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Essential spectrum and Weyl asymptotics for discrete Laplacians

Michel Bonnefont[(1)], Sylvain Golénia[(2)]

Résumé. — Dans cet article, nous étudions le spectre de Laplaciens discrets. Notre travail est basé sur l'inégalité de Hardy et l'étude des fonctions super-harmoniques. Nous retrouvons et améliorons des bornes inférieures pour le bas du spectre et le bas du spectre essentiel. Dans certains cas, nous obtenons des asymptotiques de Weyl pour les valeurs propres. Nous donnons aussi une représentation probabiliste des fonctions super-harmoniques, puis avec des arguments de type couplage, nous établissons des résultats de comparaison pour le bas du spectre, le bas du spectre essentiel et la complétude stochastique de différents Laplaciens discrets. Une classe de graphes faiblement symétriques est aussi étudiée en grand détail.

Abstract. — In this paper, we investigate spectral properties of discrete Laplacians. Our study is based on the Hardy inequality and the use of super-harmonic functions. We recover and improve lower bounds for the bottom of the spectrum and of the essential spectrum. In some situation, we obtain Weyl asymptotics for the eigenvalues. We also provide a probabilistic representation of super-harmonic functions. Using coupling arguments, we set comparison results for the bottom of the spectrum, the bottom of the essential spectrum and the stochastic completeness of different discrete Laplacians. The class of weakly spherically symmetric graphs is also studied in full detail.

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Article proposé par Gilles Carron.

Contents

1. Introduction

The study of discrete Laplacians on infinite graphs is at the crossroad of spectral theory and geometry. A special role is played by the bottom of the spectrum and that of the essential spectrum of discrete Laplacians. Concerning the former, a famous link is given through Cheeger/isoperimetrical inequalities, e.g., [3, 5, 10, 11, 12, 15, 22, 23, 20, 29]. For the latter, since the essential spectrum can be thought as the spectrum of the Laplacian "at infinity", the link is given through isoperimetrical inequalities at infinity, e.g., [15, 20]. In this article we tackle the question with another standpoint and establish a new link with the help of Hardy inequalities, see Section 3, and positive super-harmonic functions.

We fix briefly some notation. A weighted graph $\mathscr{G}$ is a triple $\mathscr{G} := (\mathscr{V}, \mathscr{E}, m)$, where $\mathscr{V}$ denotes a countable set (the *vertices* of $\mathscr{G}$), $\mathscr{E}$ a non-negative symmetric function on $\mathscr{V} \times \mathscr{V}$ and m a positive function on $\mathscr{V}$. We say that two points $x, y \in \mathscr{V}$ are *neighbors* and we denote $x \sim y$ if $\mathscr{E}(x,y) = \mathscr{E}(y,x) > 0$. We assume that $\mathscr{G}$ is locally finite in the sense that each point of $\mathscr{V}$ has only a finite number of neighbors.

The Laplacian then reads, for f with finite support,

$$\Delta_m f(x) = \frac{1}{m(x)} \sum_{y, y \sim x} \mathscr{E}(x,y)(f(x) - f(y)).$$

We then consider its Friedrich extension and keep the same symbol. It defines a non-negative self-adjoint operator. Its spectrum is thus included in $[0, \infty)$.

Given $W : \mathscr{V} \to (0, +\infty)$, the Hardy inequality reads as follows:

$$\langle f, \Delta_m f \rangle_m \geqslant \langle f, \frac{\tilde{\Delta}_m W}{W} f \rangle_m,$$

where $f : \mathscr{V} \to \mathbb{R}$ with finite support and $\tilde{\Delta}_m$ denotes the algebraic Laplacian (see Section 2.1). The heart of this article is to exploit this inequality for some good choice(s) of W. This method is very flexible:

- We recover and improve lower bounds for the spectrum and for the essential spectrum, see Section 4.

- We improve some criteria for the absence of the essential spectrum, see Section 4.
- We study the eigenvalues below the essential spectrum and obtain Weyl asymptotics for the eigenvalues, see Section 5.
- We state an Allegretto-Piepenbrink type theorem for the spectrum and the essential spectrum, see Section 6. This theorem links the bottom of the spectra with the existence of positive super-harmonic functions.
- We establish a probabilistic representation of super-harmonic functions, see Section 8. As a corollary, we derive a probabilistic understanding of the bottom of the spectrum and of the essential spectrum.
- For *weakly spherically symmetric graphs* we prove that the bottom of the spectrum and of the essential spectrum are governed only by the radial part of the Laplacian, see Section 9.
- Using a coupling argument, we establish new comparison results for the bottom of the spectrum and the essential spectrum of different discrete Laplacians, see Section 10.
- We derive a comparison result for the stochastic completeness, see Section 11.

A main part of our work is to provide geometric criterion to ensure the existence of positive super-harmonic functions; that is functions $W : \mathscr{V} \to (0, \infty)$ satisfying

$$\tilde{\Delta}_m W(x) \geqslant \lambda(x) W(x),$$

where $\lambda : \mathscr{V} \to [0, \infty)$ is some non-negative function. These criterion are based on the geometric properties of a 1-dimensional decomposition of the graph. In many situations, this 1-dimensional decomposition is given by the distance to a point or to a finite set. Using min-max principles, we then derive the lower bounds for the bottom of both the spectrum and the essential spectrum (see Theorems 4.8, 4.13 and Corollary 4.18). We also obtain some lower and upper bounds for the eigenvalues and get Weyl asymptotics for the eigenvalues.

The Hardy inequality was already known in this discrete setting under different names, e.g., [8, 16, 17]. Our present work is inspired by the work of [6], where the authors study diffusion operators in a continuous setting and with a finite invariant measure. They give criteria based on Lyapunov functions to show that the Super-Poincaré Inequality holds.

We mention that the Super-Poincaré Inequality was introduced by Wang (see [30, 31, 33]) and is equivalent to a lower bound of the essential spectrum. We do not rely on this approach but this point of view enlightens about the situation. For the sake of completeness, we include the proofs of their results in Appendix B.

The Hardy inequality directly gives one direction of the Allegretto-Piepenbrik theorem (Theorem 3.6). For the other direction, knowing a lower bound on the spectrum or the essential spectrum, one has to construct a positive super-harmonic function (Theorems 6.1 and 6.2). This was known for the spectrum (e.g., [17]) but seems to be new for the essential spectrum.

We then provide a probabilistic representation of super-harmonic functions (see Theorems 8.3 and 8.5). It is interesting to compare with [7]. The difference is that they control how the stochastic process returns in a compact domain whereas we control how the associated Markov process goes to infinity. An important tool is the Harnack inequality that we borrow from [17], see Section 7.

Next, we prove comparison results for the bottom of the spectrum and the essential spectrum of different weighted Laplacians. Theorem 10.4 is an improvement of Theorem 4 in [21]. The main new ingredient in the proof of Theorem 10.4 is a coupling argument between the associated stochastic processes (see Proposition 10.1). This coupling argument works under a condition we called *stronger weak-curvature growth* which is strictly weaker than the *stronger curvature growth* condition of Theorem 4 in [21]. Moreover, we treat the case of the essential spectrum. The coupling argument also provides a comparison result for stochastic completeness (see Theorem 11.2).

Besides, we study the class of *weakly spherically symmetric graphs*, see Definition 2.1. These graphs are a slight generalization of the ones introduced in [21]. We first show weakly spherically symmetric graphs are exactly the graphs such that the radial part of their associated continuous time Markov chain is also a 1-dimensional continuous time Markov chain, (see Propositions 9.2 and 9.5). We then show that both the bottom of the spectrum and the essential spectrum for the Laplacian on a weakly spherically symmetric graph coincide with the ones of their radial part (see Theorem 9.4).

The paper is organized as follows. In Section 2, we present the notation and we carefully introduce the Laplacian. Section 3 is devoted to the statement and a new proof of the Hardy inequality. The lower bounds for the spectrum and the essential spectrum are obtained in Section 4. In Section 5, we focus on eigenvalues. The estimates for the eigenvalues are very dependent of the intrinsic geometry of the graphs. Weyl asymptotics for eigenvalues of radial trees are given in Theorem 5.3. Theorem 5.4 treats the case of general weakly spherically symmetric graphs. In Section 6, we state and prove the Allegreto-Piepenbrik type theorem (Theorem 6.2). Section 7 is devoted to Harnack inequalities for positive super-harmonic functions. The construction of the discrete and continuous time Markov chain associated to the Laplacian are made in Section 8. We also provide the probabilistic representation of super-harmonic functions (see Theorems 8.3 and 8.5). Section 9 is dedicated to the study of the class of *weakly spherically symmetric graphs*. In Section 10, using a coupling argument, we compare the bottom of the spectrum and the essential spectrum of two given weighted Laplacians. Section 11 deals with stochastic completeness. The construction of the Friedrichs extension of the Laplacian is recalled in Appendix A and Appendix B is devoted to the Super-Poincaré inequality.

2. Notation

2.1. The Laplacian on a graph

Let us consider a graph $\mathscr{G} := (\mathscr{V}, \mathscr{E}, m)$ where $\mathscr{V}$ denotes a countable set of vertices of $\mathscr{G}$, $\mathscr{E}$ a non-negative symmetric function on $\mathscr{V} \times \mathscr{V}$ and m a positive function on $\mathscr{V}$. We say that two points $x, y \in \mathscr{V}$ are neighbors if $\mathscr{E}(x,y) = \mathscr{E}(y,x) > 0$. In this case we write $x \sim y$. We assume that $\mathscr{G}$ is locally finite in the sense that each point of $\mathscr{V}$ has only a finite number of neighbors. For simplicity, we also assume that each connected component of $\mathscr{G}$ is infinite.

Let $\mathcal{C}_c(\mathscr{V})$ be the set of functions $f : \mathscr{V} \to \mathbb{C}$ with finite support and let $\ell^2(\mathscr{V}, m)$ be the set of functions $f : \mathscr{V} \to \mathbb{C}$ such that

$$\|f\|_{\ell^2(\mathscr{V},m)} := \sum_{x \in \mathscr{V}} |f(x)|^2 m(x)$$

is finite. $\ell^2(\mathscr{V}, m)$ is an Hilbert space with respect to the scalar product:

$$\langle f, g \rangle_m := \sum_{x \in \mathscr{V}} \overline{f(x)} g(x) m(x) \text{ for } f, g \in \ell^2(\mathscr{V}, m).$$

For all $f, g \in \mathcal{C}_c(\mathscr{V})$, we introduce the quadratic form

$$Q(f,g) := \frac{1}{2} \sum_x \sum_y \mathscr{E}(x,y) \overline{(f(x) - f(y))} \,(g(x) - g(y)) \,.$$

Note that Q is well-defined since the graph is locally finite. This quadratic form is non-negative, i.e., $Q(f,f) \geqslant 0$ for all $f \in \mathcal{C}_c(\mathscr{V})$ and closable. There is a unique self-adjoint operator $\Delta_{\mathscr{G}}$ such that

$$Q(f,f) = \langle f, \Delta_{\mathscr{G}} f\rangle_m$$

for all $f \in \mathcal{C}_c(\mathscr{V})$ and $\mathcal{D}(\Delta_{\mathscr{G}}{}^{1/2})$ is the completion of $\mathcal{C}_c(\mathscr{V})$ under the norm $\|\cdot\| + Q(\cdot,\cdot)^{1/2}$. We refer to Appendix A for its construction. We have:

$$\Delta_{\mathscr{G}} f(x) = \frac{1}{m(x)} \sum_{y\in\mathscr{V}} \mathscr{E}(x,y)\,(f(x)-f(y)), \text{ for all } f \in \mathcal{C}_c(\mathscr{V}). \tag{2.1}$$

We call this operator the *Laplacian* associated to the graph $\mathscr{G}$. According to the context, we will also denote it by $\Delta_{\mathscr{G},m}$ or Δ_m. We mention that $\Delta_{\mathscr{G}}$ is the Friedrichs extension of $\Delta_{\mathscr{G}}|_{\mathcal{C}_c(\mathscr{V})}$.

Note that $\Delta_{\mathscr{G}}|_{\mathcal{C}_c(\mathscr{V})}$ is not necessarily essentially self-adjoint. We refer to [16] for a review of this matter.

We write with the symbol $\tilde{\Delta}_{\mathscr{G}}$ the algebraic version of $\Delta_{\mathscr{G}}$, i.e.,

$$\tilde{\Delta}_{\mathscr{G}} f(x) = \frac{1}{m(x)} \sum_{y\in\mathscr{V}} \mathscr{E}(x,y)\,(f(x)-f(y)), \text{ for all } f : \mathscr{V} \to \mathbb{C}.$$

Recall that $\tilde{\Delta}$ is well-defined since $\mathscr{G}$ is locally finite.

2.2. The Dirichlet Laplacian and Persson's lemma

Let $\mathscr{U}$ be any subset of $\mathscr{V}$. First, we define $\operatorname{Int}\mathscr{U} := \{x \in \mathscr{U}, y \sim x \Rightarrow y \in \mathscr{U}\}$ the interior of $\mathscr{U}$ and $\partial\mathscr{U} := \{x \in \mathscr{U}, \exists y \in \mathscr{U}^c, y \sim x\}$ the boundary of $\mathscr{U}$.

We call $\mathscr{G}^{\mathscr{U}:=(\mathscr{U},\mathscr{E}^{\mathscr{U}},m)}$ the *induced graph* on $\mathscr{U}$ where $\mathscr{E}^{\mathscr{U}}$ is defined on $\mathscr{U}\times\mathscr{U}$ by $\mathscr{E}^{\mathscr{U}(x,y):=\mathscr{E}(x,y),x,y\in\mathscr{U}}$.

We denote by $\Delta_{\mathscr{G}}^{\mathscr{U}}$ the associated *Dirichlet Laplacian.* It is defined as follows: for $f : \mathscr{U} \to \mathbb{C}$ with finite support, we define $\tilde{f} : \mathscr{V} \to \mathbb{C}$ by $\tilde{f}(x) = f(x)$, if $x \in \mathscr{U}$ and $\tilde{f} = 0$ otherwise, we set:

$$\Delta_{\mathscr{G}}^{\mathscr{U}} f(x) := \Delta_{\mathscr{G}} \tilde{f}(x), \text{ for all } x \in \mathscr{U}.$$

Note that $\Delta_{\mathscr{G}}^{\mathscr{U}}$ is a self-adjoint operator acting in $\ell^2(\mathscr{U})$. It is the Friedrichs extension of $\Delta_{\mathscr{G}}^{\mathscr{U}}|_{\mathcal{C}_c(\mathscr{U})}$. Note also that $\Delta_{\mathscr{G}}^{\mathscr{U}}$ and $\Delta_{\mathscr{G}^{\mathscr{U}}}$ define different operators.

The infimum of the essential spectrum of $\Delta_{\mathscr{G}}$ is classically described by the Persson Lemma, e.g., [19, Proposition 18]. One reads:

$$\begin{aligned}\inf \sigma_{\text{ess}}(\Delta_{\mathscr{G}}) &= \sup_{\mathscr{K}\subset\mathscr{V}\text{finite}} \inf \sigma\left(\Delta_{\mathscr{G}}{}^{\mathscr{K}^c}\right) \\ &= \sup_{\mathscr{K}\subset\mathscr{V}\text{finite}} \inf_{f\in\mathcal{C}_c(\mathscr{V}\setminus\mathscr{K}),\|f\|=1} \langle f, \Delta_{\mathscr{G}} f\rangle. \end{aligned} \tag{2.2}$$

Note that if $\Delta_{\mathscr{G}}$ is bounded from above we also have

$$\begin{aligned}\sup \sigma_{\text{ess}}(\Delta_{\mathscr{G}}) &= \inf_{\mathscr{K}\subset\mathscr{V}\text{finite}} \sup \sigma\left(\Delta_{\mathscr{G}}{}^{\mathscr{K}^c}\right) \\ &= \inf_{\mathscr{K}\subset\mathscr{V}\text{finite}} \sup_{f\in\mathcal{C}_c(\mathscr{V}\setminus\mathscr{K}),\|f\|=1} \langle f, \Delta_{\mathscr{G}} f\rangle. \end{aligned}$$

2.3. 1-dimensional decomposition, distance function and degrees

A *1-dimensional decomposition* of the graph $\mathscr{G} := (\mathscr{V}, \mathscr{E})$ is a family of finite sets $(S_n)_{n\geqslant 0}$ which forms a partition of $\mathscr{V}$, that is $\mathscr{V} = \sqcup_{n\geqslant 0} S_n$, and such that for all $x \in S_n, y \in S_m$,

$$\mathscr{E}(x,y) > 0 \implies |n-m| \leqslant 1.$$

Given such a 1-dimensional decomposition, we write $|x| := n$ if $x \in S_n$. We also write $B_N := \cup_{0\leqslant i\leqslant N} S_i$. A function $f : \mathscr{V} \to \mathbb{R}$ is said to be *radial* if $f(x)$ depends only on $|x|$.

Typical examples of such a 1-dimensional decomposition are given by level sets of the graph distance function to a finite set S_0 that is

$$S_n := \{x \in \mathscr{V}, d_{\mathscr{G}}(x, S_0) = n\}, \tag{2.3}$$

where the *graph distance function* $d_{\mathscr{G}}$ is defined by

$$d_{\mathscr{G}}(x,y) := \min\{n \in \mathbb{N}, x \sim x_1 \sim \cdots \sim x_n = y, x_i \in \mathscr{V}, i = 1, \dots, n\}. \tag{2.4}$$

Note that for a general 1-dimensional decomposition, one has solely $d_{\mathscr{G}}(x, S_0) \geqslant n$, for $x \in S_n$. Given $x \in S_n$ and $k \geqslant -n$, we shall denote by

$$S_{k,x} := \{y \in S_{n+k},\, d_{\mathscr{G}}(x,y) = |k|\}. \tag{2.5}$$

We introduce the following *unweighted degrees* of $x \in S_n$:

$$\eta_0(x) := \sum_{y\in S_n} \mathscr{E}(x,y), \quad \eta_\pm(x) := \sum_{y\in S_{n\pm 1}} \mathscr{E}(x,y),$$

with the convention that $S_{-1} = \emptyset$, i.e., $\eta_-(x) = 0$ when $x \in S_0$. The *total unweighted degree* of x is defined by:

$$\eta(x) := \eta_0(x) + \eta_-(x) + \eta_+(x) = \sum_{y \in \mathscr{V}} \mathscr{E}(x, y).$$

We stress that contrary to $\eta_\pm$ and η_0, η is independent of the choice of 1-dimensional decomposition. Moreover, η only depends on $(\mathscr{V}, \mathscr{E})$ and not on the weight m.

We now divide by the weight and obtain new quantities of interest. We call them the *(weighted) degrees* and denote them by:

$$\begin{aligned} \deg_a(x) &:= \frac{\eta_a(x)}{m(x)}, \qquad \text{where } a \in \{0, -, +\}, \\ \deg(x) &:= \frac{\eta(x)}{m(x)} = \deg_-(x) + \deg_0(x) + \deg_+(x). \end{aligned}$$

Again, note that deg is independent of the choice of a 1-dimensional decomposition. When $m(x) = \eta(x)$, we have $\deg(x) \equiv 1$ and we also write $p_+(x), p_0(x), p_-(x)$ for $\deg_+(x), \deg_0(x), \deg_-(x)$, respectively.

In the same spirit, we also define:

$$\deg(x, y) := \frac{\mathscr{E}(x, y)}{m(x)}.$$

We say that the graph $\mathscr{G}$ is *simple* when $\mathscr{E} : \mathscr{V} \times \mathscr{V} \to \{0, 1\}$. This definition is independent of the choice of the weight m. In this case, when $m = 1$, the operator Δ_1 is usually called the *combinatorial Laplacian* on the graph $\mathscr{G}$ whereas when $m(x) = \eta(x)$, or equivalently when $\deg \equiv 1$, the operator Δ_η is usually called the *normalized Laplacian.*

In the case of the combinatorial Laplacian Δ_1 on a simple graph, one has

$$\deg_\pm(x) = \eta_\pm(x) = \#\{y, y \sim x, |y| = |x| \pm 1\}$$

and

$$\deg_0(x) = \eta_0(x) = \#\{y, y \sim x, |y| = |x|\}.$$

Given a function $V : \mathscr{V} \to \mathbb{C}$, we denote by $V(\cdot)$ the operator of multiplication by V. It is elementary that $\mathcal{D}(\deg^{1/2}(\cdot)) \subset \mathcal{D}(\Delta_m^{1/2})$. Indeed, one has:

$$\langle f, \Delta_m f\rangle_m = \frac{1}{2}\sum_{x\in\mathscr{V}}\sum_{y\sim x} \mathscr{E}(x,y)|f(x)-f(y)|^2$$
$$\leqslant \sum_{x\in\mathscr{V}}\sum_{y\sim x} \mathscr{E}(x,y)(|f(x)|^2+|f(y)|^2) = 2\langle f, \deg(\cdot) f\rangle_m, \quad (2.6)$$

for $f \in \mathcal{C}_c(\mathscr{V})$. This inequality also gives a necessary condition for the absence of essential spectrum for Δ_m (see [16, Corollary 2.3]). In [16, Proposition 4.5], we also prove that, in general, the constant 2 cannot be improved. It is also easy to see that Δ_m is bounded if and only if deg is (e.g. [16, 19, 21]).

2.4. Weakly spherically symmetric graphs

We introduce the class of weakly spherically symmetric graphs. Their associated Laplacian will be studied deeply in Section 9 and 10.

Definition 2.1. — *Let $\mathscr{G} := (\mathscr{V}, \mathscr{E}, m)$ be a weighted graph and let $(S_n)_{n\geqslant 0}$ be a 1-dimensional decomposition on $\mathscr{G}$. We say that $\mathscr{G}$ is* weakly spherically symmetric *with respect to $(S_n)_{n\geqslant 0}$ if the quantities* $\deg_+(x)$ *and* $\deg_-(x)$ *only depend on the quantity $|x|$.*

It is easy to see that for a weakly spherically symmetric graph, the 1-dimensional decomposition corresponds to the one obtained by taking the level sets of the distance function to the set S_0; that is we have

$$S_n = \{x \in \mathscr{V}, d_{\mathscr{G}}(x, S_0) = n\}.$$

This due to the fact that for $x \in S_n$, obviously, one has $\deg_-(x) > 0$ thus $d_{\mathscr{G}}(y, S_{n-1}) = 1$.

If $S_0 = \{x_0\}$, we also say that $\mathscr{G} = (\mathscr{V}, \mathscr{E}, m)$ is *weakly spherically symmetric around* x_0. The definition 2.1 is a slight generalization of the one in [21] where the authors only consider the case of weakly spherically symmetric around a point x_0. In [21], the author shows that weakly spherically symmetric graphs with $S_0 = \{x_0\}$ are exactly the graphs such that the heat kernel associated to Δ_m, $p_t(x_0, \cdot)$, is a radial function.

The interest of our definition is that if $\mathscr{G} := (\mathscr{V}, \mathscr{E}, m)$ is weakly spherically symmetric with respect to some 1-dimensional decomposition $(S_n)_{n\geqslant 0}$ then so is the induced graph with vertex set $\mathscr{V} - B_n$, $n \geqslant 0$. In particular, in Proposition 9.2, we prove that with our definition, weakly spherically symmetric graphs correspond exactly to the graphs such that the radial part of the associated continuous time Markov chain is still a continuous time Markov chain.

2.5. Decomposition of the Laplacian and bipartite graphs

We fix a weighted graph $\mathscr{G} := (\mathscr{V}, \mathscr{E}, m)$ and $(S_n)_{n\in\mathbb{N}}$ a 1-dimensional decomposition of $\mathscr{V}$. We decompose the Laplacian in the following way:

$$\Delta_m = \deg(\cdot) - A_{m,\mathrm{bp}} - A_{m,\mathrm{sp}}, \tag{2.7}$$

where

$$A_{m,\mathrm{bp}} f(x) := \frac{1}{m(x)} \sum_{y, |y| \neq |x|} \mathscr{E}(x,y) f(y)$$

and

$$A_{m,\mathrm{sp}} f(x) := \frac{1}{m(x)} \sum_{y, |y| = |x|} \mathscr{E}(x,y) f(y).$$

Here *bp* and *sp* stand for bi-partite and spherical, respectively.

Let U be unitary operator defined by $Uf(x) := (-1)^{|x|} f(x)$, then

$$U \Delta_m U = \deg(\cdot) + A_{m,\mathrm{bp}} - A_{m,\mathrm{sp}} = 2\deg(\cdot) - \Delta_m - 2A_{m,\mathrm{sp}}. \tag{2.8}$$

Note if $\eta_0 \equiv 0$ thus $\Delta_m = \deg(\cdot) - A_{m,\mathrm{bp}}$ and

$$U \Delta_m U = \deg(\cdot) + A_{m,\mathrm{bp}} = 2\deg(\cdot) - \Delta_m.$$

In particular, when $m = \eta$,

$$U \Delta_\eta U = 1 + A_{\eta,\mathrm{bp}} = 2 - \Delta_\eta.$$

In this last case, this directly yields:

Proposition 2.2. — *Let $\mathscr{G} := (\mathscr{V}, \mathscr{E})$. Let $\mathscr{W}$ be any subset of $\mathscr{V}$, then the spectrum $\sigma\left(\Delta_\eta^{\mathscr{W}}\right)$ is symmetric with respect to 1.*

2.6. Upside-Down lemma

We adapt the Upside-Down-Lemma of [2] which was inspired from [13].

Lemma 2.3 (Upside-Down-Lemma). — *Let $\mathscr{G} := (\mathscr{V}, \mathscr{E}, m)$ be a weighted graph, $q : \mathscr{V} \to \mathbb{R}$ and $\mathscr{U} \subset \mathscr{V}$. Assume there are $a \in (0,1)$, $k \geqslant 0$ such that for all $f \in \mathcal{C}_c(\mathscr{U})$,*

$$(1-a)\langle f, (\deg + q)(\cdot) f\rangle_m - k\|f\|_m^2 \leqslant \langle f, \Delta_{\mathscr{G}}^{\mathscr{U}} f + q(\cdot) f\rangle_m,$$

then for all $f \in \mathcal{C}_c(\mathscr{U})$, we also have

$$\langle f, \Delta_{\mathscr{G}}^{\mathscr{U}} f + q(\cdot) f\rangle_m \leqslant (1+a)\langle f, (\deg + q)(\cdot) f\rangle_m + k\|f\|_m^2.$$

Proof. — By a direct calculation we find for $f \in \mathcal{C}_c(\mathscr{U})$

$$\begin{aligned}
&\langle f, (2\deg(\cdot) - \Delta_{\mathscr{G}}^{\mathscr{U}})f\rangle_m \\
&\qquad = \frac{1}{2}\sum_{x,y\in\mathscr{V}, x\sim y} \mathscr{E}(x,y)(2|f(x)|^2 + 2|f(y)|^2) - |f(x)-f(y)|^2) \\
&\qquad = \frac{1}{2}\sum_{x,y,x\sim y} \mathscr{E}(x,y)|f(x)+f(y)|^2 \\
&\qquad \geqslant \frac{1}{2}\sum_{x,y,x\sim y} \mathscr{E}(x,y)\,\big||f(x)|-|f(y)|\big|^2 \\
&\qquad = \langle |f|, \Delta_{\mathscr{G}}^{\mathscr{U}}|f|\rangle_m.
\end{aligned}$$

Using the assumption gives after reordering

$$\begin{aligned}
\langle f, \Delta_{\mathscr{G}}^{\mathscr{U}} f\rangle_m - \langle f, (2\deg + q)(\cdot)f\rangle_m &\leqslant -\langle |f|, \Delta_{\mathscr{G}}^{\mathscr{U}}|f|\rangle_m - \langle |f|, q(\cdot)|f|\rangle_m \\
&\leqslant -(1-a)\langle |f|, (\deg + q)(\cdot)|f|\rangle_m + k\langle |f|, |f|\rangle_m \\
&= -(1-a)\langle f, (\deg + q)(\cdot)f\rangle_m + k\langle f, f\rangle_m
\end{aligned}$$

which yields the assertion. □

Combining the upside-down Lemma and the Persson criteria we derive immediately the following proposition.

Proposition 2.4. — *Let $\mathscr{G} := (\mathscr{V}, \mathscr{E})$ be a graph. Assume that there is $a > 0$ such that*

$$\inf \sigma_{\mathrm{ess}}(\Delta_\eta) \geqslant 1 - a$$

then

$$\sup \sigma_{\mathrm{ess}}(\Delta_\eta) \leqslant 1 + a.$$

3. Hardy inequality and its links with super-harmonic functions

In this paper, one major tool is the following Hardy inequality. The terminology comes from [16]. The idea is to bound the Laplacian from below by a potential and to reduce its analysis to it. This technique has already be successfully used in [17] for some ground state related problem and in [16] to obtain some Weyl asymptotic.

Proposition 3.1 (Hardy inequality). — *Let W be a positive function on $\mathscr{V}$, then for all $f \in \mathcal{C}_c(\mathscr{V})$,*

$$\mathcal{Q}(f,f) = \langle f, \Delta_m f\rangle_m \geqslant \langle f, \frac{\tilde{\Delta}_m W}{W} f\rangle_m. \tag{3.1}$$

Here we recall that $\tilde{\Delta}_m W$ has to be understood in a algebraical sense since W is generally not a ℓ^2 function. We mention that there are other techniques to bound the Laplacian from below by a potential and refer to [8, 9, 24].

The inequality of Proposition 3.1 is well-known in the continuous setting. It can be seen as an integrated version of Picone's identity (see for example [1]). It also appears in the work [6].

We point out that the formulation of (3.1) is equivalent to the one used in [17, 16]. We shall present an alternative proof, which is closer to the one of [6]. We shall only use the reversibility of the measure m.

Proof.. — Take $f \in \mathcal{C}_c(\mathscr{V})$,

$$\begin{aligned}\left\langle f, \frac{\tilde{\Delta}_m W}{W} f\right\rangle_m &= \sum_x \sum_y \mathscr{E}(x,y)\left(|f|^2(x) - \frac{W(y)}{W(x)}|f|^2(x)\right)\\ &= \sum_x \sum_y \mathscr{E}(x,y)\left(|f|^2(x) - \frac{1}{2}\left(\frac{W(y)}{W(x)}|f|^2(x) + \frac{W(x)}{W(y)}|f|^2(y)\right)\right).\\ &\leqslant \sum_x \sum_y \mathscr{E}(x,y)\left(|f|^2(x) - \Re\left(\overline{f}(x) f(y)\right)\right)\\ &= \frac{1}{2}\sum_x \sum_y \mathscr{E}(x,y)\left|f(x) - f(y)\right|^2 = \mathcal{Q}(f,f).\end{aligned}$$

This is the announced result. $\square$

The aim of this work is to investigate the links between some properties of the spectrum of the Laplacian and the existence of some positive function W which satisfies

$$\frac{\tilde{\Delta}_m W}{W}(x) \geqslant \lambda(x), \tag{3.2}$$

for all $x \in \mathscr{V}$ and for some function λ which is non-negative away from a compact. Clearly, given $m, m' : \mathscr{V} \to (0, +\infty)$, a function W satisfies (3.2) for m if and only if it satisfies (3.2) for m' where:

$$\frac{\tilde{\Delta}_{m'} W}{W}(\cdot) \geqslant \psi(\cdot), \text{ with } \psi(\cdot) = \frac{m(\cdot)}{m'(\cdot)}\lambda(\cdot).$$

This simple fact enlightens about the flexibility of our method.

Note that in the literature, when λ is constant, these functions W are sometimes called positive λ-super-harmonic functions. In a different field, they are also called Lyapunov functions. We rely on the next definition.

Definition 3.2. — *A positive function W is called a* Lyapunov function *if there exist λ a positive function, $b > 0$, and a finite set B_{r_0} such that for all $x \in \mathscr{V}$,*

$$\frac{\tilde{\Delta}_m W}{W}(x) \geqslant \lambda(x) - b\mathbf{1}_{B_{r_0}}(x). \tag{3.3}$$

A positive function W is called a super-harmonic function *if there exists λ a non-negative function such that for all $x \in \mathscr{V}$,*

$$\frac{\tilde{\Delta}_m W}{W}(x) \geqslant \lambda(x). \tag{3.4}$$

Remark 3.3. — Usually, for Lyapunov functions, the condition $W \geqslant 1$ is also required and they are used to control the return time in a compact region (see [6]). Here we do the contrary and our Lyapunov functions control how the process goes to infinity (see section 8). They are non-increasing in our applications. Therefore, we shall not impose that $W \geqslant 1$.

Moreover, in some situations, we will have to consider family of super-harmonic functions. We set:

Definition 3.4. — *Given a graph $\mathscr{G} := (\mathscr{V}, \mathscr{E})$, we call a sequence $(\mathscr{V}_n)_n$ of finite and connected subsets of $\mathscr{V}$* exhaustive *if $\mathscr{V}_n \subset \mathscr{V}_{n+1}$, and $\cup_n \mathscr{V}_n = \mathscr{V}$.*

Definition 3.5. — *Set a graph $\mathscr{G} := (\mathscr{V}, \mathscr{E}, m)$. A family of positive functions $(W_n)_n$ is called* a family of super-harmonic functions relative to an exhaustive sequence $(\mathscr{V}_n)_n$ *if there exists a non-zero and non-negative function $\lambda : \mathscr{V} \to \mathbb{R}^+$ such that*

$$\Delta_{\mathscr{G}} W_n(x) \geqslant \lambda(x) W_n(x), \tag{3.5}$$

for all $x \in \mathscr{V}_n$.

A direct link between super-harmonic functions and spectral properties of the Laplacian is given by:

Theorem 3.6. — *Let $\mathscr{G} := (\mathscr{V}, \mathscr{E}, m)$ be a weighted graph.*

(a) Assume there exist $\lambda \geqslant 0$ and a function $W : \mathscr{V} \to (0, +\infty)$ such that, for all $x \in \mathscr{V}$,

$$\Delta_{\mathscr{G}} W(x) \geqslant \lambda W(x).$$

Then

$$\inf \sigma(\Delta_{\mathscr{G}}) \geqslant \lambda.$$

(b) Assume there exist two functions $\lambda : \mathscr{V} \to \mathbb{R}$ and $W : \mathscr{V} \to (0, +\infty)$ such, that for all $x \in \mathscr{V}$,

$$\Delta_{\mathscr{G}} W(x) \geqslant \lambda(x) W(x).$$

Then

$$\inf \sigma_{\mathrm{ess}}(\Delta_{\mathscr{G}}) \geqslant \liminf_{|x| \to +\infty} \lambda(x).$$

Proof. — We do only the proof only for the bottom of the essential spectrum. Let $\mathscr{K}$ be a finite set and let $f \in \mathcal{C}_c(\mathscr{V})$ be a function on $\mathscr{V}$ whose support is included in $\mathscr{K}^{\mathrm{c}}$. By Hardy inequality (3.1), we get:

$$\langle f, \Delta_{\mathscr{G}} f \rangle_m \geqslant \langle f, \frac{\tilde{\Delta}_{\mathscr{G}} W}{W} f \rangle_m \geqslant \left(\inf_{x \in \mathscr{K}^c} \lambda(x) \right) \langle f, f \rangle_m.$$

By the Persson Lemma (see (2.2)), we infer;

$$\inf \sigma_{\mathrm{ess}}(\Delta_{\mathscr{G}}) \geqslant \sup_{\mathscr{K} \subset \mathscr{V} \text{finite}} \left(\inf_{x \in \mathscr{K}^c} \lambda(x) \right).$$

This last quantity is precisely $\liminf_{|x| \to +\infty} \lambda(x)$ which ends the proof. $\square$

4. Super-harmonic functions, essential spectrum, and minoration of eigenvalues

In this section, we precise Theorem 3.6 and construct Lyapunov and super-harmonic functions for the Laplacian on some weighted graphs and study the (essential) spectrum of the associated Laplacian. We compare our approach with the ones obtained by isoperimetrical techniques and provide some minoration of the eigenvalues which are below the essential spectrum.

4.1. A few words about the isoperimetrical approach

Given a function $m : \mathscr{V} \to (0, \infty)$ and $U \subseteq \mathscr{V}$, we define the *isoperimetric constant* as follows:

$$\alpha_m(U) := \inf_{K, K \subseteq U \subset \mathscr{V}} \frac{L(\partial K)}{m(K)},$$

$$\text{where } L(\partial K) := \langle \mathbf{1}_{\partial K}, \Delta_m \mathbf{1}_{\partial K} \rangle_m = \langle \mathbf{1}_K, \Delta_m \mathbf{1}_K \rangle_m.$$

Note that $L(\partial K)$ is independent of m. Trivially, one has that $\alpha_m(U) \geqslant \inf \sigma(\Delta_m^U)$. However, it is important to notice that this quantity is also useful to estimate from below the Laplacian. One obtains in [20, Proposition 15] (see also [10, 12, 18] and references therein), the following result.

Proposition 4.1 (Keller-Lenz). — *Given $\mathscr{G} := (\mathscr{V}, \mathscr{E}, m)$, then*

$$\inf \sigma\left(\Delta_m^U\right) \geqslant d_U\left(1 - \sqrt{1 - \alpha_\eta(U)^2}\right), \tag{4.1}$$

where $d_U := \inf_{x \in U} \deg(x)$. Moreover, if $D_U := \sup_{x \in U} \deg(x) < +\infty$, we obtain:

$$\inf \sigma\left(\Delta_m^U\right) \geqslant \left(D_U - \sqrt{D_U^2 - \alpha_m(U)^2}\right).$$

It remains to estimate the isoperimetric constant. We adapt straightforwardly the proof of [29, Theorem 4.2.2], where the author considered the case $w = \eta$.

Proposition 4.2. — *Take a graph $\mathscr{G} := (\mathscr{V}, \mathscr{E}, w)$, $w : \mathscr{V} \to (0, \infty)$ and $U \subset \mathscr{V}$. Suppose that there are a 1-dimensional decomposition of $\mathscr{G}$ and $a > 0$ such that*

$$\eta_+(x) - \eta_-(x) \geqslant a w(x),$$

for all $x \in U$, then one obtains that $\alpha_w(U) \geqslant a$.

Proof. — Set $r(x) := |x|$. We have $\Delta_w r(x) \leqslant -a$ for $x \in U$. By the Green Formula and since $r(x) - r(y) \in \{0, \pm 1\}$ for $x \sim y$, for any $K \subseteq U$, we have:

$$\begin{aligned} L(\partial K) &= \sum_{x \in K, y \sim x, y \notin K} \mathscr{E}(x,y) \geqslant \left| \sum_{x \in K, y \sim x, y \notin K} \mathscr{E}(x,y)(r(x) - r(y)) \right| \\ &= |\langle \mathbf{1}_K, \Delta_w r\rangle_w| \geqslant a w(K). \end{aligned}$$

This yields the result. □

4.2. Lower estimates of eigenvalues

In the continuous setting, it is possible from the Hardy inequality and the Super-Poincaré Inequality (see the Appendix) to obtain some estimates of the heat semigroup and then to obtain some eigenvalues comparison. Here in this discrete setting, the situation is simpler since bounding from below the Laplacian by a non-negative multiplication operator directly give information on eigenvalues. In all this section we denote by

$$0 \leqslant \lambda_1(\Delta_m) \leqslant \lambda_2(\Delta_m) \leqslant \cdots \leqslant \lambda_n(\Delta_m) \leqslant \cdots < \inf \sigma_{\mathrm{ess}}(\Delta_m)$$

the eigenvalues of Δ_m which are located below the infimum of the essential spectrum of Δ_m. A priori this number of eigenvalues can be finite. We recall some well-known results. We refer to [26, Chapter XIII.1] and [14] for more details and to [26, Chapter XIII.15] for more applications. We start with the form-version of the standard variational characterization of the n-th eigenvalue.

Theorem 4.3. — *Let A be a non-negative self-adjoint operator with form-domain $\mathscr{Q}(A)$. For all $n \geqslant 1$, we define:*

$$\mu_n(A) := \sup_{\varphi_1,\ldots,\varphi_{n-1}} \inf_{\psi\in[\varphi_1,\ldots,\varphi_{n-1}]^\perp} \langle \psi, A\psi\rangle,$$

where $[\varphi_1,\ldots,\varphi_{n-1}]^\perp = \{\psi \in \mathscr{Q}(A)$, so that $\|\psi\| = 1$ and $\langle\psi,\varphi_i\rangle = 0$, with $i = 1,\ldots,n-1\}$. Note that φ_i are not required to be linearly independent.

We define also:

$$\nu_n(A) := \inf_{E_n\subset \mathscr{Q}(A),\dim E_n = n} \sup_{\psi\in E_n,\|\psi\|=1} \langle \psi, A\psi\rangle.$$

Then, one has $\mu_n(A) = \nu_n(A)$ and if $\mu_n(A) = \nu_n(A)$ is (strictly) below the essential spectrum of A, it is the n-th eigenvalue, counted with multiplicity, $\lambda_n(A)$. Moreover, we have that:

$$\dim \operatorname{Ran} \mathbf{1}_{[0,\mu_n(A)]}(A) = n.$$

Otherwise, $\mu_n(A) = \nu_n(A)$ is the infimum of the essential spectrum. Moreover, $\mu_j(A) = \nu_j(A) = \mu_n(A) = \nu_n(A)$, for all $j \geqslant n$ and there are at most $n-1$ eigenvalues, counted with multiplicity, below the essential spectrum. In that case,

$$\dim \operatorname{Ran} \mathbf{1}_{[0,\mu_n(A)+\varepsilon]}(A) = +\infty, \text{ for all } \varepsilon > 0.$$

This ensures the following useful criteria.

Proposition 4.4. — *Let A, B be two self-adjoint operators, with form-domains $\mathscr{Q}(A)$ and $\mathscr{Q}(B)$, respectively. Suppose that*

$$\mathscr{Q}(A) \supset \mathscr{Q}(B) \text{ and } 0 \leqslant \langle\psi, A\psi\rangle \leqslant \langle\psi, B\psi\rangle,$$

for all $\psi \in \mathscr{Q}(B)$. Then one has $\inf\sigma_{\mathrm{ess}}(A) \leqslant \inf\sigma_{\mathrm{ess}}(B)$ and

$$\mathscr{N}_\lambda(A) \geqslant \mathscr{N}_\lambda(B), \text{ for } \lambda \in [0,\infty)\setminus\{\inf\sigma_{\mathrm{ess}}(B)\}, \tag{4.2}$$

where $\mathscr{N}_\lambda(A) := \dim \operatorname{Ran} \mathbf{1}_{[0,\lambda]}(A)$.

In particular, if A and B have the same form-domain, then $\sigma_{\mathrm{ess}}(A) = \emptyset$ if and only if $\sigma_{\mathrm{ess}}(B) = \emptyset$ and $\lambda_n(A) \leqslant \lambda_n(B)$, $n \geqslant 1$.

Proof. — It is enough to notice that $\mu_n(A) \leqslant \mu_n(B)$, for all $n \geqslant 0$. Theorem 4.3 permits us to conclude for the first part. Supposing now they have the same form-domain, by the uniform boundedness principle, there are $a, b > 0$ such that:

$$\langle \psi, A\psi \rangle \leqslant a\langle \psi, B\psi \rangle + b\|\psi\|^2 \text{ and } \langle \psi, B\psi \rangle \leqslant a\langle \psi, A\psi \rangle + b\|\psi\|^2$$

for all $\psi \in \mathscr{Q}(A) = \mathscr{Q}(B)$. By using the previous statement twice we get the result. □

We start with a direct application. We shall present examples in the next section.

Corollary 4.5. — *Let ψ be a non-decreasing non-negative radial function on $\mathscr{V}$. Assume that*

$$\langle f, \Delta_m f \rangle_m \geqslant \langle f, \psi f \rangle_m,$$

for all $f \in \mathcal{C}_c(\mathscr{V})$. Then,

$$\inf \sigma_{\mathrm{ess}}(\Delta_m) \geqslant \lim_{|x| \to \infty} \psi(x)$$

and when $\lambda_{|B_{n-1}|+k}(\Delta_m)$ exists, we have:

$$\lambda_{|B_{n-1}|+k}(\Delta_m) \geqslant \psi(n), \quad \text{for } k = 1, \ldots, |S_n|. \tag{4.3}$$

4.3. Upper estimates of eigenvalues

It is also possible to obtain some upper bounds for the eigenvalues. Our method here is based on the following well-known Proposition, see [33][Proposition 5.1] for example.

Proposition 4.6. — *Let $\mathscr{G} := (\mathscr{V}, \mathscr{E}, m)$ be a graph and let Δ_m be the associated Laplacian. Let $g_1, \ldots, g_n \in \mathcal{D}(\Delta_m^{1/2})$ be n orthonormal functions ($\langle g_i, g_j \rangle_m = \delta_{ij}$)). Let $\lambda_n(M_g)$ be the largest eigenvalue of the symmetric matrix:*

$$M_g := \left(\langle g_i, \Delta_m g_j \rangle_m\right)_{1 \leqslant i,j \leqslant n}.$$

Then if $\lambda_n(\Delta_m)$ exists we have:

$$\lambda_n(\Delta_m) \leqslant \lambda_n(M_g). \tag{4.4}$$

In particular, for all non identically zero functions g_i, $i = 1, \ldots, n$ such that

$$\langle g_i, g_j \rangle_m = \langle g_i, \Delta_m g_j \rangle_m = 0 \text{ for } i \neq j, \tag{4.5}$$

$$\lambda_n(\Delta_m) \leqslant \max_{i=1,\ldots,n} \frac{\langle g_i, \Delta_m g_i \rangle_m}{\langle g_i, g_i \rangle_m}. \tag{4.6}$$

Moreover if $\lambda_n(M_g) < \inf \sigma_{\mathrm{ess}}(\Delta_m)$ then $\lambda_n(\Delta_m)$ exists.

Proof. — This is a direct consequence of Theorem 4.3. One just has to note that for E_n a subspace of dimension n of $\mathcal{D}(\Delta_m)$ and $(g_1, \dots, g_n)$ an orthonormal basis of E_n, one has: $\max_{h \in E_n, \|h\|=1} \langle h, \Delta_m h\rangle = \lambda_n(M_g)$. □

As a corollary, we obtain:

Corollary 4.7. —

(a) *Let* $g_1, \dots, g_n \in \mathcal{D}(\Delta_m^{1/2})$ *be such that* $d_{\mathscr{G}}(\operatorname{supp} g_i, \operatorname{supp} g_j) \geqslant 2$*, for* $i \neq j$ *and where* supp *denotes the support. Then if* $\lambda_n(\Delta_m)$ *exists we have:*

$$\lambda_n(\Delta_m) \leqslant \max_{1=1,\dots,n} \frac{\langle g_i, \Delta_m g_i\rangle_m}{\langle g_i, g_i\rangle_m}.$$

(b) *Let* $\mathscr{G}_i := (\mathscr{V}_i, \mathscr{E}_i, m)$*,* $i = 1, \dots, n$ *be* n *induced sub-graphs of* $\mathscr{G}$ *such that for* $i \neq j$*,* $d_{\mathscr{G}}(\mathscr{V}_i, \mathscr{V}_j) \geqslant 2$*, then if* $\lambda_n(\Delta_{\mathscr{G},m})$ *exists we have:*

$$\lambda_n(\Delta_{\mathscr{G},m}) \leqslant \max_{i=1,\dots,n} \left\{\inf \sigma\left(\Delta_{\mathscr{G},m}^{\mathscr{G}_i}\right)\right\}$$

where $\Delta_{\mathscr{G},m}^{\mathscr{G}_i}$ *denotes the Dirichlet Laplacian of* $\mathscr{G}_i$ *in* $\mathscr{G}$*.*

4.4. The approach with super-harmonic functions

In this section we improve a result of [29] and prove that the weighted Laplacian Δ_m has empty essential spectrum for a certain class of graph and give some estimation on the eigenvalues.

Theorem 4.8. — *Take* $\mathscr{G} := (\mathscr{V}, \mathscr{E}, m)$ *and assume there is a* 1*-dimensional decomposition and a constant* $c > 1$ *such that*

$$l := \liminf_{|x|\to\infty} \left(\deg_+(x) - c\deg_-(x)\right) > 0 \tag{4.7}$$

Set $n_0 := \inf\{n \in \mathbb{N}, \deg_+(x) - c\deg_-(x) \geqslant 0$ *with* $|x| \geqslant n\}$*. Then there exists a super-harmonic function* W *such that*

$$\tilde{\Delta}_m W(x) \geqslant \phi_c W(x), \quad \text{for all } x \in \mathscr{V}, \tag{4.8}$$

with

$$\phi_c(x) := \frac{c-1}{c}(\deg_+(x) - c\deg_-(x))\mathbf{1}_{B^c_{n_0}} \geqslant 0. \tag{4.9}$$

In particular, we obtain that $\sigma_{\mathrm{ess}}(\Delta_m) \geqslant l(c-1)/c$ *and* $\sigma_{\mathrm{ess}}(\Delta_m) = \emptyset$ *if* $l = \infty$*.*

Proof. — We construct a suitable Lyapunov function. Set $\tilde{W}(x) := c^{-|x|}$. We have:

$$\psi_c(x) := \frac{\tilde{\Delta}_m \tilde{W}(x)}{\tilde{W}(x)} = \deg_+(x)\left(1 - \frac{1}{c}\right) + \deg_-(x)\,(1-c)\,. \tag{4.10}$$

Now since $\deg_+(x) - c\deg_-(x)$ is positive outside a given ball B_{n_0}, $\tilde{W}$ is a Lyapunov function, which satisfies

$$\tilde{\Delta}_m \tilde{W}(x) \geqslant \phi_c(x)\tilde{W}(x) - C\mathbf{1}_{B_{n_0}}$$

with ϕ_c defined as in (4.9) and for some constant C.

Set now

$$W(x) = \begin{cases} c^{-n_0}, & \text{if } x \in B_{n_0} \\ c^{-|x|}, & \text{if } x \in B_{n_0}^c, \end{cases} \tag{4.11}$$

then it satisfies (4.8). Finally, since $\phi_c(x)$ tends to $l(c-1)/c$ when $|x| \to \infty$, Theorem 3.6 gives the statement about the essential spectrum. □

Remark 4.9. — Note that condition (4.7) with $l = +\infty$ is equivalent to the following one: $\deg_+(x) \to \infty$ as $|x| \to \infty$ and there exist a ball B_{n_0} and a constant $c' > 0$ such that for all x outside the ball B_{n_0},

$$\frac{\deg_+(x) - \deg_-(x)}{\deg_-(x)} \geqslant c' > 0.$$

Thus, when $m = 1$, this is better than the one of [29, Theorem 4.2.2] which asserts: $\eta_+(x) \to \infty$ as $|x| \to \infty$ and there exist a ball B_{n_0} and a constant $c > 0$ such that for all x outside the ball B_{n_0},

$$\frac{\eta_+(x) - \eta_-(x)}{\eta(x)} \geqslant c > 0.$$

His result follows for instance by Propositions 4.1, 4.2 and the Persson Lemma (2.2).

An example where our criterion is satisfied and the one of Wojciechowski is not satisfied is the following.

Example 4.10. — Let $\mathscr{G} := (\mathscr{V}, \mathscr{E}, 1)$ be the simple graph with weight $m \equiv 1$, whose set of vertices is

$$\mathscr{V} := \{(1, i_1, i_2, \dots, i_k), k \geqslant 0, i_j \in [\![1, j]\!] \text{ for } j \in [\![1, k]\!]\}$$

and where $\mathscr{E}(x,y) := 1$ if and only if $x \neq y$ and

$$\{x,y\} = \{(1,i_1,\ldots,i_k),(1,i_1,\ldots,i_k,i_{k+1})\}$$

or $x = (1,i_1,\ldots,i_k)$ and $y = (1,i'_1,\ldots,i'_k)$.

For $x = (1,i_1,\ldots,i_k) \in \mathscr{V}, k \geqslant 1$, we have $\eta_+(x) = k+1$, $\eta_-(x) = 1$ and $\eta(x) = k+2+k!-1$.

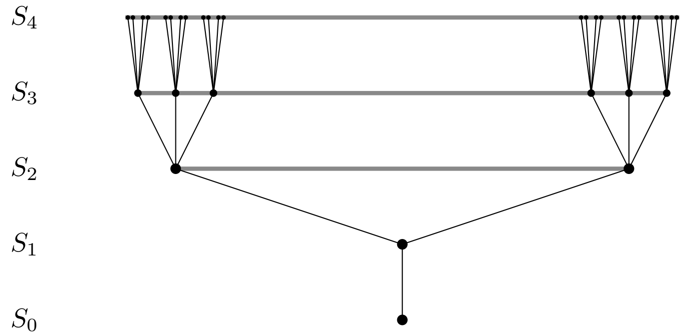

Figure 1. — Growing tree with complete graph on spheres

We provide an example of a weakly spherically symmetric graph. On a weakly spherically symmetric graph, $\inf \sigma_{\mathrm{ess}}(\Delta_1)$ does not depend on the edges inside the spheres S_n (see Corollary 10.8). Therefore, it is a good point that our criterion 4.7 does not depend on $\deg_0$.

Theorem 4.8 can also be useful to compute the asymptotics of eigenvalues. We improve partially the main result of [16] where one considered some perturbation of weighted trees.

Theorem 4.11. — *Take $\mathscr{G} := (\mathscr{V}, \mathscr{E}, m)$ and assume there is a 1-dimensional decomposition such that*

$$\lim_{|x|\to\infty} \deg_+(x) = \infty, \qquad \text{and} \qquad \max(\deg_-(x), \deg_0(x)) = o(\deg_+(x)), \tag{4.12}$$

as $|x| \to \infty$, then $\mathcal{D}(\Delta_m^{1/2}) = \mathcal{D}(\deg^{1/2}(\cdot))$, $\sigma_{\mathrm{ess}}(\Delta_m) = \emptyset$, and

$$\lim_{n\to\infty} \frac{\lambda_n(\Delta_m)}{\lambda_n(\deg(\cdot))} = 1. \tag{4.13}$$

Proof. — We apply Theorem 4.8. Note first that $l = \infty$ for all $c > 1$. The essential spectrum of Δ_m is therefore empty. Using (4.9), (3.1) and (4.12) we obtain that for all $\varepsilon > 0$ there are $c_\varepsilon, c'_\varepsilon > 0$ such that:

$$\langle f, \Delta_m f\rangle \geqslant (1-\varepsilon)\langle f, \deg(\cdot) f\rangle_m - c_\varepsilon \|f\|_m^2, \tag{4.14}$$

for all $f \in \mathcal{C}_c(\mathscr{V})$. Combined with (2.6), we get the equality of the form domains. Using Lemma 2.3 we derive:

$$\langle f, \Delta_m f\rangle \leqslant (1+\varepsilon)\langle f, \deg(\cdot) f\rangle_m + c_\varepsilon \|f\|_m^2,$$

for all $f \in \mathcal{C}_c(\mathscr{V})$. This yields:

$$1-\varepsilon \leqslant \liminf_{n\to\infty} \frac{\lambda_n(\Delta_m)}{\lambda_n(\deg(\cdot))} \leqslant \limsup_{n\to\infty} \frac{\lambda_n(\Delta_m)}{\lambda_n(\deg(\cdot))} \leqslant 1+\varepsilon.$$

By letting ε go to zero we obtain the Weyl asymptotic (4.13) for Δ_m. □

Remark 4.12. — Inequalities (4.14) was studied in full detail in [2]. It turns out that the graphs which satisfy (4.14) are exactly the so-called *almost sparse graphs* (see the definition in [2]). Combining Proposition 4.2 and [2, Theorem 5.5] we can also reprove Theorem 4.11.

With the same method, we also obtain a result when the inner and outer degrees are bounded.

Theorem 4.13. — *Take* $\mathscr{G} := (\mathscr{V}, \mathscr{E}, m)$ *and assume there is a* 1-*dimensional decomposition such that there exist* $n_0 \in \mathbb{N}$ *and two constants* a *and* D *with*

$$\deg_+(x) - \deg_-(x) \geqslant a \text{ for all } x \in B_{n_0}^c \tag{4.15}$$

and

$$\sup_{x\in B_{n_0}^c} \deg_+(x) + \deg_-(x) \leqslant D < +\infty. \tag{4.16}$$

Then there exists a positive function W *such that*

$$\tilde{\Delta}_m W(x) \geqslant \phi_c W(x), \quad \text{for all } x \in \mathscr{V},$$

with

$$\phi_c(x) := \left(D - \sqrt{D^2 - a^2}\right) \mathbf{1}_{B_{n_0}^c} \geqslant 0.$$

In particular, $\inf \sigma\left(\Delta_m^{B_n^c}\right) \geqslant D - \sqrt{D^2-a^2}$, *for all* $n \geqslant n_0$ *and*

$$\inf \sigma_{\mathrm{ess}}(\Delta_m) \geqslant D - \sqrt{D^2 - a^2}. \tag{4.17}$$

Proof. — Let $\tilde{W}(x) = c^{-|x|}$ for some $c > 1$ which will be precised later. Take ψ_c as in (4.10). For $x \in B^c_{n_0}$, conditions (4.15) and (4.16) imply that:

$$\begin{aligned}\psi_c(x) &= \frac{1}{2}\left[\left(c-\frac{1}{c}\right)(\deg_+(x)-\deg_-(x))-\left(c+\frac{1}{c}-2\right)(\deg_+(x)+\deg_-(x))\right]\\ &\geqslant \frac{1}{2}\left[\left(c-\frac{1}{c}\right)a-\left(c+\frac{1}{c}-2\right)D\right] = \frac{1}{2}\left[2D - c(D-a) - \frac{1}{c}(D+a)\right]\\ &\geqslant D - \sqrt{D^2-a^2},\end{aligned}$$

by taking $c = \sqrt{\frac{D+a}{D-a}}$. Then by choosing W as in (4.11), Theorem 3.6 ends the proof. □

An example where our criterion is satisfied and Proposition 4.1 does not apply is the following:

Example 4.14. — Given $d \geqslant 2$, let $\mathscr{G} := (\mathscr{V}, \mathscr{E}, 1)$ be the simple graph given by the d-ary tree with the complete graph on each sphere and with weight $m \equiv 1$, see Figure 2. The graph is constructed as follows. The set of vertices is

$$\mathscr{V} := \{(1, i_1, i_2, \dots, i_k), k \geqslant 1, i_j \in [\![1, d]\!] \text{ and } j \in [\![1, k]\!]\}$$

and $\mathscr{E}(x, y) := 1$ if and only if $x \neq y$ and

$$\{x, y\} = \{(1, i_1, \dots, i_k), (1, i_1, \dots, i_k, i_{k+1})\}$$

or

$$x = (1, i_1, \dots, i_k) \text{ and } y = (1, i'_1, \dots, i'_k).$$

We have $\#S_k = d^k$, for $k \geqslant 0$. Moreover, for $x = (i_0, \dots, i_k) \in \mathscr{V}, k \geqslant 1$, we have $\deg_+(x) = \eta_+(x) = d$, $\deg_-(x) = \eta_-(x) = 1$, and $\deg(x) = \eta(x) = \#S_k + d$. By Theorem 4.13, we get:

$$\inf \sigma(\Delta_1) \geqslant d + 1 - 2\sqrt{d},$$

whereas the lower bound given by (4.1) is 0. Indeed, for U the complement of a ball, by considering $K = S_n$ for n large enough, one sees that $\alpha_\eta(U) = 0$. Note also that the second part of Proposition 4.1 does not apply since $D_U = \infty$.

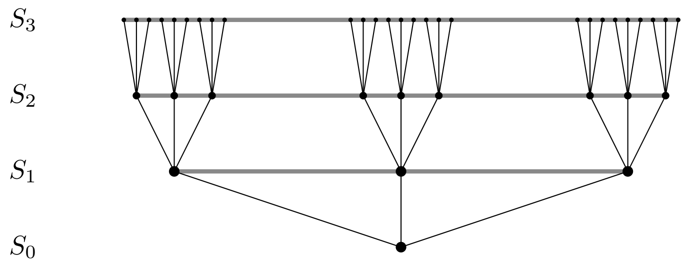

Figure 2. — 3-ary tree with complete graphs on spheres

Actually, the result: $\inf \sigma(\Delta_1) = d + 1 - 2\sqrt{d}$ was already known for the above example. Since it is a weakly spherically symmetric graph, by Corollary 6.7 in [21], the quantity $\inf \sigma(\Delta_1)$ does not depend on the edges inside the spheres S_n. Therefore one can reduce to the case of the ordinary d-ary tree.

Remark 4.15. — In the case of the *normalized* Laplacian, $m(x) = \eta(x) = \sum_y \mathscr{E}(x, y)$, that is $\deg \equiv 1$, we bring some new light to [20, Corollary 16] (which improves the original result of [12]) :

$$\inf \sigma_{\mathrm{ess}}(\Delta_\eta) \geqslant 1 - \sqrt{1 - a^2}$$

For the d-ary tree, we also recover the sharp estimate:

$$\inf \sigma(\Delta_\eta) \geqslant 1 - \frac{2\sqrt{d}}{d+1}.$$

4.5. Rapidly branching graphs

We now discuss the result of Fujiwara and Higushi (see [15, p 196]) concerning rapidly branching graphs. In [15, Corollary 4] under the hypothesis that $\sigma_{\mathrm{ess}}(\Delta_\eta) = \{1\}$, the author proves the existence of an infinite sequence of eigenvalues $\lambda_i \neq 1$ that converges to 1. Fujiwara asks if there exist two sequences of eigenvalues that tends respectively to 1^- and to 1^+. The answer is yes:

Proposition 4.16. — *Let $\mathscr{G} := (\mathscr{V}, \mathscr{E})$ be a graph such that $\sigma_{\mathrm{ess}}(\Delta_\eta) = \{1\}$. Then there exist two infinite sequences of eigenvalues $(\lambda_n^+)_{n\in\mathbb{N}}$ and $(\lambda_n^-)_{n\in\mathbb{N}}$ such that $\lambda_n^+ > 1$ and $\lambda_n^- < 1$ for all $n \in \mathbb{N}$.*

We refer to [15] for the question of 1 being an eigenvalue in the case of a radial tree.

Proof. — Given $x_0 \sim y_0$ set $g_\alpha = \delta_{x_0} + \alpha\delta_{y_0}$. We have

$$\frac{\langle g_\alpha, \Delta_\eta g_\alpha\rangle}{\langle g_\alpha, g_\alpha\rangle_\eta} = 1 - \operatorname{sign}(\alpha)\frac{\mathscr{E}(x_0, y_0)}{\sqrt{\eta(x_0)\eta(y_0)}},$$

where α is chosen to be $\pm\sqrt{\eta(x_0)/\eta(y_0)}$. The result follows from Corollary 4.7 (applied to $\pm\Delta_\eta$). □

Remark 4.17. — Note that $\sigma_{\mathrm{ess}}(\Delta_\eta) = \{1\}$ implies that that $(\lambda_n^+)_{n\in\mathbb{N}}$ and $(\lambda_n^-)_{n\in\mathbb{N}}$ tend to 1 by definition of the essential spectrum. Moreover our choice of test-functions and Corollary 4.7 also imply that:

$$\lim_{|x|\to\infty} \inf_{y\sim x} \frac{\eta(x)\eta(y)}{\mathscr{E}^2(x,y)} = +\infty,$$

where $|x|$ is defined with respect to any choice of 1-dimensional decomposition. We point out that with the help of Corollary 4.18 it is easy to construct a simple graph such that $\sigma_{\mathrm{ess}}(\Delta_\eta) = \{1\}$ and such that $\liminf_{|x|\to\infty} \eta(x) < +\infty$, see Figure 3.

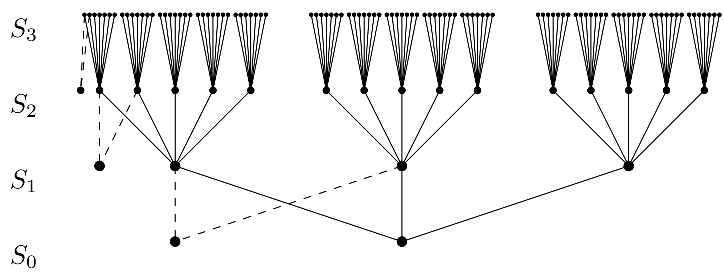

Figure 3. — Graph with $\sigma_{\mathrm{ess}}(\Delta_\eta) = \{1\}$ and $\liminf_{|x|\to\infty} \eta(x) < +\infty$

In the setting of simple graphs, the main result of [15] is the equivalence between an isoperimetry at infinity and the fact that $\sigma_{\mathrm{ess}}(\Delta_\eta) = \{1\}$. We give a sufficient condition for the latter.

Corollary 4.18. — *Take $\mathscr{G} := (\mathscr{V}, \mathscr{E})$ be a graph and assume there is a 1-dimensional decomposition such that :*

$$p_+(x) \to 1, \quad \text{as } |x| \to \infty, \tag{4.18}$$

then $\sigma_{\mathrm{ess}}(\Delta_\eta) = \{1\}$.

Proof. — Let $\varepsilon > 0$. Since $p_+(x) \to 1$ as $|x| \to \infty$, there exists a ball n_ε such that for all $n \geqslant n_\varepsilon$,

$$p_+(x) - p_-(x) \geqslant 1 - \varepsilon \text{ for } x \in B_n^c.$$

Thus, by Theorem 4.13, we obtain $\inf \sigma\left(\Delta_\eta^{B_n^c}\right) \geqslant 1-\sqrt{\varepsilon}$ and Proposition 2.4 concludes. $\square$

5. Eigenvalues Comparison

5.1. The case of trees

We turn to the case of a tree $\mathscr{T} := (\mathscr{V}, \mathscr{E})$. First we fix $v \in \mathscr{V}$ and set $S_0 := \{v\}$ and S_n given by (2.3). Let $x \in \mathscr{V}$, we denote by $\mathscr{T}'_x$ the induced tree in $\mathscr{T}$ whose set of vertices is $\mathscr{V}_{\mathscr{T}'_x} = \cup_{k\geqslant 0} S_{k,x}$, see (2.5). This corresponds to the sub-tree of $\mathscr{T}$ whose root is x. We also consider $\mathscr{T}_x$ the induced tree in $\mathscr{T}$ whose set of vertices is $\mathscr{V}_{\mathscr{T}'_x} = \cup_{k\geqslant -1} S_{k,x}$. We denote by $\overline{x}$ the unique point in $S_{-1,x}$, it is the father of x.

Let $f \in \mathcal{C}_c(\mathscr{V}_{\mathscr{T}_x})$ such that $f(\overline{x}) = 0$, we can extend f in a function $\tilde{f}$ on the all tree $\mathscr{T}$ by setting $\tilde{f}(y) = 0$ for all $y \in \mathscr{V} - \mathscr{V}_{\mathscr{T}_x}$ and we have:

$$\Delta^{\mathscr{T}'_x}_{\mathscr{T}_x,m} f(z) = \Delta_{\mathscr{T},m}\tilde{f}(z) \text{ for } z \in \mathscr{V}_{\mathscr{T}'_x}.$$

Corollary 4.7 yields:

THEOREM 5.1. — *Let $\mathscr{T}$ be a tree. Then, with the above notations:*

$$\lambda_{\#S_n}(\Delta_m) \leqslant \max_{x\in S_n} \inf \sigma\left(\Delta^{\mathscr{T}'_x}_{\mathscr{T}_x,m}\right). \tag{5.1}$$

We now turn to the case of *radial simple trees* (see the definition just below). We give in this situation a quantitative way to estimate the eigenvalues for the normalized and the combinatorial Laplacians.

A *radial simple tree* is a simple tree such that the function η depends only on the distance $|\cdot|$. By abuse of notation, we denote $\eta(n) := \eta(x), x \in S_n$. If $\mathscr{T}$ is a radial tree, for all x which belongs in a same sphere S_n, all the sub-trees $\mathscr{T}_x$ (and $\mathscr{T}'_x$) are the same. We denote by $\mathscr{T}_n$ one of them (and $\mathscr{T}'_n$ respectively).

If $\mathscr{T}$ is a radial simple tree, then (5.1) writes

$$\lambda_{\#S_n}(\Delta_m) \leqslant \inf \sigma\left(\Delta^{\mathscr{T}'_n}_{\mathscr{T}_n,m}\right). \tag{5.2}$$

The next proposition gives an estimate of the bottom of the spectrum of $\Delta^{\mathscr{T}'_n}_{\mathscr{T}_n,m}$.

Proposition 5.2. — *Let $\mathscr{T} := (\mathscr{V}, \mathscr{E})$ be a radial simple tree. Let $n \geqslant 1$, then with the above notation:*

$$\inf \sigma\left(\Delta^{\mathscr{T}'_n}_{\mathscr{T}_n,\eta}\right) \leqslant 1 - \sqrt{\frac{1 - \frac{1}{\eta(n)}}{\eta(n+1)}}$$

and

$$\inf \sigma\left(\Delta^{\mathscr{T}'_n}_{\mathscr{T}_n,1}\right) \leqslant \max(\eta(n), \eta(n+1)) \left(1 - \sqrt{\frac{1 - \frac{1}{\eta(n)}}{\eta(n+1)}}\right).$$

Proof. — We treat first the case of the normalized Laplacian. Let $x \in S_n$. Let g be the function on $\mathscr{V}_{\mathscr{T}_x}$ defined by $g(x) := 1$, $g(y) := \alpha$ for $y \in S_{1,x}$, and $g(y) := 0$ otherwise. Clearly,

$$\begin{aligned}
\inf \sigma\left(\Delta^{\mathscr{T}'_x}_{\mathscr{T}_x,\eta}\right) &\leqslant \frac{\langle g, \Delta_\eta g\rangle_\eta}{\langle g, g\rangle_\eta} \\
&= \frac{1 + (\eta(n) - 1)(1-\alpha)^2 + (\eta(n) - 1)(\eta(n+1) - 1)\alpha^2}{\eta(n) + (\eta(n) - 1)\eta(n+1)\alpha^2} \\
&= 1 - \frac{2\alpha(\eta(n) - 1)}{\eta(n) + \alpha^2(\eta(n) - 1)\eta(n+1)} \\
&= 1 - \frac{\sqrt{1 - \frac{1}{\eta(n)}}}{\sqrt{\eta(n+1)}},
\end{aligned}$$

where in the last line we have made the choice $\alpha = \frac{\sqrt{\eta(n)}}{\sqrt{(\eta(n)-1)\eta(n+1)}}$.

The same computation gives also

$$\inf \sigma\left(\Delta^{\mathscr{T}'_x}_{\mathscr{T}_x,1}\right) \leqslant \frac{\langle g, \Delta_1 g\rangle_1}{\langle g, g\rangle_1} \leqslant \max(\eta(n), \eta(n+1)) \frac{\langle g, \Delta_\eta g\rangle_\eta}{\langle g, g\rangle_\eta}.$$

This ends the proof. □

We now precise the result of Fujiwara and Higushi (see [15, p 196] and Corollary 4.18) by estimating the eigenvalues for the normalized Laplacian Δ_η. We also discuss the case of the combinatorial Laplacian.

Theorem 5.3. — *Let $\mathscr{T} := (\mathscr{V}, \mathscr{E})$ be a radial simple tree.*

(a) Let $m = \eta$. Assume $\eta(n)$ is non-decreasing and tends to $+\infty$ as n tends to ∞. Then

$$\sigma_{\mathrm{ess}}(\Delta_\eta) = \{1\} \text{ and } \sigma(\Delta_\eta) = \{1\} \cup \{\lambda_i(\Delta_\eta), 2 - \lambda_i(\Delta_\eta), i \geqslant 1\},$$

where $(\lambda_i(\Delta_\eta))_{i\geqslant 1}$ is an infinite sequence of eigenvalues converging to 1*. Moreover, for $\varepsilon > 0$, and $n \geqslant n(\varepsilon)$ we have:*

$$1 - 2\sqrt{\frac{1}{\eta(n)}} \leqslant \lambda_{(\#B_{n-1}+1)}(\Delta_\eta) \leqslant \lambda_{\#S_n}(\Delta_\eta) \leqslant 1 - \frac{1-\varepsilon}{\sqrt{\eta(n+1)}}.$$

(b) *Let $m = 1$. Assume that $\eta(n)\left(1 - 2\sqrt{\frac{1}{\eta(n)}}\right)$ is non-decreasing and that $\eta(n)$ tends to $+\infty$ as n tends to ∞. Then,*

$$\sigma_{\text{ess}}(\Delta_1) = \emptyset \text{ and } \sigma(\Delta_1) = \{\lambda_i(\Delta_1), i \geqslant 1\},$$

where $\lambda_i(\Delta_1)$ is an infinite sequence of eigenvalues which tends to $+\infty$. Moreover, one has:

$$\eta(n)\left(1 - 2\sqrt{\frac{1}{\eta(n)}}\right) \leqslant \lambda_{(\#B_{n-1}+1)}(\Delta_1) \leqslant \eta(n).$$

Proof. — We begin by the left inequality for the normalized Laplacian. Note that by hypothesis, for all $x \in B^c_{n-1}$,

$$\deg_+(x) - \deg_-(x) \geqslant \frac{\eta(n) - 2}{\eta(n)}.$$

Therefore by Theorem 4.13 and by Corollary 4.5, if the corresponding eigenvalue exists:

$$\begin{aligned}\lambda_{(\#B_{n-1}+1)}(\Delta_\eta) &\geqslant 1 - \left(\sqrt{1 - \left(\frac{\eta(n)-2}{\eta(n)}\right)^2}\right) = 1 - 2\frac{\sqrt{1-\frac{1}{\eta(n)}}}{\sqrt{\eta(n)}}\\ &\geqslant 1 - 2\sqrt{\frac{1}{\eta(n)}}.\end{aligned}$$

Since η tends to ∞, we have $\#B_{n-1}+1 \leqslant \#S_n$, for n large enough. Next the right inequality for $\lambda_{\#S_n}(\Delta_\eta)$, (if this eigenvalue exists) is a straightforward application of Theorem 5.1 and Proposition 5.2. Finally, since the upper estimate is strictly lower than 1, the min-max Theorem 4.3 ensures the existence of a infinite number of eigenvalue under the essential spectrum.

We turn to Δ_1. The left inequality is obtained by taking $c = \sqrt{\eta(n) - 1}$ in (4.10) since ψ_c can be written as

$$\psi_c(x) = \eta(n) - \left(c + \frac{\eta(n)-1}{c}\right), \quad x \in S_n.$$

Corollaries 4.18 and 4.5 give the desired result for the essential spectrum. Then we have:

$$\lambda_{(\#B_{n-1}+1)} \leqslant \lambda_{(\#S_n)} \leqslant \eta(n)$$

by taking Dirac test functions. □

5.2. The case of general weakly spherically symmetric graphs

In this section, we investigate the case of general weakly spherically symmetric graphs.

Proposition 5.4. —

(a) Let $\mathscr{G} := (\mathscr{V}, \mathscr{E}, m)$ be a weakly spherically symmetric graph with $m = \eta$. Assume that $p_+(n)(1-p_+(n))$ is non-increasing. Then, if the corresponding eigenvalues exist, we have:

$$\lambda_{(\#B_{n-1}+1)}(\Delta_\eta) \geqslant 1 - 2\sqrt{p_+(n)(1-p_+(n))}$$

and

$$\lambda_n(\Delta_\eta) \leqslant 1 - \sqrt{p_+(3n-2)p_-(3n-2)}.$$

(b) Let $\mathscr{G} := (\mathscr{V}, \mathscr{E}, m)$ be a weakly spherically symmetric graph with $m = 1$. Assume that both $\eta(n)$ and $\eta(n) - 2\sqrt{\eta_+(n)\eta_-(n)}$ are non-decreasing. Then we have, if the corresponding eigenvalues exist:

$$\lambda_{(\#B_{n-1}+1)}(\Delta_1) \geqslant \eta(n) - 2\sqrt{\eta_+(n)\eta_-(n)}$$

and

$$\lambda_n(\Delta_1) \leqslant \eta(2n-1).$$

Corollary 5.5. — *Under the hypothesis of Proposition 5.4, if moreover $p_+(n) \to 1$ as $n \to +\infty$, then $\sigma_{\mathrm{ess}}(\Delta_\eta) = \{1\}$ and the min-max Theorem 4.3 implies the existence of an infinite number of eigenvalues. Thus we have*

$$\sigma(\Delta_\eta) = \{1\} \cup \{\lambda_i^-, \lambda_i^+, i \geqslant 1\}$$

where $(\lambda_i^-)_{i\geqslant 1}$ and $(\lambda_i^+)_{i\geqslant 1}$ are infinite sequences of eigenvalues converging to 1 from below and from above respectively. Similarly, if $\eta(n)-2\sqrt{\eta_+(n)\eta_-(n)} \to +\infty$ as $n \to \infty$, then $\sigma_{\mathrm{ess}}(\Delta_1) = \emptyset$ and there is an infinite sequence of eigenvalues tending to $+\infty$.

Proof. — For the first inequality, note that by hypothesis, for all $x \in B_{n-1}^c$,

$$p_+(x) - p_-(x) \geqslant 2p_+(n) - 1.$$

By Theorem 4.13 and by Corollary 4.5,

$$\lambda_{(\# B_{n-1}+1)}(\Delta_\eta) \geqslant 1 - \sqrt{1-(2p_+(n)-1)^2} = 1 - 2\sqrt{p_+(n)(1-p_+(n))}.$$

For the right inequality, let g_n be the function defined on $\mathscr{V}$ by $g_n(x) = 1$ if $x \in S_n$, $g_n(x) = \alpha_n$ if $x \in S_{n+1}$ and $g_n(x) = 0$ otherwise, where α_n will be chosen later. Since for $|i-j| \geqslant 3$, $d_{\mathscr{G}}(\operatorname{supp} g_i, \operatorname{supp} g_j) \geqslant 2$, then

$$\lambda_n(\Delta_\eta) \leqslant \max_{i \in \{1,4,\dots,3n-2\}} \frac{\langle g_i, \Delta_\eta g_i\rangle_\eta}{\langle g_i, g_i\rangle_\eta}.$$

Now a computation gives

$$\begin{aligned}
\frac{\langle g_n, \Delta_\eta g_n\rangle_\eta}{\langle g_n, g_n\rangle_\eta} &= \frac{\eta_-(S_n)1^2 + \eta_+(S_n)(1-\alpha_n)^2 + \eta_+(S_{n+1})\alpha_n^2}{\eta(S_n) + \alpha_n^2\eta(S_{n+1})} \\
&\leqslant \frac{\eta(S_n) + \alpha_n^2\eta(S_{n+1}) - 2\alpha_n\eta_+(S_n)}{\eta(S_n) + \alpha_n^2\eta(S_{n+1})} \\
&= 1 - \frac{2\alpha_n\eta_+(S_n)}{\eta(S_n) + \alpha_n^2\eta(S_{n+1})} \\
&= 1 - \frac{\eta_+(S_n)}{\sqrt{\eta(S_n)\eta(S_{n+1})}} \\
&= 1 - \sqrt{\frac{\eta_+(S_n)}{\eta(S_n)}\frac{\eta_-(S_{n+1})}{\eta(S_{n+1})}} \\
&= 1 - \sqrt{p_+(n)p_-(n)},
\end{aligned}$$

with the choice $\alpha_n = \sqrt{\frac{\eta(S_n)}{\eta(S_{n+1})}}$ and since $\eta_+(S_n) = \eta_-(S_{n+1})$. For the Laplacian Δ_1, as before, we obtain

$$\lambda_{(\# B_{n-1}+1)}(\Delta_1) \geqslant \eta(n) - 2\sqrt{\eta_+(n)(\eta(n)-\eta_+(n))}.$$

For the right inequality, let g_n be the function defined on $\mathscr{V}$ by $g_n(x) = 1$ if $x \in S_n$ and $g_n(x) = 0$ otherwise. Since for $|i-j| \geqslant 2$, $d_{\mathscr{G}}(\operatorname{supp} g_i, \operatorname{supp} g_j) \geqslant 2$, then

$$\lambda_n(\Delta_1) \leqslant \max_{i \in \{1,3,\dots,2n-1\}} \frac{\langle g_i, \Delta_1 g_i\rangle_1}{\langle g_i, g_i\rangle_1} \leqslant \frac{\eta_+(S_{2n-1}) + \eta_-(S_{2n-1})}{|S_{2n-1}|} \leqslant \eta(2n-1).$$

This ends the proof. $\square$

5.3. The case of antitrees

A simple graph $\mathscr{G}$ is an *antitree* if there exists a 1-dimensional decomposition $(S_n)_{n\geqslant 0}$ of $\mathscr{V}$ such that $\eta_+(x) = \#S_{n+1}, \eta_-(x) = \#S_{n-1}$ and $\eta_0(x) = 0$ for all $x \in S_n, n \geqslant 0$. Antitrees are bipartite graphs. The spectral decomposition of the Laplacian on antitrees is made in [4]. It is shown that the spectrum of Δ_η is the union of $\{1\}$ and the spectrum of a Jacobi matrix.

This comes from the fact that if f is orthogonal to radial functions then $A_m f = 0$ and then $\Delta_m f = \deg(\cdot)f$. Therefore for Δ_η, 1 is an eigenvalue with infinite multiplicity. The Jacobi matrix corresponds to the action of the Laplacian on radial functions. The upper estimate for the eigenvalues in Proposition 5.4 is in fact an estimate for the eigenvalues associated to this radial part of the Laplacian. Therefore, in general, the upper estimate for the eigenvalues of Δ_m in Proposition 5.4 is reasonable.

6. An Allegretto-Piepenbrink type theorem for the essential spectrum

In this section, we prove a reverse part of Theorem 3.6. The result for the bottom of the spectrum is well-known and is sometimes called an Allegretto-Piepenbrink type theorem (see Theorem 3.1 in [17]). As far as we know, the result for the bottom of the essential spectrum is new.

First, we recall Theorem 3.1 in [17].

Theorem 6.1. — *Let $\mathscr{G} := (\mathscr{V}, \mathscr{E}, m)$ be a weighted graph. Let $\lambda^0 := \inf \sigma(\Delta_m)$ and $\lambda \leqslant \lambda^0$. Then there exists a positive function W on $\mathscr{V}$ such that*

$$\tilde{\Delta}_m W(x) \geqslant \lambda W(x).$$

We turn now to the case of the essential spectrum.

Theorem 6.2. — *Let $\mathscr{G} := (\mathscr{V}, \mathscr{E}, m)$ be a weighted graph. Let $\lambda^0_{ess} := \inf \sigma_{\mathrm{ess}}(\Delta_m)$. Then we have:*

(a) For all $\varepsilon > 0$, there exist $N_1 := N_1(\varepsilon) \geqslant 1$, $C := C(\varepsilon) > 0$, and a positive function W on $\mathscr{V}$ such that

$$\tilde{\Delta}_m W(x) \geqslant \left(\lambda^0_{ess} - \varepsilon\right) W(x) - C\mathbf{1}_{B_{N_1}}(x).$$

(b) If moreover:

$$\inf\{m(x), x \in \mathscr{V}\} > 0, \tag{6.1}$$

or if $\mathscr{G}$ is a weakly spherically symmetric graph such that $m(\mathscr{G}) = +\infty$; then for all $\varepsilon > 0$, there exist $N_2 := N_2(\varepsilon) \geqslant 1$ and a positive function W on $\mathscr{V}$ such that

$$\tilde{\Delta}_m W(x) \geqslant \left(\lambda^0_{ess} - \varepsilon\right) \mathbf{1}_{B^c_{N_2}}(x) W(x).$$

Remark 6.3. — The condition $\inf\{m(x), x \in \mathscr{V}\} > 0$ is equivalent to the inclusion $\ell^2(\mathscr{V}, m) \subset \ell^\infty(\mathscr{V}, m)$.

Proof. — Since $\lambda^0_{ess} := \inf \sigma_{\rm ess}(\Delta_m)$, by Persson lemma, there exists $K \geqslant 1$ such that the infimum of the spectrum of the Dirichlet operator $\Delta_m^{B^c_K}$ is larger than $\lambda^0_{ess} - \varepsilon$.

The operator $\left(\Delta_m^{B^c_K} - (\lambda^0_{ess} - \varepsilon)\right)^{-1}$ is thus well defined on B^c_K. Moreover this operator is positive improving. The positivity improveness is proven in [20] [Corollary 2.9] for the operator $\left(\Delta_m^{B^c_K} - \alpha\right)^{-1}$ only for $\alpha < 0$ but actually by general principles it holds for all $\alpha < \inf \sigma\left(\Delta_m^{B^c_K}\right)$ (see [26] [Chapter XIII.12] p 204 and [27]).

Let ψ be a non-negative (non trivial) function in $\mathcal{C}_c(B^c_K)$ and consider ϕ the function defined on B^c_K by $\phi := \left(\Delta_m^{B^c_K} - (\lambda^0_{ess} - \varepsilon)\right)^{-1} \psi$. By positivity improveness, $\phi > 0$ and ϕ satisfies $\Delta_m^{B^c_K} \phi(x) \geqslant (\lambda^0_{ess} - \varepsilon)\phi(x)$ for $x \in B^c_K$. Considering W to be any (positive) extension of the function ϕ on the all set $\mathscr{V}$ gives the first point.

Now, assume that $\inf\{m(x), x \in \mathscr{V}\} > 0$. Since $\phi \in \ell^2(B^c_K, m)$, this implies that $\phi(x) \to 0$ when $|x| \to \infty$. Let $\varepsilon' = \frac{1}{2} \min\{\phi(x), x \in S_{K+1}\}$, then the set $A_{\varepsilon'} = B_K \cup \{x \in B^c_K, \phi(x) > \varepsilon'\}$ is finite.

Recall that $\delta(A^c_{\varepsilon'})$ is the set of points x in $A^c_{\varepsilon'}$ who have a neighbor which belongs to $A_{\varepsilon'}$. Let u be the harmonic function in $A_{\varepsilon'} \cup \delta(A^c_{\varepsilon'})$ such that

$$\begin{cases} \Delta_m u = 0 \text{ on } A_{\varepsilon'} \\ u = \phi \text{ on } \delta(A^c_{\varepsilon'}) \end{cases}$$

Define then the function W on $\mathscr{V}$ as:

$$\begin{cases} W(x) = u(x) \text{ on } A_{\varepsilon'} \\ W(x) = \phi(x) \text{ on } A^c_{\varepsilon'} \end{cases}$$

Clearly, $\Delta_m W(x) = 0$ for $x \in A_{\varepsilon'}$ and $\Delta_m W(x) \geqslant (\lambda^0_{ess} - \varepsilon) W(x)$ for $x \in \operatorname{Int}(A^c_{\varepsilon'})$. It remains to look at the points x in $\delta(A^c_{\varepsilon'})$. Let $x \in \delta(A^c_{\varepsilon'})$.

For $y \in A^c_{\varepsilon'}$, we have $W(y) = \phi(y)$ and for $y \in \delta A_{\varepsilon'}$, since $\delta A_{\varepsilon'} \subset B^c_K \cap A_{\varepsilon'} = \{x \in B^c_K, \phi(x) > \varepsilon'\}$, by a maximum principle for harmonic functions $W(y) = u(y) \leqslant \varepsilon' \leqslant \phi(y)$, therefore

$$\tilde{\Delta}_m W(x) \geqslant \tilde{\Delta}_m \phi(x) = \tilde{\Delta}_m^{B^c_K} \phi(x) \geqslant (\lambda^0_{ess} - \varepsilon)\phi(x) = (\lambda^0_{ess} - \varepsilon) W(x).$$

Now we turn to the case of weakly spherically symmetric graphs. We use here some properties that will be established in Section 9. We repeat the same construction as before and set $\phi^{\mathbb{N}} = \left(\Delta_m^{B^c_K} - (\lambda^0_{ess} - \varepsilon)\right)^{-1} \psi^{\mathbb{N}}$ with $\psi^{\mathbb{N}}$ a radial non-negative and non trivial function in B^c_K with finite support. Since the Laplacian $\Delta_m^{B^c_K}$ preserves radial functions, $\phi^{\mathbb{N}}$ is also a radial function. By abuse of notation we write $\phi^{\mathbb{N}}(n) := \phi^{\mathbb{N}}(x), x \in S_n$. As before, $\phi^{\mathbb{N}}$ is a positive function and belongs to $\ell^2(B^c_K, m)$ and we have

$$\|\phi^{\mathbb{N}}\|^2_{\ell^2(B^c_K, m)} = \sum_{n \geqslant 0} m(S_n) \phi^{\mathbb{N}}(n)^2.$$

Since $m(\mathscr{G}) = \sum_{n \geqslant 0} m(S_n) = +\infty$, $\phi^{\mathbb{N}}$ can not be non-decreasing. Therefore there exists $n_0 \geqslant K + 1$, such that $\phi^{\mathbb{N}}(n_0 + 1) \leqslant \phi^{\mathbb{N}}(n_0)$. We can now perform the cut and paste procedure by taking W to be the function on $\mathscr{V}$ defined by

$$\begin{cases} W(x) = \phi^{\mathbb{N}}(n_0) \text{ if } |x| \leqslant n_0 \\ W(x) = \phi^{\mathbb{N}}(n) \text{ if } |x| = n \text{ with } n \geqslant n_0 + 1. \end{cases}$$

Clearly, W is the desired super-harmonic function. □

Remark 6.4. — Note that, in the above proof, since $\phi^{\mathbb{N}}$ can not have any local minimum, if we have $\phi^{\mathbb{N}}(n + 1) \leqslant \phi^{\mathbb{N}}(n)$ for some n, then ϕ_N is non-increasing on $[\![n, +\infty)$.

7. Harnack inequality and limiting procedures

In this section, we recall how to obtain a super-solution on the entire set of vertices $\mathscr{V}$ given a sequence of super-solution defined on a exhaustive sequence of finite sets. We recall that the graph is supposed to be connected. The results of this section are taken from [17]. The only difference is that, here, we consider a non-negative function λ in place of a constant. The proofs adapt straightforwardly and will not be presented.

First we begin by the Harnack inequality for non-negative super-solutions.

Theorem 7.1. — *Let $\mathscr{W} \subset \mathscr{V}$ be a finite and connected set. Let $\lambda : \mathscr{V} \to \mathbb{R}$ be a non-negative function. There exists a constant $C_{\mathscr{W}}$ such that for all non-negative function $W : \mathscr{V} \to [0,_+ \infty)$ satisfying $(\Delta - \lambda(x))W(x) \geqslant 0$ for all $x \in \mathscr{W}$, we have*

$$\max_{x \in \mathscr{W}} W(x) \leqslant C_{\mathscr{W}} \min_{x \in \mathscr{W}} W(x).$$

As Corollary we obtain:

Corollary 7.2. — *Let $\mathscr{W} \subset \mathscr{V}$ be a connected set. Let $x_0 \in \mathscr{V}$ and let $\lambda : \mathscr{V} \to \mathbb{R}$ be a non-negative function. For all $x \in \mathscr{W}$, there exists a constant $C_x := C_x(x_0, \mathscr{W})$ such that for all non-negative $W : \mathscr{W} \to [0, +\infty)$ satisfying $W(x_0) = 1$ and $(\Delta - \lambda(x))W(x) \geqslant 0$ for all $x \in \mathscr{W}$, we have*

$$C_x^{-1} \leqslant W(x) \leqslant C_x.$$

Remark 7.3. — Obviously, the last corollary can be used with $\mathscr{W} = \mathscr{V}$.

We now turn to the main result of this section.

Theorem 7.4. — *Let $x_0 \in \mathscr{V}$ and let $\lambda : \mathscr{V} \to \mathbb{R}$ be a non-negative function. Let $(\mathscr{W}_n)_n$ be an exhausting sequence of $\mathscr{V}$. Assume that there exists a sequence of non-negative functions $W_n : \mathscr{W}_n \to [0, +\infty)$ satisfying $W_n(x_0) = 1$ and $(\Delta_n - \lambda(x))W_n(x) \geqslant 0$ (respectively $(\Delta_n - \lambda(x))W_n(x) = 0$) for all $x \in \mathscr{W}_n$. Then there exists a positive function $W : \mathscr{V} \to (0, +\infty)$ such that $(\Delta_n - \lambda(x))W(x) \geqslant 0$ (respectively $(\Delta_n - \lambda(x))W(x) = 0$) for all $x \in \mathscr{V}$.*

8. Probabilistic representation of positive super-harmonic functions

In the classical situation of Poincaré inequality, there is a strong link between the linear Lyapunov functions and the hitting times of some compact sets for a stochastic process, see [7]. Here we develop an analogy of these results. In all this section, $(\mathscr{W}_n)_{n \geqslant 0}$ will denote an exhaustive sequence of $\mathscr{V}$, see Definition 3.4.

8.1. Discrete and continuous time Markov chains

In this section, we present the Markov processes whose generator is given by (minus) the Laplacian on the graph. In the case of a general weighted Laplacian, we can associate a continuous time Markov chain. In the case of the normalized Laplacian, we can associate both a continuous time and a discrete time Markov chain. More details about the construction and the properties of these Markov process can be found in the monograph [25].

8.1.1. The discrete time Markov chain associated to Δ_η

We begin by the simplest case of the normalized Laplacian. Consider the Markov chain $(X_k^{x_0})_{k\geqslant 1}$ starting in x_0 on the graph whose transition probabilities are given by

$$p(x,y) := \frac{\mathscr{E}(x,y)}{\eta(x)},$$

for all $x,y \in \mathscr{V}$. Then, set $Pf(x) = \sum_y p(x,y)f(y)$ for all $f \in \ell^\infty(\mathscr{V})$. For $k \geqslant 0$ and $x \in \mathscr{V}$, one has

$$P^k f(x) = \mathbb{E}\left[f(X_k^x)\right].$$

The generator of the above discrete time Markov chain random walk is given by $P - \mathrm{Id}$ and then equals $-\Delta_\eta$; that is for $f \in \ell^\infty(\mathscr{V})$,

$$\mathbb{E}\left[f(X_1^x)\right] - f(x) = -\Delta_\eta f(x), \quad x \in \mathscr{V}.$$

The measure η satisfies $\eta(x)p(x,y) = \eta(y)p(y,x)$ for $x,y \in \mathscr{V}$. It is symmetric (and hence invariant) for the Markov chain.

8.1.2. The continuous time Markov chain associated to Δ_m

Now we turn to the general case. With the above notation, the Laplacian $-\Delta_m$ can be written as

$$\Delta_m f(x) = \deg(x)\sum_y p(x,y)(f(x) - f(y)).$$

We construct here the *minimal* right continuous Markov chain $(X_t)_{t\geqslant 0}$ associated to $-\Delta_m$. It corresponds to the process killed at infinity. We denote by $e(X)$ its *explosion time* (recall that X depends on the choice of the initial law). We recall two useful constructions of the continuous time Markov chain when the initial law is δ_{x_0}. We denote it by $(X_t^{x_0})_{t\geqslant 0}$.

First we can construct $(X_t^{x_0})_{t\geqslant 0}$ as follows: At time $t=0$, $X_0^{x_0} = x_0$. It stays in x_0 during an exponential random time of parameter $\deg(x_0)$ and then jumps in a point y chosen with probability $p(x_0,y)$. We then iterate this procedure.

Another useful equivalent construction of the process $(X_t^{x_0})_{t\geqslant 0}$ is the following. At time $t=0$, $X_0^{x_0} := x_0$. For each, neighbor y of x_0, we let E_y be an independent exponential random clock variable of parameter $\deg(x_0,y)$. Consider $T := \min\{E_y, y \sim x_0\}$. Let z be the neighbor of x_0 such that

$E_z = \min\{E_y, y \sim x_0\}$, z is unique almost surely. We set $X_t := x_0$ for $0 \leqslant t < T$, $X_T := z$ and repeat this construction.

Using the memorylessness property of the exponential distribution and Lemma 8.1 below, it is easy to see that both constructions are equivalent and that $(X_t^x)_{t\geqslant 0}$ is a Markov process. Moreover, the jump chain associated to $(X_t^x)_{t\geqslant 0}$ is the discrete time Markov chain of generator $-\Delta_\eta$.

Lemma 8.1. — *Let $(E_i)_{1\leqslant i\leqslant n}$ be n independent exponential random variables of parameter $c_i > 0$, then the variable $\min\{E_i, 1 \leqslant i \leqslant n\}$ is also an exponential random variable of parameter $c_1 + \cdots + c_n$. Moreover, for all $1 \leqslant r \leqslant n$ we have:*

$$\mathbb{P}\left(\min\{E_i, 1 \leqslant i \leqslant n\} = E_r\right) = \frac{c_r}{c_1 + \cdots + c_n}.$$

The next lemma concerns also the memorylessness property of the exponential distribution. It will be useful to add some "artificial jumps" in the construction of the process $(X_t)_{t\geqslant 0}$.

Lemma 8.2. — *Let $n \geqslant 1$ and $c_1, \ldots, c_n > 0$. Let $(E_{i,j})_{i\geqslant 1, 1\leqslant j\leqslant n}$ be independent exponential random variables such that the parameter of $E_{i,j}$ is c_j. Let $(A_i)_{i\geqslant 1}$ be independent random variables such that almost surely*

$$A_i > 0 \text{ and } \sum_{i=1}^{\infty} A_i = +\infty \tag{8.1}$$

Let k be defined by

$$k := \inf\{i \geqslant 1, \min(E_{i,1}, \ldots, E_{i,n}, A_i) \neq A_i\}$$

Then k is finite almost surely and the random variable $B := A_1 + \cdots + A_{k-1} + \min(E_{k,1}, \ldots, E_{k,n})$ is also an exponential random variable of parameter $c := c_1 + \cdots + c_n$. Moreover for all $1 \leqslant r \leqslant n$, we have:

$$\begin{aligned}\mathbb{P}\left(\min(E_{k,1}, \ldots, E_{k,n}) = E_{k,r}\right) &= \mathbb{P}\left(\min(E_{1,1}, \ldots, E_{1,n}) = E_{1,r}\right)\\ &= \frac{c_r}{c_1 + \cdots + c_n}.\end{aligned}$$

The Lemma 8.2 allows us to add some "artificial jumps" in the construction of the Markov process $(X_t)_{t\geqslant 0}$. Indeed, it implies that we can also construct (X_t) as follows: If at time t, $X_t = x$, then as before for each neighbor y of x, we let E_y be an independent exponential random clock variable of parameter $\deg(x, y)$. We let also E_x be another independent exponential random clock variable. Let $\deg(x,x)$ be its parameter. Consider $\tilde{T} := \min\{E_y, y \sim x \text{ or } y = x\}$. Let z be the unique vertex such that $E_z = \min\{E_y, y \sim x \text{ or } y = x\}$. We set $X_s := x$ for $t \leqslant s < t + \tilde{T}$, $X_{\tilde{T}} := z$ and repeat this construction. Moreover at each step, the choice of the parameter $\deg(x,x)$ can change (with the restriction that it has to satisfy condition (8.1)). The only difference with the previous construction is that $\tilde{T}$ does not really correspond anymore to a physical jump of the process.

This modification of the construction will be useful in the coupling arguments of section 10.

In the above constructions, the sequence of the (random) times of the jumps of the Markov process X is increasing, thus has a limit in $(0,\infty]$. This limit is called the *explosion time* of Markov process X and is denoted by $e(X)$.

We can now associate a *continuous time semigroup* P_t for $f \in \ell^\infty(\mathscr{V})$ by

$$P_t f(x) := \mathbb{E}\left[f(X_t^x)\mathbf{1}_{\{t<e(X^x)\}}\right], \quad t \geqslant 0.$$

For t small, using exponential distributions, it is easy and well-known to compute explicitly the first order expansion of law of X_t^x. One gets

$$\begin{aligned}&\mathbb{P}(X_t^x = x) = 1 - \deg(x)t + o(t)\\ &\mathbb{P}(X_t^x = y) = \deg(x)p(x,y)t + o(t), \text{ for } y \neq x.\end{aligned}$$

In particular for $f \in \ell^\infty(\mathscr{V})$, the following pointwise convergences hold:

$$\lim_{t\to 0^+} P_t f(x) = f(x)$$

and

$$\lim_{t\to 0^+} \frac{P_t f(x) - f(x)}{t} = -\tilde{\Delta}_{\mathscr{G},m} f(x).$$

Actually, in the sequel, we only need to consider the above Markov process stopped outside a finite set. Let B be a finite subset of $\mathscr{V}$ and let

$$T_{B^c} := \inf\{t \geqslant 0, X_t \in B^c\}$$

the *hitting time* of the set B^c for the continuous time Markov chain $(X_t)_{t\geqslant 0}$. Clearly, since each connected component of $\mathscr{G}$ is infinite and B is finite, by classical result on transience, T_{B^c} is almost surely finite. Moreover, we also have $T_{B^c} < e(X)$. We can now define a new continuous time semigroup $P_t^{D_B}$ for $f : \mathscr{V} \to \mathbb{R}$ by

$$P_t^{D_B} f(x) := \mathbb{E}\left[f(X^x_{t\wedge T_{B^c}})\right], \quad t \geqslant 0, \quad x \in \mathscr{V}.$$

A computation similar to the above shows that for $f : \mathscr{V} \to \mathbb{R}$ and $x \in \mathscr{V}$, pointwise,

$$\lim_{t\to 0^+} \frac{P_t^{D_B} f(x) - f(x)}{t} = \begin{cases} -\tilde{\Delta}_{\mathscr{G},m} f(x) & \text{if} \quad x \in B \\ 0 & \text{if} \quad x \in B^c. \end{cases}$$

It is not symmetric on $\mathcal{C}_c(\mathscr{V})$. Indeed its generator can be written as $-\Pi_B\Delta$ where Π_B is the projection defined by $\Pi_B f(x) := f(x)\mathbf{1}_B(x)$, for all $f : \mathscr{V} \to \mathbb{R}$.

For $f \in \ell^\infty(\mathscr{V})$ we could also define the semigroup:

$$P_t^{D'_B} f(x) := \mathbb{E}\left[f(X_t^x)\mathbf{1}_{\{t<T_{B^c}\}}\right], \quad t \geqslant 0.$$

It corresponds to the usual Dirichlet semigroup. As before, one can compute that its generator is: $-\Pi_B\Delta\Pi_B$. More precisely, for all $f : \mathscr{V} \to \mathbb{R}$, pointwise, one has:

$$\lim_{t\to 0^+} \frac{P_t^{D'_B} f(x) - f(x)}{t} = -\Pi_B\Delta\Pi_B f(x).$$

8.2. The normalized Laplacian in the discrete time setting

For simplicity, we begin with the case of the normalized Laplacian Δ_η. Actually, Theorem 8.3 below can also be seen as a corollary of the general Theorem 8.5. A direct proof is included for the reader more familiar with Markov chains than continuous time Markov chains.

Theorem 8.3. — *Let $\lambda : \mathscr{V} \to [0,1)$ and let $\Lambda(x) := \frac{1}{1-\lambda(x)}$. The following assertions are equivalent:*

(i) There exists a positive function W on $\mathscr{V}$ such that $\tilde{\Delta}_\eta W(x) = \lambda(x)W(x)$ for all $x \in \mathscr{V}$.

(ii) There exists a positive function W on $\mathscr{V}$ such that $\tilde{\Delta}_\eta W(x) \geqslant \lambda(x)W(x)$ for all $x \in \mathscr{V}$.

(iii) For all $N \geqslant 1$, there exists a positive functions W_N on $\mathscr{V}$ such that $\tilde{\Delta}_\eta W_N(x) \geqslant \lambda(x)W_N(x)$ for $x \in \mathscr{W}_N$.

(iv) For all $N \geqslant 1$ and all $x \in \mathscr{V}$, we have

$$\mathbb{E}\left[\prod_{k=0}^{T_N-1} \Lambda(X_k^x)\right] < +\infty, \tag{8.2}$$

where $T_N := T_N^x := \inf\{n \geqslant 0, X_n^x \in \mathscr{W}_N^c\}$ is the hitting time of the set $\mathscr{W}_N^c$ for X_n^x the Markov chain on $\mathscr{G}$ starting in x and whose generator is $-\Delta_\eta$.

When the function λ is constant, one has $\Lambda := \frac{1}{1-\lambda}$ and the function in (4.) reads also $\mathbb{E}_x[\Lambda^{T_N}] := \mathbb{E}[\Lambda^{T_N^x}]$.

Remark 8.4. — In this situation, one can not have $\lambda(x) \geqslant 1$ for some x since for all positive function W and all $x \in \mathscr{V}$,

$$\tilde{\Delta}_\eta W(x) = \sum_y p(x,y)(W(x) - W(y)) < \sum_y p(x,y)W(x) = W(x).$$

Proof. — Clearly, (1.) implies (2.). The equivalence between (2.) and (3.) is given by Theorem 7.4. We now show that (3.) implies (4.). Set

$$A_n := \prod_{k=0}^{n-1} \Lambda(X_k^x), \text{ for } n \geqslant 1$$

and $A_0 := 1$. Let $N \geqslant 1$, $x \in \mathscr{W}_N$ and $n \geqslant 0$, we have

$$\mathbb{E}_x\left[A_{n\wedge T_N}\right] \leqslant \frac{1}{\min\{W_N(x), x \in \mathscr{W}_N \cup \delta\mathscr{W}_N} \mathbb{E}_x\left[A_{n\wedge T_N} W_N(X^x_{n\wedge T_N})\right]. \tag{8.3}$$

Using the Abel transform
$u_n v_n = u_0 v_0 + \sum_{k=0}^{n-1}\left((u_{k+1} - u_k)v_{k+1} + u_k(v_{k+1} - v_k)\right)$, we get

$$\begin{aligned}
&\mathbb{E}_x\left[A_{n\wedge T_N} W_N(X^x_{n\wedge T_N})\right] \\
&\quad = W_N(x) + \sum_{k=0}^{(n\wedge T_N)-1} \mathbb{E}_x\left[A_{k+1}\left(W_N(X^x_{k+1}) - W_N(X^x_k)\right)\right] \\
&\qquad + \sum_{k=0}^{(n\wedge T_N)-1} \mathbb{E}_x\left[(A_{k+1} - A_k) W_N(X^x_k)\right].
\end{aligned}$$

The event A_{k+1} is measurable with respect to the σ-algebra $\sigma(X_1^x, \ldots, X_k^x)$, thus

$$\begin{aligned}
\mathbb{E}_x&\left[A_{k+1}\left(W_N(X^x_{k+1}) - W_N(X^x_k)\right)\right] = \\
&= \mathbb{E}_x\left[A_{k+1}\mathbb{E}\left[W_N(X^x_{k+1}) - W_N(X^x_k) | \sigma(X_1^x, \ldots, X_k^x)\right]\right] \\
&= -\mathbb{E}_x\left[A_{k+1}\tilde{\Delta}_\eta W_N(X^x_k)\right]
\end{aligned}$$

and therefore

$$\mathbb{E}_x\left[A_{n\wedge T_N} W_N(X^x_{n\wedge T_N})\right] = W_N(x) + \sum_{k=0}^{(n\wedge T_N)-1} \mathbb{E}_x\left[-A_{k+1}\tilde{\Delta}_\eta W_N(X^x_k)\right]$$
$$+ \sum_{k=0}^{(n\wedge T_N)-1} \mathbb{E}_x\left[A_{k+1}\left(1-\frac{1}{\Lambda(X^x_k)}\right) W_N(X^x_k)\right]$$
$$\leqslant W_N(x),$$

where we have used $\tilde{\Delta}_\eta W_N(x) \geqslant \lambda(x) W_N(x) = \left(1 - \frac{1}{\Lambda(x)}\right) W_N(x)$ for $x \in \mathscr{W}_N$. Finally, since T_N is almost surely finite and recalling (8.3), by letting $n \to \infty$ we obtain (4.).

We turn to (4.) implies (1.). Set $U_N(x) := \mathbb{E}_x[A_{T_N}]$ for $N \geqslant 1$. By hypothesis, it is finite for all N and all $x \in \mathscr{V}$. Let $x \in \mathscr{W}_N$, by Markov property, we have

$$U_N(x) = \sum_{y, y\sim x} p(x,y)\Lambda(x)\mathbb{E}_y[A_{T^y_N}] = \Lambda(x) \sum_{y, y\sim x} p(x,y) U_N(y),$$

thus

$$\Delta_\eta U_N(x) = U_N(x) - \sum_{y, y\sim x} p(x,y) U_N(y)$$
$$= \left(1 - \frac{1}{\Lambda(x)}\right) U_N(x) = \lambda(x) U_N(x).$$

Theorem 7.4 ends the proof. □

8.3. The general case: the continuous time setting

In the case of a general weight m, we obtain the analogous of Theorem 8.3 for the continuous time Markov process associated to Δ_m.

Theorem 8.5. — *Let λ a non-negative function on $\mathscr{V}$. The following assertions are equivalent.*

(i) There exists a positive function W such that: $\tilde{\Delta}_m W(x) = \lambda(x) W(x)$ for all $x \in \mathscr{V}$.

(ii) There exists a positive function W such that: $\tilde{\Delta}_m W(x) \geqslant \lambda(x) W(x)$ for all $x \in \mathscr{V}$.

(iii) For all $N \geqslant 1$, there exists a positive function W_N on $\mathscr{V}$ such that: $\tilde{\Delta}_m W_N(x) \geqslant \lambda(x) W_N(x)$ for $x \in \mathscr{W}_N$.

(iv) For all $N \geqslant 1$ and all $x \in \mathscr{V}$, the positive function

$$U_N(x) := \mathbb{E}\left[\exp\left(\int_0^{T_N} \lambda(X_s^x)ds\right)\right] \tag{8.4}$$

is finite where $T_N := \inf\{t \geqslant 0, X_t^x \in \mathscr{W}_N^c\}$ is the hitting time of the set $\mathscr{W}_N^c$ for the continuous time Markov chain $(X_t^x)_{t\geqslant 0}$ starting in x and whose generator is $-\Delta_m$.

Proof. — We focus on the implications: (iii) implies (iv) and (iv) implies (i). We start with (iii) implies (iv). Let

$$A_t(x) := \mathbb{E}\left[\exp\left(\int_0^{t} \lambda(X_s^x)ds\right)\right].$$

We have:

$$A_{t\wedge T_N}(x) \leqslant \frac{1}{\min\{W_N(z), z \in \mathscr{W}_N \cup \delta\mathscr{W}_N\}}\mathbb{E}\left[\exp\left(\int_0^{t\wedge T_N} \lambda(X_s^x)ds\right) W_N(X^x_{t\wedge T_N})\right].$$

By the Dynkin formula we get:

$$\begin{aligned}
&\mathbb{E}\left[\exp\left(\int_0^{t\wedge T_N} \lambda(X_s^x)ds\right) W_N(X^x_{t\wedge T_N})\right]\\
&= W_N(x) + \mathbb{E}\left[\int_0^{t\wedge T_N} \exp\left(\int_0^{u} \lambda(X_v^x)dv\right)\left(\lambda(X_u^x)W_N(X_u^x) - \tilde{\Delta}_m W_N(X_u^x)\right)du\right]\\
&\leqslant W_N(x) < +\infty
\end{aligned}$$

since by hypothesis $\tilde{\Delta}_m W(X_u^x) - \lambda(X_u^x)W(X_u^x) \geqslant 0$. As T_N is finite almost surely, letting $t \to \infty$ gives (4.).

Finally we assume (4.). Using the strong Markov property, one has, for $x \in \mathscr{W}_N$ and $0 < h \leqslant 1$,

$$\begin{aligned}
P_h^{D\mathscr{W}_N} U_N(x) &= \mathbb{E}\left[U_N\left(X^x_{h\wedge T_N^x}\right)\right]\\
&= \mathbb{E}\left[\exp\left(\int_0^{T_N^{X_h^x}} \lambda\left(X_s^{X_h^x}\right)ds\right)\mathbf{1}_{\{h<T_N^x\}} + \mathbf{1}_{\{h\geqslant T_N^x\}}\right]\\
&= \mathbb{E}\left[\exp\left(\int_{h\wedge T_N^x}^{T_N^x} \lambda\left(X_u^x\right)du\right)\right].
\end{aligned}$$

Therefore, by dominated convergence theorem, since λ is bounded on $\mathscr{W}_N$,

$$\begin{aligned}\frac{P_h^{D_{\mathscr{W}_N}} U_N(x) - U_N(x)}{h} \\ &= \mathbb{E}\left[\exp\left(\int_0^{T_N^x} \lambda\left(X_u^x\right) du\right)\left(\frac{\exp\left(-\int_0^{h\wedge T_N^x} \lambda\left(X_u^x\right) du\right) - 1}{h}\right)\right] \\ &\to -\lambda(x) U_N(x), \text{ as } h \to 0^+.\end{aligned}$$

But the above limit was already compute to be $-\tilde{\Delta}_m U_N(x)$; thus

$$\tilde{\Delta}_m U_N(x) = \lambda(x) U_N(x), \text{ for } x \in \mathscr{W}_N.$$

Theorem 7.4 implies (1.). □

Remark 8.6. — For the normalized Laplacian Δ_η, both quantities (8.2) and (8.4) coincide. Indeed, with the above notation, if Z is an exponential random variable of parameter 1 and if $0 \leqslant \lambda < 1$, then

$$\mathbb{E}\left[\exp(\lambda Z)\right] = \frac{1}{1-\lambda}.$$

9. Weakly spherically symmetric graphs

In this section, we assume that the graph $\mathscr{G}$ is weakly spherically symmetric with respect to a 1-dimensional decomposition $(S_n)_{n\in\mathbb{N}}$, see Definition 2.1. We prove that the bottom of the spectrum and the bottom of the essential spectrum are the same as that of a 1-dimensional Laplacian. The key point behind this result is that on a weakly spherically symmetric graph, the radial part of the Markov process associated to the Laplacian on $\mathscr{G}$ is still a Markov process. We finally construct more explicitly the global super-solution of Theorem 7.4.

First in the next lemma, we collect some useful known results for weakly spherically symmetric graphs.

Lemma 9.1. — *Let $\mathscr{G}$ be a weakly spherically symmetric graph with respect to a 1-dimensional decomposition $(S_n)_{n\in\mathbb{N}}$ and let λ a be radial function on $\mathscr{V}$. The following assertions hold.*

(a) *For $n \geqslant 0$,*

$$m(S_n)\deg_+(n) = m(S_{n+1})\deg_-(n+1),$$

where $\deg_a(n) := \deg_a(x)$, where $x \in S_n$ and $a \in \{-, 0, +\}$.

(b) *Given* $f : \mathscr{V} \to \mathbb{C}$ *we define* $\tilde{M}$ *to be the* averaging operator *by:*

$$\tilde{M}f(x) := \frac{1}{m(S_n)} \sum_{\tilde{x} \in S_n} f(\tilde{x})m(\tilde{x}), \quad x \in S_n.$$

We have the following algebraic commutation

$$\tilde{\Delta}_{\mathscr{G}} \tilde{M} f = \tilde{M} \tilde{\Delta}_{\mathscr{G}} f.$$

(c) *If there exists a positive function* W *which satisfies*

$$\tilde{\Delta}_{\mathscr{G}} W(x) = \lambda(x) W(x), \text{ for all } x \in \mathscr{V}, \tag{9.1}$$

then there also exists a positive radial function which satisfies (9.1)*.*

(d) *Assuming that* $\deg_+(n) \neq 0$ *for all* $n \in \mathbb{N}$*, then the vector space of radial functions* W *which satisfy the algebraic relations* (9.1) *is a* 1*-dimensional vector space.*

(e) *Moreover, if* W *is a radial function which satisfies* (9.1) *and if both* λ *and* W *are non-negative on* $\mathscr{V}$*, then* W *is a non-increasing radial function.*

Proof. — The point a) is a direct consequence of the relation:

$$\begin{aligned} \sum_{x \in S_n} m(x) \deg_+(x) &= \sum_{x \in S_n} \sum_{y \in S_{n+1}} \mathscr{E}(x, y) \\ &= \sum_{y \in S_{n+1}} \sum_{x \in S_n} \mathscr{E}(y, x) = \sum_{y \in S_{n+1}} m(y) \deg_-(y) \end{aligned}$$

and the definition of weakly spherically graphs. b) and c) were already proven in Lemma 3.2 in [21] and Lemma 3.2.1 in [29], respectively. Let now W be a radial function; W satisfies (9.1) if and only it satisfies

$$\begin{cases} \deg_+(0)W(1) = (\deg_+(0) - \lambda(0))W(0) \\ \deg_+(n)W(n+1) = (\deg_+(n) + \deg_-(n) - \lambda(n))W(n) - \deg_-(n)W(n-1) \end{cases}$$

for $n \geqslant 1$. Thus W is determined by its value in 0. This gives d). If moreover W and λ are non-negative, one has that $W(1) \leqslant W(0)$ and writing

$$\deg_+(n)(W(n+1) - W(n)) = \deg_-(n)(W(n) - W(n-1) - \lambda(n))W(n)$$

for $n \geqslant 1$, by immediate induction, e) holds. □

We now study the radial part of the Markov process associated to a weakly spherically symmetric graph.

Proposition 9.2. — *Let $\mathscr{G} := (\mathscr{V}, \mathscr{E}, m)$ be a weighted graph and let $(S_n)_{n\geqslant 0}$ be a 1-dimensional decomposition of $\mathscr{G}$. Let $(X_t)_{t\geqslant 0}$ be the minimal continuous time Markov chain on $\mathscr{V}$ associated to $-\Delta_{\mathscr{G}}$ (see section 8.1). Then the graph $\mathscr{G}$ is weakly spherically symmetric with respect to $(S_n)_{n\geqslant 0}$ if and only if the process $(|X_t|)_{t\geqslant 0}$ is a continuous time Markov chain on $\mathbb{N}$.*

Moreover in this case, the generator $L^{\mathbb{N}}$ of the Markov process $(|X_t|)_{t\geqslant 0}$ is given by the formula

$$L^{\mathbb{N}} f(n) = \deg_+(n)(f(n+1) - f(n)) + \deg_-(n)(f(n-1) - f(n)) \tag{9.2}$$

for $f \in \ell^\infty(\mathbb{N})$. It corresponds exactly to $-\tilde{\Delta}_{\mathscr{G}_{\mathbb{N}}}$ where $\mathscr{G}_{\mathbb{N}} := (\mathbb{N}, \mathscr{E}_{\mathbb{N}}, m_{\mathbb{N}})$ with

$$\begin{aligned} \mathscr{E}_{\mathbb{N}}(n,m) &:= \begin{cases} m(S_n)\deg_\pm(n), & \text{when } m = n \pm 1 \\ 0, & \text{otherwise} \end{cases} \\ m_{\mathbb{N}}(n) &:= m(S_n), \end{aligned} \tag{9.3}$$

for all $n, m \in \mathbb{N}$.

Note that $\mathscr{E}_{\mathbb{N}}$ is symmetric by Lemma 9.1 a).

Proof. — First we assume that $\mathscr{G}$ is weakly spherically symmetric. We provide an explicit construction of the process $(X_t)_{t\geqslant 0}$. The desired properties for the process $(|X_t|)_{t\geqslant 0}$ will follow. Let $x := X_0$, $T := 0$ and $k := 0$. We begin to describe the iteration procedure:

1) We let run three independent (and independent of all the possible previous steps) random exponential clock variables $E_+(|x|)$, $E_0(x)$, $E_-(|x|)$ of parameter $\deg_+(|x|)$, $\deg_0(x)$, $\deg_-(|x|)$, respectively. We then replace T by the time given by $T + \min(E_+(|x|), E_0(x), E_-(|x|))$.

2) If the above minimum equals $E_+(|x|)$ or $E_-(|x|)$, we set $T_{k+1} := T$ and replace k by $k+1$. We let the process X stay in x until the time $(T_{k+1})^-$ and jump at time T_{k+1} in a point $z \in S_{|x|\pm 1}$ whether the minimum equals $E_\pm(|x|)$. We then go to 3).

If the above minimum equals $E_0(x)$, we let the process X stay in x until the time T^- and jump in a point $\tilde{x} \in S_{|x|}$ at this time T and repeat 1) with x replaced by $\tilde{x}$.

3) Replace x by z and repeat 1).

With the above construction, the sequence $(T_k)_{k\geqslant 0}$ corresponds exactly to the sequence of times of the jumps associated to the process $(|X_t|)_{t\geqslant 0}$ in $\mathbb{N}$.

By Lemma 8.2, each time T_k in the algorithm is almost surely finite. Indeed for each $n \geqslant 0$, since S_n is finite, $\sup_{x\in S_n} \deg_0(x) < \infty$; this ensures that hypothesis (8.1) is satisfied. Moreover $T_{k+1} - T_k$ corresponds to the minimum of two independent random exponential variables Z_+, Z_- of parameter $\deg_+(|X_{T_k}|), \deg_-(|X_{T_k}|)$, respectively.

It is then clear that the process $(|X_t|)_{t\geqslant 0}$ is a continuous time Markov chain whose generator is given by (9.2).

Now assume $(|X_t|)_{t\geqslant 0}$ is a continuous time Markov chain on $\mathbb{N}$. Since $(|X_t|)_{t\geqslant 0}$ can only make jumps of size 1, the generator $L^{\mathbb{N}}$ reads

$$L^{\mathbb{N}} f(n) = \alpha_+(n)(f(n+1) - f(n)) + \alpha_-(n)(f(n-1) - f(n))$$

for $f \in \ell^\infty(\mathbb{N})$ and some constants $\alpha_\pm(n) \geqslant 0, n \in \mathbb{N}$ (and $\alpha_-(0) = 0$). Let P_t and $P_t^{\mathbb{N}}$ the semigroup on associated to $(X_t)_{t\geqslant 0}$ and $(|X_t|)_{t\geqslant 0}$, respectively. Let $x \in \mathscr{V}$ and set $n := |x|$. Consider $f := \mathbf{1}_{n+1} \in \mathcal{C}_c(\mathbb{N})$ and $g := f \circ |\cdot| \in \mathcal{C}_c(\mathscr{V})$, one has

$$P_t(g)(x) = \mathbb{E}\left[g(X_t^x)\right] = \mathbb{E}\left[f(|X_t^x|)\right] = P_t^{\mathbb{N}}(f)(n).$$

Taking derivative at $t = 0^+$ gives

$$\deg_+(x) = -\Delta_{\mathscr{G}}\, g(x) = L^{\mathbb{N}} f(n) = \alpha_+(n).$$

This shows that for $x \in \mathscr{V}$, the quantity $\deg_+(x)$ depends only on $|x|$. Similarly, one has that $\deg_-(x)$ depends also only on $|x|$; that is $\mathscr{G}$ is weakly spherically symmetric. $\square$

Remark 9.3. — It is a remarkable fact that the quantity $\deg_0$ does not appear in the generator of the process $(|X_t|)_{t\geqslant 0}$ on a weakly spherically symmetric graph.

We now show that the bottom of the spectrum and the essential spectrum for the two Laplacians coincide.

THEOREM 9.4. — *Let $\mathscr{G} := (\mathscr{V}, \mathscr{E}, m)$ be a weakly symmetric weighted graph such that $m(\mathscr{V}) = +\infty$. With the above notation, we have*

$$\inf \sigma(\Delta_{\mathscr{G}}) = \inf \sigma(\Delta_{\mathscr{G}_{\mathbb{N}}}) \text{ and } \inf \sigma_{\mathrm{ess}}(\Delta_{\mathscr{G}}) = \inf \sigma_{\mathrm{ess}}(\Delta_{\mathscr{G}_{\mathbb{N}}}).$$

Proof. — We start with a general fact. Given $f : \mathbb{N} \to \mathbb{C}$, let $g : \mathscr{V} \to \mathbb{C}$ be defined by $g(x) := f(|x|)$, then for $x \in S_n$, observe that

$$\tilde{\Delta}_{\mathscr{G}}\, g(x) = \tilde{\Delta}_{\mathscr{G}_{\mathbb{N}}} f(n) \text{ and } \|g\|_{\ell^2(\mathscr{G}, m)} = \|f\|_{\ell^2(\mathbb{N}, m_{\mathbb{N}})}.$$

It follows easily that $\mathcal{D}(\Delta_{\mathscr{G}}) \cap (\mathbb{C}^{\mathscr{V}})_{\mathrm{rad}=\mathcal{D}(\Delta_{\mathscr{G}_{\mathscr{N}}})}$, where $(\mathbb{C}^{\mathscr{V}})_{\mathrm{rad}}$ denotes the set of radial (w.r.t. the 1-dimensional decomposition) functions $f : \mathscr{V} \to \mathbb{C}$. It easily follows that $\sigma(\Delta_{\mathscr{G}_{\mathbb{N}}}) \subset \sigma(\Delta_{\mathscr{G}})$ and $\sigma_{\mathrm{ess}}(\Delta_{\mathscr{G}_{\mathbb{N}}}) \subset \sigma_{\mathrm{ess}}(\Delta_{\mathscr{G}})$. This gives that $\inf \sigma(\Delta_{\mathscr{G}}) \leqslant \inf \sigma(\Delta_{\mathscr{G}_{\mathbb{N}}})$ and $\inf \sigma_{\mathrm{ess}}(\Delta_{\mathscr{G}}) \leqslant \inf \sigma_{\mathrm{ess}}(\Delta_{\mathscr{G}_{\mathbb{N}}})$.

For the reverse inequality, we do the proof only for the bottom of the essential spectrum. The proof for the bottom of the spectrum is similar and uses Theorem 6.1. Let $\lambda^0_{\mathbb{N},\mathrm{ess}} := \inf \sigma_{\mathrm{ess}}(\Delta_{\mathscr{G}_{\mathbb{N}}})$. By Theorem 6.2, for all $\varepsilon > 0$, there exist $n_0 := n_0(\varepsilon)$ and a positive function W on $\mathbb{N}$ such that

$$\tilde{\Delta}_{\mathscr{G}_{\mathbb{N}}} W(n) \geqslant (\lambda^0_{\mathbb{N},\mathrm{ess}} - \varepsilon)\mathbf{1}_{n \geqslant n_0} W(n).$$

Let $U : \mathscr{G} \to (0, \infty)$ be the function defined by $U(x) := W(|x|)$. Since $\mathscr{G}$ is a weakly symmetric graph, for $x \in S_n$, we have $\tilde{\Delta}_{\mathscr{G}} U(x) = \tilde{\Delta}_{\mathscr{G}_{\mathbb{N}}} W(n)$. Therefore,

$$\tilde{\Delta}_{\mathscr{G}} U(x) \geqslant (\lambda^0_{\mathbb{N},\mathrm{ess}} - \varepsilon)\mathbf{1}_{|x| \geqslant n_0} U(x).$$

Finally Theorem 3.6 and letting $\varepsilon \to 0$ give the conclusion. $\square$

We turn to the case of the normalized Laplacian. Note that for weakly spherically symmetric graphs, since $\deg \equiv 1$ and since $\deg_\pm$ are radial, $\deg_0$ is also radial. The next proposition is the discrete analogous of Proposition 9.2. The proof is straightforward. We omit it.

Proposition 9.5. — *Let $\mathscr{G} := (\mathscr{V}, \mathscr{E})$ be a graph and let $(S_n)_{n \geqslant 0}$ be a 1-dimensional decomposition of $\mathscr{G}$. Let $(X_k)_{k \in \mathbb{N}}$ be the discrete time Markov chain on $\mathscr{V}$ whose generator is $-\Delta_{\mathscr{G},\eta}$ (as defined in section 8.1). Then the graph $(\mathscr{V}, \mathscr{E}, \eta)$ is weakly spherically symmetric with respect to $(S_n)_{n \geqslant 0}$ if and only if the process $(|X_k|)_{k \in \mathbb{N}}$ is a Markov chain on $\mathbb{N}$.*

Moreover, in this case, the transition probabilities of the Markov chain $(|X_k|)_{k \in \mathbb{N}}$ on $\mathbb{N}$ are given by

$$\begin{cases} p(n, n+1) & := & p_+(n) \\ p(n, n-1) & := & p_-(n), \\ p(n, n) & := & p_0(n). \end{cases}$$

The generator of the Markov chain $(|X_k|)_{k \in \mathbb{N}}$ corresponds exactly to $-\tilde{\Delta}_{\mathscr{G}_{\mathbb{N}}}$ where $\mathscr{G}_{\mathbb{N}} := (\mathbb{N}, \mathscr{E}_{\mathbb{N}}, m_{\mathbb{N}})$ with

$$\mathscr{E}_{\mathbb{N}}(n, m) := \begin{cases} m(S_n)p_\pm(n), & \text{when } m = n \pm 1, \\ 0 & \text{otherwise.} \end{cases}$$
$$m_{\mathbb{N}}(n) := m(S_n).$$

For $f \in \mathcal{C}_c(\mathbb{N})$, $-\Delta_{\mathscr{G}_{\mathbb{N}}}$ can be written by as ,

$$-\Delta_{\mathscr{G}_{\mathbb{N}}} f(n) = p_+(n)(f(n+1) - f(n)) + p_-(n)(f(n-1) - f(n)). \quad (9.4)$$

We go back to the general setting and provide a more explicit construction of the super-harmonic function of Theorem 8.5.

Proposition 9.6. — *Assume the graph $\mathscr{G} := (\mathscr{V}, \mathscr{E}, m)$ is weakly spherically symmetric with respect to a 1-dimensional decomposition $(S_n)_{n\geqslant 0}$. Let $\lambda : \mathscr{V} \to [0,1)$ a radial function which satisfies one of the assertions of Theorem 8.5. Then the unique radial function W which satisfies $W(x_0) = 1$ for all $x_0 \in S_0$ and $\tilde{\Delta}_m W(x) = \lambda(x) W(x)$ is given by*

$$W(x) = \frac{\mathbb{E}_x \left[\exp\left(\int_0^{T_N} \lambda(X_s^x) ds\right)\right]}{\mathbb{E}_\nu \left[\exp\left(\int_0^{T_N} \lambda(X_s^\nu) ds\right)\right]}, \text{ for } |x| \leqslant N \text{ and } x_0 \in S_0,$$

where ν is any probability measure supported on S_0 and the hitting time $T_N := \inf\{t \geqslant 0, X_t^\nu \in B_N^c\}$ of the set B_N^c for the continuous Markov process $(X_t^\nu)_{t\geqslant 0}$ on $\mathscr{V}$ whose generator is $-\Delta_m$ and initial law is ν.

In particular, this function W is a non-increasing radial positive function.

Proof. — Actually, the only thing to prove is that the function W in the proposition is well-defined. For $N \geqslant 1$, consider the functions $W_N(x) := \mathbb{E}_x \left[\exp\left(\int_0^{T_N} \lambda(X_s^x) ds\right)\right]$. By hypothesis, these functions are well-defined. Since $\mathscr{G}$ is weakly spherically symmetric, Proposition 9.2 ensures that $(|X_t|)_{t\geqslant 0}$ is a continuous time Markov process. Therefore W_N is a radial function. Moreover, it is constant on S_0, thus we have $W_N(0) = \mathbb{E}_\nu \left[\exp\left(\int_0^{T_N} \lambda(X_s^\nu) ds\right)\right]$ for any probability measure ν supported on S_0.

Write also

$$\tilde{W}_N := \frac{W_N}{W_N(0)} \text{ and } \tilde{W}_{N+1} := \frac{W_{N+1}}{W_{N+1}(0)}$$

so that $\tilde{W}_N(0) = \tilde{W}_{N+1}(0) = 1$. Previous computations show that

$$\tilde{\Delta}_m \tilde{W}_{N+1}(x) = \lambda(x) \tilde{W}_{N+1}(x) \text{ for all } x \in B_N$$

and

$$\tilde{\Delta}_m \tilde{W}_N(x) = \lambda(x) \tilde{W}_N(x) \text{ for all } x \in B_N$$

By lemma 9.1, we have $\tilde{W}_{N+1}(x) = \tilde{W}_N(x)$ for all $x \in B_N$.

It follows that the function W in the proposition is well-defined and satisfies $W > 0$ and $\tilde{\Delta} W(x) = \lambda(x) W(x)$. It is clearly radial as a limit of radial functions and non-increasing by Lemma 9.1. □

Remark 9.7. — In the case of the normalized Laplacian, the function W in Proposition 9.6 can also be written as:

$$W(x) = \frac{\mathbb{E}_x[A_{T_N}]}{\mathbb{E}_\nu[A_{T_N}]} \text{ for } |x| \leqslant N$$

with $A_n := \prod_{k=0}^{n-1} \Lambda(X_k^x)$, $\Lambda(x) := \frac{1}{1-\lambda(x)}$, ν any probability measure on S_0, and $T_N := \inf\{n \geqslant 0, X_n^x \in B_N^c\}$ the hitting time of the set B_N^c for $(X_k^\nu)_{k\geqslant 0}$ the random walk on $\mathscr{V}$ whose generator is $-\Delta_\eta$ and initial law ν.

10. The bottom of the spectrum and of the essential spectrum

In this section, we compare the bottom of the spectrum and the essential spectrum of different weighted Laplacians. The idea here is to compare directly the associated stochastic Markov processes (see Proposition 10.1). We then obtain a general comparison result (see Theorem 10.4). This result is an important improvement of Theorem 4 in [21]

First, we provide a coupling between the Markov processes on two different weighted graphs.

Proposition 10.1. — *Let $\mathscr{G} := (\mathscr{V}^{\mathscr{G}}, \mathscr{E}^{\mathscr{G}}, m^{\mathscr{G}})$ and $\mathscr{H} := (\mathscr{V}^{\mathscr{H}}, \mathscr{E}^{\mathscr{H}}, m^{\mathscr{H}})$ be two weighted graphs. Let $(S_n^{\mathscr{G}})_{n\geqslant 0}$ and $(S_n^{\mathscr{H}})_{n\geqslant 0}$ be 1-dimensional decompositions for respectively $\mathscr{G}$ and $\mathscr{H}$. Let $x_0 \in \mathscr{G}$ and $y_0 \in \mathscr{H}$ be such that $|x_0|^{\mathscr{G}} = |y_0|^{\mathscr{H}}$. Let $(X_t)_{t\geqslant 0}$ and $(Y_t)_{t\geqslant 0}$ be the minimal continuous time Markov chains on $\mathscr{G}$ and $\mathscr{H}$ associated to $\Delta_{\mathscr{G}}$ and $\Delta_{\mathscr{H}}$, respectively and starting in x_0 and y_0, respectively.*

Assume that for all $n \geqslant 0, x \in S_n^{\mathscr{G}}, y \in S_n^{\mathscr{H}}$ there exist $\deg_{0,0}^{\mathscr{G}}(x) \geqslant 0$ *and* $\deg_{0,0}^{\mathscr{H}}(y) \geqslant 0$ *such that*

$$\tilde{p}_+^{\mathscr{G}}(x) \geqslant \tilde{p}_+^{\mathscr{H}}(y), \tilde{p}_-^{\mathscr{G}}(x) \leqslant \tilde{p}_-^{\mathscr{H}}(x) \text{ and } \widetilde{\deg}^{\mathscr{G}}(x) \geqslant \widetilde{\deg}^{\mathscr{H}}(y); \tag{10.1}$$

where

$$\widetilde{\deg}^{\mathscr{A}}(z) := \deg^{\mathscr{A}}(z) + \deg_{0,0}^{\mathscr{A}}(z) \text{ and } \tilde{p}_l^{\mathscr{A}}(z) := \frac{\deg_l^{\mathscr{A}}(z)}{\widetilde{\deg}^{\mathscr{A}}(z)},$$

for $z = x, y; l = +, -$ and $\mathscr{A} = \mathscr{G}, \mathscr{H}$. Then there exists a coupling of the processes $(X_t)_{t\geqslant 0}$ and $(Y_t)_{t\geqslant 0}$ such that, almost surely,

$$|X_{G_i}|^{\mathscr{G}} = |Y_{H_i}|^{\mathscr{H}},$$

$$|X_t|^{\mathscr{G}} \geqslant |Y_s|^{\mathscr{H}} \text{ for } t \in [G_i, G_{i+1}[, s \in [H_i, H_{i+1}[, \ i \geqslant 0,$$

where $[G_i, G_{i+1}[$ and $[H_i, H_{i+1}[$ are random intervals such that, almost surely, $G_i \to e(X), H_i \to e(Y)$ as $i \to +\infty$ where $e(\cdot)$ denotes the explosion time of the minimal markov chains and

$$H_{i+1} - H_i \geqslant G_{i+1} - G_i \geqslant 0,\ i \geqslant 0.$$

Since $G_0 = H_0 = 0$, almost surely, we have

$$e(Y) \geqslant e(X), \tag{10.2}$$

$$T_N(X) \leqslant T_N(Y) \tag{10.3}$$

where $T_N(Z) := \inf\{t \geqslant 0, |Z_t|^{\mathscr{A}} > N\}$, $\mathscr{A} = \mathscr{G}, \mathscr{H}; Z_t = X_t, Y_t$; and

$$L_N^X(n) \leqslant L_N^Y(n),\ 1 \leqslant n \leqslant N \tag{10.4}$$

where $L_N^Z(n) := \int_0^{T_N(Z)} \mathbf{1}_{S_n^{\mathscr{A}}}(Z_s)ds$, $Z_t = X_t, Y_t; \mathscr{A} = \mathscr{G}, \mathscr{H}$; is the time spent in the sphere $S_n^{\mathscr{A}}$ by the process $(Z_t)_{t\geqslant 0}$ before it reaches $S_{N+1}^{\mathscr{A}}$.

Proof. — We proceed by induction on $i \geqslant 0$. Assume $X_{G_i} = x, Y_{H_i} = y$ with $|x|^{\mathscr{G}} = |y|^{\mathscr{H}}$. Let us add the artificial jumps $\deg_{0,0}^{\mathscr{G}}(x)$ and $\deg_{0,0}^{\mathscr{H}}(y)$. Consider G an independent exponential random variable of parameter $\widetilde{\deg}^{\mathscr{G}}(x)$ and set $H := \frac{\widetilde{\deg}^{\mathscr{G}}(x)}{\widetilde{\deg}^{\mathscr{H}}(y)}G$. H is thus an exponential random variable of parameter $\widetilde{\deg}^{\mathscr{H}}(y)$. Clearly by construction $H \geqslant G$. Moreover, we can couple X_t and Y_s in such a way that after the jumps

$$|X_{G_i+G}|^{\mathscr{G}} \geqslant |Y_{H_i+H}|^{\mathscr{H}}.$$

The construction will be explained below. We then set $G_{i+1} := G_i + G$. If $|Y_{H_i+H}|^{\mathscr{H}} = |X_{G_{i+1}}|^{\mathscr{G}}$ we set $H_{i+1} = H_i + H$. Otherwise if $|Y_{H_i+H}|^{\mathscr{H}} < |X_{G_{i+1}}|^{\mathscr{G}}$, we freeze the process X in $X_{G_{i+1}}$ and let evolve independently the process Y until the time s' defined by $s' := \inf\{u \geqslant H_i+H, |Y_u|^{\mathscr{H}} = |X_t|^{\mathscr{G}}\}$. s' is thus the hitting time of the sphere $S_{|X_t|^{\mathscr{G}}}^{\mathscr{H}}$. Since B_N is a finite set s' is finite almost surely. We then set $G_{i+1} = s'$.

Now we turn to the construction of the coupling of the jumps. Label the neighbors of x and y by $x_1, \ldots, x_r$ and $y_1, \ldots, y_{r'}$ in such a way that $|x_k|^{\mathscr{G}}$ and $|y_k|^{\mathscr{H}}$ are non-increasing with k. Note that if $\deg_{0,0}^{\mathscr{A}}(z) > 0$ then z is a neighbor of z, $\mathscr{A} = \mathscr{G}, \mathscr{H}; z = x, y$. Let U be an independent random variable with uniform law on $[0,1]$. Set $X_{G_i+G} = x_j$ and $Y_{H_i+H} = y'_j$ where j and j' are the unique integer in $\{1, \ldots, r\}$ and $\{1, \ldots, r'\}$ such that

$$\tilde{p}^{\mathscr{G}}(x, x_1) + \cdots + \tilde{p}^{\mathscr{G}}(x, x_{j-1}) \leqslant U < \tilde{p}^{\mathscr{G}}(x, x_1) + \cdots + \tilde{p}^{\mathscr{G}}(x, x_j)$$

and

$$\tilde{p}^{\mathscr{H}}(y, y_1) + \cdots + \tilde{p}^{\mathscr{H}}(y, y_{j'-1}) \leqslant U < \tilde{p}^{\mathscr{H}}(y, y_1) + \cdots + \tilde{p}^{\mathscr{H}}(y, y_{j'}).$$

Since by hypothesis $\tilde{p}_+^{\mathscr{G}}(x) \geqslant \tilde{p}_+^{\mathscr{H}}(y)$ and $\tilde{p}_-^{\mathscr{G}}(x) \leqslant \tilde{p}_-^{\mathscr{H}}(x)$, it is clear that

$$|X_{G_{i+1}}|^{\mathscr{G}} \geqslant |Y_{H_i+H}|^{\mathscr{H}}.$$

By using Lemma 8.2, $(X_t)_{t\geqslant 0}$ and $(Y_s)_{s\geqslant 0}$ are the Markov processes associated to $\Delta_{\mathscr{G}}$ and $\Delta_{\mathscr{H}}$. The other statements are then immediate. □

Actually, there is a simpler characterization of condition (10.1).

Definition 10.2. — *Let $\mathscr{G} := (\mathscr{V}^{\mathscr{G}}, \mathscr{E}^{\mathscr{G}}, m^{\mathscr{G}})$ and $\mathscr{H} := (\mathscr{V}^{\mathscr{H}}, \mathscr{E}^{\mathscr{H}}, m^{\mathscr{H}})$ be two weighted graphs. We say that $\mathscr{G}$ has a* stronger weak-curvature growth *than $\mathscr{H}$ if*

$$\deg_+^{\mathscr{G}}(x) \geqslant \deg_+^{\mathscr{H}}(y) \text{ and } \frac{\deg_+^{\mathscr{G}}(x)}{\deg_-^{\mathscr{G}}(x)} \geqslant \frac{\deg_+^{\mathscr{H}}(y)}{\deg_-^{\mathscr{H}}(y)} \tag{10.5}$$

for $x \in \mathscr{V}^{\mathscr{G}}, y \in \mathscr{V}^{\mathscr{H}}$, $|x|^{\mathscr{G}} = |y|^{\mathscr{H}}$.

Proposition 10.3. — *Let $\mathscr{G} := (\mathscr{V}^{\mathscr{G}}, \mathscr{E}^{\mathscr{G}}, m^{\mathscr{G}})$ and $\mathscr{H} := (\mathscr{V}^{\mathscr{H}}, \mathscr{E}^{\mathscr{H}}, m^{\mathscr{H}})$ be two weighted graphs. Then* (10.1) *holds true if and only if $\mathscr{G}$ has a stronger weak-curvature growth than $\mathscr{H}$.*

Proof. — Indeed, condition (10.1) in Proposition 10.1 is equivalent to: there exist $z_1 \geqslant \deg^{\mathscr{G}}(x)$ and $z_2 \geqslant \deg^{\mathscr{H}}(y)$ such that

$$\frac{\deg_+^{\mathscr{G}}(x)}{z_1} \geqslant \frac{\deg_+^{\mathscr{H}}(y)}{z_2}, \ \frac{\deg_-^{\mathscr{G}}(x)}{z_1} \leqslant \frac{\deg_-^{\mathscr{H}}(y)}{z_2} \text{ and } z_1 \geqslant z_2.$$

The above line is equivalent to

$$\max\left(\frac{\deg_-^{\mathscr{G}}(x)}{\deg_-^{\mathscr{H}}(y)}, 1\right) \leqslant \frac{z_1}{z_2} \leqslant \frac{\deg_+^{\mathscr{G}}(x)}{\deg_+^{\mathscr{H}}(y)}$$

Therefore condition (10.1) implies condition (10.5). Reciprocally, if condition (10.5) holds, then it is possible to find $\alpha \geqslant 1$ such that

$$\max\left(\frac{\deg_-^{\mathscr{G}}(x)}{\deg_-^{\mathscr{H}}(y)}, 1\right) \leqslant \alpha \leqslant \frac{\deg_+^{\mathscr{G}}(x)}{\deg_+^{\mathscr{H}}(y)}$$

It is then easy to see that one can choose $z_1 \geqslant \deg^{\mathscr{G}}(x)$ and $z_2 \geqslant \deg^{\mathscr{H}}(y)$ in such a way that $\frac{z_1}{z_2} = \alpha$. □

We then state a general comparison result for the bottom of the spectra.

THEOREM 10.4. — *Let $\mathscr{G} := (\mathscr{V}^{\mathscr{G}}, \mathscr{E}^{\mathscr{G}}, m^{\mathscr{G}})$ and $\mathscr{H} := (\mathscr{V}^{\mathscr{H}}, \mathscr{E}^{\mathscr{H}}, m^{\mathscr{H}})$ two weighted graphs. Assume $\mathscr{G}$ has stronger weak-curvature growth than $\mathscr{H}$, then*

$$\inf \sigma(\Delta_{\mathscr{G}}) \geqslant \inf \sigma(\Delta_{\mathscr{H}}).$$

If moreover $\inf\{m^{\mathscr{H}}(x), x \in \mathscr{V}^{\mathscr{H}}\} > 0$ *or $\mathscr{H}$ is a weakly symmetric graph and $m^{\mathscr{H}}(\mathscr{V}^{\mathscr{H}}) = +\infty$, then*

$$\inf \sigma_{\mathrm{ess}}(\Delta_{\mathscr{G}}) \geqslant \inf \sigma_{\mathrm{ess}}(\Delta_{\mathscr{H}}).$$

Proof. — We keep the notation of the proof of Proposition 10.1. Let $(X_t^x)_{t\geqslant 0}$ and $(Y_t^y)_{t\geqslant 0}$ be the coupled continuous time Markov chains of generator $\Delta_{\mathscr{G}}$ and $\Delta_{\mathscr{H}}$ starting in $x \in \mathscr{V}^{\mathscr{G}}$ and $y \in \mathscr{V}^{\mathscr{H}}$ such that $|x|^{\mathscr{G}} = |y|^{\mathscr{H}}$, respectively.

Let $\lambda^0_{\mathscr{H},\mathrm{ess}} := \inf \sigma_{\mathrm{ess}}(\Delta_{\mathscr{H}})$ and let $\varepsilon > 0$. With the hypothesis in Theorem 10.4, by Theorem 6.2, there exist $n_0 := n_0(\varepsilon)$ and a positive function W on $\mathscr{V}^{\mathscr{H}}$ such that,

$$\tilde{\Delta}_{\mathscr{H}} W(x) \geqslant \lambda(|x|^{\mathscr{H}}) W(x).$$

where $\lambda(n) := (\lambda^0_{\mathscr{H},\mathrm{ess}} - \varepsilon)\mathbf{1}_{n\geqslant n_0}, n \geqslant 0$. The probabilistic representation of Theorem 8.5 of super-harmonic functions gives that for all $N \geqslant 1$,

$$\mathbb{E}\left[\exp\left(\int_0^{T_N^{\mathscr{H}}} \lambda(|Y_s^y|)ds\right)\right] < +\infty.$$

By Proposition 10.1, for $N \geqslant 1$ we have:

$$L_N^{X^x}(n) \leqslant L_N^{Y^y}(n), \quad 0 \leqslant n \leqslant N.$$

By noticing that

$$\mathbb{E}\left[\exp\left(\int_0^{T_N^{Z^z}} \lambda(|Z_s^z|)ds\right)\right] = \mathbb{E}\left[\exp\left(\sum_{k=0}^{N} \lambda(n) L_N^{Z^z}(n)\right)\right],$$

with $Z^z = X^x$ or Y^y, this yields:

$$\mathbb{E}\left[\exp\left(\int_0^{T_N^{\mathscr{G}}} \lambda(|X_s^x|)ds\right)\right] \leqslant \mathbb{E}\left[\exp\left(\int_0^{T_N^{\mathscr{H}}} \lambda(|Y_s^y|)ds\right)\right] < +\infty.$$

Then the probabilistic representation of Theorem 8.5 gives the existence of a positive super-harmonic $\tilde{W}$ on $\mathscr{V}^{\mathscr{G}}$ satisfying $\tilde{\Delta}_{\mathscr{G},m}\tilde{W}(x) \geqslant \lambda(|x|)\tilde{W}(x), x \in \mathscr{V}$. Theorem 3.6 and letting $\varepsilon \to 0$ finally imply

$$\inf \sigma_{\mathrm{ess}}(\Delta_{\mathscr{G}}) \geqslant \inf \sigma_{\mathrm{ess}}(\Delta_{\mathscr{H}}).$$

The proof for the bottom of the spectrum is similar. $\square$

Remark 10.5. — Theorem 10.4 is an improvement of [21, Theorem 4] in three directions. First, contrary to the latter, our result applies also to the bottom of the essential spectrum. Second, they suppose that one of the graph is weakly spherically symmetric. Third and mainly, our hypothesis is weaker, even for the comparison of the bottom of spectra. Therefore, the condition called *stronger curvature growth* introduced in [21] and which can be written as

$$\deg_+^{\mathscr{G}}(x) \geqslant \deg_+^{\mathscr{H}}(y) \text{ and } \deg_-^{\mathscr{G}}(x) \leqslant \deg_-^{\mathscr{H}}(y),\ x \in \mathscr{V}^{\mathscr{G}}, y \in \mathscr{V}^{\mathscr{H}},\ |x|^{\mathscr{G}} = |y|^{\mathscr{H}} \tag{10.6}$$

is not the optimal one to compare the bottom of the spectrum.

Remark 10.6. — When $\mathscr{H}$ is a weakly spherically symmetric graph, we give a direct proof of Theorem 10.4. Let $\varepsilon > 0$. By Theorem 6.2, there exist $n_0 := n_0(\varepsilon)$ and a positive non-increasing function W on $\mathbb{N}$ such that,

$$\tilde{\Delta}_{\mathscr{H}} W(|y|^{\mathscr{H}}) \geqslant \lambda(|y|^{\mathscr{H}}) W(|y|^{\mathscr{H}}).$$

where $\lambda(n) := (\lambda^0_{\mathscr{H},\mathrm{ess}} - \varepsilon)\mathbf{1}_{n \geqslant n_0}, n \geqslant 0$. For $\mathscr{A} = \mathscr{G}, \mathscr{H}$, let $\Delta_{\mathrm{bd},\mathscr{A}}$ be defined by

$$\Delta_{\mathrm{bd},\mathscr{A}} := \frac{1}{\widetilde{\deg}^{\mathscr{A}}(\cdot)} \Delta_{\mathscr{A}}.$$

For f a radial function on $\mathscr{H}$, we have

$$\tilde{\Delta}_{\mathrm{bd},\mathscr{H}} f(n) := \tilde{p}_+^{\mathscr{H}}(n)(f(n) - f(n+1)) + \tilde{p}_-^{\mathscr{H}}(n)(f(n) - f(n-1)).$$

Therefore, for $x \in \mathscr{V}^{\mathscr{G}}$, $y \in \mathscr{V}^{\mathscr{H}}$ such that $|x|^{\mathscr{G}} = |y|^{\mathscr{H}} = n$,

$$\begin{aligned}\tilde{\Delta}_{\mathscr{H}} W(|y|^{\mathscr{H}}) &= \widetilde{\deg}^{\mathscr{H}}(y) \tilde{\Delta}_{\mathrm{bd},\mathscr{H}} W(|y|^{\mathscr{H}}) \leqslant \widetilde{\deg}^{\mathscr{G}}(x) \tilde{\Delta}_{\mathrm{bd},\mathscr{G}} W(|x|^{\mathscr{G}}) \\ &= \tilde{\Delta}_{\mathscr{G}} W(|x|^{\mathscr{G}}).\end{aligned}$$

Indeed we have $\widetilde{\deg}^{\mathscr{H}}(y) \leqslant \widetilde{\deg}^{\mathscr{G}}(x)$ and $\tilde{\Delta}_{\mathrm{bd},\mathscr{H}} W(|y|^{\mathscr{H}}) \leqslant \tilde{\Delta}_{\mathrm{bd},\mathscr{G}} W(|x|^{\mathscr{G}})$ since $\tilde{p}_+^{\mathscr{G}}(x) \geqslant \tilde{p}_+^{\mathscr{H}}(y)$, $\tilde{p}_-^{\mathscr{G}}(x) \leqslant \tilde{p}_-^{\mathscr{H}}(y)$ and W is non-increasing. The end of the proof is now the same as before.

Here, we first recall that

$$\inf \sigma(\Delta_{\mathcal{T}_d,1}) = \inf \sigma_{\mathrm{ess}}(\Delta_{\mathcal{T}_d,1}) = d + 1 - 2\sqrt{d}$$

and

$$\inf \sigma(\Delta_{\mathcal{T}_d,\eta}) = \inf \sigma_{\mathrm{ess}}(\Delta_{\mathcal{T}_d,\eta}) = 1 - \frac{2\sqrt{d}}{d+1}.$$

where $\mathcal{T}_d$ denotes the simple d-ary tree.

Example 10.7. — Let $\mathscr{G} := (\mathscr{V}, \mathscr{E})$ be a simple tree such that each vertex x satisfies $\eta_+(x) \in \{\alpha, \beta\}$ with $\alpha \leqslant \beta$, see Figure **??**. A direct application of Theorem 10.4 gives

$$\inf \sigma(\Delta_{\mathcal{T}_\alpha,1}) \leqslant \inf \sigma(\Delta_{\mathscr{G},1}) \leqslant \inf \sigma_{\mathrm{ess}}(\Delta_{\mathscr{G},1}) \leqslant \inf \sigma_{\mathrm{ess}}(\Delta_{\mathcal{T}_\beta,1}) = \inf \sigma(\Delta_{\mathcal{T}_\beta,1})$$

and

$$\inf \sigma(\Delta_{\mathcal{T}_\alpha,\eta}) \leqslant \inf \sigma(\Delta_{\mathscr{G},\eta}) \leqslant \inf \sigma_{\mathrm{ess}}(\Delta_{\mathscr{G},\eta}) \leqslant \inf \sigma_{\mathrm{ess}}(\Delta_{\mathcal{T}_\beta,\eta}) = \inf \sigma(\Delta_{\mathcal{T}_\beta,\eta}),$$

where $\mathcal{T}_\alpha$ (resp. $\mathcal{T}_\beta$) denotes the simple α-ary (resp. β-ary) tree. Only the part with $\Delta_{\mathscr{G},1}$ was covered by [21, Theorem 4].

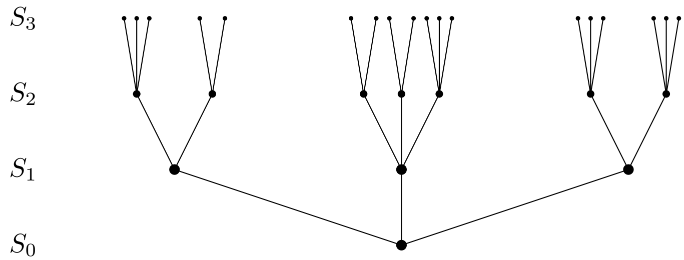

Figure 4. — Tree with 2 or 3 sons at each generation

Note that in the setting of the previous example, it is possible to have:

$$\inf \sigma(\Delta_{\mathscr{G},\eta}) < \inf \sigma_{\mathrm{ess}}(\Delta_{\mathscr{G},\eta})$$

and an infinite number of eigenvalues below the essential spectrum can occur (see [28]).

As a corollary of Theorem 10.4, we extend Corollary 6.7 in [21] to the case of the essential spectrum.

Corollary 10.8. — *Let $\mathscr{G}$ and $\mathscr{H}$ be two weakly spherically symmetric graphs which have the* same curvature growth *in the sense of [21]; that is:*

$$\deg_+^{\mathscr{G}}(x) = \deg_+^{\mathscr{H}}(y) \text{ and } \deg_-^{\mathscr{G}}(x) = \deg_-^{\mathscr{H}}(y),\ x \in \mathscr{V}^{\mathscr{G}}, y \in \mathscr{V}^{\mathscr{H}},\ |x|^{\mathscr{G}} = |y|^{\mathscr{H}}.$$

Then

$$\inf \sigma(\Delta_{\mathscr{G}}) = \inf \sigma(\Delta_{\mathscr{H}}) \text{ and } \inf \sigma_{\mathrm{ess}}(\Delta_{\mathscr{G}}) = \inf \sigma_{\mathrm{ess}}(\Delta_{\mathscr{H}}).$$

In particular, on a fixed simple weakly spherically symmetric graph, the bottoms of the spectrum and the essential spectrum of the combinatorial Laplacian Δ_1 do not change if one adds or removes edges inside the spheres S_n.

Example 10.9. — Let $\mathscr{G} := (\mathscr{V}, \mathscr{E})$ be a simple infinite bipartite graph and $x_0 \in \mathscr{V}$ such that $\eta_+(x_0) = 2, \eta_-(x_0) = 0$ and

$$(\eta_+(x) = 2, \eta_-(x) = 1) \text{ or } (\eta_+(x) = 4, \eta_-(x) = 2),\ x \in \mathscr{V}, x \neq x_0. \quad (10.7)$$

Then $(\mathscr{V}, \mathscr{E}, \eta)$ is a weakly spherically symmetric graph. Thus Corollary 10.8 gives that:

$$\inf \sigma(\Delta_{\mathcal{T}_2,\eta}) = \inf \sigma(\Delta_{\mathscr{G},\eta}) = \inf \sigma_{\mathrm{ess}}(\Delta_{\mathscr{G},\eta}) = \inf \sigma_{\mathrm{ess}}(\Delta_{\mathcal{T}_2,\eta}).$$

and Theorem 10.4 that:

$$\inf \sigma(\Delta_{\mathcal{T}_2,1}) \leqslant \inf \sigma(\Delta_{\mathscr{G},1}) \leqslant \inf \sigma_{\mathrm{ess}}(\Delta_{\mathscr{G},1}) \leqslant 2 \inf \sigma_{\mathrm{ess}}(\Delta_{\mathcal{T}_2,1}).$$

Note that, if moreover one supposes that

$$\min\{\deg_+(x), x \in S_n\} \leqslant \max\{\deg_-(x), x \in S_n\},$$

a direct application of [21, Theorem 4] gives only that $\inf \sigma(\Delta_{\mathscr{G},1}) \geqslant 0$.

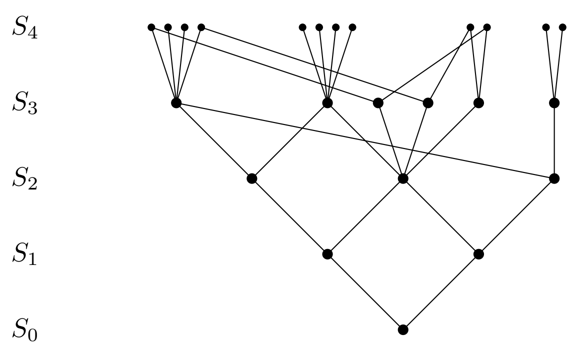

Figure 5. — A weakly spherically symmetric graph satisfying (10.7)

11. Stochastic completeness

By definition, a graph $\mathscr{G} := (\mathscr{V}^{\mathscr{G}}, \mathscr{E}^{\mathscr{G}}, m^{\mathscr{G}})$ is said to be *stochastically complete* if for all $x \in \mathscr{V}$, $\mathbb{P}(e(X^x) < +\infty) = 0$ where X^x is the minimal right continuous Markov process constructed in section 8.1. Otherwise, it is said to be *stochastically incomplete*. First we consider weakly spherically symmetric graph. The following result was already included in Theorem 5 in [21] (except that we slightly generalize their notion of spherically symmetric graphs). We provide a direct stochastic proof.

Theorem 11.1. — *Let* $\mathscr{G} := (\mathscr{V}^{\mathscr{G}}, \mathscr{E}^{\mathscr{G}}, m^{\mathscr{G}})$ *be a* weakly spherically symmetric graph *and let* $\mathscr{G}_{\mathbb{N}} := (\mathbb{N}, \mathscr{E}_{\mathbb{N}}, m_{\mathbb{N}})$ *where* $\mathscr{E}_{\mathbb{N}}$ *and* $m_{\mathbb{N}}$ *are defined as in Proposition 9.2. Then* $\mathscr{G}$ *is* stochastically complete *if and only if* $\mathscr{G}_{\mathbb{N}}$ *is.*

Proof. — Let $n \geqslant 0$ and $x \in S_n$ and let $(X_t^x)_{t\geqslant 0}$ be the minimal right continuous Markov process associated to $\mathscr{G}$. By Proposition 9.2, $(|X_t^x|)_{t\geqslant 0}$ is the minimal right continuous Markov process associated to $\mathscr{G}_{\mathbb{N}}$ and starting in n. Clearly, one has $e(X^x) = e(|X^x|)$. The conclusion of the theorem follows by the definition of stochastic completeness. □

The coupling argument of Proposition 10.1 implies the following comparison result. It is an improvement of Theorem 6 in[21].

Theorem 11.2. — *Let* $\mathscr{G} := (\mathscr{V}^{\mathscr{G}}, \mathscr{E}^{\mathscr{G}}, m^{\mathscr{G}})$ *and* $\mathscr{H} := (\mathscr{V}^{\mathscr{H}}, \mathscr{E}^{\mathscr{H}}, m^{\mathscr{H}})$ *two weighted graphs. Assume* $\mathscr{G}$ *has stronger weak-curvature growth than* $\mathscr{H}$*. If* $\mathscr{H}$ *is stochastically incomplete then so is* $\mathscr{G}$*. If* $\mathscr{G}$ *is stochastically complete then so is* $\mathscr{H}$*.*

Proof. — Let $n \geqslant 0$, $x \in S_n(\mathscr{G})$ and $y \in S_n(\mathscr{H})$. Proposition 10.1 provides a coupling $(X_t^x)_{t\geqslant 0}$ and $(Y_t^y)_{t\geqslant 0}$ of the two minimal right continuous Markov chains on $\mathscr{G}$ and $\mathscr{H}$ starting in x and y, respectively, such that $e(X^x) \leqslant e(Y^y)$. The conclusion of the theorem follows by the definition of stochastic completeness. □

Appendix

A. The Friedrichs extension

In this section, we recall the construction of the Friedrichs extension of a positive symmetric densely defined operator. Given a dense subspace $\mathscr{D}$ of a Hilbert space $\mathscr{H}$ and a non-negative symmetric operator H on $\mathscr{D}$, let $\mathscr{H}_1$ be the completion of $\mathscr{D}$ under the norm given by $\mathscr{Q}(\varphi)^2 = \langle H\varphi, \varphi\rangle + \|\varphi\|^2$. The domain of the Friedrichs extension of H is given by

$$\begin{aligned}\mathcal{D}(H^{\mathscr{F}}) &= \{f \in \mathscr{H}_1 \mid \mathscr{D} \ni g \mapsto \langle Hg, f\rangle + \langle g, f\rangle \text{ extends to a norm} \\ &\qquad \text{continuous function on } \mathscr{H}\} \\ &= \mathscr{H}_1 \cap \mathcal{D}(H^*).\end{aligned}$$

For each $f \in \mathcal{D}(H^{\mathscr{F}})$, there is a unique u_f such that $\langle Hg, f\rangle + \langle g, f\rangle = \langle g, u_f\rangle$, by Riesz' Theorem. The Friedrichs extension of H, is given by $H^{\mathscr{F}} f := u_f - f$. It is a self-adjoint extension of H, e.g., [26, Theorem X.23].

We now describe the domain of the adjoint of the discrete Laplacian. This is well-known, e.g., [8, 20]. Let $\mathscr{G} = (\mathscr{V}, \mathscr{E}, m)$ be a weighted graph. We have:

$$\mathcal{D}\big((\Delta_{\mathscr{G}}|_{\mathcal{C}_c(\mathscr{V})})^*\big) = \Big\{f \in \ell^2(\mathscr{V}, m),\\ x \mapsto \frac{1}{m(x)} \sum_{y\in\mathscr{V}} \mathscr{E}(x,y)(f(x)-f(y)) \in \ell^2(\mathscr{V}, m)\Big\}.$$

Then, given $f \in \mathcal{D}((\Delta_{\mathscr{G}}|_{\mathcal{C}_c(\mathscr{V})})^*)$, one has:

$$\big((\Delta_{\mathscr{G}}|_{\mathcal{C}_c(\mathscr{V})})^* f\big)(x) = \frac{1}{m(x)} \sum_{y\in\mathscr{V}} \mathscr{E}(x,y)(f(x)-f(y)),$$

for all $x \in \mathscr{V}$. We recall that $\mathscr{H}_1$ here is the completion of $\mathcal{C}_c(\mathscr{V})$ under the norm:

$$\|f\|^2_{\mathscr{H}_1} := \frac{1}{2} \sum_{x,y\in\mathscr{V}} \mathscr{E}(x,y)|f(x)-f(y)|^2 + \|f\|^2.$$

By definition, the operator $\Delta_{\mathscr{G}}|_{\mathcal{C}_c(\mathscr{V})}$ is *essentially self-adjoint* if its closure is equal to its adjoint. A review of recent developments of essential self-adjointness may be found in [16].

B. Lyapunov functions, Super-Poincaré inequality and the infimum of the essential spectrum

In this section, we explain how to use the above Lyapunov functions so as to obtain a lower bound on the infimum of the essential spectrum and in some cases its emptiness. This is a straightforwardly adaptation of the continuous setting. In our discrete setting, this approach is not strictly necessary to obtain our results (see Remark B.7) on the essential spectra. We have included it for the sake of completeness and because it provides some insights and a characterization for the lower bound of the essential spectrum. We shall rely on the following Super-Poincaré Inequality, which was introduced by Wang (see [30, 31, 33]).

Definition B.1. — *We say that* Super-Poincaré Inequality of parameter $s_0 \in \mathbb{R}$ *holds true, if there is a function* $h : \mathscr{V} \to (0,\infty)$ *such that* $m(|h|^2) = 1$ *and some positive non-increasing functions* $\beta_h : (s_0, \infty) \to (0,\infty)$*, such that*

$$SPI\ (s_0) \qquad m(|f|^2) \leqslant s\, m\big(\overline{f}\, \Delta_m f\big) + \beta_h(s)\, m(|f|h)^2 \text{ for all } s > s_0 \quad \text{(B.1)}$$

and $f \in \mathcal{C}_c(\mathscr{V})$.

Where, by abuse of notation, we wrote:

$$m(f) := \sum_{x\in\mathscr{V}} f(x)m(x).$$

Remark B.2. — If SPI (s_0) holds for some not necessarily non-increasing function β_h, then, for any $\mu > s_0$, SPI (μ) holds for the non-increasing function $\beta'_h(s) = \inf_{\mu\leqslant s}\beta_h(s)$.

We start by showing how the existence of Lyapunov functions implies SPI.

THEOREM B.3. — *Take ψ a non-negative function, $b \geqslant 0$, and some finite set B_{r_0}. Denote by*

$$\Psi(r) = \inf\{\psi(x), x \in B_r^c\} \text{ and } s_0 = \left(\lim_{r\to\infty}\Psi(r)\right)^{-1} \in [0,\infty).$$

Suppose that $\Psi(r) > 0$ for r large enough and that W is a positive function such that

$$\Delta_m W \geqslant \psi \times W - b\mathbf{1}_{B_{r_0}} \tag{B.2}$$

Let $h : \mathscr{V} \to (0,\infty)$ such that $\|h\|_m = 1$ and let $a_h(r) = \inf\{h(x)^2 m(x), x \in B_r\}$. Then SPI (s_0) holds true with

$$\beta_h(s) = \left(1 + \frac{bs}{\alpha_{r_0}}\right)\frac{1}{a_h(\Psi^{-1}(\frac{1}{s}))},$$

where $\alpha_{r_0} = \inf\{W(x), x \in B_{r_0}\}$.

We follow the proof in [6, Section 2].

Proof. — Set $r \geqslant r_0$ such that $\Psi(r) > 0$,

$$\begin{aligned}
m(|f|^2) =& m(\frac{|f|^2\psi}{\psi}\mathbf{1}_{B_r^c}) + m(|f|^2\mathbf{1}_{B_r}) \leqslant \frac{1}{\Psi(r)}m(|f|^2\psi\mathbf{1}_{B_r^c}) + m(|f|^2\mathbf{1}_{B_r})\\
\leqslant& \frac{1}{\Psi(r)}m(|f|^2\psi) + m(|f|^2\mathbf{1}_{B_r})\\
\leqslant& \frac{1}{\Psi(r)}m\left(|f|^2\left(\frac{\Delta_m W}{W} + \frac{b\mathbf{1}_{B_{r_0}}}{W}\right)\right) + m(|f|^2\mathbf{1}_{B_r})\\
\leqslant& \frac{1}{\Psi(r)}m\left(|f|^2\left(\frac{\Delta_m W}{W}\right)\right) + \left(\frac{b}{\Psi(r)\inf\{W(x), x\in B_{r_0}\}} + 1\right)m(|f|^2\mathbf{1}_{B_r}).
\end{aligned}$$

We concentrate on the second term. Let $h : \mathscr{V} \to (0,\infty)$ such that $m(h^2) = 1$. Since the set B_r is finite,

$$\begin{aligned} m(|f|h\mathbf{1}_{B_r})^2 = \left(\sum_{x\in B_r} |f(x)|h(x)m(x)\right)^2 &\geqslant \sum_{x\in B_r} |f(x)|^2 h(x)^2 m(x)^2 \\ &\geqslant \left(\inf_{x\in B_r} h(x)^2 m(x)\right) m(|f|^2\mathbf{1}_{B_r}) \end{aligned}$$

which gives the following local Super-Poincaré Inequality:

$$m(|f|^2\mathbf{1}_{B_r}) \leqslant \left(\inf_{x\in B_r} m(x)h(x)^2\right)^{-1} m(|f|h)^2.$$

Therefore, by combining the above estimate and the Hardy inequality (3.1), we get:

$$\begin{aligned} m(|f|^2) \leqslant &\frac{1}{\Psi(r)} m(\overline{f}\Delta_m f) \\ &+ \left(\frac{b}{\Psi(r)\inf\{W(x), x\in B_{r_0}\}} + 1\right)\left(\inf_{x\in B_r} m(x)h(x)^2\right)^{-1} m(|f|h)^2. \end{aligned}$$

Finally, this yields the SPI (s_0) with $\beta_h(s)$ defined as in the theorem. □

Remark B.4. — Note that if the constant $b = 0$ in the above, the function $\beta_h(s)$ is then given by

$$\beta_h(s) = \frac{1}{a_h(\Psi^{-1}(\frac{1}{s}))}.$$

We turn now to the equivalence between the Super-Poincaré Inequality and the infimum of the essential spectrum of the operator (see Theorem 2.2 in [33]).

Theorem B.5. — *Let $s_0 > 0$. Then the following assertions are equivalent:*

(a) $\sigma_{\mathrm{ess}}(\Delta_m) \subset [\frac{1}{s_0}, \infty)$.

(b) There exists a positive function h such that $m(h^2) = 1$ and a non-increasing function $\beta_h : (s_0,\infty) \to (0,\infty)$ such that (B.1) holds for all $s > s_0$.

(c) For any positive function h such that $m(h^2) = 1$, there exists a non-increasing function $\beta_h : (s_0,\infty) \to (0,\infty)$ such that (B.1) holds for all $s > s_0$.

In particular, $\sigma_{\mathrm{ess}}(\Delta_m) = \emptyset$ if and only if there exists some functions h and β_h for which (B.1) holds for $s > 0$.

For the seek of completeness, we will give the proof of this result. We follow the proof of Theorem 3.2 in [31] which was originally made in the case of probability measure. The slight difference comes from the fact that we consider the Friedrichs extension and we are not in a essentially self-adjoint setting.

Proof. — It is is clear that (c) implies (b). We show first that (b) implies (a). Let h be the positive function such that $m(h^2) = 1$ and let $f \in \mathcal{C}_c(\mathscr{V})$. Let $0 < \varepsilon < 1$, $\eta > 0$ and let B_r such that $m(\mathbf{1}_{B_r^c} h^2) \leqslant \varepsilon$, then, for $f \in \mathcal{C}_c(\mathscr{V})$ such that $m(|f|^2) = 1$ and $f|_{B_r} = 0$,

$$\begin{aligned} 1 = m(|f|^2) &\leqslant (s_0+\eta)\langle f, \Delta_m f\rangle_m + \beta(s_0+\eta) m(|f|h)^2 \\ &\leqslant (s_0+\eta)\langle f, \Delta_m f\rangle_m + \beta(s_0+\eta) m(|f|^2) m(\mathbf{1}_{B_r^c} h^2) \\ &\leqslant (s_0+\eta)\langle f, \Delta_m f\rangle_m + \beta(s_0+\eta)\varepsilon. \end{aligned}$$

Using (2.2), we get

$$\inf \sigma_{\mathrm{ess}}(\Delta_m) \geqslant \sup_{\varepsilon>0,\eta>0} \frac{1-\beta(s_0+\eta)\varepsilon}{s_0+\eta} = \frac{1}{s_0}.$$

Now we show that (a) implies (c). Let h be a positive function such that $m(h^2) = 1$. Let $r' > r > s_0$. Since $r > s_0$, $\sigma(\Delta_m) \cap [0, \frac{1}{r}]$ is given by a finite number of finite dimensional eigenvalues. Let $0 \leqslant \lambda_1 \leqslant \cdots \leqslant \lambda_{n_r}$ be these eigenvalues (including multiplicity), $g_1, \ldots, g_{n_r}$ be some associated orthonormalized eigenfunctions, and $\mathcal{H}_r$ the corresponding spanned vector space. Let $f \in \mathcal{C}_c(\mathscr{V})$ and consider $g := \mathbf{1}_{[0,\frac{1}{r}]}(\Delta_m) f = \sum_{i=1}^{n_r} m(\overline{g}_i f) g_i$ and $k := \mathbf{1}_{(\frac{1}{r},\infty)}(\Delta_m) f$. By construction $m(|f|^2) = m(|g|^2) + m(|k|^2)$ and

$$m(|k|^2) \leqslant r \cdot m(\overline{k}\Delta_m k) \leqslant r \cdot m(\overline{f}\Delta_m f).$$

Moreover, since $\mathcal{H}_r$ is finite dimensional, there is a finite $\beta_1(r)$ such that

$$m(|u|^2) \leqslant \beta_1(r) m(|uh|)^2, \text{ for all } u \in \mathcal{H}_r.$$

Let $c_r > 0$ be a constant to be precised later. Then using several times the

Cauchy-Schwarz inequality,

$$\begin{aligned} m(|gh|) &\leqslant \sum_{i=1}^{n_r} |m(f\overline{g}_i)| m(|hg_i|) \leqslant \sum_{i=1}^{n_r} m(|fg_i|) \\ &\leqslant \sum_{i=1}^{n_r} \left(c_r m(|f|h) + m\left(|fg_i|\mathbf{1}_{\{|g_i|\geqslant c_r h\}}\right)\right) \\ &\leqslant n_r c_r m(|f|h) + n_r \varepsilon_r^{1/2} m(|f|^2)^{1/2}, \end{aligned}$$

where $\varepsilon_r := \sup_{i=1,\dots,n_r} m\left(|g_i|^2\mathbf{1}_{\{|g_i|\geqslant c_r h\}}\right)$. By dominated convergence theorem $\varepsilon_r \to 0$ when $c_r \to \infty$. Therefore

$$m(|g|^2) \leqslant 2\beta_1(r)\left(n_r^2 c_r^2 m(|fh|)^2 + n_r^2 \varepsilon_r m(|f|^2)\right)$$

and

$$\left(1 - 2\beta_1(r) n_r^2 \varepsilon_r\right) m(|f|^2) \leqslant r \cdot m(\overline{f}\Delta_m f) + 2\beta_1(r) n_r^2 c_r^2 m(|fh|)^2.$$

Taking $r' > r$ and c_r large enough such that $\varepsilon_r \leqslant \frac{r'-r}{2\beta_1(r) n_r^2 r'}$ gives

$$m(|f|^2) \leqslant r' m(\overline{f}\Delta_m f) + 2\beta_1(r) n_r^2 c_r^2 \frac{r'}{r} m(|fh|)^2.$$

Taking $\beta_h(r') = \inf_{s_0 < r < r'} 2\beta_1(r) n_r^2 c_r^2 \frac{r'}{r}$ ends the proof. □

The conjunction of Theorem B.3 and Theorem B.5 gives the following result.

Corollary B.6. — *Assume there exists W a positive function such that*

$$\tilde{\Delta}_m W \geqslant \psi \times W - b\mathbf{1}_{B_{r_0}}$$

for some non-negative function ψ, some constant $b \geqslant 0$ and some finite set B_{r_0}.

If $\liminf \psi(x) = l$, as $|x| \to \infty$, then $\sigma_{\mathrm{ess}}(\Delta_m) \subset [l, \infty)$. In particular, if $\lim \psi(x) = +\infty$, then $\sigma_{\mathrm{ess}}(\Delta_m) = \emptyset$.

Proof. — Indeed, with our assumptions, Theorem B.3 gives SPI $(1/l)$ and Theorem B.5 implies in turn that $\sigma_{\mathrm{ess}}(\Delta_m) \subset [l, \infty)$. □

Remark B.7. — One can avoid the use of Super-Poincaré Inequality in our setting and give a direct proof of Corollary B.6 by using the Hardy inequality 3.1 and either the Persson Lemma or the min-max principle (see Theorem 3.6).

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