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Essential self-adjointness of Schrödinger operators with potentials singular along affine subspaces

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1. Introduction

The aim of this paper is to study the essential self-adjointness of a Schrödinger operator $-\Delta + q(x)$ acting in $L^2(\mathbf{R}^m)$, $m \geqq 1$, with the domain $C_0^\infty(\mathbf{R}^m \backslash F)$, where F is the union of at most countable number of k_α-dimensional $(0 \leqq k_\alpha \leqq m-1)$ affine subspaces S_α $(\alpha \in A)$ in $\mathbf{R}^m$ which satisfy

$$r = \inf \{\operatorname{dist}(S_\alpha, S_\beta);\ \alpha, \beta \in A, \alpha \neq \beta\} > 0.$$

Here dist (S_α, S_β) denotes the distance from S_α to S_β.

This study is motivated by a theorem proved by B. Simon [6], which is a generalization of the results of H. Kalf and J. Walter [1] and U. W. Schmincke [5]. In this theorem of Simon, which corresponds to the case of $F = \{0\}$, it is assumed that the potential $q = q_1 + q_2$ is a real-valued function with $q_1 \in L^2_{\mathrm{loc}}(\mathbf{R}^m \backslash \{0\})$ and $q_2 \in L^\infty(\mathbf{R}^m)$ such that

$$q_1(x) \geqq -(1/4)m(m-4)|x|^{-2} \qquad (x \in \mathbf{R}^m \backslash \{0\}).$$

We extend this result to the case of the general F as stated above. The following is our theorem.

THEOREM. *Set $\Omega = \mathbf{R}^m \backslash F$ and let $a_j \in C^1(\Omega)$ $(1 \leqq j \leqq m)$, $q_1 \in L^2_{\mathrm{loc}}(\Omega)$ and $q_2 \in L^\infty(\mathbf{R}^m)$ be real-valued functions. Assume that for some ε $(0 < \varepsilon < r/2)$, q_1 satisfies the following conditions*:

(C.1) *For each $\alpha \in A$*

$$q_1(x) \geqq -(1/4)(m-k_\alpha)(m-k_\alpha-4)[\operatorname{dist}(x, S_\alpha)]^{-2}$$

whenever $0 < \operatorname{dist}(x, S_\alpha) < \varepsilon$.

(C.2) *q_1 is bounded from below on*

$$\bigcap_{\alpha \in A} \{x \in \mathbf{R}^m;\ \varepsilon \leqq \operatorname{dist}(x, S_\alpha)\}.$$

Let $q = q_1 + q_2$. Then the symmetric operator T acting in $L^2(\mathbf{R}^m)$ defined by

$$T = -\sum_{j=1}^m (\partial/\partial x_j - i a_j(x))^2 + q(x), \quad D(T) = C_0^\infty(\Omega),$$

is essentially self-adjoint.

For the proof of this theorem, we apply the method given in Simon [6] and Kalf-Walter [2].

2. Basic lemmas

Let us first recall Kato's inequality. Set $L=\sum_{j=1}^{m}(\partial/\partial x_j-ia_j(x))^2$. If $u\in L^1_{loc}(\Omega)$ and $Lu\in L^1_{loc}(\Omega)$, then we have the following distributional inequality (see [3], [4], [7], [8]):

$$\Delta|u| \geqq \mathrm{Re}\,[(\mathrm{sgn}\,\bar{u})Lu]\,.$$

By the aid of this inequality, we obtain the following lemma as in [6] and [2].

LEMMA 1. *Let Ω and T be as in the theorem, and suppose that there exist functions Q, Φ and Φ_n $(n=1, 2,\dots)$ which satisfy the following conditions:*

(P.1) $Q\in C^0(\Omega)$, $\Phi\in C^2(\Omega)\cap L^2(\Omega)$, $(-\Delta+Q)\Phi\in L^2(\Omega)$ *and* $\Phi_n\in C_0^2(\Omega)$ $(n=1, 2,\dots)$.

(P.2) $\Phi_n\to\Phi$ *weakly in* $L^2(\Omega)$ *and* $(-\Delta+Q)\Phi_n\to(-\Delta+Q)\Phi$ *weakly in* $L^2(\Omega)$ *as* $n\to\infty$.

(P.3) $q_1\geqq Q$ *on* Ω, $\Phi_n\geqq 0$ *on* Ω $(n=1, 2,\dots)$ *and* $(-\Delta+Q+\delta)\Phi>0$ *on* Ω *for some* $\delta\in\mathbf{R}$.

Then the assertion of the theorem holds.

Before stating Lemma 2 we introduce some functions.

Let $\alpha(t)$ be a non-increasing function in $C^\infty(\mathbf{R})$ such that

$$\begin{aligned} &\alpha(t)=1 \quad \text{for} \quad t\leqq 0, \qquad \alpha(t)=0 \quad \text{for} \quad t\geqq 1,\\ &0<\alpha(t)<1 \quad \text{for} \quad 0<t<1, \qquad\qquad (2.1)\\ &\sup_{0<t<1}|\alpha'(t)|<3 \quad \text{and} \quad \sup_{0<t<1}|\alpha''(t)|<5. \end{aligned}$$

Let f and f_n $(n=1, 2,\dots)$ be functions which satisfy the following conditions (1)~(4):

(1) $f\in\mathscr{S}(\mathbf{R}^m)$ and $f_n\in C_0^\infty(\mathbf{R}^m)$ $(n=1, 2,\dots)$, where $\mathscr{S}(\mathbf{R}^m)$ is the Schwartz space of C^∞-functions of rapid decrease.

(2) $f(x)>0$ and $0\leqq f_n(x)\leqq f_{n+1}(x)\leqq f(x)$ for any $x\in\mathbf{R}^m$ and $n=1, 2,\dots$.

(3) If we set $D_n=\{x\in\mathbf{R}^m; f_n(x)=f(x)\}$ $(n=1, 2,\dots)$, then $D_n\subseteqq\mathrm{Int}\,D_{n+1}$ $(n=1, 2,\dots)$ and $\bigcup_{n=1}^{\infty}D_n=\mathbf{R}^m$, where $\mathrm{Int}\,D_{n+1}$ is the interior of D_{n+1}.

(4) For any $r>0$, x, y, σ, $\tau\in\mathbf{R}^m$ with $|x-y|<r$ and $|\sigma|=|\tau|=1$, the following estimates hold:

(2.2) $$|D_\sigma f(x)| \leqq f(x) \leqq e^r f(y), \quad |D_\sigma D_\tau f(x)| \leqq 3f(x),$$
$$|D_\sigma f_n(x)| \leqq 4f(x) \quad \text{and} \quad |D_\sigma D_\tau f_n(x)| \leqq 20 f(x) \qquad (n=1, 2, \ldots),$$

where D_σ denotes the directional derivative in the direction σ.

An example of a set of f and f_n is given by (cf. [2])

$$f(x) = \exp(-(1+|x|^2)^{1/2}), \quad f_n(x) = \alpha((|x|/n)-1)\cdot\exp(-(1+|x|^2)^{1/2}).$$

Let f and f_n satisfy (1)$\sim$(4), P be an orthogonal transformation acting in $\mathbf{R}^m$, and $a \in \mathbf{R}^m$. If we define $\tilde{f}$ and $\tilde{f}_n$ $(n=1, 2,\ldots)$ by $\tilde{f}(x)=f(Px+a)$ and $\tilde{f}_n(x)=f_n(Px+a)$, then $\tilde{f}$ and $\tilde{f}_n$ also satisfy (1)$\sim$(4). We use this fact in the proof of Lemma 2.

LEMMA 2. *Let v be an arbitrary positive constant, S be a k-dimensional affine subspace in* $\mathbf{R}^m$ $(0 \leqq k \leqq m-1)$, *and f, f_n $(n=1, 2,\ldots)$ be functions which satisfy* (1)$\sim$(4) *stated above. Set* $V=\{x \in \mathbf{R}^m;\ 0<\operatorname{dist}(x, S)<v\}$.

Then there exist functions ψ and ψ_n $(n=1, 2,\ldots)$ which satisfy the following conditions (i)$\sim$(v):

(i) $\psi \in C^\infty(\mathbf{R}^m \backslash S)$ *and* $\psi_n \in C_0^\infty(\mathbf{R}^m \backslash S)$ $(n=1, 2,\ldots)$.

(ii) $\psi(x)>0$ *and* $0 \leqq \psi_n(x) \leqq \psi_{n+1}(x) \leqq \psi(x)$ *for all* $x \in \mathbf{R}^m$ *and* $n=1, 2,\ldots$.

(iii) *If we set* $E_n=\{x \in V;\ \psi_n(x)=\psi(x)\}$ $(n=1, 2,\ldots)$, *then* $E_n \subseteqq \operatorname{Int} E_{n+1}$ $(n=1, 2,\ldots)$ *and* $\bigcup_{n=1}^\infty E_n = V$.

(iv) $\psi(x)=f(x)$ *and* $\psi_n(x)=f_n(x)$ $(n=1, 2,\ldots)$ *for* $x \in \mathbf{R}^m \backslash S \backslash V$.

(v) *There is a constant $c>0$ depending only on v and m such that the following estimates* (v–a), (v–b) *and* (v–c) *hold*:

(v–a) $$\int_V |\psi|^2 dx \leqq c \int_V |f|^2 dx.$$

(v–b) $$|(-\Delta - (1/4)(m-k)(m-k-4)[\operatorname{dist}(x, S)]^{-2})\psi(x)| < c\psi(x)$$
for any $x \in V$.

(v–c) $$\int_V |(-\Delta - (1/4)(m-k)(m-k-4)[\operatorname{dist}(x, S)]^{-2})\psi_n|^2 dx \leqq c\int_V |f|^2 dx \qquad \text{for any} \quad n=1, 2,\ldots.$$

PROOF. We prove this lemma only for $k \neq 0$; our proof is valid for $k=0$ under some modification.

By a coordinate transformation remarked just before Lemma 2, we may assume that $S=\mathbf{R}^k \times \{0\}$ from the beginning. Then $\operatorname{dist}(x, S)=|x_2|$ for any $x=(x_1, x_2) \in \mathbf{R}^m = \mathbf{R}^k \times \mathbf{R}^{m-k}$.

Set $\beta(x_2)=\alpha(2-(2/v)|x_2|)$, $x_2 \in \mathbf{R}^{m-k}$ and define ψ and ψ_n $(n=1, 2,\ldots)$ by

$$\psi(x) = f(x)\beta(x_2) + |x_2|^{(4-m+k)/2} f(x_1, 0)(1 - \beta(x_2)),$$
$$\psi_n(x) = f_n(x)\beta(x_2) + |x_2|^{(4-m+k)/2} f_n(x_1, 0)\beta(nx_2)(1 - \beta(x_2))$$

for $x=(x_1, x_2) \in \mathbf{R}^m = \mathbf{R}^k \times \mathbf{R}^{m-k}$ and $n=1, 2, \dots$.

Let us verify that ψ and ψ_n defined as above satisfy the conditions (i)$\sim$(v). Since by definition (i), (ii), (iii) and (iv) hold evidently, we have only to prove (v). In what follows we use c_j ($j=1, 2, 3, 4$) to denote constants depending only on v and m.

First we remark that for any integer $s > -m+k$

$$(2.3)\quad \int_V |x_2|^s |f(x_1, 0)|^2 dx = (m-k)(m-k+s)^{-1} v^s \int_V |f(x_1, 0)|^2 dx \leqq m v^s e^{2v} \int_V |f|^2 dx.$$

By this inequality we have

$$\int_V |\psi|^2 dx \leqq 2\int_V |f|^2 dx + 2\int_V |x_2|^{4-m+k} |f(x_1, 0)|^2 dx \leqq 2(1 + m v^{4-m+k} e^{2v}) \int_V |f|^2 dx,$$

which implies (v–a).

We proceed to prove (v–b). Let us set

$$I(x) = (-\Delta - (1/4)(m-k)(m-k-4)|x_2|^{-2})\psi(x),$$
$$\Delta_1 = \textstyle\sum_{i=1}^k \partial^2/\partial x_i^2 \quad \text{and} \quad \Delta_2 = \Delta - \Delta_1.$$

We first note that

$$(2.4)\qquad (\Delta_2 + (1/4)(m-k)(m-k-4)|x_2|^{-2})|x_2|^{(4-m+k)/2} = 0.$$

If $0 < |x_2| \leqq v/2$, then $\psi(x) = |x_2|^{(4-m+k)/2} f(x_1, 0)$, so that

$$|I(x)| \leqq |x_2|^{(4-m+k)/2} |(\Delta_1 f)(x_1, 0)| + |(\Delta_2 + (1/4)(m-k)(m-k-4)|x_2|^{-2})|x_2|^{(4-m+k)/2}| \cdot f(x_1, 0) = |x_2|^{(4-m+k)/2} |(\Delta_1 f)(x_1, 0)|$$

by (2.4). Since

$$|(\Delta_1 f)(x_1, 0)| \leqq 3k f(x_1, 0) < 3m f(x_1, 0),$$

in view of condition (2.2), it follows that $|I(x)| < 3m\psi(x)$ for $0 < |x_2| \leqq v/2$. We next consider the case $v/2 < |x_2| < v$. Noting that

$$(2.5)\qquad |(\partial\beta/\partial x_i)(x_2)| < 6/v \quad \text{and} \quad |(\partial^2\beta/\partial x_i^2)(x_2)| < 44/v^2$$

for $k+1\leqq i\leqq m$ and using (2.2) we can see that there is a constant c_1 such that $|I(x)|<c_1 f(x)$. Combining this with the fact that

$$f(x)=f(x)\beta(x_2)+f(x)(1-\beta(x_2))\leqq(1+e^v \sup_{v/2<t<v} t^{(m-k-4)/2})\psi(x),$$

we obtain $|I(x)|<c_2\psi(x)$ for $v/2<|x_2|<v$. Thus (v–b) is satisfied.

Finally we show (v–c). For simplicity we prove (v–c) only for $n=3, 4,\dots$. Let us set $\gamma_n(x_2)=\beta(nx_2)(1-\beta(x_2))$ for $x_2\in\mathbf{R}^{m-k}$. Then by (2.5) we have

$$|(\partial\gamma_n/\partial x_i)(x_2)|\leqq\begin{cases}(6/v)n & \text{if } v/(2n)<|x_2|<v/n\\ 6/v & \text{if } v/2<|x_2|<v\\ 0 & \text{elsewhere,}\end{cases}\tag{2.6}$$

$$|(\partial^2\gamma_n/\partial x_i^2)(x_2)|\leqq\begin{cases}(44/v^2)n^2 & \text{if } v/(2n)<|x_2|<v/n\\ 44/v^2 & \text{if } v/2<|x_2|<v\\ 0 & \text{elsewhere}\end{cases}\tag{2.7}$$

for $i=k+1,\dots, m$. Thus we have

$$\begin{aligned}&\left\{\int_V|(-\Delta-(1/4)(m-k)(m-k-4)|x_2|^{-2})\psi_n|^2dx\right\}^{1/2}\\ &\leqq\left\{\int_V|\Delta(f_n(x)\beta(x_2))|^2dx\right\}^{1/2}\\ &\quad+(1/4)(m-k)|m-k-4|\left\{\int_V|x_2|^{-4}(f_n(x)\beta(x_2))^2dx\right\}^{1/2}\\ &\quad+\left\{\int_V|x_2|^{4-m+k}(\gamma_n(x_2))^2((\Delta_1 f_n)(x_1, 0))^2dx\right\}^{1/2}\\ &\quad+\left\{\int_V|(\Delta_2+(1/4)(m-k)(m-k-4)|x_2|^{-2})(|x_2|^{(4-m+k)/2}\gamma_n(x_2))|^2\right.\\ &\qquad\qquad\left.\times(f_n(x_1, 0))^2dx\right\}^{1/2}\\ &=I_1+I_2+I_3+I_4,\end{aligned}$$

where I_j ($j=1, 2, 3, 4$) denotes the j-th term respectively.

By virtue of (2.2), (2.3) and (2.5), we can easily check that there is a constant c_3 such that

$$I_1+I_2+I_3\leqq c_3\left\{\int_V|f|^2dx\right\}^{1/2}.\tag{2.8}$$

Now we estimate I_4. By virtue of (2.4), (2.6) and (2.7),

$$I_4=\left\{\int_V|2\textstyle\sum_{i=k+1}^m \partial/\partial x_i(|x_2|^{(4-m+k)/2})\cdot(\partial\gamma_n/\partial x_i)(x_2)\right.$$

$$+ |x_2|^{(4-m+k)/2}(\Delta_2\gamma_n)(x_2)|^2(f_n(x_1, 0))^2dx\Big\}^{1/2}$$

$$\leqq (m-k)|4-m+k|\Big\{\int_{V_n} |x_2|^{2-m+k}(6n)^2v^{-2}(f(x_1, 0))^2dx$$

$$+\int_{V_1} |x_2|^{2-m+k}(6/v)^2(f(x_1, 0))^2dx\Big\}^{1/2}$$

$$+ (m-k)\Big\{\int_{V_n} |x_2|^{4-m+k}(44n^2)^2v^{-4}(f(x_1, 0))^2dx$$

$$+ \int_{V_1} |x_2|^{4-m+k}(44/v^2)^2(f(x_1, 0))^2dx\Big\}^{1/2},$$

where we set $V_n=\{x=(x_1, x_2)\in \mathbf{R}^m=\mathbf{R}^k\times\mathbf{R}^{m-k};\ v/(2n)<|x_2|<v/n\}$ $(n=1, 2,\dots)$. Since

$$n^2\int_{V_n} |x_2|^{2-m+k}(f(x_1, 0))^2dx = \int_{V_1} |x_2|^{2-m+k}(f(x_1, 0))^2dx,$$

$$n^4\int_{V_n} |x_2|^{4-m+k}(f(x_1, 0))^2dx = \int_{V_1} |x_2|^{4-m+k}(f(x_1, 0))^2dx$$

for any $n=1, 2,\dots$, it follows from (2.3) that

$$I_4 \leqq (6/v)m^2\Big\{2\int_{V_1} |x_2|^{2-m+k}(f(x_1, 0))^2dx\Big\}^{1/2}$$

$$+ (44/v^2)m\Big\{2\int_{V_1} |x_2|^{4-m+k}(f(x_1, 0))^2dx\Big\}^{1/2}$$

$$\leqq c_4\Big\{\int_V |f|^2dx\Big\}^{1/2}.$$

Combining this with (2.8), we obtain

$$I_1 + I_2 + I_3 + I_4 \leqq (c_3 + c_4)\Big\{\int_V |f|^2dx\Big\}^{1/2},$$

which completes the proof of (v–c). q.e.d.

3. Proof of the theorem

Now we fix a set of f and f_n $(n=1, 2,\dots)$ satisfying (1)~(4). For each $\alpha\in A$ we apply Lemma 2 with $S=S_\alpha$ and $v=\varepsilon/2$, and put

$$\psi^\alpha = \psi,\quad \psi^\alpha_n = \psi_n \quad\text{and}\quad E^\alpha_n = E_n, \qquad n = 1, 2,\dots.$$

Let Q be a real-valued function in $C^0(\Omega)$ which satisfies the following conditions (a), (b) and (c):

(a) $q_1(x) \geqq Q(x)$ for any $x\in\Omega$.

(b) For each $\alpha \in A$

$$Q(x) = -(1/4)(m-k_\alpha)(m-k_\alpha-4)[\text{dist}(x, S_\alpha)]^{-2},$$
$$\text{whenever} \quad 0 < \text{dist}(x, S_\alpha) < \varepsilon/2.$$

(c) Q is bounded on $\bigcap_{\alpha\in A}\{x \in \mathbf{R}^m;\ \varepsilon/2 \leqq \text{dist}(x, S_\alpha)\}$.

Define Φ and Φ_n $(n=1, 2,\dots)$ by

$$\Phi(x) = \begin{cases} \psi^\alpha(x) & \text{if} \quad 0<\text{dist}(x, S_\alpha)<\varepsilon/2 \quad \text{for some} \quad \alpha \\ f(x) & \text{elsewhere,} \end{cases}$$

$$\Phi_n(x) = \begin{cases} \psi^\alpha_n(x) & \text{if} \quad 0<\text{dist}(x, S_\alpha)<\varepsilon/2 \quad \text{for some} \quad \alpha \\ f_n(x) & \text{elsewhere.} \end{cases}$$

We now prove that the conditions (P.1), (P.2) and (P.3) in Lemma 1 are satisfied with these Q, Φ and Φ_n. Let us set $V(\alpha)=\{x\in\mathbf{R}^m;\ 0<\text{dist}(x, S_\alpha)<\varepsilon/2\}$ for each $\alpha\in A$, $W=\bigcup_{\alpha\in A} V(\alpha)$ and $\Xi=\Omega\backslash W$.

To verify (P.1) we have only to examine that $\Phi\in L^2(\Omega)$ and $(-\Delta+Q)\Phi\in L^2(\Omega)$ since the other conditions in (P.1) are obvious. Using (v–a) and (v–b) in Lemma 2, we have

$$\int_\Omega |\Phi|^2 dx = \int_\Xi |f|^2 dx + \sum_{\alpha\in A}\int_{V(\alpha)} |\psi^\alpha|^2 dx \leqq \int_\Xi |f|^2 dx + c\int_W |f|^2 dx < +\infty$$

and

$$\int_\Omega |(-\Delta+Q)\Phi|^2 dx = \int_\Xi |(-\Delta+Q)f|^2 dx$$
$$+ \sum_{\alpha\in A}\int_{V(\alpha)} |(-\Delta-(1/4)(m-k_\alpha)(m-k_\alpha-4)[\text{dist}(x, S_\alpha)]^{-2})\psi^\alpha|^2 dx$$
$$\leqq \int_\Xi |(-\Delta+Q)f|^2 dx + c^2\sum_{\alpha\in A}\int_{V(\alpha)} |\psi^\alpha|^2 dx < +\infty.$$

We proceed to verify (P.2). Since $0\leqq\Phi_n\leqq\Phi$ on Ω $(n=1, 2,\dots)$ and $\lim_{n\to\infty}\Phi_n(x)=\Phi(x)$ $(x\in\Omega)$, it follows from Lebesgue's covergence theorem that $\Phi_n\to\Phi$ strongly in $L^2(\Omega)$. Let u be an arbitrary element of $L^2(\Omega)$. Then we have

$$\text{(3.1)} \qquad \left|\int_\Omega \bar{u}\cdot(-\Delta+Q)(\Phi_n-\Phi)dx\right|$$
$$\leqq \left\{\int_{\Pi(n)} |u|^2 dx\right\}^{1/2}\left\{\int_\Omega |(-\Delta+Q)(\Phi_n-\Phi)|^2 dx\right\}$$

where we set $\Pi(n)=\{x\in\Omega;\ (-\Delta+Q(x))(\Phi_n(x))\neq(-\Delta+Q(x))(\Phi(x))\}$ $(n=1, 2,\dots)$. Since, from the condition (3) imposed on f and f_n and (iii) in Lemma 2, $\Pi(n+1)\subseteqq(\mathbf{R}^m\backslash D_n)\cup\{\bigcup_{\alpha\in A}(V(\alpha)\backslash E_n)\}$ for any $n=1, 2,\dots$, we obtain

(3.2)
$$\lim_{n\to\infty}\int_{\Pi(n)}|u|^2dx = 0.$$

On the other hand, by (v–c) and (2.2), we see that

$$\int_{\Omega}|(-\Delta+Q)(\Phi_n-\Phi)|^2dx \leqq c'\int_{\Omega}|f|^2dx$$

for some constant c' which is independent of n. Applying this fact and (3.2) to (3.1), we conclude that

$$\lim_{n\to\infty}\int_{\Omega}\bar{u}\cdot(-\Delta+Q)(\Phi_n-\Phi)dx = 0.$$

Finally let us verify (P.3). We define δ by

$$\delta = 20m + c + \sup\{|Q(x)|;\ x\in\Xi\},$$

where c is the constant given in Lemma 2 for $\nu=\varepsilon/2$. If $x\in\Xi$, then

$$\begin{aligned}(-\Delta+Q(x)+\delta)\Phi(x) &= -(\Delta f)(x) + Q(x)f(x) + \delta f(x)\\ &\geqq -20mf(x) - \sup\{|Q(x)|;\ x\in\Xi\}\cdot f(x) + \delta f(x) > 0.\end{aligned}$$

If $x\in V(\alpha)$ for some $\alpha\in A$, then by (v–b) in Lemma 2

$$\begin{aligned}&(-\Delta+Q(x)+\delta)\Phi(x)\\ &= (-\Delta-(1/4)(m-k_\alpha)(m-k_\alpha-4)[\mathrm{dist}(x, S_\alpha)]^{-2} + \delta)\psi^\alpha(x)\\ &\geqq (-c\psi^\alpha(x) + \delta\psi^\alpha(x)) > 0.\end{aligned}$$

This completes the proof of (P.3). q. e. d.

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