PATRICIO G. HERBST[1]

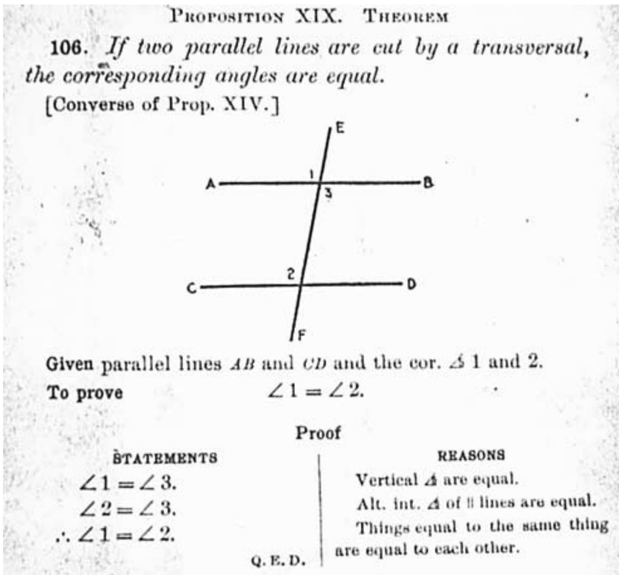
ESTABLISHING A CUSTOM OF PROVING IN AMERICAN SCHOOL GEOMETRY: EVOLUTION OF THE TWO-COLUMN PROOF IN THE EARLY TWENTIETH CENTURY

ABSTRACT. Having high school students prove geometrical propositions became the norm in the United States with the reforms of the 1890's – when geometry was designated as the place for students to learn the 'art of demonstration.' A custom of asking students to produce and write proofs in a 'two-column format' of statements and reasons developed as the teaching profession responded to the demands of reform. I provide a historical account for how proving evolved as a task for students in school geometry, starting from the time when geometry became a high school subject and continuing to the time when proof became the centerpiece of the geometry curriculum. I use the historical account to explain how the two-column proof format brought stability to the course of studies in geometry by making it possible for teachers to claim that they were teaching students how to prove and for students to demonstrate that their work involved proving. I also uncover what the nature of school geometry came to be as a result of the emphasis in students' learning to prove by showing that students' acquisition of a generic notion of proof was made possible at the expense of reducing students' participation in the development of new ideas. I draw connections between that century-old reform and current reform emphases on reasoning and proof. I use observations from history to suggest that as we carve a place for proof in present-day school mathematics we must be leery of isolating issues of proving from issues of knowing.

1. INTRODUCTION: STUDENT PROVING IN AMERICAN MATHEMATICS EDUCATION

Current recommendations for mathematics education, as expressed in the *Principles and Standards for School Mathematics* (NCTM, 2000), include the expectation that students at all levels be involved in reasoning and proving. "Part of the beauty of mathematics is that when interesting things happen, it is usually for good reason. Students should understand this." More specifically, "[by] the end of secondary school, students should be able to understand and produce mathematical proofs" (NCTM, 2000, p. 56). The rationale for including proving among the main tasks for students takes into consideration what is involved in the work of mathematicians (Lakatos, 1976; Rav, 1999). As proof is intimately connected to the construction of mathematical ideas, proving should be as natural an activity

Educational Studies in Mathematics **49:** 283–312, 2002.



PROPOSITION XIX. THEOREM

106. *If two parallel lines are cut by a transversal, the corresponding angles are equal.*

[Converse of Prop. XIV.]

Given parallel lines *AB* and *CD* and the cor. ∠s 1 and 2.

To prove $\angle 1 = \angle 2$.

Proof

STATEMENTS	REASONS
$\angle 1 = \angle 3$.	Vertical ∠s are equal.
$\angle 2 = \angle 3$.	Alt. int. ∠s of ∥ lines are equal.
$\therefore \angle 1 = \angle 2$.	Things equal to the same thing are equal to each other.

Q. E. D.

Figure 1. A two-column proof (photographed by the author from Schulze and Sevenoak, 1913, p. 53, Macmillan publishers).

for students as defining, modeling, representing, or problem solving. Yet, important questions that must be raised concern what it takes to organize classrooms where students can be expected to produce arguments and proofs and what proof may look like in school classrooms.

The notion that students should prove propositions is not completely foreign to American teachers. It has been traditional to use the high school geometry course to help students develop the skill of 'doing proofs.' This custom has been in place for more than a century and has had an enduring influence on how Americans think about mathematical proof. For most people, proofs are sequences of steps composed of two columns of statements and reasons that show why the premises of a proposition (the 'givens') lead to a conclusion (see Figure 1). Under this *two-column proving custom*, proving activities for students have often been closer to exercising logic to validate obvious and inconsequential statements (Kline, 1965), than to building compelling arguments for the reasonableness of important mathematical ideas (Ball and Bass, 2000; Schoenfeld, 1987)

In spite of those criticisms, two things are important to note about two-column proving. One, that however reductive we may find that custom, its endurance for about a century has been effective in maintaining a place for 'proof' among the academic work that students do in school mathematics (hereafter, *student proving*). Two, that the two-column proving custom developed at a time when curriculum discussions had brought to the fore concerns about school's responsibility for students' intellectual activity (see Eliot et al., 1893; Nightingale et al., 1899). On account of those two observations, it is conceivable that the two-column proving custom – this reduction of mathematical reasoning to its logical, formal dimensions – had developed as a viable way for instruction to meet the demand that every student should be able to do proofs.

The investigation that I report herein explored that possibility. I account for the correspondence between changes in how curriculum documents, scholarly commentaries, and texts for teachers referred to proof in the study of geometry and changes in the instructional practices related to proof in school geometry (as these practices are prescribed in the textbooks of the time). In analyzing those correspondences, the present article does a historical reconstruction of how the two-column proving custom evolved to become a part of geometry instruction in the period from the 1850's to the 1910's. I show how geometry evolved as a subject of school studies in synchrony with the dominant conceptions of the students as learners, the notion also developed that students might be asked to craft proofs. As that notion became an expectation for all students and a criterion for curriculum development, the two-column proof format helped stabilize the geometry curriculum by melding together the proofs given by the text and the proofs expected from students.

An in-depth consideration of how concomitant questions about the subject of studies and about the students as learners became central and had an effect in the development of a custom of proving can help us grapple with the challenges of present day reform. Whereas different issues than that century-old reform are at stake now, those issues also involve shaping the course of studies and the activities expected of students. It is conceivable that for the teaching profession to make room for the goals of present-day reform, other issues also need to be considered and that systemic effects analogous to those of the past century may be produced concomitantly. Thus as we think about current recommendation for change in instructional practices, the history of how instruction responded to past curriculum change efforts can serve as a source of information, encouragement, and caution.

I thus propose to look at history with the interest of drawing some lessons for the present. In section 2, I start this consideration of history with a brief account of the reform movement that made student proving a standard for geometry instruction a century ago. Then I situate the emergence of that standard for student proving within a historical account of geometry instruction. In section 3, I describe a baseline period when student proving was not an explicit issue in geometry instruction. In section 4, I move to a discussion of a transition period when crafting proofs emerged as a possible task for students. In section 5, I come back to examine the ideas of the reform proposed to the Committee of Ten (Eliot et al., 1893) by an appointed committee of mathematics specialists (the Mathematics Conference). In section 6, I show how in the years that followed, the study of geometry changed dramatically as it became mandatory that students learn the art of proving. Finally, in section 7, I propose an explanation for why the two-column proof format brought stability to the study of geometry in school.

2. The Report of the Committee of Ten and the Study of Geometry

The imperative that the geometry course should be a place where students should learn the 'art' of proving traces back to the report of the Committee of Ten (Eliot et al., 1893). As the nineteenth century was coming to an end, issues related to the secondary school population (who the clientele was and what they were going to do after finishing school) coalesced to set the stage for a discussion of curriculum that continued for many years (Kliebard, 1987). The Committee of Ten was a group of educational leaders, chaired by Harvard's President Charles Eliot, called to serve by the National Educational Association with the mandate to study the problems related to college requirements for admissions. At that point, it was acknowledged that there was a great diversity of courses being offered by high schools, many of which were allocated so "short periods [of time] that little training could be derived from them" (Eliot et al., 1893, p. 5). The Committee had the mission of examining whether high school should be different for students according to their interests and, in any case, what should be the high school course of studies and best methods of teaching each subject.[2]

Historians have written extensively about the work of the Committee of Ten and about its influence shaping American education (Kliebard, 1987; Krug, 1964; Ravitch, 2000; Sizer, 1964). George Stanic (1983, 1987) has shown how the curriculum debates then started have imposed a continuous need for the justification of school mathematical studies. Eileen Donoghue (1987) has elaborated on some of the consequences of those debates in shaping mathematics teacher education programs. As far as geometry teaching, histories have been written which cover, to varying degree of detail, the period of influence of the report (Quast, 1968; Shibli, 1932) though they do not examine in detail the evolution of proof in instruction. My aim is not to write a comprehensive history of a period. Instead, I use the available histories along with my own examination of sources (curriculum documents, geometry textbooks, and scholarly articles[3]) to provide a historical reconstruction of how the place of proof in geometry instruction evolved as the teaching profession responded to the curriculum issues raised by the Committee of Ten. A tension was present in the work of the Committee of Ten between the notion of a curriculum that would discipline the mental faculties and a curriculum that would transmit the knowledge of the academic disciplines (Baker, 1893/1969; Eliot et al., 1893/1969; Eliot, 1905; Harris, 1894). The Committee of Ten consulted with the Mathematics Conference, an appointed committee chaired by Simon Newcomb, a mathematics professor at Johns Hopkins, and including five other mathematics professors, two school principals, and two mathematics teachers. This Conference produced a report including important observations and recommendations for the teaching of geometry. The Conference's recommendations made provisions for the transmission of culturally valued geometrical knowledge, but used the notion that academic studies were done in order to provide mental discipline as the criterion for deciding what the high school curriculum should be. Drawing on those recommendations, the Committee of Ten identified the high school geometry course as a vehicle for students to acquire the *art of demonstration* (or proving), as "geometrical demonstration is to be chiefly prized [as] a discipline in complete, exact, and logical statement" (Eliot et al., 1893/1969, p. 25).

The Committee argued that, as students had thus far been used to memorizing the demonstrations of a geometry text, the mental discipline that geometry made possible was being lost. Instructional changes were needed in order to enable geometry to do its job. The fact that students were memorizing the text was not new and would not disappear easily; what was new was the institutional recognition that such a state of affairs was problematic, that it posed a challenge for geometry instruction. To bring to the fore the changes in the share of labor reserved to students that led to the custom of two-column proving, I have characterized three periods in the study of geometry. I identify a baseline period characterized by students replicating the proofs given by a text, a transitional period of students crafting proofs for propositions, and a final stage of students learning how to do proofs. To understand the magnitude of the changes recommended by the Committee of Ten, it is worthwhile to go back in time and start with a brief characterization of geometry instruction at the time when high schools started to teach it.

3. The Era of Text: Replicating proofs

American high schools started to offer geometry courses in the 1840's as universities started to make it a requisite for admission (Quast, 1968, p. 36). During this period, which I call the *Era of Text,* the study of geometry entailed mastering the Euclidean body of knowledge as developed by a text. Texts that served that purpose included Robert Simson's (1756) *Elements of Euclid* John Playfair's (1795/1860) *Elements of Geometry*, and Adrien-Marie Legendre's *Elements of Geometry* (translated and edited by John Farrar in 1819). There are notable differences in how the three authors – Simson, Playfair, and Legendre – develop the Euclidean body of knowledge (Jones, 1944), but as far as what proving could have meant for students who studied those texts, the three texts are similar. The student was seen as a reader whose relationship with proof was built primarily on values that could be called mathematical. Concerns about the demands that the study imposed on students and how they would be able to meet those demands were not apparent. For example, all three texts expose the material in accordance with the notion that it is valuable to develop as much as possible of the geometrical body of knowledge with the fewest possible postulates and only call for more concessions when those are deemed mathematically indispensable. The length or difficulty of a proof was subordinate to that value. Thus Farrar's 1841 edition of Legendre's book proudly offered a proof of the triangle sum theorem much longer and complicated than Euclid's on the presumption that it did not use the theory of parallel lines (see Farrar, 1841, p. iv). It was considered valuable for students to know (and be able to show) that the truth of the triangle sum theorem might not depend on the theorems about parallel lines. How much more effort it might take for them to learn this proof rather than the classic Euclidean one that depended on parallelism was not an issue.

Playfair's introduction to his Elements indicates what kind of pedagogical concerns would support that choice of exposition. "The end of Mathematical Demonstration... is not only to prove the truth of a certain proposition, but to show its necessary connection with other propositions, and its dependence on them" (Playfair, 1795/1860, pp. xiv).

Thus, geometry was conceived as an organic body of knowledge to be acquired as it was developed by a text. The purpose of studying geometry in school was to grasp the necessary character of the relationships between geometrical objects. It was essential to that acquisition to study the proofs of those relationships. The word *demonstration* was used in the text, but it was neither used as the name of an object to be studied nor used as the name of a singular skill to be trained on. One would grasp the meaning of

THEOREM.

70. *If two straight lines* AB, CD (fig. 36), *make, with a third* EF, *two interior angles* BEF, DFE, *the sum of which is equal to two right angles, the lines* AB, CD, *will be parallel.* Fig. 36.

Demonstration. If the angles *BEF*, *DFE*, are equal, they will each be right angles, and we fall upon the case of the preceding proposition. Let us suppose, then, that they are unequal; and that, through the point *F*, the vertex of the greater, we let fall upon *AB* a perpendicular *FG*.

In the triangle *EFG*, the sum of the two acute angles *FEG* + *EFG* is equal to a right angle (61); this sum being subtracted from the sum *BEF* + *DFE*, equal by hypothesis to two right angles, there will remain the angle *DFG* equal to a right angle. Therefore the two lines *AB*, *CD*, are perpendicular to the third line *FG*, and are consequently parallel (69).

* A re-entering angle is one whose vertex is directed inward, as *CDE* (*fig.* 43), while a saliant angle has its vertex directed outward, as *ABC*. Fig. 43.

Geom. 3

Figure 2. A proof from the Era of Text (photographed by the author from Legendre, 1841, p. 17, Hilliard and Gray publishers.

demonstration through exposure to the particular demonstrations used for various theorems stated in the text: Each proof was unique and expressed the argument needed to show why the proposition had to be true on account of other things that were already held as true. One can say that each proof established its own norm for how it should be done, as proofs depended heavily on the substance of the proposition at stake. Texts included neither general descriptions of proof nor methods of proving. Proofs were written in paragraph form and made up of sentences that could be long and complex. Only some of those statements would make explicit references to the postulates, axioms, or previous theorems that justified them; most statements would rather perform an application of a previous proposition leaving the statement of that proposition implicit (see Figure 2). It seems as though *statements* would be backed by an explicitly stated *reason* when communicating this reason was part of the insight to be gotten through the reading of the proof.

The study of geometry was done through reading and reproducing a text; such work would train the reasoning faculties of students. But, the texts do not hint at the existence of official mechanisms to verify or steer the evolution of students' reasoning. To know geometry and to be able to prove the theorems of geometry were indistinguishable. And the difference between knowing geometry and remembering the text was immaterial in school (see Quast, 1968, p. 40). As the geometry course became more common in high schools, geometry instruction began to move away from simply reproducing a geometry text.

4. The Era of Originals: Crafting proofs

Geometry textbooks multiplied in numbers as more high schools took charge of the teaching of geometry. Changes in geometry textbooks started to become apparent some decades before the work of the Committee of Ten (Shibli, 1932, p. 107). Benjamin Greenleaf was one of the first American authors of a geometry text written primarily for high school teaching. In his preface, he cited "the aim [of adapting] the work fully to the latest and most approved modes of instruction" (Greenleaf, 1858, p. iii).

Changes in the geometry for school studies responded in part to evolving considerations about the students. The notion developed that students were, as Charles Davies said, 'untrained intellects' who had to acquire the facts of geometry as well as the logical reasoning that connected those facts (Davies, 1850, p. 256–258). In addition to being accountable for replicating the proofs of the propositions in the course of studies, students were given opportunities to craft proofs for 'original' propositions. In the following sections I show how these originals emerged and what other elements were developed concomitantly to accommodate their presence in the curriculum.

4.1. *The emergence of original propositions for students to prove*

The texts by Greenleaf (1858) and later by William Chauvenet (1870) differed from those of the Era of Text in that they included exercises at the end. As these exercises were spoken of as opportunities for students to do original work, I call this transitional period in the evolution of proving *the Era of Originals*. These *originals* were corollaries of propositions proved in the main text and additional theorems that might not have deserved a place in the main text. Thus, they afforded opportunities for students to use their reasoning to further their geometric knowledge. In his preface to a later edition of Chauvenet's (1887) text, William Byerly said the purpose of originals was to "compel the student to think and to reason for himself" beyond just learning "to understand and demonstrate a few set propositions" (p. 5). Changes in the way later texts would give original exercises seem to respond to that purpose of training students' "power of grasping and proving a simple geometrical truth," a power that "can never be gained by memorizing demonstrations" (Chauvenet, 1887, p. 5). In that spirit, George Wentworth (1878) would give long lists of original exercises at the end of each chapter, and Wentworth (1888) would also interject some of those original exercises in between the propositions of a given chapter.

The presence of originals presumed that students would learn to reason by reasoning. But what they would be reasoning about (what kind of propositions they would be proving) initially depended on what aspects of the subject matter needed to be covered and what ideas the student should know about and be able to use. To prove these originals could require as much ingenuity as that needed to come up with proofs for the theorems given in the text. As the time of the Committee of Ten approached, the goal of disciplining the students' minds took priority over that of having them explore the subject of studies. Thus, suggesting a contrast with older texts, Wentworth (1888) indicated that the originals he proposed were "not so difficult. . . but well adapted to afford an effectual test of degree in which [the student] is mastering the subject of his reading" (p. iv).

As these originals became more prominent in textbooks and in the discourse about the study of geometry, other features began to appear along with them. On the one hand, students would not just be given opportunities to prove propositions; they would also be afforded resources that enabled them to prove. In the following section I exemplify this by noting changes observed in the way diagrams were given. On the other hand, students' production of proofs was guided and monitored by increasingly explicit characterizations of what proving entailed. In the section after next I show how texts developed the bare bones of a norm for proving. In both cases – resources and norms – I note how these features had an impact on the nature of the subject of studies.

4.1.1. *Diagrams that enabled student proving: Hypothetical constructions*

For authors to come up with new conventions to draw diagrams had been common since Legendre's text. Davies (1848), Greenleaf (1958), and Chauvenet (1870) used dotted lines to represent constructions done in the course of a demonstration – to distinguish them from lines given by the conditions of the proposition to prove. For the exercises he gave, Greenleaf provided no diagrams, whereas Chauvenet did, indicating his purpose thus,

> In order to make these exercises progressive as to difficulty, and to bring them fairly within the grasp of the student at the successive stages of his progress, many of them are accompanied by diagrams in which the necessary auxiliary lines are drawn. (Chauvenet, 1870, p. 293)

Wentworth also made changes to the diagrams he would give with propositions. In addition to the full and dotted lines, Wentworth (1878, p. iii) added a third way of drawing lines: "The *given* lines of the figures are full lines, the lines employed as *aids* in the demonstrations are short-dotted, and the *resulting* lines are long-dotted." The origin of those aids or auxiliary lines was sometimes problematic. In his critique of the then-so-called-new methods of geometry, Eugene Richards, a professor of mathematics at Yale, targeted the changes in "arrangement and order of propositions" (Richards, 1892, p. 33) in modern geometries. In Richards' opinion it was this "[interference] with the natural order of development of the subject" (p. 33) that had made necessary some hypothetical constructions which were problematic:

> The pupil is told to make a line equal to another line, an angle equal to another angle. . . without being instructed how to make these necessary constructions. . .. [Modern geometry does not] inform the pupil what constructions it is possible for him to make. . .. Having proved certain propositions by means of "hypothetical constructions," they then take up the problems on which the theorems really depend. They prove the problems to be done, according to theorems already proved by means of the problems. This may all be good classification, but it is not geometry. Neither is it good logic. (Richards, 1892, p. 34; see also Halsted, 1893)

But, in a rejoinder to Richards' article, George Shutts (1892) defended the use of hypothetical constructions because they were "represent[ing] to the eye, for convenience in demonstration, the relations desired to be considered" (p. 265). On the one hand, it seems apparent that what was at stake for the 'moderns' in the study of geometry was no longer the conditions under which certain assertions were true or certain constructions were possible, but rather,

> Modern geometry seeks the shortest path to results. . .. Discipline of mind is acquired through mental exercise in gaining truth, and the particular subject-matter studied is of less importance than that the pupil succeeds in detecting truth and eliminating error, in holding logically to a line of thought until he has reached a conclusion. (Shutts, 1892, p. 264)

On the other hand, the hypothetical constructions were used not (necessarily) because of ignorance of geometry on the part of textbook writers but in response to a felt instructional need: To afford students some of the elements that they would need in furnishing a proof. The emergence of these 'hypothetical constructions' is therefore important for this inquiry because they exemplify how the educational system chose to afford resources for student proving, in spite of the non trivial changes that those provisions imposed in the nature of school geometry. Indeed, school geometry was becoming the logical treatment of a universe of objects whose existence and truth were taken as factual, independent of the reasoning involved in dealing with them. This separation had been instrumental to classify the content, carve a share of labor for students, and provide resources for students to craft proofs. At the same time that hypothetical constructions in diagrams were included as resources for student proving, a norm was being shaped for teachers to monitor the work that students were to do.

4.1.2. *The beginnings of a generic norm for proving*

In his text about the teaching of mathematics, Davies (1850) had suggested that teachers quote or spell out reasons for statements as they presented the proofs of the theorems in the course of studies. But neither Davies' (1848) edition of Legendre's text nor Greenleaf's or Chauvenet's textbooks changed much the style of the era of text: They kept writing proofs in paragraphs and providing only some of the reasons for statements. It was Wentworth (1878) who first changed several of the features of the printed page in a geometry text (Shibli, 1932, p. 144). Wentworth (1878) committed himself to displaying every theorem and its proof in a single page to facilitate its understanding, and he introduced normative considerations about the format in which demonstrations would be written,

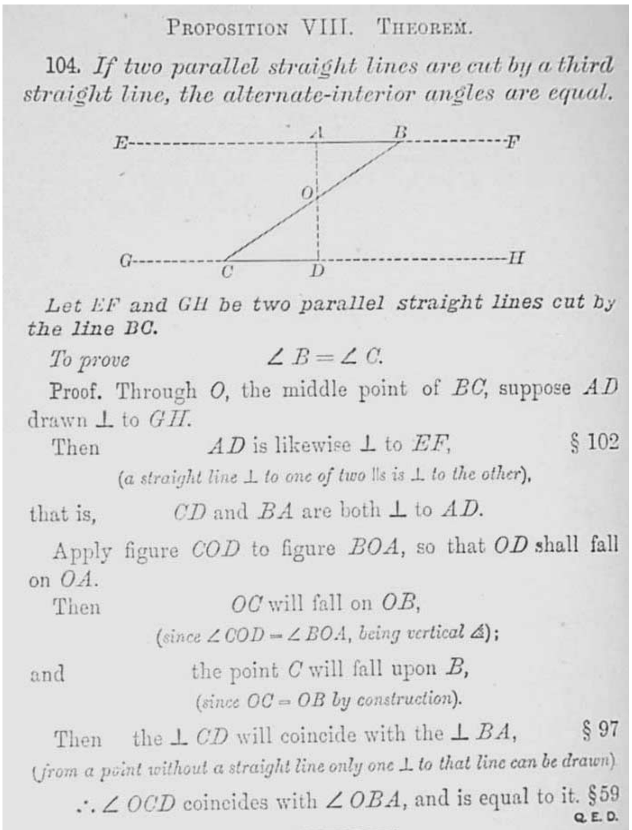
> The reason for each step is indicated in small type between that step and the one following, thus preventing the necessity of interrupting the process of the argument by referring to a previous section. . .. *Moreover, each distinct assertion in the demonstrations*. . . *begins a new line*. . .. This arrangement presents obvious advantages. The pupil. . . readily refers to the figure at every step, . . . acquires facility in simple and accurate expression, rapidly *learns to reason*.
>
> (p. iv, italics in original)

The demonstrations given by the book were developed in more succinct sentences as a result of the condition to give them in steps or lines (see Figure 3). Each of those lines was accompanied by reasons: A number to the right of each line referred to the paragraph (usually a theorem) that supported the statement made, and a sentence below each statement indicated how the previous paragraph applied.

In contrast with the proofs of the Era of Text, the proofs given by Wentworth (1878) and subsequent authors would thus adhere to a more general norm of exposition that tended to make explicit what logical reasoning was. Wentworth's identification of 'the process of the argument' as a proper part of the demonstration (which included the argument and the references) points at the use of that general norm. The particular proof (argument) is provided within a normative frame (lines, references) by which the spelling out of the proof may (if needed) be judged as understood. The fact that the student provided reasons for each statement upon request would mean, for the teacher, that the student had understood the argument as having the status of proof (see also Wells, 1894, p. viii).

4.2. *Student proving*

The elements were then in place to make it possible for students to engage in proving. The student had to be trained in reasoning and absolute truth, and this training could be done through the study of geometry. But that was



Proposition VIII. Theorem.

104. *If two parallel straight lines are cut by a third straight line, the alternate-interior angles are equal.*

Let EF **and** GH **be two parallel straight lines cut by the line** BC**.**

To prove $\angle B = \angle C$.

Proof. Through O, the middle point of BC, suppose AD drawn $\perp$ to GH.

Then AD is likewise $\perp$ to EF, § 102

(*a straight line* $\perp$ *to one of two* ||s *is* $\perp$ *to the other*),

that is, CD and BA are both $\perp$ to AD.

Apply figure COD to figure BOA, so that OD shall fall on OA.

Then OC will fall on OB,

(*since* $\angle COD = \angle BOA$, *being vertical* ∡);

and the point C will fall upon B,

(*since* $OC = OB$ *by construction*).

Then the $\perp$ CD will coincide with the $\perp$ BA, § 97

(*from a point without a straight line only one* $\perp$ *to that line can be drawn*)

$\therefore \angle OCD$ coincides with $\angle OBA$, and is equal to it. § 59

Q. E. D.

Figure 3. A proof from the Era of Originals (photographed by the author from Wentworth, 1888, p. 24, Ginn publishers).

not seen as an automatic byproduct of exposure to the study of a text. To effect such training, the knowledge of geometry had had to be transposed into the subject-matter of school geometry, particularly yielding an image of school geometry as the organization of a collection of factual truths about plane and space. Each of those truths was justified according to the logical principles of deductive reasoning, though the notion that such reasoning actually had a share in producing those truths was farther from view than in the Era of Text. Student proving was a means by which the expected training could be monitored and it happened in two kinds of situations: (1) demonstrating (understanding of the proofs of) the propositions of the main text and (2) proving original propositions. Those situations were not essentially different, in that both entailed exercising the reasoning faculties and acquiring and demonstrating acquisition of new knowledge.

As the time of the Committee of Ten approached, student proving was not only suggested as convenient for mental discipline but it was also beginning to be fostered and surveyed. When informing the Committee of Ten's recommendation to make student proving a requirement, the Mathematics Conference drew heavily on the instructional adaptations done during the Era of Originals.

5. The Mathematics Conference's Recommendations on Geometry Teaching

The main challenge for the Committee of Ten was to find a way to accommodate the diversity of available studies into an academic curriculum meant for all students (Ravitch, 2000, p. 42). The notion of exposure to the liberal arts, those arts that would liberate the intellect, was not enough. New fields of knowledge could use it to find their way into the curriculum, but reformers were aware that an overloaded curriculum would make that exposure trivial. The doctrine of mental discipline became a tool for general educators to handle the problem of how those possible studies could be compared and contrasted in the interest of specifying a curriculum of optimal educational value (see Harris, 1894). The "educational value of mathematics" would thus be identified in the "training to the mind's powers of conceiving, judging, and reasoning" (Hill, 1895, p. 353). And, whereas the physical sciences would train students in inductive methods, "in formal geometry we have the best possible arena for training in deductive reasoning" (Hill, 1895, p. 354). Hence, mental discipline, that had been used before to provide a justification for the study of mathematics, was now being officially used as a criterion to decide what should be included in the study of mathematics (Newcomb et al., 1893, p. 105).

The Mathematics Conference (Newcomb et al., 1893), recommended changes to the geometry curriculum that intended to accommodate the tension between training the mental faculties and transmitting the culturally valued geometrical knowledge. They recommended a new course of studies in *concrete geometry* for the elementary school. Such a course would "familiarize the pupil with the facts of plane and solid geometry and with those geometrical conceptions to be subsequently employed in abstract reasoning" (p. 106). In envisioning the high school geometry course, the Conference could thus count on the students knowing basic facts from the previous concrete geometry course. They concentrated on envisioning a course on "demonstrative geometry [as] the most elaborate illustration of the mechanism of formal logic in the entire curriculum of the student" (p. 115). In the years before, school geometry had been conceived of as the logical treatment of truths about the plane and space – separating questions about how geometric objects came to exist and have properties from the deductive reasoning used to justify those properties. The reforms proposed by the Committee of Ten were able to keep the privileged place of geometry in high school by going farther than Davies in the same direction – arguing that it was not the geometric truths but their logic that was most important. 'The art of demonstration' (i.e., proving) thus became a main objective of the study of geometry. It was understood that the way to acquire the art of demonstration would be by demonstrating, by actually dealing with the geometric truths as a pretext to acquire the art. Those geometric truths that had served as anchors for the development of reasoning would now be taught as factual truths before any attempt at deductive reasoning (see Halsted, 1893). But such changes responded to the awareness that logical reasoning and memorization of a logically reasoned text were not the same – and students had to learn the former.

So, whereas giving students the opportunity to prove had been part of the study of geometry in high school for some time, the Committee of Ten made it the centerpiece of the course. The normative impetus to get students to prove and to reject "demonstrations that are not exact and formally perfect" because they did not provide the training sought (Eliot et al., 1893/1969, p. 25) posed a crucial challenge for instruction. The challenge was one of how to ensure that the high school geometry course became a place where students not merely *could* but rather *would* prove.

6. THE ERA OF EXERCISE: LEARNING TO DO PROOFS

Important changes in geometry texts resulted from the recommendations of the Committee of Ten. Opportunities for students to craft proofs increased in number and changed in nature as one contrasts the textbooks published after the recommendations of the Committee of Ten with those of the Era of Originals. Because most of the changes observed in the course of studies seemed to be geared toward enabling and supporting students' performance in those proof 'exercises,' I call this period the Era of Exercise. In the sections 6.1, 6.2, and 6.3, I account for the changes that occurred in the study of geometry during this period.

6.1. *Teaching about proof*

Whereas it had already been customary to ask students to craft proofs, textbooks had not delved at length into what a proof was. Also, whereas methods and strategies for proving had been used to prove propositions in the text, it had not been customary for textbooks to detail those methods. A notable change during the Era of Exercise had to do with the making explicit of what a proof was considered to be. Textbooks also began to instruct on methods and strategies for coming up with proofs.

6.1.1. *From descriptions of proof to the two-column format*

Textbooks from the Era of Originals had conveyed the sense of what a proof was by giving proofs for the theorems of the course of studies. Following the report of the Committee of Ten, more explicit definitions or descriptions of proof began to appear. For example, Wooster Beman and David Eugene Smith started a series of 'New plane geometry' textbooks in 1895. They developed a section on 'Preliminary propositions' to "show to the beginner the nature of a geometric proof and to lead him by easy steps to appreciate the logic of geometry" (Beman and Smith, 1899, p. 13). And, in addition to giving the proofs of some propositions as models, Beman and Smith also included a section on 'the nature of a logical proof' (ibid., p. 19) that explicitly described what a proof should look like:

> Every statement in a proof must be based upon a postulate, an axiom, a definition, or some proposition previously considered of which the student is prepared to give the proof again when he refers to it.... No statement is true simply because it appears to be true from a figure.... [In a proof] are set forth, in concise steps, the statements to prove the conclusion ... asserted. If the proof is written on the blackboard, the steps should be numbered for convenient reference by class and teacher. The teacher will state how much in the way of written or indicated authorities shall be required after each step. (Beman and Smith, 1899, pp. 19–20)

Beman and Smith put on paper things that were known and already in use by teachers in inspecting the formal aspects of student work. But making those stipulations explicit amounted to indicating to students that what was important in their work was the logical arrangement of statements.

The next remarkable change in making proof explicit was pioneered by the second edition of Arthur Schultze and Frank Sevenoak's geometry textbook (1913). Shibli (1932) says those authors were the first to write proofs in two columns of *statements* and *reasons* divided by a vertical line (see Figure 1). Shibli (1932) adds,

> [This arrangement in two columns] seems to emphasize more strongly the necessity of giving a reason for each statement made, and it saves time when the teacher is inspecting and correcting written work. (p. 145)

After using the two-column format for the first time, Schultze and Sevenoak described it as follows:

> Every proof consists of a number of statements, each of which is supported by a definite reason. The only admissible reasons are: a previously proved proposition; an axiom; a definition; or the hypothesis. (Schultze and Sevenoak, 1913, p. 19)

It is intriguing how widely used this format became and how long it has endured as an icon of proof in school mathematics. The graphical arrangement that the format imposed on the elements of a proof involved yet another move toward establishing a norm for the production and control of proofs by students. The format also established more clearly what was considered important in students' performance by providing an image of what the finished product should look like.

As far as proof is concerned, the educational literature of the time (e.g. Slaught et al., 1912) was focused on the contrast between *informal* and *formal* proofs – where the latter referred to proofs for which every statement was given an explicit reason. In line with the aims of the Committee of Ten, a premium was put on students learning to write formal proofs, even if some propositions in a text were only 'informally' proved. The fact that the two-column format emphasized in so evident ways the formal aspects of proving, enforcing the notion that a proof consisted of steps of statements and reasons, made it useful at the time. I return to this issue after describing how other aspects of the geometry course changed accordingly.

6.1.2. *Teaching methods and strategies for proving propositions*

The Committee of Ten had recommended teaching students methods for discovering proofs. Popular textbooks like Wentworth's and Wells' were revised accordingly (Wells, 1908; Wentworth, 1899). Wentworth (1899) added a section on 'Methods of proving theorems' to the end of the first chapter and before the exercises where he exposed the synthetic, analytic, and indirect methods of proving.

Some of the many new texts developed during that time also included more specific points about method. For example, in their first edition, Shultze and Sevenoak (1901) wanted to "introduce the student *systematically* into original geometrical work." Hence, they complemented the exposition of theorems with "remarks" about the strategies used in proving those theorems and their import for student work. Thus, after proving all the conditions for parallelism they said,

> Lines are demonstrated to be parallel by proving that (a) two alternate interior angles are equal, (b) two corresponding angles are equal, or (c) two interior angles on the same side of a transversal are supplementary. (p. 20)

Exercises were often placed near theorems where those strategies had been used. The nature of these exercises and their relationship with the theorems of the course of studies deserves more elaboration.

6.2. *Ensuring that students would do proofs: From originals to exercises*

More precise directions about how to organize the geometry course were provided in a document issued by a new Committee (Nightingale et al., 1899). This new Committee was also informed by the report of a group of mathematics specialists, whose chair was Jacob W. A. Young, a professor of mathematical pedagogy in the University of Chicago, and that included seven mathematics professors and two school principals. (Young et al., 1899). They suggested that the geometry course should be composed of two different kinds of propositions: *Fundamental* propositions and *exercises*. The fundamental propositions would be "the minimum which all pupils alike should know" (p. 142) – they would serve to develop the subject of studies. But additionally, the proofs of those fundamentals would exemplify what it meant to prove a proposition, and the propositions themselves would be the information that students would use as they did the proof 'exercises.' These exercises were extremely important. The 1899 Committee had insisted that to make the study of mathematics valuable, the student had to think and produce; "Not to learn proofs, but to prove, must be his task" (Young et al., 1899, p. 136).

As shown in section 4, it was not new that students be given the opportunity to craft proofs. But whereas a change in names from 'originals' to 'exercises' might seem a trivial consequence of the popularity of mental discipline jargon, the change was not just a change of names. 'Originals' were then regarded as having been too few and too difficult. The 'exercises' where students would produce proofs, would instead have to be "many, easy, and carefully graded" (Young et al., 1899, p. 136) and their proofs would have "to be based on the fundamental propositions" (ibid., p. 142).

In contrast with the Era of Originals, students would not just be given the opportunity to craft proofs; students would actually "learn to demonstrate by demonstrating" (Young, 1906, p. 259). Schultze (1912, p. 98) thus pointed out that it was not the same to just devote time for original work and to do so in a way "consistent with sound pedagogical principle." And Young (1906, p. 260) said that students were expected to "actually succeed in discovering" proofs. To give students the opportunity to prove was therefore just the first step – that opportunity had to be actually taken and its purpose had to be accomplished for the course to fulfill its purpose as the place to acquire the art of proving.

Teachers had to take proactive steps to ensure that the course served its purpose. The argument was common at the time and addressed what educators thought of students as learners. D.E. Smith (1911, p. 70) suggested that to give the opportunity to prove might not be enough because the diversity of students in geometry classes made it unrealistic for teachers to expect all students in their classes to be enthusiastic “over a logical sequence of proved propositions.” But whereas it was not reasonable to expect that all students would “discover new truths,” proving truths stated by somebody else was something that all students should be able to do (ibid., p. 160). The task of ensuring that all students would do proofs was one that the teaching profession had to take on.

Authors of texts for teachers – like Schultze, D.E. Smith, and Young – gave suggestions on what could be done to induce success in student proving. Young (1906, p. 259) recommended that “the matter [be] duly simplified, and cut up into small portions for [the student].” Later, the Committee of Fifteen on the Geometry Syllabus would also make recommendations for easier and more concrete exercises formulated in connection with the specific theorems that the exercises were an application of (Slaught et al., 1912, p. 93). For the “rank and file” not “to become discouraged and hopelessly lost in the so-called ‘originals’. . . the grading [had to be] carefully done, and steps of difficulty [had to be] kept down to a very reasonable lower limit” (Slaught et al., 1912, p. 95).

The purpose of these exercises for students was still, in part, to “give opportunity for quiet thinking.” But they were now also to “drill on operations whose theory [was] understood” and had the mission of "completing the class work of the previous day, not of preparing for the next” (Slaught et al., 1912, p. 132). Thus the exercises provided after a few theorems tended to call in one way or another for application of those theorems. Smith (1911) had said that these simpler, sometimes ‘one-step’ (p. 70) exercises would “create the interest that comes from independent work, from a feeling of conquest, and from a desire to do something original” (p. 112).

Therefore, the ‘exercises’ were similar to the ‘originals’ in that they required students to exercise their reasoning skills to produce proofs. They were different, however, in that they were no longer meant as opportunities for students to develop new ideas that pertained to the course of studies. Rather, exercises were to be used to practice what had been already learned. The transition in contexts for student proving, from originals to exercises, thus involved continuity in the formal aspects of the work that they called for – crafting proofs. But it also involved rupture in the substantial aspects of that work – from proving as means to know new things to proving to practice using known things in proofs. In the next section I discuss how the course of studies responded to those changes in the meaning of student proving.

6.3. *A course of studies that would teach students how to prove*

The distinction between fundamental propositions and exercises played an important role in making possible the teaching of the art of proving. Fundamentals were to develop the subject matter as well as provide tools for students to do their proof exercises, but it was imperative that students spend time working and succeeding in proof exercises. The textbooks of the time show that authors confronted difficult questions when conceiving of a geometry course that would adequately balance attention to fundamentals and exercises. Which propositions should be the fundamental ones? In which sequence should they be placed in the text? What proofs should be envisioned for them? How should the 'discovery' of these proofs be reported in the text? A remarkable diversity in the exposition of these fundamental propositions can be observed in geometry textbooks from the beginning of the century (Smith, 1909; Schultze and Sevenoak, 1913; Shibli, 1932). In different ways, authors tried to balance respect for the integrity of the discipline of geometry with usefulness of the exposition to enable and support students' performance in exercises. They also endeavored to reduce the number of fundamental propositions and increase the number of exercises. The following sections describe ways in which the course of studies increased its capacity to enable and support students' production of proofs. As I describe those changes, I also discuss how those changes impacted the nature of the geometry being studied.

6.3.1. *Changes in what must be proved and what can be taken as an axiom*

Pedagogical concerns were invoked by the 1899 Committee to suggest changes in the way axioms were given. The Committee recommended reserving for a later stage the introduction of those axioms that were "more nearly self-evident" (Nightingale et al., 1899, p. 142) and directly omitting self-evident propositions like "all right angles are equal."

At about the same time, mathematicians had been making important progress in establishing Euclidean geometry on a solid axiomatic basis, for example making explicit as axioms things that Euclid had assumed without comment (e.g., Hilbert, 1899/1971). But mainstream recommendations for geometry instruction in the United States were apparently taking the opposite direction. The subsequent Committee of Fifteen weighed the mathematical recommendation of an "irreducible minimum of assumptions" against pedagogical considerations that "such a list would be unintelligible to pupils" (Slaught et al., 1912, p. 81). The axioms they chose balanced mathematical considerations with considerations about the learners:

> The principle is that a logical sequence should be maintained, and formal proofs of propositions necessary to the sequence should be required, so far as this is consonant with the educational principle of adapting the matter to the mind of the learner. (Slaught et al., 1912, p. 81)

Schultze (1912) also suggested that the import for geometry instruction of the progress in axiomatics was that it had warranted choosing for school geometry axioms different than Euclid's:

> We do not need to strive for absolute completeness when compiling lists of axioms. A statement may be considered an axiom even if it can be deduced from other axioms; and a common sense reason may sometimes be given when technically an axiom should be quoted. (p. 65)

Similarly, George Carson suggested turning into assumptions all properties that it was "possible to induce the child to accept without the aid of numerical measurement" (Carson, 1913, p. 98).

The possibility of choosing what to propose as axioms served a purpose. The emphasis of the Committee of Ten on using geometry as a context for learning to prove had helped establish a specific notion of proof in geometry that had already been latent in the Era of Originals. According to this notion, the existence and properties of geometric objects were true facts, independent of the proofs given for them. To the background had gone the notion that it is the existence of a proof what makes properties true, by stipulating the theoretical conditions under which those properties should be taken as facts. As a result of that move, to have many axioms was not seen as any different than to have a few, as long as they all stated things that were true. And it helped reduce the number of fundamental propositions that all students should know how to prove. The fundamental propositions to be proved would be those most central to the development of the subject matter, those not so evident to be held as axioms, and those that could illustrate methods of proving. This reduction in the number of fundamentals would likely allow more class time for students to craft proofs for exercises. At the same time, the existence of a long list of propositions taken as axioms would enable students to use what those propositions asserted without requiring them to know how those propositions came to be true. Thus, an assumption could be referred to as reason in a proof because it was an accepted truth, without the need for questioning whether it was a proposition provable with the available tools. Finally, the fact that concrete geometry (including how to construct geometric objects) would be taught before demonstrative geometry would help dispel questions regarding the existence of geometric objects. On this matter it is important to note that during the first decade of the century, geometry textbooks started to enlarge the introductory section (traditionally used to define some geometric objects and state the postulates) incorporating elements of concrete geometry (particularly constructions). Hence, even if students had not actually seen concrete geometry in elementary school, the secondary course would separate the geometry studies between a first encounter with the "facts" and a second encounter with their logical organization as proposed by the Committee of Ten.

6.3.2. *Changes in notations and in the way diagrams were given*
Innovations in notation and language use were also geared toward supporting student proving. The notion that a proof was composed of 'concise steps' (Beman and Smith, 1899, p. 20) could operate normatively to prevent students from interjecting unwarranted arguments. Yet, to do so it had to be supported by available language that permitted such conciseness. Hence, it became customary to write proofs using notation more flexibly (e.g., various ways of denoting angles). It also became customary to shorten the statements of special propositions (e.g., '*a. s. a'* for the theorem that asserts triangle congruence based on the congruence of two angles and their common side). It seems that such abbreviations would support students' random access to previous theorems to quote them as reasons when producing a proof. All of these changes would help students use productively the notion that a proof was composed of statements and reasons. A diversity of notation and a plurality of technical words would protect from the risk of having to shift the activity of proof production into one of searching for the canonical way of writing a statement or quoting a reason. Instead, random and rapid availability of propositions that could be used as *reasons* could be used heuristically to think of *statements* that might be possible to make.

Another feature that came to support student proving was the way diagrams were provided. Newcomb et al. (1893) had recommended that students do oral exercises at the blackboard assisted only by a figure. Young et al. (1899) had also said that teachers should give students "frequent drills in seeing relations in a given figure..." (p. 142).

To facilitate students' inspection of diagrams, Schultze (1912) provided "graphic methods for presenting geometric facts," devised to "remove all external difficulties" for "students who can reason logically" but who forget "parts of the proof, and hence are unable to continue" (p. 110). Like some others before him, Schultze used small numbers to indicate which angles of a figure were of interest for a proof. He added the suggestion that a small square be used to indicate whether an angle was right and that equal number of arcs inside angles be used to signify that those were congruent. Other symbols were devised to indicate parallel lines or congruent segments in a diagram. These 'graphic methods' would not just dress the figure to express visually the same information contained in the premises of the proposition to prove. The figure so dressed would unpack what the premises involved (e.g., to give a triangle with equal number of hash marks on two sides and equal number of arcs on two angles made explicit more than to merely say "consider an isosceles triangle"). In doing so, the figure would indicate to the student that those pieces of information had to be used in the proof.

7. The two-column proof format: Bringing stability to the geometry course

Sections 3 through 6 have accounted for the evolution of a custom of student proving, developed as the teaching profession and the textbook authors tried to meet the expectation that the course of studies in geometry teach students the art of proving. The evolution that took place during the Era of Exercise included further changes to the geometry course, some of which went in a direction different from those of the Era of Originals. The notion that all students had to learn to prove, practice it, and succeed on it, but that not all of them should be expected to generate new knowledge, was behind the change from originals to exercises. To make room and streamline support for work on exercises, the propositions of the course were reduced to a smaller number of 'fundamentals,' whose proofs students would learn in order to know how to prove and what to use in their proofs. In this section I use that historical account to provide an interpretation of the role that the two-column format played in the establishment of that custom. I intend to put forward the hypothesis that the two-column proof format brought stability to the geometry curriculum by providing a way to meld the proofs given by the text and the proofs asked from students. In the following subsections I elaborate that hypothesis.

7.1. *Fundamentals and exercises: Similar and different*

The words 'fundamentals' and 'exercises' were brought in to refer to the two categories of *propositions* to be proved. I contend that the function assigned to propositions in each of those categories had the effect of increasing the distance between the proving work of the student and the proving work of the teacher (or the text). For example, whereas proofs for self-evident propositions were not to be stressed in the course of studies, exercises were supposed to call for easy proofs whose building blocks could be discovered by looking at a figure. Also, whereas fundamentals were to be the important building blocks of the subject-matter and were chosen because they were useful in proving other propositions, the main criterion for choosing exercises was the chances for students' practice and success, not how interesting or consequential were their results.

However, both kinds of propositions, no matter how different in substance, were instrumental in supporting the new identity of the course, centered on student proving. For students to have 'reasons' to justify all their statements as they proved, they needed the powerful results provided by the fundamentals. The more compelling the proofs for those fundamentals the better students would see "the chain of reasoning by which the proofs [are] discovered" (Young, 1906, p. 261). Proofs of fundamentals would thus illustrate why mathematical proofs are necessary and show what kind of reasoning goes into proving. To give those proofs was essential for geometry teachers not just as a way of justifying the propositions under study but also as a warrant for them to be able to claim that they were indeed *teaching* the art of proving. Yet, in order to ensure that students were actually *learning* such art of proving, it was required that students succeed in crafting proofs. Exercises would have to target simpler arguments as a way to aim at students' practice in the formal aspects of proving, without making their success contingent on their abilities to come up with the ideas for an argument. Thus the identity of the geometry course was established on the *interdependence* between fundamentals and exercises, but the viability of the course was to be obtained by *distancing* exercises from fundamentals. This simultaneous dependence and distance created a problematic situation. I explain why such state of affairs was problematic in the next section.

7.2. *Making fundamentals and exercises refer to the same object*

As the fundamentals and exercises distinction came to the fore, the problem emerged of how to make so different activities as proving fundamentals and solving exercises refer to a same object of learning, namely the art of proving. Exercises were very different from fundamentals in regard to their substance. Whereas fundamentals were consequential, often important geometric theorems, exercises were obvious or immediate consequences of propositions, inconsequential for further activity. To prove the fundamentals would have been mathematically in order (they did not go without saying, to prove them was mathematically relevant). But to prove the propositions stated as exercises would have been mathematically uninteresting (even a diehard formalist would have agreed that it was evident they could be proved, so why bother?). To hold fundamentals and exercises as comparable objects may have been an action needed for the viability of instruction, but an action that created instability in the subject of studies.

To solve that problem, *proof*, as an object of the discourse of geometry instruction, would need to evolve, beyond the traditional implicit notion that proof was the process by which the truth of a proposition is demonstrated (Legendre, 1841, p. 3). To do so would help ease the differences between fundamentals and exercises and make them serve together the purpose of the high school geometry class. As soon as it could be acknowledged that fundamentals and exercises were specimens of the same kind, the former would be justified as preliminary training to the latter, and the latter would be a valid criterion to assess the acquisition of the former. The two column proof format would do that job at the expense of making *proof* come to the foreground as an object of teaching and learning whose main thrust was in logical form rather than in the geometric substance.

The move from the implicit norms for proving provided by model, complete proofs (e.g., Wentworth, 1888), through descriptions of proof (e.g., Beman and Smith, 1899), and eventually to the two-column format (e.g., Schultze and Sevenoak, 1913) had the effect of stabilizing the geometry course. It gave students an 'objective' representation that enabled smoother recognition of the similarities between such different activities as proving fundamental propositions and solving proof exercises.

The emphasis on formal aspects of proving permitted a sort of trade between teacher and students in regard to proving fundamentals and exercises. It was understood that a mature reader would not need reasons for every statement to understand a proof. However, it was expected that teachers and textbooks present proofs of fundamentals in the form of statements and reasons: The argument was that only in that way the 'beginner' would understand how the argument had been put together (Young, 1906, p. 262).

Young was suggesting that giving the reason that justified a statement was equivalent to spelling out the conditions that made such statement relevant. By holding him or herself accountable for sharing those heuristics, a teacher would enable students to discover the proofs for fundamentals. The teacher would be able to use the same argument to hold the student to the formal norm of proving embodied in the two-column format – the teacher could pretend that the proof done by the student would not be understandable unless all reasons had been given. Thus, on the one hand the teacher would use the two column format to *explain* (in the sense of help understand) *the proof* to students. On the other hand the students would use the two column format to *explain* (in the sense of show their work on) *the proof* to the teacher.

The two-column proving custom was an accomplishment of geometry instruction in the sense that it helped comply with a mandate. But that accomplishment did not come for free. It brought to the fore the logical aspects of a proof at the expense of the substantive role of proof in knowledge construction. Questions about the relevance or the strength of the propositions proved, or about the accomplishments and potential use of the theories being developed, were left in the background. This happened not because those issues lacked advocates (Dewey, 1903; Moore, 1902; Smith, 1909), but perhaps because those issues were difficult to reconcile, in practice, with the more pressing mandate that the course would teach all students the art of proving.

8. Conclusion: What can we learn from the history of proving in geometry?

The history of how the high school geometry course accommodated the mandate to teach the art of demonstration gives us an important analogy to think about the renewed emphasis on proof in mathematics education. We learn that incorporating student proving as a normative element of the curriculum involves important work that is systemic in nature. Many aspects of geometry instruction were involved in responding to the mandate of the Committee of Ten. Notably, changes in the conception of students as learners effected and were affected by changes on the nature of the subject of studies. But also, to make student proving possible, a system of resources had to be developed and coordinated with a norm for accomplished proofs. The integration of all those elements produced a stable geometry course oriented toward students' learning the art of proving embodied in the two-column format. However, that stability came with a price – that of dissociating the doing of proofs from the construction of knowledge. Whereas notable attempts to change the role of proof in school geometry have taken place during the 20th century (e.g., Fawcett, 1938), the reduction of proof to its logical, formal aspects has endured for about a century now. The comment of an undergraduate student of mine speaks about the same issue: "We did proofs in school, but we never proved anything."

When we think about the present recommendations that students should be engaged in proving across the school years and in all subjects (NCTM, 2000), we should take into consideration those lessons from history. First of all, recommendations for practice are not viable by decree; to make them viable requires work, and that work is systemic in nature. In order to make proof a salient element of school mathematics, several other elements must be acted upon. We must ask how does our notion of proof cohere with our conceptions of the student as learner and with the course of mathematical studies where that learning is to take place. Further, what kind of resources and norms are needed in order to enable students to do proofs that deserve a place within that course of studies and that cohere with the goals we have for students' learning in the context of mathematical work in classrooms?

A second lesson is that whereas engaging students in proving is something we must talk about, it may be dangerous to talk about it outside of the context of developing knowledge of specific mathematical ideas and finding out things that are interesting about those ideas. We learn from Lakatos (1976, 1978) that mathematical proof is not a generic logical process, but a substantive methodological tool for developing and shaping concepts, for finding out which theorems may be true and why. If this valuable role of proof as a tool to create knowledge merits a place in classrooms – and I think it does – we need to be leery of thinking of proof as something independent of the domain of mathematical activity where it is being used. Proving and proofs may look different according to the questions being asked and the knowledge and tools available to those who address those questions in classrooms. Proof is essential in mathematics education not only as a valuable process for students to engage in (such as developing their capacity for mathematical reasoning) but, more importantly, as a necessary aspect of knowledge construction. This historical investigation on the evolution of proof in American education suggests that making proof a separate object of study will not empower children to use proof as a means to know with, but will rather separate the practices of proving from the practices of knowing.

It would be unfair to judge the evolution of the two-column proving custom with the criteria of the present. It would be equally unfair to interpret the renewed emphasis on proof as a warrant to insist on teaching and learning any 'art of proving.' The *Principles and Standards* (NCTM, 2000) provide the elements to envision classrooms where students' active participation in making knowledge claims is accompanied by responsibility to argue why those claims are reasonable. Thus we need to develop means to address the Standard of Reasoning and Proof so as to keep a balance between substance and form as we ask students to craft arguments and proofs. The crucial question should not be whether students' arguments are expressed in a form that a logician would approve of, but whether they are adapted to the nature of the mathematical objects that a community of knowers wants to know more about.

NOTES

1. The author acknowledges valuable comments from Daniel Chazan, Humberto Alagia, and two anonymous reviewers.
2. Reform ideas in mathematics (geometry in particular) were emerging in other countries as well (see Howson, 1982, pp. 134–136, 155–159; Kilpatrick, 1992, Poincaré, 1899). In writing this paper I have limited the corpus of archival sources to those that influenced directly American classrooms, though several of those indeed make reference to trends that were taking place internationally (see, for example, Moore, 1902; Schulze, 1912; Slaught et al., 1912).
3. I have not come across records of what actual geometry instruction looked like in the period of interest. The choice to focus on textbooks as records of instruction is thus a simplification warranted on pragmatic reasons.

REFERENCES

Baker, J.: 1969, 'Minority report to the national council of education', in National Education Association, *Report of the Committee on secondary school studies*, Arno Press, New York, pp. 56–59. (Original work published 1893)

Ball, D.L. and Bass, H.: 2000, 'Making believe: The collective construction of public mathematical knowledge in the elementary classroom', in D. Phillips (ed.), *Constructivism in Education: Yearbook of the National Society for the Study of Education*, University of Chicago Press, Chicago, pp. 193–224.

Beman, W. and Smith, D.E.: 1899, *New Plane and Solid Geometry*, Ginn, Boston.

Carson, G.: 1913, *Essays on Mathematical Education*, Ginn, London.

Chauvenet, W.: 1870, *A Treatise on Elementary Geometry with Appendices containing a Collection of Exercises for Students and an Introduction to Modern Geometry*, Lippincot, Philadelphia.

Chauvenet, W.: 1898, *Treatise on Elementary Geometry*, W. Byerly (ed.), Lippincot, Philadelphia. (Original work published 1887)

Davies, C.: 1850, *The Logic and Utility of Mathematics, with the Best Methods of Instruction Explained and Illustrated*, Barnes, New York.

Dewey, J.: 1903, 'The psychological and the logical in teaching geometry', *Educational Review* 25, 387–399.

Donoghue, E.: 1987, *The Origins of a Professional Mathematics Education Program at Teachers College*, Unpublished doctoral dissertation. Columbia University Teachers College.

Eliot, C. et al.: 1969, 'Report of the Committee of Ten to the National Education Association', in National Education Association, *Report of the Committee on secondary school studies*, Arno Press, New York, pp. 3–5. (Original work published 1893)

Eliot, C.: 1905, 'The fundamental assumptions in the report of the Committee of Ten (1893)', *Educational Review* 30, 325–343.

Fawcett, H.: 1938, *The Nature of Proof – The National Council of Teachers of Mathematics Thirtheenth Yearbook*, Bureau of Publications of Teachers College, Columbia University, New York.

Greenleaf, B.: 1858, *Elements of Geometry with Practical Applications to Mensuration,* Robert Davis, Boston.

Halsted, G.: 1893, 'The old and the new geometry', *Educational Review* 6, 144–157.
Harris, W.T.: 1894, 'The committee of ten on secondary schools', *Educational Review* 7, 1–10.
Hilbert, D.: 1971, *Foundations of Geometry*, L. Unger, Trans., P. Bernays, Rev.. Open Court, La Salle, IL. (Original work published in German in 1899)
Hill, F.: 1895, 'The educational value of mathematics', *Educational Review* 9, 349–358.
Howson, G.: 1982, *A History of Mathematics Education in England*, Cambridge University Press, Cambridge.
Jones, P.: 1944, 'Early American geometry', *The Mathematics Teacher* 37, 3–11.
Kilpatrick, J.: 1992, 'A history of research in mathematics education', in D. Grouws (ed.), *Handbook of Research in Mathematics Teaching and Learning*, Macmillan, New York, pp. 3–38.
Kliebard, H.: 1986, *The Struggle for the American Curriculum 1893–1958*, Routledge and Kegan Paul, Boston.
Kline, M.: 1965, 'View of the new math', in E. Moise, A. Calandra, R. Davis, M. Kline and H. Bacon (eds.), *Five Views of the "New Math'*, (Council for Basic Education, Washington, DC, pp. 13–16.
Krug, E.: 1964, *The Shaping of the American High School*, Harper and Row, New York.
Lakatos, I.: 1976, *Proofs and Refutations: The Logic of Mathematical Discovery*, J. Worrall and E. Zahar (eds.), Cambridge University Press, Cambridge.
Lakatos, I.: 1978, 'A renaissance of empiricism in the recent philosophy of mathematics?' in J. Worrall and G. Currie (eds.), *Imre Lakatos. Mathematics, Science and Epistemology: Philosophical Papers*, Cambridge University Press, Cambridge, Vol. 2, pp. 24–42. (Original work published in 1967)
Legendre, A.-M.: 1819, *Elements of Geometry*, J. Farrar (ed. and trans.), Hilliard and Metcalf, Cambridge, New England.
Legendre, A.-M.: 1841, *Elements of Geometry*, J. Farrar (ed. and trans.), Hilliard Gray, Boston.
Legendre, A.-M.: 1848, *Elements of Geometry and Trigonometry*, D. Brewster (trans.), C. Davies (rev. and ed.), Barnes, New York.
Moore, E.H.: 1926, 'On the foundations of mathematics', in C. Austin, H. English, W. Betz, W. Eells and F. Touton (eds.), *A General Ssurvey of Progress in the Last Twenty-Five Years: First Yearbook*, National Council of Teachers of Mathematics, Washington, DC, pp. 32–57. (Original speech delivered in 1902)
NCTM: 2000, *Principles and Standards for School Mathematics*, Author, Reston, VA.
Newcomb, S. et al.: 1893, 'Reports of the conferences: Mathematics', in National Education Association, *Report of the Committee on secondary school studies*, Arno Press, New York, pp. 104–116. (Original work published 1893)
Nightingale, A. et al.: 1899, 'Report of the Committee on College entrance requirements', in National Education Association, *Report of Committee on College Entrance Requirements – July, 1899*, NEA, Chicago, pp. 5–49.
Playfair, J.: 1860, *Elements of Geometry; Containing the First Six Books of Euclid, with a Supplement on the Quadrature of the Circle, and the Geometry of Solids. To which are Added Elements of Plane and Spherical Trigonometry*, Collins and Hannay, New York. (Original work published 1795)
Poincaré, H.: 1899, 'La logique et l'intuition dans la science mathématique et dans l'enseignement', *L'Enseignement Mathématique* 1, 157–162.

Quast, W.G.: 1968, *Geometry in the High Schools of the United States: An Historical Analysis from 1890 to 1966*, Unpublished doctoral dissertation. Rutgers – The State University of New Jersey, New Brunswick.

Rav, Y.: 1999, 'Why do we prove theorems?' *Philosophia Mathematica* 7, 5–41.

Ravitch, D.: 2000, *Left Back: A Century of Failed School Reforms*, Simon and Shuster, New York.

Richards, E.: 1892, 'Old and new methods in elementary geometry', *Educational Review* 3, 31–39.

Schoenfeld, A.: 1987, 'On having and using geometrical knowledge', in J. Hiebert (ed.), *Conceptual and Procedural Knowledge: The Case of Mathematics*, Erlbaum, Hillsdale, NJ, pp. 225–264.

Schultze, A.: 1912, *The Teaching of Mathematics in Secondary Schools*, MacMillan, New York.

Schultze, A. and Sevenoak, F.: 1901, *Plane and Solid Geometry*, MacMillan, New York.

Schulze, A. and Sevenoak, F.: 1913, *Plane Geometry*, A. Schultze, (rev.), MacMillan, New York.

Shibli, J.: 1932, *Recent Developments in the Teaching of Geometry*, Author, State College, PA.

Shutts, G.: 1892, 'Old and new methods in geometry', *Educational Review* 3, 264–266.

Simson, R.: 1756, *The Elements of Euclid, viz. the First Six Books together with the Eleventh and Twelfth. In this edition, the Errors, by which Theon, or others, have long ago vitiated these Books, are corrected, and some of Euclid's Demonstrations are Restored*, Foulis, Glasgow.

Sizer, T.: 1964, *Secondary Schools at the Turn of the Century*, Yale University Press, New Haven.

Slaught, H. et al.: 1912, 'Final report of the National Committee of Fifteen on geometry syllabus', *The Mathematics Teacher* 5, 46–131.

Smith, D.E.: 1911, *The Teaching of Geometry*, Ginn, Boston.

Smith, E.R.: 1909, *Plane Geometry Developed by the Syllabus Method*, American Book Co, New York.

Stanic, G.M.A.: 1983, *Why Teach Mathematics? A Historical Study of the Justification Question*, Unpublished doctoral dissertation. University of Wisconsin, Madison.

Stanic, G.M.A.: 1987, 'Mathematics education in the United States at the beginning of the twentieth century', in T. Popkewitz (ed.), *The Formation of School Subjects: The Struggle for Creating an American Institution*, Falmer, New York, pp. 145–175.

Wells, W.: 1887, *The Elements of Geometry*, Leach, Shewell, and Sanborn, Boston.

Wells, W.: 1908, *New Plane and Solid Geometry*, Heath, Boston.

Wentworth, G.: 1878, *Elements of Geometry*, Ginn and Heath, Boston.

Wentworth, G.: 1888, *A Text-Book of Geometry*, Ginn, Boston.

Wentworth, G.: 1899, *Plane and Solid Geometry*, Ginn, Boston.

Young, J.W.A. et al.: 1899, 'Report of the Committee of the Chicago section of the American Mathematical Society', in National Education Association, *Report of Committee on College Entrance Requirements – July, 1899*, NEA, Chicago, pp. 135–149.

Young, J.W.A.: 1906, *The Teaching of Mathematics in the Elementary and the Secondary School*, Longmans, Green, and Co, New York.

The University of Michigan,
School of Education,
610 East University Avenue,
Ann Arbor, MI 48109-1259, USA,
E-mail: pgherbst@umich.edu