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ESSENTIAL DIMENSION OF FINITE p -GROUPS

NIKITA A. KARPENKO AND ALEXANDER S. MERKURJEV

ABSTRACT. We prove that the essential dimension and p -dimension of a p -group G over a field F containing a primitive p -th root of unity is equal to the least dimension of a faithful representation of G over F .

The notion of the essential dimension $\text{ed}(G)$ of a finite group G over a field F was introduced in [5]. The integer $\text{ed}(G)$ is equal to the smallest number of algebraically independent parameters required to define a Galois G -algebra over any field extension of F . If V is a faithful linear representation of G over F then $\text{ed}(G) \leq \dim(V)$ (cf. [2, Prop. 4.15]). The essential dimension of G can be smaller than $\dim(V)$ for every faithful representation V of G over F . For example, we have $\text{ed}(\mathbb{Z}/3\mathbb{Z}) = 1$ over \mathbb{Q} or any field F of characteristic 3 (cf. [2, Cor. 7.5]) and $\text{ed}(S_3) = 1$ over \mathbb{C} (cf. [5, Th. 6.5]).

In this paper we prove that if G is a p -group and F is a field of characteristic different from p containing p -th roots of unity, then $\text{ed}(G)$ coincides with the least dimension of a faithful representation of G over F (cf. Theorem 4.1).

We also compute the essential p -dimension of a p -group G introduced in [15]. We show that $\text{ed}_p(G) = \text{ed}(G)$ over a field F containing p -th roots of unity.

In the paper the word “scheme” means a separated scheme of finite type over a field and “variety” an integral scheme.

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1. PRELIMINARIES

1.1. **Severi-Brauer varieties.** (cf. [1]) Let A be a central simple algebra of degree n over a field F . The *Severi-Brauer variety* $P = \text{SB}(A)$ of A is the variety of right ideals in A of dimension n . For a field extension L/F , the algebra A is split over L if and only if $P(L) \neq \emptyset$ if and only if $P_L \simeq \mathbb{P}_L^{n-1}$.

The change of field map $\text{deg} : \text{Pic}(P) \rightarrow \text{Pic}(P_L) = \mathbb{Z}$ for a splitting field extension L/F identifies $\text{Pic}(P)$ with $e\mathbb{Z}$, where e is the exponent (period) of A . In particular, P has divisors of degree e . The algebra A is split over L if and only if P_L has a prime divisor of degree 1 (a hyperplane).

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1.2. Groupoids and gerbes. (cf. [4]) Let \mathcal{X} be a groupoid over F in the sense of [19]. We assume that for any field extension L/F , the isomorphism classes of objects in the category $\mathcal{X}(L)$ form a set which we denote by $\widehat{\mathcal{X}}(L)$. We can view $\widehat{\mathcal{X}}$ as a functor from the category \mathbf{Fields}/F of field extensions of F to \mathbf{Sets} .

Example 1.2.1. If G is an algebraic group over F , then the groupoid BG is defined as the category of G -torsors over a scheme over F . Hence the functor \widehat{BG} takes a field extension L/F to the set of all isomorphism classes of G -torsors over L .

Special examples of groupoids are *gerbes banded by a commutative group scheme* C over F . There is a bijection between the set of isomorphism classes of gerbes banded by C and the Galois cohomology group $H^2(F, C)$ (cf. [7, Ch. 4] and [13, Ch. 4, §2]). The split gerbe BC corresponds to the trivial element of $H^2(F, C)$.

Example 1.2.2. (Gerbes banded by μ_n) Let A be a central simple F -algebra and n an integer with $[A] \in \mathrm{Br}_n(F) = H^2(F, \mu_n)$. Let P be the Severi-Brauer variety of A and S a divisor on P of degree n . Denote by \mathcal{X}_A the gerbe banded by μ_n corresponding to $[A]$. For a field extension L/F , the set $\widehat{\mathcal{X}}_A(L)$ has the following explicit description (cf. [4]): $\widehat{\mathcal{X}}_A(L)$ is nonempty if and only if P is split over L . In this case $\widehat{\mathcal{X}}_A(L)$ is the set of equivalence classes of the set

$$\{f \in L(P)^\times : \mathrm{div}(f) = nH - S_L, \text{ where } H \text{ is a hyperplane in } P_L\},$$

and two functions f and f' are equivalent if $f' = fh^n$ for some $h \in L(P)^\times$.

1.3. Essential dimension. Let $T : \mathbf{Fields}/F \rightarrow \mathbf{Sets}$ be a functor. For a field extension L/F and an element $t \in T(L)$, the *essential dimension of t* , denoted $\mathrm{ed}(t)$, is the least $\mathrm{tr. deg}_F(L')$ over all subfields $L' \subset L$ over F such that t belongs to the image of the map $T(L') \rightarrow T(L)$. The *essential dimension of the functor T* is the supremum of $\mathrm{ed}(t)$ over all $t \in T(L)$ and field extensions L/F .

Let p be a prime integer and $t \in T(L)$. The *essential p -dimension of t* , denoted $\mathrm{ed}_p(t)$, is the least $\mathrm{tr. deg}_F(L')$ over all subfields $L' \subset L$ over F , where L' is a finite field extension of L of degree prime to p such that the image of t in $T(L')$ belongs to the image of the map $T(L'') \rightarrow T(L')$. The *essential p -dimension of the functor T* is the supremum of $\mathrm{ed}_p(t)$ over all $t \in T(L)$ and field extensions L/F . Clearly, $\mathrm{ed}(T) \geq \mathrm{ed}_p(T)$.

Let G be an algebraic group over F . The *essential dimension of G* (respectively the *essential p -dimension of G*) is the essential dimension (respectively the essential p -dimension) of the functor taking a field extension L/F to the set of isomorphism classes of G -torsors over $\mathrm{Spec} L$.

If G is a finite group, we view G as a constant group over a field F . Every G -torsor over $\mathrm{Spec} L$ has the form $\mathrm{Spec} K$ where K is a Galois G -algebra over L . Therefore, $\mathrm{ed}(G)$ is the essential dimension of the functor taking a field L to the set of isomorphism classes of Galois G -algebras over L .

Example 1.3.1. Let \mathcal{X} be a groupoid over F . The *essential dimension* of \mathcal{X} , denoted by $\text{ed}(\mathcal{X})$, is the essential dimension $\text{ed}(\widehat{\mathcal{X}})$ of the functor $\widehat{\mathcal{X}}$ defined in §1.2. The *essential p -dimension* of $\text{ed}_p(\mathcal{X})$ is defined similarly. In particular, $\text{ed}(BG) = \text{ed}(G)$ and $\text{ed}_p(BG) = \text{ed}_p(G)$ for an algebraic group G over F .

1.4. Canonical dimension. (cf. [3], [11]) Let F be a field and \mathcal{C} a class of field extensions of F . A field $E \in \mathcal{C}$ is called *generic* if for any $L \in \mathcal{C}$ there is an F -place $E \rightsquigarrow L$.

The *canonical dimension* $\text{cdim}(\mathcal{C})$ of the class \mathcal{C} is the minimum of the $\text{tr. deg}_F E$ over all generic fields $E \in \mathcal{C}$.

Let p be a prime integer. A field E in a class \mathcal{C} is called *p -generic* if for any $L \in \mathcal{C}$ there is a finite field extension L' of L of degree prime to p and an F -place $E \rightsquigarrow L'$. The *canonical p -dimension* $\text{cdim}_p(\mathcal{C})$ of the class \mathcal{C} is the least $\text{tr. deg}_F E$ over all p -generic fields $E \in \mathcal{C}$. Obviously, $\text{cdim}(\mathcal{C}) \geq \text{cdim}_p(\mathcal{C})$.

Let $T : \mathbf{Fields}/F \rightarrow \mathbf{Sets}$ be a functor. Denote by \mathcal{C}_T the class of *splitting fields of T* , i.e., the class of field extensions L/F such that $T(L) \neq \emptyset$. The *canonical dimension (p -dimension) of T* , denoted $\text{cdim}(T)$ (respectively $\text{cdim}_p(T)$), is the canonical dimension (p -dimension) of the class \mathcal{C}_T .

If X is a scheme over F , we write $\text{cdim}(X)$ and $\text{cdim}_p(X)$ for the canonical dimension and p -dimension of X viewed as a functor $L \mapsto X(L) = \text{Mor}_F(\text{Spec } L, X)$.

Example 1.4.1. Let \mathcal{X} be a groupoid over F . We define the *canonical dimension* $\text{cdim}(\mathcal{X})$ and *p -dimension* $\text{cdim}_p(\mathcal{X})$ of \mathcal{X} as the canonical dimension and p -dimension of the functor $\widehat{\mathcal{X}}$.

Example 1.4.2. If X is a regular and complete variety over F viewed as a functor then $\text{cdim}(X)$ is equal to the smallest dimension of a closed subvariety $Z \subset X$ such that there is a rational morphism $X \dashrightarrow Z$ (cf. [11, Cor. 4.6]). If p is a prime integer then $\text{cdim}_p(X)$ is equal to the smallest dimension of a closed subvariety $Z \subset X$ such that there are dominant rational morphisms $X' \dashrightarrow X$ of degree prime to p and $X' \dashrightarrow Z$ for some variety X' (cf. [11, Prop. 4.10]).

Remark 1.4.3. (A relation between essential and canonical dimension) Let $T : \mathbf{Fields}/F \rightarrow \mathbf{Sets}$ be a functor. We define the “contraction” functor $T^c : \mathbf{Fields}/F \rightarrow \mathbf{Sets}$ as follows. For a field extension L/F , we have $T^c(L) = \emptyset$ if $T(L)$ is empty and $T^c(L)$ is a one element set otherwise. If X is a regular and complete variety over F viewed as a functor then one can show that $\text{ed}(X^c) = \text{cdim}(X)$ and $\text{ed}_p(X^c) = \text{cdim}_p(X)$.

1.5. Valuations. Let K/F be a regular field extension, i.e., for any field extension L/F , the ring $K \otimes_F L$ is a domain. We write KL for the quotient field of $K \otimes_F L$.

Let v be a valuation on L over F with residue field R . Let O be the associated valuation ring and M its maximal ideal. As $K \otimes_F R$ is a domain, the ideal $\widetilde{M} := K \otimes_F M$ in the ring $\widetilde{O} := K \otimes_F O$ is prime. The localization ring $\widetilde{O}_{\widetilde{M}}$ is

a valuation ring in KL with residue field KR . The corresponding valuation \tilde{v} of KL is called the *canonical extension of v on KL* . Note that the groups of values of v and \tilde{v} coincide.

We shall need the following lemma.

Lemma 1.1 (cf. [11, Lemma 3.2]). *Let v be a discrete valuation (of rank 1) of a field L with residue field R and L'/L a finite field extension of degree prime to p . Then v extends to a discrete valuation of L' with residue field R' such that the ramification index and the degree $[R' : R]$ are prime to p .*

Proof. If L'/L is separable and v_1, \dots, v_k are all the extensions of v on L' then $[L' : L] = \sum e_i [R_i : R]$ where e_i is the ramification index and R_i is the residue field of v_i (cf. [20, Ch. VI, Th. 20 and p. 63]). It follows that the integer $e_i [R_i : R]$ is prime to p for some i .

If L'/L is purely inseparable of degree q then the valuation v' of L' defined by $v'(x) = v(x^q)$ satisfies the desired properties. The general case follows. \square

2. CANONICAL DIMENSION OF A SUBGROUP OF $\text{Br}(F)$

Let F be an arbitrary field, p a prime integer and D a finite subgroup of $\text{Br}_p(F)$ of dimension r over $\mathbb{Z}/p\mathbb{Z}$. In this section we determine the canonical dimension $\text{cdim } D$ and the canonical p -dimension $\text{cdim}_p D$ of the class of common splitting fields of all elements of D . We say that a basis $\{a_1, a_2, \dots, a_r\}$ of D is *minimal* if for any $i = 1, \dots, r$ and any element $d \in D$ outside of the subgroup generated by a_1, \dots, a_{i-1} , we have $\text{ind } d \geq \text{ind } a_i$.

One can construct a minimal basis of D by induction as follows. Let a_1 be a nonzero element of D of minimal index. If the elements a_1, \dots, a_{i-1} are already chosen for some $i \leq r$, we take for the a_i an element of D of the minimal index among the elements outside of the subgroup generated by a_1, \dots, a_{i-1} .

In this section we prove the following

Theorem 2.1. *Let F be an arbitrary field, p a prime integer, $D \subset \text{Br}_p(F)$ a subgroup of dimension r and $\{a_1, a_2, \dots, a_r\}$ a minimal basis of D . Then*

$$\text{cdim}_p(D) = \text{cdim}(D) = \left(\sum_{i=1}^r \text{ind } a_i \right) - r.$$

We prove Theorem 2.1 in several steps.

Let $\{a_1, a_2, \dots, a_r\}$ be a minimal basis of D . For every $i = 1, 2, \dots, r$, let P_i be the Severi-Brauer variety of a central division F -algebra A_i representing the element $a_i \in \text{Br}_p F$. We write P for the product $P_1 \times P_2 \times \dots \times P_r$. We have

$$\dim P = \sum_{i=1}^r \dim P_i = \left(\sum_{i=1}^r \text{ind } a_i \right) - r.$$

Moreover, the classes of splitting fields of P and D coincide, hence $\text{cdim}(D) = \text{cdim}(P)$ and $\text{cdim}_p(D) = \text{cdim}_p(P)$. Thus, the statement of Theorem 2.1 is equivalent to the equality $\text{cdim}_p(P) = \text{cdim}(P) = \dim(P)$.

Let $r \geq 1$ and $0 \leq n_1 \leq n_2 \leq \dots \leq n_r$ be integers and $K = K(n_1, \dots, n_r)$ the subgroup of the polynomial ring $\mathbb{Z}[x]$ in r variables $x = (x_1, \dots, x_r)$ generated by the monomials $p^{e(j_1, \dots, j_r)} x_1^{j_1} \dots x_r^{j_r}$ for all $j_1, \dots, j_r \geq 0$, where the exponent $e(j_1, \dots, j_r)$ is 0 if all the j_1, \dots, j_r are divisible by p , otherwise $e(j_1, \dots, j_r) = n_k$ with the maximum k such that j_k is not divisible by p . In fact, K is a subring of $\mathbb{Z}[x]$.

Remark 2.2. Let A_1, \dots, A_r be central division algebras over some field such that for any non-negative integers j_1, \dots, j_r , the index of the tensor product $A_1^{\otimes j_1} \otimes \dots \otimes A_r^{\otimes j_r}$ is equal to $p^{e(j_1, \dots, j_r)}$. The group K can be interpreted as the colimit of the Grothendieck groups of the product over $i = 1, \dots, r$ of the Severi-Brauer varieties of the matrix algebras $M_{l_i}(A_i)$ over all positive integers l_1, \dots, l_r .

We set $h = (h_1, \dots, h_r)$ with $h_i = 1 - x_i \in \mathbb{Z}[x]$.

Proposition 2.3. *Let $bh_1^{i_1} \dots h_r^{i_r}$ be a monomial of the lowest total degree of a polynomial f in the variables h lying in K . Assume that the integer b is not divisible by p . Then $p^{n_1} \mid i_1, \dots, p^{n_r} \mid i_r$.*

Proof. We recast the proof for $r = 1$ given in [8, Lemma 2.1.2] to the case of arbitrary r .

We proceed by induction on $m = r + n_1 + \dots + n_r$. The case $m = 1$ is trivial. If $m > 1$ and $n_1 = 0$, then $K = K(n_2, \dots, n_r)[x_1]$ and we are done by induction applied to $K(n_2, \dots, n_r)$. In what follows we assume that $n_1 \geq 1$.

Since $K(n_1, n_2, \dots, n_r) \subset K(n_1 - 1, n_2, \dots, n_r)$, by the induction hypothesis $p^{n_1 - 1} \mid i_1, p^{n_2} \mid i_2, \dots, p^{n_r} \mid i_r$. It remains to show that i_1 is divisible by p^{n_1} .

Consider the additive operation $\varphi: \mathbb{Z}[x] \rightarrow \mathbb{Q}[x]$ which takes a polynomial $g \in \mathbb{Z}[x]$ to the polynomial $p^{-1}x_1 \cdot g'$, where g' is the partial derivative of g with respect to x_1 . We have

$$\varphi(K) \subset K(n_1 - 1, n_2 - 1, \dots, n_r - 1) \subset K(n_1 - 1)[x_2, \dots, x_r]$$

and

$$\varphi(h_1^{j_1} h_2^{j_2} \dots h_r^{j_r}) = -p^{-1} j_1 h_1^{j_1 - 1} h_2^{j_2} \dots h_r^{j_r} + p^{-1} j_1 h_1^{j_1} h_2^{j_2} \dots h_r^{j_r}.$$

Since $bh_1^{i_1} \dots h_r^{i_r}$ is a monomial of the lowest total degree of the polynomial f , it follows that $-bp^{-1}i_1 h_1^{i_1 - 1} h_2^{i_2} \dots h_r^{i_r}$ is a monomial of $\varphi(f)$ considered as a polynomial in h . As

$$\varphi(f) \in K(n_1 - 1)[x_2, \dots, x_r],$$

we see that $-bp^{-1}i_1 h_1^{i_1 - 1}$ is a monomial of a polynomial from $K(n_1 - 1)$. It follows that $p^{-1}i_1$ is an integer and by Lemma 2.4 below, this integer is divisible by $p^{n_1 - 1}$. Therefore $p^{n_1} \mid i_1$. \square

Lemma 2.4. *Let g be a polynomial in h_1 lying in $K(m)$ for some $m \geq 0$. Let bh_1^{i-1} be a monomial of g such that i is divisible by p^m . Then b is divisible by p^m .*

Proof. We write h for h_1 and x for x_1 . Note that $h^i \in K(m)$ since i is divisible by p^m . Moreover, the quotient ring $K(m)/(h^i)$ is additively generated by $p^{e(j)}x^j$ with $j < i$. Indeed, the polynomial $x^i - (-h)^i = x^i - (x-1)^i$ is a linear combination with integer coefficients of $p^{e(j)}x^j$ with $j < i$. Consequently, for any $k \geq 0$, multiplying by $p^{e(k)}x^k$, we see that the polynomial $p^{e(i+k)}x^{i+k} = p^{e(k)}x^{i+k}$ modulo the ideal (h^i) is a linear combination with integer coefficients of the $p^{e(j)}x^j$ with $j < i+k$.

Thus, $K(m)/(h^i)$ is additively generated by $p^{e(j)}(1-h)^j$ with $j < i$. Only the generator $p^{e(i-1)}(1-h)^{i-1} = p^m(1-h)^{i-1}$ has a nonzero h^{i-1} -coefficient and that coefficient is divisible by p^m . \square

Let Y be a scheme over the field F . We write $\text{CH}(Y)$ for the Chow group of Y and set $\text{Ch}(Y) = \text{CH}(Y)/p\text{CH}(Y)$. We define $\text{Ch}(\overline{Y})$ as the colimit of $\text{Ch}(Y_L)$ where L runs over all field extensions of F . Thus for any field extension L/F , we have a canonical homomorphism $\text{Ch}(Y_L) \rightarrow \text{Ch}(\overline{Y})$. This homomorphism is an isomorphism if $Y = P$, the variety defined above, and L is a splitting field of P .

We define $\overline{\text{Ch}}(Y)$ to be the image of the homomorphism $\text{Ch}(Y) \rightarrow \text{Ch}(\overline{Y})$.

Proposition 2.5. *We have $\overline{\text{Ch}}^j(P) = 0$ for any $j > 0$.*

Proof. Let $K_0(P)$ be the Grothendieck group of P . We write $K_0(\overline{P})$ for the colimit of $K_0(P_L)$ taken over all field extensions L/F . The group $K_0(\overline{P})$ is canonically isomorphic to $K_0(P_L)$ for any splitting field L of P . Each of the groups $K_0(P)$ and $K_0(\overline{P})$ is endowed with the topological filtration. The subsequent factor groups $G^j K_0(P)$ and $G^j K_0(\overline{P})$ of these filtrations fit into the commutative square

$$\begin{array}{ccc} \text{CH}^j(\overline{P}) & \longrightarrow & G^j K_0(\overline{P}) \\ \uparrow & & \uparrow \\ \text{CH}^j(P) & \longrightarrow & G^j K_0(P) \end{array}$$

where the top map is an isomorphism. Therefore it suffices to show that the image of the homomorphism $G^j K_0(P) \rightarrow G^j K_0(\overline{P})$ is divisible by p for any $j > 0$.

The ring $K_0(\overline{P})$ is identified with the quotient of the polynomial ring $\mathbb{Z}[h]$ by the ideal generated by $h_1^{\text{ind } a_1}, \dots, h_r^{\text{ind } a_r}$. Under this identification, the element h_i is the pull-back to P of the class of a hyperplane in P_i over a splitting field and the j -th term $K_0(\overline{P})^{(j)}$ of the filtration is generated by the classes of monomials of degree at least j . The group $G^j K_0(\overline{P})$ is identified with the group of all homogeneous polynomials of degree j .

The group $K_0(P)$ is isomorphic to the direct sum of $K_0(B)$, where $B = A_1^{\otimes j_1} \otimes \dots \otimes A_r^{\otimes j_r}$, over all j_i with $0 \leq j_i < \text{ind } a_i$ (cf. [14, §9]). The image of the natural map $K_0(B) \rightarrow K_0(B_L) = \mathbb{Z}$, where L is a splitting field of B , is equal to $\text{ind}(a_1^{j_1} \dots a_r^{j_r})\mathbb{Z}$. The image of the homomorphism $K_0(P) \rightarrow K_0(\overline{P})$

(which is in fact an injection) is generated by

$$\text{ind}(a_1^{j_1} \cdots a_r^{j_r})(1 - h_1)^{j_1} \cdots (1 - h_r)^{j_r}$$

over all $j_1, \dots, j_r \geq 0$.

We embed $K_0(\overline{P})$ into the polynomial ring $\mathbb{Z}[x] = \mathbb{Z}[x_1, \dots, x_r]$ as a subgroup by identifying a monomial $h_1^{j_1} \cdots h_r^{j_r}$ where $0 \leq j_i < \text{ind } a_i$ with the polynomial $(1 - x_1)^{j_1} \cdots (1 - x_r)^{j_r}$. As the elements a_1, \dots, a_r form a minimal basis of D , the index $\text{ind}(a_1^{j_1} \cdots a_r^{j_r})$ is a power of p with the exponent at least $e(\log_p \text{ind } a_1, \dots, \log_p \text{ind } a_r)$. Therefore,

$$K_0(P) \subset K(\log_p \text{ind } a_1, \dots, \log_p \text{ind } a_r) \subset \mathbb{Z}[x].$$

An element of $K_0(P)^{(j)}$ with $j > 0$ is a polynomial f in h of degree at least j . The image of f in $G^j K_0(\overline{P})$ is the j -th homogeneous part f_j of f . As the degree of f with respect to h_i is less than $\text{ind } a_i$, it follows from Proposition 2.3 that all the coefficients of f_j are divisible by p . \square

Let $d = \dim P$ and $\alpha \in \text{CH}^d(P \times P)$. The *first multiplicity* $\text{mult}_1(\alpha)$ of α is the image of α under the push-forward map $\text{CH}^d(P \times P) \rightarrow \text{CH}^0(P) = \mathbb{Z}$ given by the first projection $P \times P \rightarrow P$ (cf. [10]). Similarly, we define the *second multiplicity* $\text{mult}_2(\alpha)$.

Corollary 2.6. *For any element $\alpha \in \text{CH}^d(P \times P)$, we have*

$$\text{mult}_1(\alpha) \equiv \text{mult}_2(\alpha) \pmod{p}.$$

Proof. We follow the proof of [9, Th. 2.1]. The homomorphism

$$f: \text{CH}^d(P \times P) \rightarrow (\mathbb{Z}/p\mathbb{Z})^2,$$

taking an $\alpha \in \text{CH}^d(P \times P)$ to $(\text{mult}_1(\alpha), \text{mult}_2(\alpha))$ modulo p , factors through the group $\overline{\text{Ch}}^d(P \times P)$. Since for any i , any projection $P_i \times P_i \rightarrow P_i$ is a projective bundle, the Chow group $\overline{\text{Ch}}^d(P \times P)$ is a direct sum of several copies of $\overline{\text{Ch}}^i(P)$ for some i 's and the value $i = 0$ appears once. By Proposition 2.5, the dimension over $\mathbb{Z}/p\mathbb{Z}$ of the vector space $\overline{\text{Ch}}^d(P \times P)$ is equal to 1 and consequently the dimension of the image of f is at most 1. Since the image of the diagonal class under f is $(1, 1)$, the image of f is generated by $(1, 1)$. \square

Corollary 2.7. *Any rational map $P \dashrightarrow P$ is dominant.*

Proof. Let $\alpha \in \text{CH}^d(P \times P)$ be the class of the closure of the graph of a rational map $P \dashrightarrow P$. We have $\text{mult}_1(\alpha) = 1$. Therefore, by Corollary 2.6, $\text{mult}_2(\alpha) \neq 0$, and it follows that the rational map is dominant. \square

Corollary 2.8. $\text{cdim}_p P = \text{cdim } P = \dim P$.

Proof. As $\text{cdim}_p P \leq \text{cdim } P \leq \dim P$, it suffices to show that $\text{cdim}_p P = \dim P$. Let $Z \subset P$ be a closed subvariety and $f: P' \dashrightarrow P$ and $g: P' \dashrightarrow Z$ dominant rational morphisms such that $\deg f$ is prime to p . Let α be the class in $\text{CH}^d(P \times P)$ of the closure in $P \times P$ of the image of $f \times g: P' \dashrightarrow P \times Z$.

As $\text{mult}_1(\alpha) = \deg f$ is prime to p , by Corollary 2.6, we have $\text{mult}_2(\alpha) \neq 0$, i.e., $Z = P$. By Example 1.4.2, $\text{cdim}_p P = \dim P$. \square

The corollary completes the proof of Theorem 2.1.

Remark 2.9. Theorem 2.1 can be generalized to the case of any finite subgroup $D \subset \text{Br}(F)$ consisting of elements of p -primary orders. Let $\{a_1, a_2, \dots, a_r\}$ be elements of D such that their images $\{a'_1, a'_2, \dots, a'_r\}$ in D/D^p form a minimal basis, i.e., for any $i = 1, \dots, r$ and any element $d \in D$ with the class in D/D^p outside of the subgroup generated by a'_1, \dots, a'_{i-1} , the inequality $\text{ind } d \geq \text{ind } a_i$ holds. In particular, $\{a_1, a_2, \dots, a_r\}$ generate D . Then, as in Theorem 2.1, we have

$$\text{cdim}_p(D) = \text{cdim}(D) = \left(\sum_{i=1}^r \text{ind } a_i \right) - r .$$

Indeed, the group D and the variety $P = P_1 \times \dots \times P_r$, where P_i for every $i = 1, \dots, r$ is the Severi-Brauer variety of a central division algebra representing the element a_i , have the same splitting fields. Therefore, $\text{cdim}(D) = \text{cdim}(P)$ and $\text{cdim}_p(D) = \text{cdim}_p(P)$. Corollaries 2.6, 2.7 and 2.8 hold for P since $K_0(P) \subset K(\log_p \text{ind } a_1, \dots, \log_p \text{ind } a_r)$.

Remark 2.10. One can compute the canonical p -dimension of an arbitrary finite subgroup of $D \subset \text{Br}(F)$ as follows. Let D' be the Sylow p -subgroup of D . Write $D = D' \oplus D''$ for a subgroup $D'' \subset D$ and let L/F be a finite field extension of degree prime to p such that D'' is split over L . Then $D_L = D'_L$ and $\text{cdim}_p(D) = \text{cdim}_p(D_L) = \text{cdim}_p(D'_L) = \text{cdim}_p(D') = \text{cdim}(D')$.

3. ESSENTIAL AND CANONICAL DIMENSION OF GERBES BANDED BY $(\mu_p)^s$

In this section we relate the essential and canonical (p -)dimensions of gerbes banded by $(\mu_p)^s$ where $s \geq 0$. The following statement is a generalization of [4, Th. 7.1].

Theorem 3.1. *Let p be a prime integer and \mathcal{X} a gerbe banded by $(\mu_p)^s$ over an arbitrary field F . Then*

$$\text{ed}(\mathcal{X}) = \text{ed}_p(\mathcal{X}) = \text{cdim}_p(\mathcal{X}) + s = \text{cdim}(\mathcal{X}) + s .$$

Proof. The gerbe \mathcal{X} is given by an element in $H^2(F, (\mu_p)^s) = \text{Br}_p(F)^s$, i.e., by an s -tuple of central simple algebras A_1, A_2, \dots, A_s with $[A_i] \in \text{Br}_p(F)$. Let P be the product of the Severi-Brauer varieties $P_i := \text{SB}(A_i)$ and D the subgroup of $\text{Br}_p(F)$ generated by the $[A_i]$, $i = 1, \dots, s$. As the classes of splitting fields for \mathcal{X} , D and P coincide, we have

$$(1) \quad \text{cdim}(\mathcal{X}) = \text{cdim}(P) = \text{cdim}(D) = \text{cdim}_p(D) = \text{cdim}_p(P) = \text{cdim}_p(\mathcal{X})$$

by Theorem 2.1. We shall prove the inequalities $\text{ed}_p(\mathcal{X}) \geq \text{cdim}(P) + s \geq \text{ed}(\mathcal{X})$.

Let S_i be a divisor on P_i of degree p . Let L/F be a field extension and $f_i \in L(P_i)^\times$ with $\text{div}(f_i) = pH_i - (S_i)_L$, where H_i is a hyperplane in $(P_i)_L$ for $i = 1, \dots, s$. We write $\langle f_i \rangle_{i=1}^s$ for the corresponding element in $\widehat{\mathcal{X}}(L)$ (cf. §1.2).

By Example 1.4.2, there is a closed subvariety $Z \subset P$ and a rational dominant morphism $P \dashrightarrow Z$ with $\dim(Z) = \text{cdim}(P) = \text{cdim}_p(P)$. We view $F(Z)$ as a subfield of $F(P)$. As $P(L) \neq \emptyset$ and P is regular, there is an F -place $\gamma : F(P) \rightsquigarrow L$ (cf. [11, §4.1]). Since Z is complete, the valuation ring of the restriction $\gamma|_{F(Z)} : F(Z) \rightsquigarrow L$ dominates a point in Z . It follows that $Z(L) \neq \emptyset$. Choose a point $y \in Z$ such that $F' := F(y) \subset L$.

Since $P(F') \neq \emptyset$, the P_i are split over F' , hence $\text{Pic}(P_i)_{F'} = \mathbb{Z}$ and there are functions $g_i \in F'(P_i)^\times$ with $\text{div}(g_i) = pH'_i - (S_i)_{F'}$, where H'_i is a hyperplane in P_i for $i = 1, \dots, s$. As $\text{Pic}(P_i)_L = \mathbb{Z}$, there are functions $h_i \in L(P_i)^\times$ with $\text{div}(h_i) = (H'_i)_L - H_i$. We have

$$\text{div}(g_i)_L = \text{div}(f_i) + \text{div}(h_i^p),$$

hence

$$a_i g_i = f_i h_i^p$$

for some $a_i \in L^\times$. It follows that $\langle f_i \rangle_{i=1}^s = \langle a_i g_i \rangle_{i=1}^s$ in $\mathcal{X}(L)$, therefore $\langle f_i \rangle_{i=1}^s$ is defined over the field $F'(a_1, a_2, \dots, a_s)$. Hence

$$\text{ed}\langle f_i \rangle_{i=1}^s \leq \text{tr. deg}_F(F') + s \leq \dim(Z) + s = \text{cdim}(P) + s,$$

and therefore $\text{ed}(\mathcal{X}) \leq \text{cdim}(P) + s$.

We shall prove the inequality $\text{ed}_p(\mathcal{X}) \geq \text{cdim}(P) + s$. As $P(F(Z)) \neq \emptyset$, there are functions $f_i \in F(Z)(P_i)^\times$ with $\text{div}(f_i) = pH_i - (S_i)_{F(Z)}$, where H_i is a hyperplane in $(P_i)_{F(Z)}$. Let $L := F(Z)(t_1, t_2, \dots, t_s)$, where the t_i are variables, and consider the point $\langle t_i f_i \rangle_{i=1}^s \in \widehat{\mathcal{X}}(L)$.

We claim that $\text{ed}_p\langle t_i f_i \rangle_{i=1}^s \geq \text{cdim}(P) + s$. Let L' be a finite extension of L of degree prime to p and $L'' \subset L'$ a subfield such that the image of $\langle t_i f_i \rangle_{i=1}^s$ in $\widehat{\mathcal{X}}(L')$ is defined over L'' , i.e., there are functions $g_i \in L''(P_i)^\times$ and $h_i \in L'(P_i)^\times$ with $t_i f_i = g_i h_i^p$. We shall show that $\text{tr. deg}_F(L'') \geq \text{cdim}(P) + s$.

Let $L_i := F(Z)(t_i, \dots, t_s)$ and v_i be the discrete valuation of L_i corresponding to the variable t_i for $i = 1, \dots, s$. We construct a sequence of field extensions L'_i/L_i of degree prime to p and discrete valuations v'_i of L'_i for $i = 1, \dots, s$ by induction on i as follows. Set $L'_1 = L'$. Suppose the fields L'_1, \dots, L'_i and the valuations v'_1, \dots, v'_{i-1} are constructed. By Lemma 1.1, there is a valuation v'_i of L'_i with residue field L'_{i+1} extending the discrete valuation v_i of L_i with the ramification index e_i and the degree $[L'_{i+1} : L_{i+1}]$ prime to p .

The composition v' of the discrete valuations v'_i is a valuation of L' with residue field of degree over $F(Z)$ prime to p . A choice of prime elements in all the L'_i identifies the group of values of v' with \mathbb{Z}^s . Moreover, for every $i = 1, \dots, s$, we have

$$v'(t_i) = e_i \varepsilon_i + \sum_{j>i} a_{ij} \varepsilon_j$$

where the ε_i 's denote the standard basis elements of \mathbb{Z}^s and $a_{ij} \in \mathbb{Z}$.

Write v'' for the restriction of v' on L'' . Let $K = F(P)$. We extend canonically the valuations v' and v'' to valuations \tilde{v}' and \tilde{v}'' of KL' and KL'' respectively (cf. §1.5). Note that $f_i \in K(Z)^\times$, $g_i \in (KL'')^\times$ and $h_i \in (KL')^\times$. We

have

$$e_i \varepsilon_i + \sum_{j>i} a_{ij} \varepsilon_j = v'(t_i) = \tilde{v}'(t_i f_i) \equiv \tilde{v}''(g_i) \pmod{p}.$$

Since e_i are prime to p , the elements $\tilde{v}''(g_i)$ generate a subgroup of \mathbb{Z}^s of finite index. It follows that the value group of \tilde{v}'' is of rank s , hence $\text{rank}(v'') = \text{rank}(\tilde{v}'') = s$.

Let R'' and R' be residue fields of v'' and v' respectively. We have the inclusions $R'' \subset R' \supset F(Z)$ and $[R' : F(Z)]$ is prime to p . By [20, Ch. VI, Th. 3, Cor. 1],

$$(2) \quad \text{tr. deg}_F(L'') \geq \text{tr. deg}_F(R'') + \text{rank}(v'') = \text{tr. deg}_F(R'') + s.$$

As $P(L'') \neq \emptyset$, there is an F -place $F(P) \rightsquigarrow L''$. Composing it with the place $L'' \rightsquigarrow R''$ given by v'' , we get an F -place $F(P) \rightsquigarrow R''$. As P is complete, we have $P(R'') \neq \emptyset$, i.e., R'' is a splitting field of P .

We prove that R'' is a p -generic splitting field of P . Let M be a splitting field of P . A regular system of parameters at the image of a morphism $\alpha : \text{Spec } M \rightarrow P$ yields an F -place $F(P) \rightsquigarrow M$ that is a composition of places associated with discrete valuations (cf. [11, §1.4]). By [11, Lemma 3.2] applied to the restriction of α to $F(Z)$, there is a finite field extension M' of M and an F -place $R' \rightsquigarrow M'$. Restricting to R'' we get an F -place $R'' \rightsquigarrow M'$, i.e., R'' is a p -generic splitting field of P .

By the definition of the canonical p -dimension,

$$\text{cdim}(P) = \text{tr. deg}_F F(Z) = \text{tr. deg}_F R' \geq \text{tr. deg}_F(R'') \geq \text{cdim}_p(P).$$

It follows that $\text{tr. deg}_F(R'') = \text{cdim}(P)$ by (1) and therefore, $\text{tr. deg}_F(L'') \geq \text{cdim}(P) + s$ by (2). The claim is proved.

It follows from the claim that $\text{ed}_p(\mathcal{X}) \geq \text{cdim}(P) + s$. \square

4. MAIN THEOREM

The main result of the paper is the following

Theorem 4.1. *Let G be a p -group and F a field of characteristic different from p containing a primitive p -th root of unity. Then $\text{ed}_p(G)$ over F is equal to $\text{ed}(G)$ over F and coincides with the least dimension of a faithful representation of G over F .*

The rest of the section is devoted to the proof of the theorem. As was mentioned in the introduction, we have $\text{ed}_p(G) \leq \text{ed}(G) \leq \dim(V)$ for any faithful representation V of G over F . We shall construct a faithful representation V of G over F with $\text{ed}_p(G) \geq \dim(V)$.

Denote by C the subgroup of all central elements of G of exponent p and set $H = G/C$, so we have an exact sequence

$$(3) \quad 1 \rightarrow C \rightarrow G \rightarrow H \rightarrow 1.$$

Let $E \rightarrow \text{Spec } F$ be an H -torsor and $\text{Spec } F \rightarrow BH$ be the corresponding morphism. Set $\mathcal{X}^E := BG \times_{BH} \text{Spec } F$. Then \mathcal{X}^E is a gerbe over F banded by

C and its class in $H^2(F, C)$ coincides with the image of the class of E under the connecting map $H^1(F, H) \rightarrow H^2(F, C)$ (cf. [13, Ch. 4, §2]). An object of \mathcal{X}^E over a field extension L/F is a pair (E', α) , where E' is a G -torsor over L and $\alpha : E'/C \xrightarrow{\sim} E_L$ is an isomorphism of H -torsors over L .

Alternatively, $\mathcal{X}^E = [E/G]$ with objects (over L) G -equivariant morphisms $E' \rightarrow E_L$, where E' is a G -torsor over L (cf. [19]).

A lower bound for $\text{ed}(G)$ was established in [4, Prop. 2.20]. We give a similar bound for $\text{ed}_p(G)$.

Theorem 4.2. *For any H -torsor E over F , we have $\text{ed}_p(G) \geq \text{ed}_p(\mathcal{X}^E)$.*

Proof. Let L/F be a field extension and $x = (E', \alpha)$ an object of $\mathcal{X}^E(L)$. Choose a field extension L'/L of degree prime to p and a subfield $L'' \subset L'$ over F such that $\text{tr. deg}(L'') = \text{ed}_p(E')$ and there is a G -torsor E'' over L'' with $E''_{L'} \simeq E'_{L'}$.

Let Z be the (zero-dimensional) scheme of isomorphisms $\text{Iso}_{L''}(E''/C, E_{L''})$ of H -torsors over L'' . The image of the morphism $\text{Spec } L' \rightarrow Z$ over L'' representing the isomorphism $\alpha_{L'}$ is a one point set $\{z\}$ of Z . The field extension $L''(z)/L''$ is algebraic since $\dim Z = 0$.

The isomorphism $\alpha_{L'}$ descends to an isomorphism of the H -torsors E''/C and E over $L''(z)$. Hence the isomorphism class of $x_{L'}$ belongs to the image of the map $\widehat{\mathcal{X}}^E(L''(z)) \rightarrow \widehat{\mathcal{X}}^E(L')$. Therefore,

$$\text{ed}_p(G) \geq \text{ed}_p(E') = \text{tr. deg}(L'') = \text{tr. deg}(L''(z)) \geq \text{ed}_p(x).$$

It follows that $\text{ed}_p(G) \geq \text{ed}_p(\mathcal{X}^E)$. \square

Let $C^* := \text{Hom}(C, \mathbf{G}_m)$ denote the character group of C . An H -torsor E over F yields a homomorphism

$$\beta^E : C^* \rightarrow \text{Br}(F)$$

taking a character $\chi : C \rightarrow \mathbf{G}_m$ to the image of the class of E under the composition

$$H^1(F, H) \xrightarrow{\partial} H^2(F, C) \xrightarrow{\chi^*} H^2(F, \mathbf{G}_m) = \text{Br}(F),$$

where ∂ is the connecting map for the exact sequence (3). Note that as $\mu_p \subset F^\times$, the intersection of $\text{Ker}(\chi^*)$ over all characters $\chi \in C^*$ is trivial. It follows that the classes of splitting fields of the gerbe \mathcal{X}^E and the subgroup $\text{Im}(\beta^E)$ coincide. It follows that

$$(4) \quad \text{cdim}_p(\mathcal{X}^E) = \text{cdim}_p(\text{Im}(\beta^E)).$$

Let $\chi_1, \chi_2, \dots, \chi_s$ be a basis of C^* over $\mathbb{Z}/p\mathbb{Z}$ such that $\{\beta^E(\chi_1), \dots, \beta^E(\chi_r)\}$ is a minimal basis of $\text{Im}(\beta^E)$ for some r and $\beta^E(\chi_i) = 1$ for $i > r$. By Theorem 2.1, we have

$$(5) \quad \text{cdim}_p(\text{Im}(\beta^E)) = \left(\sum_{i=1}^r \text{ind } \beta^E(\chi_i) \right) - r = \left(\sum_{i=1}^s \text{ind } \beta^E(\chi_i) \right) - s.$$

In view of (4) and Theorems 3.1 and 4.2, we shall find an H -torsor E (over a field extension of F) so that the integer in (5) is as large as possible. Let U be a faithful representation of H and X an open subset of the affine space $\mathbb{A}(U)$ of U where H acts freely. Set $Y := X/H$. Let E be the generic fiber of the H -torsor $\pi : X \rightarrow Y$. It is a “generic” H -torsor over the function field $L := F(Y)$.

Let $\chi : C \rightarrow \mathbf{G}_m$ be a character and $\text{Rep}^{(\chi)}(G)$ the category of all finite dimensional representations ρ of G such that $\rho(c)$ is multiplication by $\chi(c)$ for any $c \in C$. Fix a representations $\rho : G \rightarrow \mathbf{GL}(W)$ in $\text{Rep}^{(\chi)}(G)$. The conjugation action of G on $B := \text{End}(W)$ factors through an H -action. By descent (cf. [13, Ch. 1, §2]), there is (a unique up to canonical isomorphism) Azumaya algebra \mathcal{A} over Y and an H -equivariant algebra isomorphism $\pi^*(\mathcal{A}) \simeq B_X := B \times X$. Let A be the generic fiber of \mathcal{A} ; it is a central simple algebra over $L = F(Y)$.

Consider the homomorphism $\beta^E : C^* \rightarrow \text{Br}(L)$.

Lemma 4.3. *The class of A in $\text{Br}(L)$ coincides with $\beta^E(\chi)$.*

Proof. Consider the commutative diagram

$$\begin{array}{ccccccc} 1 & \longrightarrow & C & \longrightarrow & G & \longrightarrow & H & \longrightarrow & 1 \\ & & \chi \downarrow & & \rho \downarrow & & \alpha \downarrow & & \\ 1 & \longrightarrow & \mathbf{G}_m & \longrightarrow & \mathbf{GL}(W) & \longrightarrow & \mathbf{PGL}(W) & \longrightarrow & 1 \end{array}$$

The image of the H -torsor $\pi : X \rightarrow Y$ under α is the $\mathbf{PGL}(W)$ -torsor

$$E' := \mathbf{PGL}(W)_X/H \rightarrow Y$$

where $\mathbf{PGL}(W)_X := \mathbf{PGL}(W) \times X$ and H acts on $\mathbf{PGL}(W)_X$ by $h(a, x) = (ah^{-1}, hx)$. The conjugation action of $\mathbf{PGL}(W)$ on B gives rise to an isomorphism between $\mathbf{PGL}(W)_X$ and the H -torsor $\text{Iso}_X(B_X, \text{End}(W)_X)$ of isomorphisms between the (split) Azumaya \mathcal{O}_X -algebras B_X and $\text{End}(W)_X$. Note that this isomorphism is H -equivariant if H acts by conjugation on B_X and trivially on $\text{End}(W)_X$. By descent,

$$E' \simeq \text{Iso}_Y(\mathcal{A}, \text{End}(W)_Y).$$

Therefore, the image of the class of the torsor $E' \rightarrow Y$ under the connecting map for the bottom row of the diagram coincides with the class of the Azumaya algebra \mathcal{A} . Restricting to the generic fiber yields $[A] = \beta^E(\chi)$. \square

Theorem 4.4. *For any character $\chi \in C^*$, we have $\text{ind } \beta^E(\chi) = \min \dim(V)$ over all representations V in $\text{Rep}^{(\chi)}(G)$.*

Proof. We follow the approach given in [12]. Let H act on a scheme Z over F . We also view Z as a G -scheme. Denote by $\mathcal{M}(G, Z)$ the (abelian) category of left G -modules on Z that are coherent \mathcal{O}_Z -modules (cf. [18, §1.2]). In particular, $\mathcal{M}(G, \text{Spec } F) = \text{Rep}(G)$, the category of all finite dimensional representations of G .

Note that C acts trivially on Z . For a character $\chi : C \rightarrow \mathbf{G}_m$, let $\mathcal{M}^{(\chi)}(G, Z)$ be the full subcategory of $\mathcal{M}(G, Z)$ consisting of G -modules on which C acts via χ . For example, $\mathcal{M}^{(\chi)}(G, \text{Spec } F) = \text{Rep}^{(\chi)}(G)$.

We write $K_0(G, Z)$ and $K_0^{(\chi)}(G, Z)$ for the Grothendieck groups of $\mathcal{M}(G, Z)$ and $\mathcal{M}^{(\chi)}(G, Z)$ respectively.

Every M in $\mathcal{M}(G, Z)$ is a direct sum of unique submodules $M^{(\chi)}$ of M in $\mathcal{M}^{(\chi)}(G, Z)$ over all characters χ of C . It follows that

$$K_0(G, Z) = \coprod K_0^{(\chi)}(G, Z).$$

Let q be the order of G . By [17, Th. 24], every irreducible representation of G is defined over the field $F(\mu_q)$. Since F contains p -th roots of unity, the degree $[F(\mu_q) : F]$ is a power of p . Hence the dimension of any irreducible representation of G over F is a power of p . It follows by Lemma 4.3 that it suffices to show $\text{ind}(A) = \gcd \dim(V)$ over all representations V in $\text{Rep}^{(\chi)}(G)$.

The image of the map $\dim : K_0(A) \rightarrow \mathbb{Z}$ given by the dimension over L is equal to $\text{ind}(A) \cdot \dim(W) \cdot \mathbb{Z}$. To finish the proof of the theorem it suffices to construct a surjective homomorphism

$$(6) \quad K_0(\text{Rep}^{(\chi)}(G)) \rightarrow K_0(A)$$

such that the composition $K_0(\text{Rep}^{(\chi)}(G)) \rightarrow K_0(A) \xrightarrow{\dim} \mathbb{Z}$ is given by the dimension times $\dim(W)$.

First of all we have

$$(7) \quad K_0(\text{Rep}^{(\chi)}(G)) \simeq K_0^{(\chi)}(G, \text{Spec } F).$$

Recall that X an open subset of $\mathbb{A}(U)$ where H acts freely. By homotopy invariance in the equivariant K -theory [18, Cor. 4.2],

$$K_0(G, \text{Spec } F) \simeq K_0(G, \mathbb{A}(U)).$$

It follows that

$$(8) \quad K_0^{(\chi)}(G, \text{Spec } F) \simeq K_0^{(\chi)}(G, \mathbb{A}(U)).$$

By localization [18, Th. 2.7], the restriction homomorphism

$$(9) \quad K_0^{(\chi)}(G, \mathbb{A}(U)) \rightarrow K_0^{(\chi)}(G, X).$$

is surjective.

Denote by $\mathcal{M}^{(1)}(G, X, B_X)$ the category of left G -modules M on X that are coherent \mathcal{O}_X -modules and right B_X -modules such that C acts trivially on M and the G -action on M and the conjugation G -action on B_X agree. The corresponding Grothendieck group is denoted by $K_0^{(1)}(G, X, B_X)$. For any object L in $\mathcal{M}^{(\chi)}(G, X)$, the group C acts trivially on $L \otimes_F W^*$ and B acts on the right on $L \otimes_F W^*$. We have Morita equivalence

$$\mathcal{M}^{(\chi)}(G, X) \xrightarrow{\sim} \mathcal{M}^{(1)}(G, X, B_X)$$

given by $L \mapsto L \otimes_F W^*$ (with the inverse functor $M \mapsto M \otimes_B W$). Hence

$$(10) \quad K_0^{(\chi)}(G, X) \simeq K_0^{(1)}(G, X, B_X).$$

Now, as C acts trivially on X and B_X , the category $\mathcal{M}^{(1)}(G, X, B_X)$ is equivalent to the category $\mathcal{M}(H, X, B_X)$ of left H -modules M on X that are coherent \mathcal{O}_X -modules and right B_X -modules such that the G -action on M and the conjugation G -action on B_X agree. Hence

$$(11) \quad K_0^{(1)}(G, X, B_X) \simeq K_0(H, X, B_X).$$

Recall that $Y = X/H$. By descent, the category $\mathcal{M}(H, X, B_X)$ is equivalent to the category $\mathcal{M}(Y, \mathcal{A})$ of coherent \mathcal{O}_Y -modules that are right \mathcal{A} -modules. Hence

$$(12) \quad K_0(H, X, B_X) \simeq K_0(Y, \mathcal{A}).$$

The restriction to the generic point of Y gives a surjective homomorphism

$$(13) \quad K_0(Y, \mathcal{A}) \rightarrow K_0(A).$$

The homomorphism (6) is the composition of (7), (8), (9), (10), (11), (12) and (13). It takes the class of a representation V to the class in $K_0(A)$ of the generic fiber of the vector bundle $((V \otimes W^*) \times X)/H$ over Y of rank $\dim(V) \cdot \dim(W)$. \square

Remark 4.5. The theorem holds with \min replaced by the \gcd (with the same proof) in a more general context when the sequence (3) is an arbitrary exact sequence of algebraic groups with C a central diagonalizable subgroup of G .

Example 4.6 (cf. [6], [4, §14], [16, Th. 7.3.8]). Let p be a prime integer, F be a field of characteristic different from p and C_m the cyclic group $\mathbb{Z}/p^m\mathbb{Z}$. Let $K = F(t_1, \dots, t_{p^m})$ and C_m act on the variables t_1, \dots, t_{p^m} by cyclic permutations. Then K is a Galois C_m -algebra over K^{C_m} . Assume that F contains a primitive root of unity ξ_{p^k} for some k . The image of the class of K under the connecting map $H^1(F, C_m) \rightarrow H^2(F, C_k) \simeq \text{Br}_{p^k}(F)$ for the exact sequence

$$1 \rightarrow C_k \rightarrow C_n \rightarrow C_m \rightarrow 1,$$

where $n = k + m$, is the class of the cyclic algebra $A = (K/K^{C_m}, \xi_{p^k})$. The group C_n acts F -linearly on $F(\xi_{p^n})$ by multiplication by roots of unity making the F -space $F(\xi_{p^n})$ a faithful representation of C_n of the smallest dimension. By Theorem 4.4 and Remark 4.5, we have

$$\text{ind}(A) = [F(\xi_{p^n}) : F].$$

We can now complete the proof of Theorem 4.1. By Theorem 4.4, there are representations V_i in $\text{Rep}^{(\chi_i)}(G)$ such that $\text{ind} \beta^E(\chi_i) = \dim(V_i)$, $i = 1, \dots, s$. Let V be the direct sum of all the V_i . By Theorem 4.2 (applied to the group G over L and the generic torsor E), Theorem 3.1, (4) and (5), we have

$$\begin{aligned} \text{ed}_p(G) &\geq \text{ed}_p(G_L) \geq \text{ed}_p(\mathcal{X}^E) = \text{cdim}_p(\mathcal{X}^E) + s = \text{cdim}_p(\text{Im}(\beta^E)) + s \\ &= \sum_{i=1}^s \text{ind} \beta^E(\chi_i) = \sum_{i=1}^s \dim(V_i) = \dim(V). \end{aligned}$$

Since $\chi_1, \chi_2, \dots, \chi_s$ generate C^* , the restriction of V on C is faithful. As every nontrivial normal subgroup of G intersects C nontrivially, the G -representation V is faithful. We have constructed a faithful representation V of G over F with $\text{ed}_p(G) \geq \dim(V)$. The theorem is proved.

Remark 4.7. The proof of Theorem 4.1 shows how to compute the essential dimension of G over F . For every character $\chi \in C^*$ choose a representation $V_\chi \in \text{Rep}^{(\chi)}(G)$ of the smallest dimension. It appears as an irreducible component of the smallest dimension of the induced representation $\text{Ind}_C^G(\chi)$. We construct a basis χ_1, \dots, χ_s of C^* by induction as follows. Let χ_1 be a nonzero character with the smallest $\dim(V_{\chi_1})$. If the characters $\chi_1, \dots, \chi_{i-1}$ are already constructed for some $i \leq s$, then we take for χ_i a character with minimal $\dim(V_{\chi_i})$ among all the characters outside of the subgroup generated by $\chi_1, \dots, \chi_{i-1}$. Then V is a faithful representation of the least dimension and $\text{ed}(G) = \sum_{i=1}^s \dim(V_{\chi_i})$.

Remark 4.8. We can compute the essential p -dimension of an arbitrary finite group G over a field F of characteristic different from p . (We don't assume that F contains p -th roots of unity.) Let G' a Sylow p -subgroup of G . One can prove that $\text{ed}_p(G) = \text{ed}_p(G')$ and $\text{ed}_p(G')$ does not change under field extensions of degree prime to p . In particular $\text{ed}_p(G') = \text{ed}_p(G'_{F'})$ where $F' = F(\mu_p)$. It follows from Theorem 4.1 that $\text{ed}_p(G)$ coincides with the least dimension of a faithful representation of G' over F' .

5. AN APPLICATION

Theorem 5.1. *Let G_1 and G_2 be two p -groups and F a field of characteristic different from p containing a primitive p -th root of unity. Then*

$$\text{ed}(G_1 \times G_2) = \text{ed}(G_1) + \text{ed}(G_2).$$

Proof. The index j in the proof takes the values 1 and 2. If V_j is a faithful representation of G_j then $V_1 \oplus V_2$ is a faithful representation of $G_1 \times G_2$. Hence $\text{ed}(G_1 \times G_2) \leq \text{ed}(G_1) + \text{ed}(G_2)$ (cf. [5, Lemma 4.1(b)]).

Denote by C_j the subgroup of all central elements of G_j of exponent p . Set $C = C_1 \times C_2$. We identify C^* with $C_1^* \oplus C_2^*$.

For every character $\chi \in C^*$ choose a representation $\rho_\chi : G_1 \times G_2 \rightarrow \mathbf{GL}(V_\chi)$ in $\text{Rep}^{(\chi)}(G_1 \times G_2)$ of the smallest dimension. We construct a basis $\{\chi_1, \chi_2, \dots, \chi_s\}$ of C^* following Remark 4.7. We claim that all the χ_i can be chosen in one of the C_j^* . Indeed, suppose the characters $\chi_1, \dots, \chi_{i-1}$ are already constructed, and let χ_i be a character with minimal $\dim(V_{\chi_i})$ among the characters outside of the subgroup generated by $\chi_1, \dots, \chi_{i-1}$. Let $\chi_i = \chi_i^{(1)} + \chi_i^{(2)}$ with $\chi_i^{(j)} \in C_j^*$. Denote by ε_1 and ε_2 the endomorphisms of $G_1 \times G_2$ taking (g_1, g_2) to $(g_1, 1)$ and $(1, g_2)$ respectively. The restriction of the representation $\rho_{\chi_i} \circ \varepsilon_j$ on C is given by the character $\chi_i^{(j)}$. We replace χ_i by $\chi_i^{(j)}$ with j such that $\chi_i^{(j)}$ does not belong to the subgroup generated by $\chi_1, \dots, \chi_{i-1}$. The claim is proved.

Let W_j be the direct sum of all the V_{χ_i} with $\chi_i \in C_j^*$. Then the restriction of W_j on C_j is faithful, hence so is the restriction of W_j on G_j . It follows that $\text{ed}(G_j) \leq \dim(W_j)$. As $W_1 \oplus W_2 = V$, we have

$$\text{ed}(G_1) + \text{ed}(G_2) \leq \dim(W_1) + \dim(W_2) = \dim(V) = \text{ed}(G_1 \times G_2). \quad \square$$

Corollary 5.2. *Let F be a field as in Theorem 5.1. Then*

$$\text{ed}(\mathbb{Z}/p^{n_1}\mathbb{Z} \times \mathbb{Z}/p^{n_2}\mathbb{Z} \times \cdots \times \mathbb{Z}/p^{n_s}\mathbb{Z}) = \sum_{i=1}^s [F(\xi_{p^{n_i}}) : F].$$

Proof. By Theorem 5.1, it suffices to consider the case $s = 1$. This case has been done in [6]. It is also covered by Theorem 4.1 as the natural representation of the group $\mathbb{Z}/p^n\mathbb{Z}$ in the F -space $F(\xi_{p^n})$ is faithful irreducible of the smallest dimension (cf. Remark 4.6). \square

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