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ESTIMATES AND ITERATION PROCEDURES FOR PROPER VALUES OF ALMOST DECOMPOSABLE MATRICES

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Introduction. In the authors' paper [1] the following problem is considered: to estimate the spectrum of a matrix of the type

$$\begin{pmatrix} A_{11}, & A_{12} \\ A_{21}, & A_{22} \end{pmatrix}$$

where at least one of the matrices A_{12}, A_{21} is "small", in terms of the spectra of A_{11} and A_{22}. If the dimension of A_{11} is 1 and the distance between A_{11} and the spectrum of A_{22} is sufficiently large, an estimate has been obtained which contains as special cases the results of GERSHGORIN, OSTROWSKI, BRAUER and others.

In the present paper we improve further the estimate of [1] and we describe three iteration procedures which converge to the proper value near A_{11}. The estimates for the initial value in these procedures yield the generalizations mentioned. The results are formulated in one theorem the proof of which forms the contents of the present remark.

Definitions and notation. Let Y be a finite-dimensional complex vector space. The elements of Y will be considered as row-vectors $x = (x_1, \ldots, x_n)$ so that the column-vector with the same coordinates will be denoted by x'. The column vectors will be considered as elements of the adjoint space Y' so that the scalar product of an $x \in Y$ and a $y' \in Y'$ is the same as the ordinary matrix product $xy' = \sum x_i y_i$. Let g be a norm on the space Y. The adjoint norm g' on Y' is defined in the usual manner as

$$g'(y') = \sup |yy'| \quad \text{for} \quad g(y) = 1 .$$

Since we are dealing with a fixed coordinate system in Y, we shall not distinguish between a matrix and the linear operator defined by it. The operator norm corresponding to g is defined by the formula

$$g(B) = \sup g(yB) \quad \text{for} \quad g(y) = 1 .$$

We shall also need the function $\hat{g}$ defined by

$$\hat{g}(B) = \inf g(yB) \quad \text{for} \quad g(y) = 1 .$$

Note that $\hat{g}(B) = 0$ if B is singular and $\hat{g}(B) = (g(B^{-1}))^{-1}$ if B^{-1} exists. We shall frequently use the inequality

$$\hat{g}(A - B) \geqq \hat{g}(A) - g(B)$$

the proof of which is obvious.

If A is a matrix and t a complex number, we shall write simply $A - t$ for $A - tE$ where E is the unit matrix. In statements about the spectrum of a matrix the following abbreviation will be useful: if z_0 is a complex number and $\varrho > 0$ then $K(z_0; \varrho)$ will be the disk of all complex numbers z such that $|z - z_0| \leqq \varrho$. The open disk with the same center and radius is the set of all z such that $|z - z_0| < \varrho$ and will be denoted by $K^0(z_0; \varrho)$. Note that the inclusion $K(z_1; \varrho_1) \supset K(z_2; \varrho_2)$ is equivalent with $|z_1 - z_2| \leqq \varrho_1 - \varrho_2$. If $|z_1 - z_2| < \varrho_1 - \varrho_2$, the disk $K(z_2; \varrho_2)$ is contained in the interior of $K(z_1; \varrho_1)$.

Let M be the set of all realvalued lebesgue measurable functions defined on the domain

$$\Omega = \{\xi_1, \xi_2, \xi_3;\ \xi_1 \geqq 0,\ \xi_2 \geqq 0,\ \xi_3 \geqq 0 \quad \text{and} \quad \sqrt{\xi_1} + \sqrt{\xi_2} < \sqrt{\xi_3}\}\,.$$

If $x = (\xi_1, \xi_2, \xi_3)$ and $y = (\eta_1, \eta_2, \eta_3)$ both belong to this domain, we shall write $x \leqq y$ if $\xi_1 \leqq \eta_1$, $\xi_2 \leqq \eta_2$ and $\xi_3 \geqq \eta_3$. Let M^+ be the subset of M consisting of all $f \in M$ such that $x_1 \leqq x_2$ implies $f(x_1) \leqq f(x_2)$. The set M^- is defined as $-M^+$.

To simplify some expressions which will occur in the main text, let us introduce some abbreviations.

For $x = (\xi_1, \xi_2, \xi_3)$, let

$$\begin{aligned}
S_1(x) &= -\xi_1 + \xi_2 + \xi_3\,,\\
S_2(x) &= \xi_1 - \xi_2 + \xi_3\,,\\
S_3(x) &= -\xi_1 - \xi_2 + \xi_3\,,\\
W(x) &= (\xi_1^2 + \xi_2^2 + \xi_3^2 - 2\xi_1\xi_2 - 2\xi_1\xi_3 - 2\xi_2\xi_3)^{\frac{1}{2}}\,,\\
L_i(x) &= \frac{S_i(x) - W(x)}{S_i(x) + W(x)}\,, \qquad i = 1, 2, 3,\\
Q(x) &= \frac{4\xi_1\xi_2}{(S_3(x))^2}\,,\\
R_k^{(1)}(x) &= W(x)\,\frac{L_2(x)\,(L_1(x))^k}{1 - L_2(x)\,(L_1(x))^k}\,, \qquad k = 0, 1, 2, \ldots,\\
R_k^{(2)}(x) &= W(x)\,\frac{L_2(x)\,(L_3(x))^k}{1 - L_2(x)\,(L_3(x))^k}\,,\\
R_k^{(3)}(x) &= \frac{1}{2^k}\,S_3(x)\,(Q(x))^{2^k - 1}\,,\\
R(x) &= \tfrac{1}{2}(S_2(x) + W(x))\,.
\end{aligned}$$

(1,1) *The functions S_i, W, L_i, Q, $R_k^{(p)}$, R all belong to M. Moreover, for $x \in \Omega$ we have $W(x) > 0$, $R(x) > R_0^{(1)}(x)$, $L_i(x) < 1$ and $\lim_k R_k^{(p)}(x) = 0$.*

Proof. It suffices to prove the statement about W. This, however, follows immediately from the relation

$$\xi_1^2 + \xi_2^2 + \xi_3^2 - 2\xi_1\xi_2 - 2\xi_1\xi_3 - 2\xi_2\xi_3 = (\sqrt{\xi_3} - \sqrt{\xi_1} - \sqrt{\xi_2}) \cdot$$
$$\cdot (\sqrt{\xi_3} - \sqrt{\xi_1} + \sqrt{\xi_2})(\sqrt{\xi_3} + \sqrt{\xi_1} - \sqrt{\xi_2})(\sqrt{\xi_3} + \sqrt{\xi_1} + \sqrt{\xi_2}) > 0\,.$$

(1,2) *If $x \in \Omega$, put*

$$\Phi(x) = \left(\frac{\xi_1\xi_3}{\xi_3 - \xi_2}, \frac{\xi_1\xi_2}{\xi_3 - \xi_2}, \quad \xi_3 - \xi_2\right)$$

where ξ_1, ξ_2, ξ_3 are the coordinates of x. The mapping Φ is a mapping of Ω into Ω and preserves the order $\leqq$.

Further, $W(\Phi(x)) = W(x)$, $S_1(\Phi(x)) = S_3(x)$, $L_1(\Phi(x)) = L_3(x)$, $R_k^{(1)}(\Phi(x)) = R_k^{(2)}(x)$ for $k = 1, 2, \ldots$.

Proof. Denote for a moment by $\sigma_1, \sigma_2, \sigma_3$ the coordinates of $\Phi(x)$. Since $x \in \Omega$, $\xi_3 > \xi_2$ so that $\sigma_1, \sigma_2, \sigma_3$ exist and are nonnegative. Further,

$$\sqrt{\sigma_3} = \sqrt{\xi_3 - \xi_2} > \sqrt{\frac{\xi_1\xi_3}{\xi_3 - \xi_2}} + \sqrt{\frac{\xi_1\xi_2}{\xi_3 - \xi_2}} = \sqrt{\sigma_1} + \sqrt{\sigma_2}$$

since

$$\xi_3 - \xi_2 = (\sqrt{\xi_3} - \sqrt{\xi_2})(\sqrt{\xi_3} + \sqrt{\xi_2}) > \sqrt{\xi_1}\,(\sqrt{\xi_3} + \sqrt{\xi_2}) = \sqrt{\xi_1\xi_3} + \sqrt{\xi_1\xi_2}\,.$$

Thus, $\Phi(x) \in \Omega$. Since

$$\sigma_1 = \xi_1 + \frac{\xi_1\xi_2}{\xi_3 - \xi_2}, \quad \sigma_2 = \frac{\xi_1\xi_2}{\xi_3 - \xi_2}, \quad \sigma_3 = \xi_3 - \xi_2\,,$$

Φ preserves the order $\leqq$.

Further, it is easily seen that $S_1(\sigma_1, \sigma_2, \sigma_3) = S_3(\xi_1, \xi_2, \xi_3)$. It follows that

$$W(\sigma_1, \sigma_2, \sigma_3) = \sqrt{S_1^2(\sigma_1, \sigma_2, \sigma_3) - 4\sigma_2\sigma_3} = \sqrt{S_3^2(\xi_1, \xi_2, \xi_3) - 4\xi_1\xi_2} =$$
$$= W(\xi_1, \xi_2, \xi_3)\,.$$

The rest is obvious.

(1,3) *Let $x \in \Omega$, $x = (\xi_1, \xi_2, \xi_3)$. Suppose that the numbers $X_0, X_1, \ldots, X_k$ and the nonnegative numbers $Y_0, Y_1, \ldots, Y_k$ satisfy the inequalities*

$$(1) \qquad X_0 \geqq \xi_3\,, \quad Y_0 \leqq \xi_1\,,$$

$$X_j \geqq X_{j-1} - Y_{j-1}\,, \quad Y_j \leqq \xi_2\xi_3 \frac{Y_{j-1}}{X_{j-1}X_j}\,, \quad j = 1, 2, \ldots, k\,;$$

Then

$$X_j \geqq X_j^0(x),\ \ Y_j \leqq Y_j^{(0)}(x) \quad \text{for } j = 0, 1, 2, \ldots, k\,,$$

where

$$X_j^0(x) = \tfrac{1}{2}\left(S_1(x) + W(x)\right) \frac{1 - L_2(x)\left(L_1(x)\right)^{j+1}}{1 - L_2(x)\left(L_1(x)\right)^{j}}\,,$$

$$Y_j^0(x) = \tfrac{1}{2}\left(S_1(x) + W(x)\right) \frac{L_2(x)\left(1 - L_1(x)\right)^2}{\left(1 - L_2(x)\left(L_1(x)\right)^j\right)\left(1 - L_2(x)\left(L_1(x)\right)^{j+1}\right)} \left(L_1(x)\right)^j =$$

$$= W(x)\left(1 - L_1(x)\right) \frac{L_2(x)\left(L_1(x)\right)^j}{\left(1 - L_2(x)\left(L_1(x)\right)^j\right)\left(1 - L_2(x)\left(L_1(x)\right)^{j+1}\right)}$$

form the (unique) solution of the system (1) *with inequalities replaced by equalities.*

Proof. Let us show first that the numbers X_j^0 and Y_j^0 satisfy the system of equations (1). Indeed.

$$X_0^0 = \tfrac{1}{2}(S_1 + W) \frac{1 - L_2 L_1}{1 - L_2} = \tfrac{1}{2}(S_1 + S_2) = \xi_3$$

and

$$Y_0^0 = \tfrac{1}{2}(S_1 + W) \frac{L_2(1 - L_1)^2}{(1 - L_2)(1 - L_2 L_1)} = \xi_1\,.$$

Further

$$X_j^0 = \tfrac{1}{2}(S_1 + W) \frac{1 - L_2 L_1{}^{j+1}}{1 - L_2 L_1^j} =$$

$$= \tfrac{1}{2}(S_1 + W)\left(\frac{1 - L_2 L_1^j}{1 - L_2 L_1^{j-1}} - \frac{L_2(1 - L_1)^2\, L_1^{j-1}}{(1 - L_2 L_1^{j-1})(1 - L_2 L_1^j)}\right) = X_{j-1}^0 - Y_{j-1}^0\,,$$

$$Y_j^0 = \tfrac{1}{2}(S_1 + W) \frac{L_2(1 - L_1)^2}{(1 - L_2 L_1^j)(1 - L_2 L_1^{j+1})} L_1^j =$$

$$= \frac{\tfrac{1}{4}(S_1^2 - W^2)}{\tfrac{1}{4}(S_1 + W)^2} \frac{\tfrac{1}{2}(S_1 + W)\, L_2(1 - L_1)^2}{\dfrac{1 - L_2 L_1^j}{1 - L_2 L_1^{j-1}} \cdot \dfrac{1 - L_2 L_1^{j+1}}{1 - L_2 L_1^j}} \frac{L_1^{j-1}}{(1 - L_2 L_1^{j-1})(1 - L_2 L_1^j)} =$$

$$= \xi_2 \xi_3 \frac{Y_{j-1}^0}{X_{j-1}^0\, X_j^0}\,.$$

The proof of our lemma will be complete if we show that any solution X_j, Y_j of the system of inequalities satisfies

$$X_j \geqq X_j^0 \quad \text{and} \quad Y_j \leqq Y_j^0\,.$$

This can be easily done by induction.

(1,4) *The following inclusions hold:*

$X_j^0 \in M^-$, $Y_j^0 \in M^+$ *for* $j = 0, 1, 2, \ldots$; $R \in M^-$; $R_j^{(p)} \in M^+$ *for* $p = 1, 2, 3$ *and* $j = 0, 1, 2, \ldots$.

Proof. Let $x, y \in \Omega$ and $x \lesssim y$. Clearly

$$X_0^0(x) = \xi_3 \geqq \eta_3 = X_0^0(y) \quad \text{and} \quad Y_0^0(x) = \xi_1 \leqq \eta_1 = Y_0^0(y).$$

Suppose the monotonicity of both X_j^0 and Y_j^0 has already been proved up to some k. We have

$$X_{k+1}^0(x) = X_k^0(x) - Y_k^0(x) \geqq X_k^0(y) - Y_k^0(y) = X_{k+1}^0(y)$$

by induction hypothesis.

In order to show $Y_{k+1}^0(x) \leqq Y_{k+1}^0(y)$, let us prove first the inequality

$$\frac{\eta_3}{X_k^0(y)} \geqq \frac{\xi_3}{X_k^0(x)}. \tag{2}$$

Indeed,

$$X_k^0(y) = \eta_3 - \sum_{j=0}^{k-1} Y_j^0(y) \leqq \eta_3 - \sum_{j=0}^{k-1} Y_j^0(x)$$

and

$$X_k^0(x) = \xi_3 - \sum_{j=0}^{k-1} Y_j^0(x).$$

Since $\eta_3 \leqq \xi_3$, we have thus

$$\frac{\eta_3}{X_k^0(y)} \geqq \frac{\eta_3}{\eta_3 - \sum_{j=0}^{k-1} Y_j^0(x)} \geqq \frac{\xi_3}{\xi_3 - \sum_{j=0}^{k-1} Y_j^0(x)} = \frac{\xi_3}{X_k^0(x)}.$$

Using the inequality (2) together with $X_{k+1}^0(x) \geqq X_{k+1}^0(y)$, $\xi_2 \leqq \eta_2$ and $Y_k^0(x) \leqq \leqq Y_k^0(y)$ it follows that

$$Y_{k+1}^0(x) = \frac{\xi_2 \xi_3 Y_k^0(x)}{X_k^0(x)\, X_{k+1}^0(x)} \leqq \frac{\xi_2 \eta_3 Y_k^0(x)}{X_k^0(y)\, X_{k+1}^0(x)} \leqq \frac{\eta_2 \eta_3 Y_k^0(y)}{X_k^0(y)\, X_{k+1}^0(y)} = Y_{k+1}^0(y)$$

which completes the induction.

To show that $R \in M^-$ it is sufficient to observe that

$$\frac{\partial R}{\partial \xi_1} = -\frac{S_1 - W}{2W} < 0,$$

$$\frac{\partial R}{\partial \xi_2} = -\frac{S_2 + W}{2W} < 0,$$

$$\frac{\partial R}{\partial \xi_3} = \frac{S_3 + W}{2W} > 0.$$

Let us show now that $R_k^{(3)} \in M^+$. Indeed,

$$R_k^{(3)} = \frac{1}{2^k} (S_3 Q) (Q^{2^{k-1}-1})$$

and both

$$S_3 Q = \frac{4\xi_1 \xi_2}{\xi_3 - \xi_1 - \xi_2} \quad \text{and} \quad Q = \frac{4\xi_1 \xi_2}{(\xi_3 - \xi_1 - \xi_2)^2}$$

are evidently members of M^+.

To prove the monotonicity of the $R_k^{(1)}$, we observe first that, for $k = 0, 1, 2, \ldots$

$$Y_k^0(x) = R_k^{(1)}(x) - R_{k+1}^{(1)}(x).$$

This can be easily verified.

Hence $R_k^{(1)}(x) = \sum_{j=k}^{k+p-1} Y_j^0(x) + R_{k+p}^1(x)$. Evidently the series $\sum Y_j^0(x)$ is absolutely convergent for $x \in \Omega$ and $R_n^{(1)}(x)$ converges to zero for $x \in \Omega$. It follows that $R_k^{(1)}(x) = \sum_{j=k}^{\infty} Y_j^0(x)$. Since $Y_j^0 \in M^+$ we have $R_k^{(1)} \in M^+$ as well.

The monotonicity of the $R_k^{(2)}$ follows from the relation $R_k^{(2)}(x) = R_k^{(1)}(\Phi(x))$ using the inclusion $R_k^{(1)} \in M^+$ and the fact that Φ is order preserving.

The starting point of the further investigations is the following simple observation which we formulate as a lemma.

(1,5) *Let A be a matrix of the form*

$$A = \begin{pmatrix} a_{11}, & a_1 \\ a_2', & A_{22} \end{pmatrix},$$

the blocks being of dimensions 1 *and* n. *Suppose that λ is not a proper value of A_{22}. Then*

$$\det(A - \lambda) = (a_{11} - \lambda - a_1(A_{22} - \lambda)^{-1} a_2') \det(A_{22} - \lambda)$$

so that λ is a proper value of A if and only if

$$a_{11} - \lambda = a_1(A_{22} - \lambda)^{-1} a_2'.$$

Proof. An immediate consequence of the relation

$$(A - \lambda)\begin{pmatrix} 1 & , & 0 \\ -(A_{22} - \lambda)^{-1} a_2', & & (A_{22} - \lambda)^{-1} \end{pmatrix} =$$

$$= \begin{pmatrix} a_{11} - \lambda - a_1(A_{22} - \lambda)^{-1} a_2', & a_1(A_{22} - \lambda)^{-1} \\ 0 \quad , & E \end{pmatrix}.$$

Now we are able to formulate the main result.

(2,1) Theorem. *Let A be a matrix of the form*

$$A = \begin{pmatrix} a_{11}, & a_1 \\ a'_2, & A_{22} \end{pmatrix},$$

the blocks being of dimensions 1 *and n. Suppose that* $B = A_{22} - a_{11}$ *is nonsingular and that we have three numbers* $\alpha_1, \alpha_2, \alpha_3$ *which satisfy the inequalities*

(3)
$$|a_1 B^{-1} a'_2| \leqq \alpha_1,$$
$$g(a_1)\, g'(a'_2)/\hat{g}(B) \leqq \alpha_2,$$
$$\hat{g}(B) \geqq \alpha_3.$$

Suppose that the condition

(4)
$$\sqrt{\alpha_1} + \sqrt{\alpha_2} < \sqrt{\alpha_3}$$

is fulfilled. Let us denote by a the vector $(\alpha_1, \alpha_2, \alpha_3)$ *so that* $a \in \Omega$. *Then the following statements hold:*

1° *The open disk* $U^* = K^0(a_{11}; r)$, $r = R(a)$ *contains exactly one proper value of A; this will be denoted by x.*

2° *The following three iteration procedures are meaningful and convergent to x:*

(P1)
$$x_0 = a_{11},\ x_{k+1} = a_{11} - a_1(A_{22} - x_k)^{-1}\, a'_2, \quad \lim x_k = x;$$

(P2)
$$y_0 = a_{11},\ y_{k+1} = a_{11} - \frac{a_1 B^{-1} a'_2}{1 + a_1(A_{22} - y_k)^{-1}\, B^{-1} a'_2}, \quad \lim y_k = x;$$

(P3)
$$B_0 = B, \quad c'_0 = B_0^{-1} a'_2,$$
$$B_{k+1} = B_k + c'_k a_1 + a_1 c'_k, \quad c'_{k+1} = -a_1 c'_k B_{k+1}^{-1} c'_k,$$
$$\lim z_k = x \quad \text{where} \quad z_k = a_{11} - \sum_{\gamma=0}^{k-1} a_1 c'_j.$$

Further, we have the following three sequences of inclusions

(5)
$$U^* \supset K(x_0; r_0^{(1)}) \supset K(x_1; r_1^{(1)}) \supset \dots,$$
$$U^* \supset K(y_0; r_0^{(2)}) \supset K(y_1; r_1^{(2)}) \supset \dots,$$
$$U^* \supset K(z_1; r_1^{(3)}) \supset K(z_2; r_2^{(3)}) \supset \dots,$$

where $r_k^{(p)} = R_k^{(p)}(a)$, *the point x being the only point of intersection of each of these sequences.*

3° *Suppose that we replace the estimates* (3) *by less favourable ones, which again satisfy condition* (4). *More precisely, take another vector* $a' \in \Omega$, $a' \geqq a$. *Of course, all three processes remain unchanged; the estimates* (5) *will be, however, less favourable: the domain* U^* *shrinks and all the radii* $r_k^{(p)}$ *increase.*

(2,2) Corollary. *Let A be a matrix of the form*

$$A = \begin{pmatrix} a_{11}, & a_1 \\ a_2', & A_{22} \end{pmatrix},$$

the dimensions of the blocks being 1 and n. Suppose that $B = A_{22} - a_{11}$ is non-singular. Let β_1, β_2 and γ be real numbers such that

$$g(a_1) \leqq \beta_1, \quad g'(a_2') \leqq \beta_2, \quad \hat{g}(B) \geqq \gamma > 0.$$

Suppose that $\gamma^2 > 4\beta_1\beta_2$. Under these conditions the disk

$$K(a_{11}; \tfrac{1}{2}(\gamma - \sqrt{\gamma^2 - 4\beta_1\beta_2}))$$

contains exactly one proper value of A.

Proof. We intend to prove the theorem and the corollary simultaneously. The proof will be divided into several sections.

1. Let us show first that the corollary is a consequence of statement 2^0 (P1) of the theorem. To see that, let us estimate first the product $a_1 B^{-1} a_2'$. We have

$$|a_1 B^{-1} a_2'| \leqq \frac{g(a_1)\, g'(a_2')}{\hat{g}(B)} \leqq \frac{\beta_1 \beta_2}{\gamma}.$$

It follows that we may take

$$\alpha_1 = \alpha_2 = \frac{\beta_1\beta_2}{\gamma} \quad \text{and} \quad \alpha_3 = \gamma$$

in the theorem and the three required inequalities are satisfied. The condition

$$\sqrt{\alpha_1} + \sqrt{\alpha_2} < \sqrt{\alpha_3} \quad \text{becomes} \quad 2\sqrt{\frac{\beta_1\beta_2}{\gamma}} < \sqrt{\gamma}$$

which is, however, only another form of $\gamma^2 > 4\beta_1\beta_2$. The assumptions of the theorem are thus seen to be satisfied. According to 2° (P1) and (5) of the theorem the disk $K(a_{11}; r_0^{(1)})$ contains exactly one proper value of A. An easy computation yields

$$r_0^{(1)} = \tfrac{1}{2}(\gamma - \sqrt{\gamma^2 - 4\beta_1\beta_2}).$$

2. Put

$$|a_1 B^{-1} a_2'| = \varepsilon_1, \quad g(a_1)\, g'(a_2')/\hat{g}(B) = \varepsilon_2, \quad \hat{g}(B) = \varepsilon_3.$$

It follows that the vector $e = (\varepsilon_1, \varepsilon_2, \varepsilon_3)$ belongs to Ω and $e \leqq a$.

Further, we shall use the following abbreviation: If t is a complex number, we shall write $h(t)$ for $\hat{g}(B - t)$. For $h(t) > 0$ let us introduce the function $f(t) =$

$= -a_1(B - t)^{-1} a_2'$ so that $|f(0)| = \varepsilon_1$. We shall need the following simple relation for f

(6) $$f(t_1) - f(t_2) = (t_2 - t_1)\, a_1(B - t_1)^{-1} (B - t_2)^{-1} a_2' .$$

To describe the process (P1), put $u_k = x_k - a_{11}$ so that it reduces to

$$u_0 = 0\,, \quad u_{k+1} = f(u_k)\,.$$

Further, put $p_k = u_{k+1} - u_k$ so that $p_0 = u_1 = f(u_0) = f(0)$. We have $h(u_{k+1}) = \hat{g}(B - u_k - p_k) \geqq \hat{g}(B - u_k) - |p_k|$ whence

(7) $$h(u_{k+1}) \geqq h(u_k) - |p_k|\,.$$

We intend to show now that the process is meaningful. To see that, we shall prove by induction the following statement:

$$S_k \begin{cases} h(u_j) \geqq h(u_{j-1}) - |p_{j-1}| & (1 \leqq j \leqq k)\,, \\ |p_j| \leqq |p_{j-1}| \dfrac{\varepsilon_2 \varepsilon_3}{h(u_{j-1})\, h(u_j)} & (1 \leqq j \leqq k)\,, \\ h(u_j) - |p_j| > 0 & (0 \leqq j \leqq k)\,. \end{cases}$$

Clearly $h(u_0) = h(0) = \varepsilon_3 > \varepsilon_1 = |f(0)| = |p_0|$. Assume now S_k and let us prove S_{k+1}. The inequality (7) shows that

$$h(u_{k+1}) \geqq h(u_k) - |p_k| > 0$$

so that $f(u_{k+1})$ is defined. We have by (6)

$$p_{k+1} = f(u_{k+1}) - f(u_k) = -p_k a_1(B - u_{k+1})^{-1} (B - u_k)^{-1} a_2'$$

whence

$$|p_{k+1}| \leqq |p_k| \frac{g(a_1)\, g'(a_2')}{h(u_{k+1})\, h(u_k)} = \varepsilon_2 h(0) \frac{|p_k|}{h(u_{k+1})\, h(u_k)}\,.$$

According to lemma (1,3) with $X_j = h(u_j)$ and $Y_j = |p_j|$, we have

$$h(u_{k+1}) - |p_{k+1}| \geqq X_{k+1}^0(e) - Y_{k+1}^0(e)$$

and it is easy to verify that $X_{k+1}^0(e) - Y_{k+1}^0(e) > 0$. Hence S_{k+1} is proved. It follows from (1,3) that

(8) $$h(u_k) \geqq X_k^0(e) = \tfrac{1}{2}(s_1^0 + w^0) \frac{1 - \lambda_2^0(\lambda_1^0)^{k+1}}{1 - \lambda_2^0(\lambda_1^0)^k}\,,$$

$$|p_k| \leqq Y_k^0(e) = \tfrac{1}{2}(s_1^0 + w^0) \frac{\lambda_2^0(1 - \lambda_1^0)^2}{(1 - \lambda_2^0(\lambda_1^0)^k)(1 - \lambda_2^0(\lambda_1^0)^{k+1})} (\lambda_1^0)^k\,,$$

where $\lambda_i^0 = L_i(e)$, $s_i^0 = S_i(e)$, $w^0 = W(e)$.

It follows from the second of the relations (8) that the series $\sum_{j=0}^{\infty} p_j$ converges absolutely; put $x = a_{11} + \sum_{j=0}^{\infty} p_j$. Since $u_k = \sum_{j=0}^{k-1} p_j$, the sequence u_k converges (its limit is $x - a_{11}$). For the same reason, $B - u_k$ converges to the matrix $A_{22} - x$. Now, $\mathfrak{g}(B - u_k) = h(u_k) \geqq \frac{1}{2}(s_1^0 + w^0) > 0$, so that the matrix $A_{22} - x$ is nonsingular. Since $u_{k+1} = f(u_k)$ and $\lim u_k = x - a_{11}$, we have $x - a_{11} = f(x - a_{11}) = -a_1(A_{22} - x)^{-1} a_2'$ so that x is a proper value of A according to lemma (1,5). The right hand side in the second relation (8) may be written as

$$(9) \qquad w^0 \lambda_2^0 \left(\frac{(\lambda_1^0)^k}{1 - \lambda_2^0(\lambda_1^0)^k} - \frac{(\lambda_1^0)^{k+1}}{1 - \lambda_2^0(\lambda_1^0)^{k+1}} \right) = \varrho_k^{(1)} - \varrho_{k+1}^{(1)}$$

where

$$\varrho_k^{(1)} = w^0 \frac{\lambda_2^0(\lambda_1^0)^k}{1 - \lambda_2^0(\lambda_1^0)^k} = R_k^{(1)}(e)\,.$$

Hence we have the estimate

$$(10) \qquad \sum_{j=k}^{\infty} |p_j| \leqq \varrho_k^{(1)}\,.$$

Since $x_k = a_{11} + u_k$, we have $x - x_k = \sum_{j=k}^{\infty} p_j$, whence by (10) $|x_k - x| \leqq \varrho_k^{(1)}$. Further, it follows from (9) that

$$|x_k - x_{k+1}| = |u_k - u_{k+1}| = |p_k| \leqq \varrho_k^{(1)} - \varrho_{k+1}^{(1)}$$

so that $K(x_k; \varrho_k^{(1)}) \supset K(x_{k+1}; \varrho_{k+1}^{(1)})$. The convergence of the first process is thus established.

3. Let us show now that the open disk $K^0(a_{11}; \frac{1}{2}(s_2^0 + w^0))$ contains exactly one proper value of A, namely x. We show first that $A_{22} - z$ is nonsingular whenever $|z - a_{11}| < \mathfrak{g}(B)$. Indeed,

$$\mathfrak{g}(A_{22} - z) = \mathfrak{g}(B - (z - a_{11})) \geqq \mathfrak{g}(B) - |z - a_{11}|\,.$$

It follows from lemma (1,5) that each proper value z of A in the open disk $|z - a_{11}| < \mathfrak{g}(B)$ fulfills $z - a_{11} = f(z - a_{11})$. Suppose now that z is a proper value of A in $K^0(a_{11}; \frac{1}{2}(s_2^0 + w^0))$. We have $|z - a_{11}| < \frac{1}{2}(s_2^0 + w^0) = \varepsilon_3 - \frac{1}{2}(s_1^0 - w^0) < \varepsilon_3 \leqq \mathfrak{g}(B)$ so that $z - a_{11} = f(z - a_{11})$. It follows from (6) that

$$|z - x| = |(z - a_{11}) - (x - a_{11})| =$$
$$= |f(z - a_{11}) - f(x - a_{11})| \leqq |z - x| \frac{g(a_1)\, g'(a_2')}{h(z - a_{11})\, h(x - a_{11})}\,.$$

We shall show now that

$$\frac{g(a_1)\, g'(a_2')}{h(z - a_{11})\, h(x - a_{11})} < 1\,.$$

First of all, $x - a_{11} = \lim u_k$ so that $h(x - a_{11}) = \lim h(u_k) \geqq \frac{1}{2}(s_1^0 + w^0)$. Further, $z \in K^0(a_{11}; \frac{1}{2}(s_2^0 + w^0))$ whence $|z - a_{11}| < \frac{1}{2}(s_2^0 + w^0)$ and

$$h(z - a_{11}) \geqq h(0) - |z - a_{11}| > h(0) - \tfrac{1}{2}(s_2^0 + w^0) = \tfrac{1}{2}(s_1^0 - w^0).$$

It follows that

$$\frac{g(a_1)\, g'(a_2')}{h(z - a_{11})\, h(x - a_{11})} < \frac{\varepsilon_2}{\frac{1}{2}(s_1^0 + w^0)} \frac{h(0)}{\frac{1}{2}(s_1^0 - w^0)} = \frac{\varepsilon_2}{\frac{1}{2}(s_1^0 + w^0)} \frac{\varepsilon_3}{\frac{1}{2}(s_1^0 - w^0)} = 1$$

so that $|z - x|$ has to be zero. It follows that x is the only proper value of A contained in $K^0(a_{11}; \frac{1}{2}(s_1^0 + w^0))$. Since $\varrho_0^{(1)} = \frac{1}{2}(s_2^0 - w^0) < \frac{1}{2}(s_2^0 + w^0)$, we have

$$K(x_0; \varrho_0^{(1)}) \subset K^0(a_{11}; \tfrac{1}{2}(s_2^0 + w^0)).$$

4. To see that x is a simple proper value of A let us compute the derivative of the characteristic polynomial $P(\lambda)$ at the point x. Since $\hat{g}(A_{22} - \lambda) = \hat{g}(B - (\lambda - a_{11})) \geqq \hat{g}(B) - |\lambda - a_{11}|$, the matrix $A_{22} - \lambda$ is nonsingular for $|\lambda - a_{11}| < \hat{g}(B)$. By lemma (1,5) we have

$$P(\lambda) = (f(\lambda - a_{11}) - (\lambda - a_{11}))\, Q(\lambda),$$

where $Q(\lambda) = \det(A_{22} - \lambda)$ is different from zero in the whole domain $|\lambda - a_{11}| < \hat{g}(B)$. Since

$$f'(x - a_{11}) = \lim_k \frac{f(u_{k+1}) - f(u_k)}{u_{k+1} - u_k} = \lim_k \frac{p_{k+1}}{p_k}$$

we have according to a preceding estimate

$$\frac{|p_{k+1}|}{|p_k|} \leqq \frac{\varepsilon_2 \varepsilon_3}{h(u_k)\, h(u_{k+1})} \leqq \frac{\varepsilon_2 \varepsilon_3}{(\frac{1}{2}(s_1^0 + w^0))^2} = \lambda_1^0 < 1.$$

It follows that $f'(x - a_{11}) - 1 \neq 0$ so that $P'(x) \neq 0$ and the proper value x is thus seen to be simple.

5. Replace now the vector e by a. It follows from lemma (1,4) that $r = R(a) \leqq R(e) = \frac{1}{2}(s_2^0 + w^0)$ and from (1,1) that $R(a) > R_0^{(1)}(a)$. Hence $U^* = K^0(a_{11}; r) \supset K(a_{11}; r_0^{(1)})$. To prove the inclusions $K(x_k; r_k^{(1)}) \supset K(x_{k+1}; r_{k+1}^{(1)})$ it suffices to show that

$$|x_k - x_{k+1}| \leqq r_k^{(1)} - r_{k+1}^{(1)}.$$

This follows, however, from the fact that

$$|x_k - x_{k+1}| \leqq \varrho_k^{(1)} - \varrho_{k+1}^{(1)} = R_k^{(1)}(e) - R_{k+1}^{(1)}(e)$$
$$= Y_k^0(e) \leqq Y_k^0(a) = R_k^{(1)}(a) - R_{k+1}^{(1)}(a) = r_k^{(1)} - r_{k+1}^{(1)}$$

according to lemma (1,4). The statement 3° concerning the monotonicity of the radii follows from lemma (1,4) as well.

6. Let us turn now to the second process. For t such that $h(t) = \hat{g}(B - t) > \alpha_2$ let

$$m(t) = \frac{a_1 B^{-1} a_2'}{1 + a_1(B - t)^{-1} B^{-1} a_2'}.$$

Indeed, if $h(t) > \alpha_2$, we have

$$|a_1(B - t)^{-1} B^{-1} a_2'| \leqq \frac{g(a_1)\, g'(a_2')}{h(t)\, h(0)} \leqq \frac{\alpha_2}{h(t)} < 1$$

so that $m(t)$ exists. Let us prove first an estimate for $m(t)$. Clearly

$$|m(t)| \leqq \frac{\alpha_1}{1 - \dfrac{g(a_1)\, g'(a_2')}{h(t)\, h(0)}} \leqq \frac{\alpha_1}{1 - \dfrac{\alpha_2}{h(t)}} = \frac{\alpha_1\, h(t)}{h(t) - \alpha_2}.$$

An easy computation yields

$$m(t_1) - m(t_2) = (t_2 - t_1) \frac{(a_1 B^{-1} a_2')\,(a_1(B - t_1)^{-1}\,(B - t_2)^{-1}\, B^{-1} a_2')}{(1 + a_1(B - t_1)^{-1}\, B^{-1} a_2')(1 + a_1(B - t_2)^{-1}\, B^{-1} a_2^1)}$$

whence

(11)
$$|m(t_1) - m(t_2)| \leqq$$
$$\leqq |t_2 - t_1| \left\{\alpha_1 \Big/ \left(1 - \frac{\alpha_2}{h(t_1)}\right)\right\} \cdot \left\{\frac{[g(a_1)\, g'(a_2')/h(0)]\,.\,[h(t_1)\, h(t_2)]^{-1}}{1 - \alpha_2/h(t_2)}\right\} =$$
$$= |t_2 - t_1| \frac{\alpha_1 \alpha_2}{(h(t_1) - \alpha_2)\,(h(t_2) - \alpha_2)}.$$

To describe the process (P2), let us put $v_k = x_k^{(2)} - a_{11}$ so that the process reduces to $v_0 = 0$, $v_{k+1} = m(v_k)$. Put $q_k = v_{k+1} - v_k$. Since $B - v_{k+1} = B - v_k - q_k$, we have $h(v_{k+1}) \geqq h(v_k) - |q_k|$. Put $l_k = h(v_k) - \alpha_2$. We intend to show now that the second process is meaningful. To see that, we shall prove by induction the following statement S_k':

$$S_k' \begin{cases} l_j \geqq l_{j-1} - |q_{j-1}| & (j = 1, 2, \ldots, k), \\ |q_j| \leqq |q_{j-1}| \dfrac{\alpha_1 \alpha_2}{l_{j-1} l_j} & (j = 1, 2, \ldots, k), \\ l_j - |q_j| > 0 & (j = 0, 1, \ldots, k). \end{cases}$$

Clearly

$$l_0 = h(0) - \alpha_2 \geqq \alpha_3 - \alpha_2 > \frac{\alpha_1 \alpha_3}{\alpha_3 - \alpha_2} \geqq |q_0|$$

since

$$\alpha_3 - \alpha_2 = (\sqrt{\alpha_3} - \sqrt{\alpha_2})(\sqrt{\alpha_3} + \sqrt{\alpha_2}) > \sqrt{\alpha_1}\, \sqrt{\alpha_3}\,.$$

Assume now that S'_k is fulfilled. We have $l_k > |q_k| \geqq 0$ whence $h(v_k) > \alpha_2$. It follows that $m(v_k) = v_{k+1}$ is defined. Since

$$l_{k+1} = h(v_{k+1}) - \alpha_2 = h(v_k + q_k) - \alpha_2 \geqq h(v_k) - |q_k| - \alpha_2 = l_k - |q_k| > 0,$$

$m(v_{k+1})$ is also defined. We have by (11)

$$|q_{k+1}| = |v_{k+2} - v_{k+1}| = |m(v_{k+1}) - m(v_k)| \leqq |q_k| \frac{\alpha_1 \alpha_2}{l_k l_{k+1}}.$$

According to lemma (1,3) we have $l_{k+1} - |q_{k+1}| > 0$ so that S'_{k+1} is fulfilled. It follows, again from lemma (1,3) and from lemma (1,2) that

$$l_k \geqq \tfrac{1}{2}(s_3 + w) \frac{1 - \lambda_2 \lambda_3^{k+1}}{1 - \lambda_2 \lambda_3^k}, \tag{12}$$

$$|q_k| \leqq \tfrac{1}{2}(s_3 + w) \frac{\lambda_2 (1 - \lambda_3)^2}{(1 - \lambda_2 \lambda_3^k)(1 - \lambda_2 \lambda_3^{k+1})} \lambda_3^k =$$

$$= w(1 - \lambda_3) \frac{\lambda_2 \lambda_3^k}{(1 - \lambda_2 \lambda_3^k)(1 - \lambda_2 \lambda_3^{k+1})}.$$

The second of the inequalities (12) shows that the series $\sum_{k=0}^{\infty} q_k$ is absolutely convergent so that the sequence $v_k = q_0 + q_1 + \ldots + q_{k-1}$ has a limit v. Since $h(v) - \alpha_2 = \lim h(v_k) - \alpha_2 \geqq \frac{1}{2}(s_3 + w) > 0$, $m(v)$ exists and fulfills the relation $v = m(v)$. Hence

$$v(1 + a_1(B - v)^{-1} B^{-1} a'_2) = a_1 B^{-1} a'_2$$

so that

$$v - a_1 B^{-1} a'_2 = -a_1[(B - v)^{-1} - B^{-1}] a'_2$$

and

$$v = -a_1(B - v)^{-1} a'_2.$$

It follows from lemma (1,5) that $v + a_{11}$ is a proper value of A. Put

$$r_k^{(2)} = w \frac{\lambda_2 \lambda_3^k}{1 - \lambda_2 \lambda_3^k}$$

so that the right hand side in the second inequality (12) may be written as $r_k^{(2)} - r_{k+1}^{(2)}$. It follows that

$$K(a_{11} + v_k; r_k^{(2)}) \supset K(a_{11} + v_{k+1}; r_{k+1}^{(2)}).$$

Besides, we know from the study of the process (P1) that

$$U^* \supset K(a_{11}; r_0^{(1)}) = K(a_{11} + v_0; r_0^{(2)}).$$

This shows that $v = x - a_{11}$ and concludes the proof of the statement about the second process.

7. Let us take up now the third process. If $a_1 = 0$, the process is trivially meaningful since $B_k = B$ for $k = 0, 1, 2, \ldots$ and $c'_k = 0$ for $k = 1, 2, 3, \ldots$. Assume now $a_1 \neq 0$. Let us show by induction that, for $k = 1, 2, 3, \ldots$

$$g'(c'_k) \leqq \frac{1}{g(a_1)} \frac{s_3}{2^{k+1}} q^{2^{k-1}}, \tag{13}$$

$$h_k = \hat{g}(B_k) \geqq \frac{s_3}{2^{k-1}}. \tag{14}$$

Let us introduce the following abbreviation: $\omega_k = a_1 c'_k$. First, let $k = 1$. Let us note that

$$|\omega_0| \leqq \alpha_1, \quad g'(c'_0) \leqq \frac{1}{g(a_1)} \alpha_2, \quad g(c'_0 a_1) \leqq \alpha_2$$

and

$$h_1 = \hat{g}(B_1) \geqq \hat{g}(B) - g(c'_0 a_1) - |\omega_0| \geqq \alpha_3 - \alpha_1 - \alpha_2 = s_3.$$

Thus (14) is true for $k = 1$.

Since

$$g'(c'_1) = g'(-\omega_0 B_1^{-1} c'_0) \leqq \frac{\alpha_1 \alpha_2}{g(a_1)\, s_3} = \frac{s_3 q}{4 g(a_1)},$$

(13) is true for $k = 1$ as well.

Suppose now that (13) and (14) are satisfied for k. To prove (14) for $k + 1$, note first that $|\omega_k| \leqq g(a_1)\, g'(c'_k)$ whence

$$h_{k+1} \geqq h_k - g'(c'_k)\, g(a_1) - |\omega_k| \geqq h_k - 2g(a_1)\, g'(c'_k) \geqq$$

$$\geqq \frac{s_3}{2^{k-1}} - \frac{s_3}{2^k} q^{2^{k-1}} \geqq \frac{s_3}{2^k}.$$

Further

$$g'(c'_{k+1}) \leqq \frac{|\omega_k|\, g'(c'_k)}{h_{k+1}} \leqq \frac{1}{h_{k+1}} g(a_1) (g'(c'_k))^2 \leqq$$

$$\leqq \frac{2^k}{s_3} g(a_1) \frac{1}{g(a_1)^2} s_3^2 \frac{1}{2^{2k+2}} q^{2^k} = \frac{1}{g(a_1)} s_3 \frac{1}{2^{k+2}} q^{2^k}$$

and the induction is complete. The series $\sum_{j=0}^{\infty} \omega_j$ is clearly absolutely convergent. For the remainder $\sum_{j=k}^{\infty} |\omega_j|$ the following estimate may be obtained from (14) for $k \geqq 1$

$$\sum_{j=k}^{\infty} |\omega_j| \leqq \sum_{j=k}^{\infty} \frac{s_3}{2^{j+1}} q^{2^{j-1}} \leqq q^{2^{k-1}} s_3 \sum_{j=k}^{\infty} \frac{1}{2^{j+1}} = \frac{s_3}{2^k} q^{2^{k-1}}.$$

Let us recall that we have denoted $a_{11} - \sum_{j=0}^{k-1} \omega_j$ by z_k. It remains to prove that

$$\lim_k z_k = \lim_k \left(a_{11} - \sum_{j=0}^{k-1} \omega_j\right) = a_{11} - \sum_{j=0}^{\infty} \omega_j = x\,.$$

This will be done by means of corollary (2,2) applied to the sequence of matrices A_k defined as follows

$$A_0 = A, \; A_k = \begin{pmatrix} z_k, & a_1 \\ -\,\omega_{k-1} c'_{k-1}, & B_k + z_k \end{pmatrix}.$$

All these matrices are similar transforms of A: it is easy to see that $T_k A_k T_k^{-1} = A_{k+1}$ with

$$T_k = \begin{pmatrix} 1, & 0 \\ c'_k, & E \end{pmatrix}.$$

Let us now apply corollary (2,2) to the matrix A_k. Put $\beta_1 = g(a_1)$. It follows from (13) that

$$g(\omega_{k-1} c'_{k-1}) \leqq g(a_1)\,(g'(c'_{k-1}))^2 \leqq \frac{1}{g(a_1)} \frac{s_3^2}{2^{2k}}\, q^{2^{k-1}}$$

which will be denoted by β_2. If $\gamma = s_3/2^{k-1}$, we have by (14) $\hat{g}(B_k) \geqq \gamma$. It is easy to see that $\gamma^2 > 4\beta_1\beta_2$ so that corollary (2,2) may be applied. It follows that the disk

$$\tilde{D}_k = K(z_k; \tfrac{1}{2}(\gamma - \sqrt{\gamma^2 - 4\beta_1\beta_2}))$$

contains exactly one proper value of A_k and hence exactly one proper value of A. The radius of D_k is easily found to be equal

$$\frac{1}{2^k}\, s_3 \left(1 - \sqrt{1 - q^{2^{k-1}}}\right).$$

Let us consider further the disks

$$D_k = K(z_k; r_k^{(3)})$$

where $r_k^{(3)} = (s_3/2^k)\, q^{2^{k-1}}$ so that $\tilde{D}_k \subset D_k$, the center being the same. Let us show now that $D_1 \subset K(a_{11}; R')$ for some R' which is smaller than the radius of U^* so that D_1 will be contained wholly in the interior of U^*. To this end it will suffice to show that

$$|a_{11} - (a_{11} - \omega_0)| \leqq R' - r_1^{(3)}$$

for some $R' < r$. Let us show that $R' = \frac{1}{2}s_2$ will do. Indeed,

$$|\omega| \leqq \alpha_1 < \alpha_1 + \frac{w^2}{2s_3} = \frac{2\alpha_1(s_1 - 2\alpha_2) + w^2}{2s_3} =$$

$$= \frac{2\alpha_1 s_1 + w^2}{2s_3} - \frac{2\alpha_1\alpha_2}{s_3} = \frac{s_2 s_3}{2s_3} - \frac{\frac{1}{2}q s_3^2}{s_3} = R' - r_1^{(3)}\,.$$

Let us show further that $D_1 \supset D_2 \supset \ldots$. The distance of the centers of D_k and D_{k+1} being $|\omega_k|$ it suffices to show that

$$|\omega_k| \leqq \frac{s_3}{2^{k+1}} q^{2^{k-1}} \left(2 - q^{2^k - 2^{k-1}}\right) = r_k^{(3)} - r_{k+1}^{(3)} .$$

We are going to prove now that the point $a_{11} - \sum_{j=0}^{\infty} a_1 c'_j$ belongs to each D_k. This is an immediate consequence of the estimate

$$\sum_{j=k}^{\infty} |\omega_j| \leqq \frac{s_3}{2^k} q^{2^{k-1}} = r_k^{(3)} .$$

Since $\tilde{D}_k \subset D_k$ have the same center and the radii of $\tilde{D}_k$ converge to zero the point $a_{11} - \sum_{j=0}^{\infty} \omega_j$ is the intersection of both sequences D_k and $\tilde{D}_k$. Since $\tilde{D}_k \subset D_k \subset \subset D_1 \subset U^*$, we have $\tilde{D}_k \subset U^*$ for all k. Now U^* contains exactly one proper value of A, and so does $\tilde{D}_k$. It follows that x belongs to each $\tilde{D}_k$ so that $x = a_{11} - \sum_{j=0}^{\infty} \omega_j$. The proof is complete.

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Резюме

ОЦЕНКИ И ИТЕРАЦИОННЫЕ ПРОЦЕССЫ ДЛЯ СОБСТВЕННЫХ ЗНАЧЕНИЙ ПОЧТИ РАЗЛОЖИМЫХ МАТРИЦ

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Пусть дана клеточная матрица

$$A = \begin{pmatrix} a_{11}, & a_1 \\ a'_2, & A_{22} \end{pmatrix}$$

с размерностями клеток 1 и n. Если хотя бы один из векторов a_1, a'_2 „мал", можно ожидать, что матрица A будет обладать собственным значением, близким к a_{11}. Приводится круг с центром a_{11}, в котором лежит в точности одно собственное значение матрицы A. Радиус круга зависит от оценок для норм векторов a_1 и a'_2 и меры невырожденности матрицы $A_{22} - a_{11}$. Точная формулировка результата содержится в теореме (2,1).