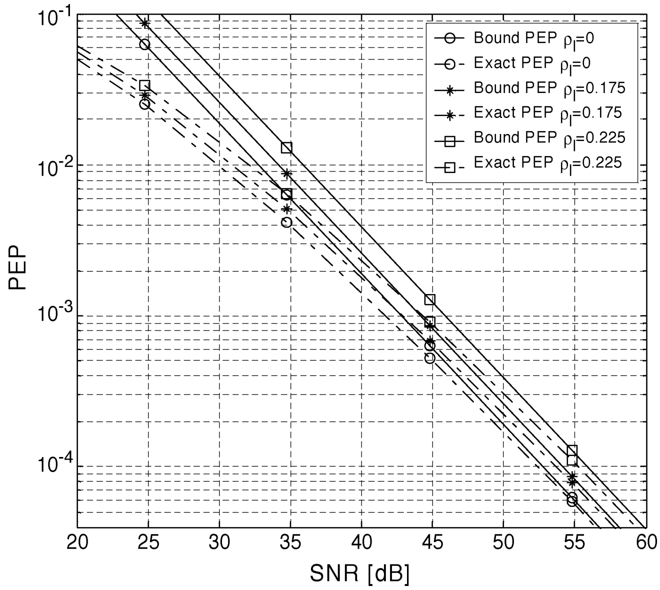

Fig. 1. Comparison of exact and derived PEPs for $\alpha = 2$.

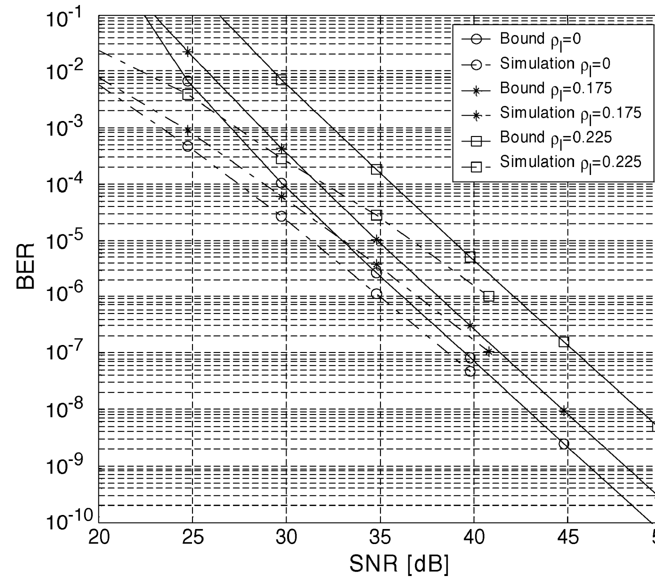

Fig. 2. Upper bounds on BER for the K channel.

the set of codewords with Hamming weight of w and n_w is the number of such error events.

IV. Numerical Results

In this section, we will first compare the derived PEP bound with the exact PEP. Then, as an example, we will consider a convolutionally coded system and use the derived PEP expression to compute upper bounds on the BER performance.

In Fig. 1, we plot the derived bounds using the approximate PEPs given by (13) for an error event of length 2 with channel parameters $\alpha = 2$ and correlation values of $\rho_I = 0$, $\rho_I = 0.175$ and $\rho_I = 0.225$. We also compute the respective exact PEPs given by (9) and provide them as a reference (illustrated by dashed lines). It is observed that the derived bounds match well with the exact PEPs for high SNRs. Although the tightness of the bound for small SNR values is somehow low (i.e. the overlapping with the exact expression occurs asymptotically), the derived bounds capture well the behavior for a large range of SNR values and for all correlation values.

To further demonstrate the usefulness of derived bounds, we take as an example the convolutional code in [13, p. 471, Fig. 8.2.2] which has a code rate of 1/3, constraint length of 3 and minimum Hamming distance of 6. The average BER results are computed based on a truncated version of (15) considering error events with lengths up to 8. The results are illustrated in Fig. 2 for the K channel with parameter $\alpha = 2$ for various correlation values. In all cases considered, upper bounds on BER are in reasonable agreement with the Monte-Carlo simulations. Although there is some discrepancy in the lower SNR region, it shows a good agreement as SNR increases. Due to the long simulation time involved, we are able to give simulation results only up to $\mathrm{BER} = 10^{-7}$. Considering $\mathrm{BER} = 10^{-9}$ is a practical performance target for a FSO link, our analytical results can serve as a simple and reliable method to estimate BER performance without resorting to lengthy simulations.

V. Conclusion

In this letter, we investigated error performance bounds for coded FSO communication systems operating over atmospheric turbulence channels based on a temporally correlated K channel model. Unlike the classically used log-normal assumption, this channel model describes strong turbulence conditions. We have derived an upper bound on the PEP for the K channel and then applied the union-bound technique in conjunction with the derived PEP bound to obtain upper bounds on the BER performance.

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