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ESTIMATES FOR A MEAN TIME OF TROUBLE-FREE WORK FOR ONE QUEUING NETWORK CLASS*

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The problem of estimation of the mean of process time for networks with queues and with synchronization of service requirements in nodes is considered. The apparatus of the idempotent algebra is used which admits to describe the dynamics of system by the stochastic generalized difference equations. For the cases that the network topology is described by an acyclic graph, the upper and lower estimates of mean time of work are obtained.

In the present work the problem of estimating a mean time of the trouble-free work for queuing networks with the synchronization of requirements in nodes is considered. The apparatus of idempotent algebra is used which permits us to describe the system's dynamics by the generalized stochastic difference equations. For the case that the net topology is described by an acyclic graph, the upper and lower estimates of a mean time of trouble-free work are obtained.

1. Idempotent algebra. Consider the set of real numbers $\overline{\mathbb{R}}$, extended by adding one element $\varepsilon = -\infty$, with the defined in it operations $\oplus$ and $\otimes$, whose definitions for any $x, y \in \overline{\mathbb{R}}$ are the following:

$$x \oplus y = \max(x, y), \qquad x \otimes y = x + y,$$

and $x \otimes \varepsilon = \varepsilon$.

The set $\overline{\mathbb{R}}$ with the operations $\oplus$ and $\otimes$ is a commutative semiring with the idempotent addition, whose zero and unit elements are the numbers ε and 0 respectively. Semirings with the described properties are usually called idempotent algebras (see, e.g., [1, 2]).

The idempotent algebra (semiring) of real matrices is introduced by the usual way: for any two matrices $A = (a_{ij})$ and $B = (b_{ij})$ of size $n \times n$ operations $\oplus$ and $\otimes$ are performed by the formulas

$$\{A \oplus B\}_{ij} = a_{ij} \oplus b_{ij} \quad \text{and} \quad \{A \otimes B\}_{ij} = \bigoplus_{k=1}^{n} a_{ik} \otimes b_{kj}.$$

A matrix $\mathcal{E}$, whose elements are all equal to ε, is a neutral element with respect to the operation $\oplus$. The matrix E with zeros on the main diagonal and ε out of it represents the unit matrix.

Obviously, the operations $\oplus$ and $\otimes$ have the property of monotony, i. e., the inequalities $A \leqslant C$ and $B \leqslant D$ imply the inequalities $A \oplus B \leqslant C \oplus D$ and $A \otimes B \leqslant C \otimes D$.

Let $A \neq \mathcal{E}$ be a square matrix. As usually, we put $A^0 = E$ and $A^k = A \otimes A^{k-1} = A^{k-1} \otimes A$ for any integer $k \geqslant 1$.

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Consider an arbitrary matrix $A = (a_{ij})$ and introduce the following magnitudes:

$$\|A\| = \bigoplus_{i,j} a_{ij}, \qquad \mu(A) = \min_{i,j}\{a_{ij} | a_{ij} \neq \varepsilon\}$$

under the condition that $\mu(\mathcal{E}) = \varepsilon$.

It is obvious that if $A \leqslant B$, then $\|A\| \leqslant \|B\|$ and $\mu(A) \leqslant \mu(B)$. Besides, for any number $c > 0$ the equalities $\|cA\| = c\|A\|$ and $\mu(cA) = c\mu(A)$ are true.

It is easy to check that for any matrices A and B such that an expression $A \otimes B$ is meaningful, the following inequality holds

$$\|A \otimes B\| \leqslant \|A\| \otimes \|B\|.$$

If $A = D$ is a diagonal matrix (with non-diagonal elements equal to ε), then the inequality holds:

$$\|D \otimes B\| \geqslant \mu(D) \otimes \|B\|.$$

2. Systems with synchronous service. Consider a network consisting of n nodes, each node having a service device and a queue. The network topology is described by an oriented acyclic graph $\mathcal{G} = (\mathbf{N}, \mathbf{A})$, where $\mathbf{N} = \{1, \dots, n\}$ is the set of graph nodes, corresponding to the network's nodes, and $\mathbf{A} = \{(i,j)\} \subset \mathbf{N} \times \mathbf{N}$ is a set of graph's edges, which define the paths of requirements motion.

For any node $i \in \mathbf{N}$, define two sets of nodes

$$\mathbf{P}(i) = \{j | (j,i) \in \mathbf{A}\}, \qquad \mathbf{S}(i) = \{j | (i,j) \in \mathbf{A}\}.$$

Each node i, for which $\mathbf{P}(i) = \emptyset$, is considered as a source of the infinite flow of requirements, entering the system. Requirements are deleted from the system after their service in the nodes i for which $\mathbf{S}(i) = \emptyset$. At the initial time all the service devices of the network are free, the queue of requirements in each source-node has infinite length, the queues of all other nodes do not contain requirements.

We assume that the processes of servicing of the requirements of the network nodes satisfy some restraints on synchronization [3]. Mechanisms of synchronization are realized by auxiliary operations "uniting" (join) and "disjunction" (fork), which are performed in each node respectively before and after that requirement is serviced. The performance of the operation "uniting" in node i is as follows: a requirement does not enter the queue until the node obtains one requirement from each node $j \in \mathbf{P}(i)$. These requirements are united into one requirement, which then enters the queue of requirements waiting for service in the node i.

The operation "disjunction" in node i is performed each time the service of some requirement is ended. In this case the requirement is replaced by new requirements, whose number is equal to the number of nodes in $\mathbf{S}(i)$. Then these new requirements simultaneously leave node i and are directed to each of the nodes $j \in \mathbf{S}(i)$. It is assumed that operations "uniting" and "disjunction", and also the movements of requirements between nodes are performed instantly.

Denote by τ_{ik} the duration and by $x_i(k)$ the time of ending of the kth service in the node i. We assume that τ_{ik} are given by the nonnegative stochastic variables and $\mathbb{E}[\tau_{ik}] < \infty$ for all $i = 1, \dots, n$, and $k = 1, 2, \dots$ Under the condition that the system begins to work at the zero time, put $x_i(0) = 0$ and $x_i(k) = \varepsilon$ for all $k < 0$, $i = 1, \dots, n$. Introduce a vector $\boldsymbol{x}(k) = (x_1(k), \dots, x_n(k))^T$, and also a matrix $\mathcal{T}_k = \mathrm{diag}(\tau_{1k}, \dots, \tau_{nk})$, whose nondiagonal elements are equal to ε. Then the dynamics of the system may be described by the equation [4]

$$\boldsymbol{x}(k) = A(k) \otimes \boldsymbol{x}(k-1) \tag{1}$$

with the matrix

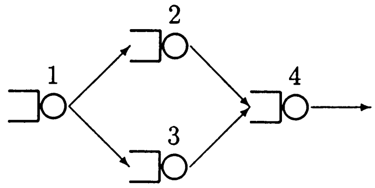
$$A(k) = (E \oplus \mathcal{T}_k \otimes G^T)^r \otimes \mathcal{T}_k, \tag{2}$$

where $G = (g_{ij})$ is a matrix, whose elements are defined in the following way

$$g_{ij} = \begin{cases} 0, & \text{if } i \in \mathbf{P}(j), \\ \varepsilon & \text{in other cases,} \end{cases}$$

r is the length of the longest path in the network graph $\mathcal{G}$.

As an example, consider a net with $n = 4$ nodes, which together with the corresponding matrix G is shown in the figure:



$$G = \begin{pmatrix} \varepsilon & 0 & 0 & \varepsilon \\ \varepsilon & \varepsilon & \varepsilon & 0 \\ \varepsilon & \varepsilon & \varepsilon & 0 \\ \varepsilon & \varepsilon & \varepsilon & \varepsilon \end{pmatrix}$$

Since for the net graph the maximal path length is $r = 2$, the matrix (2) in the equation (1) has the form

$$A(k) = (E \oplus \mathcal{T}_k \otimes G^T)^2 \otimes \mathcal{T}_k = \begin{pmatrix} \tau_{1k} & \varepsilon & \varepsilon & \varepsilon \\ \tau_{1k} \otimes \tau_{2k} & \tau_{2k} & \varepsilon & \varepsilon \\ \tau_{1k} \otimes \tau_{3k} & \varepsilon & \tau_{3k} & \varepsilon \\ \tau_{1k} \otimes (\tau_{2k} \oplus \tau_{3k}) \otimes \tau_{4k} & \tau_{2k} \otimes \tau_{4k} & \tau_{3k} \otimes \tau_{4k} & \tau_{4k} \end{pmatrix}.$$

Note that for the matrix $A(k)$

$$\|A(k)\| = \tau_{1k} \otimes (\tau_{2k} \oplus \tau_{3k}) \otimes \tau_{4k}.$$

Mean time of trouble-free work. We consider the evolution of the system as a sequence of service cycles. The first cycle begins at time zero and goes along until in each net node, one requirement will be serviced. The second cycle is accomplished when in each node two requirements are serviced and so on. It is obvious that with taking into account the condition $x(0) = 0$ the time of accomplishment of the kth cycle can be found as follows:

$$\|x(k)\| = \|A_k\|, \qquad A_k = A(k) \otimes \ldots \otimes A(1).$$

Let there exist the probability of appearance of some condition, which prohibits the performance of the current cycle and further work of the system (probability of refusal). Introduce a random magnitude ν being the number of the last successful cycle of service after which the refusal of the system occur and suppose that $\mathbb{E}[\nu] < \infty$. Then the mean time of the system work until the instant of refusal can be defined as a mathematical expectation, namely,

$$\mathbb{E}\|A_\nu\| = \mathbb{E}[\mathbb{E}\|A_k\| | \nu = k] = \sum_{k=1}^{\infty} \mathbb{E}\|A_k\| \mathbb{P}\{\nu = k\}.$$

We assume that for any k the probability of the successful completion of the kth service cycle does not depend on k. Denoting this probability by p, we obtain

$$\mathbb{E}\|A_\nu\| = (1-p) \sum_{k=1}^{\infty} \mathbb{E}\|A_k\| p^k. \tag{3}$$

4. Evaluation of the mean time of the network. To construct these estimates, we use the following auxiliary results. Note at first that for any k we have

$$A(k) = \mathcal{T}_k \oplus (\mathcal{T}_k \otimes G^T) \otimes \mathcal{T}_k \oplus \ldots \oplus (\mathcal{T}_k \otimes G^T)^r \otimes \mathcal{T}_k \geqslant \mathcal{T}_k.$$

Besides, for diagonal matrices $\mathcal{T}_1, \ldots, \mathcal{T}_k$ the obvious identity holds $\mathcal{T}_1 \otimes \ldots \otimes \mathcal{T}_k = \mathcal{T}_1 + \ldots + \mathcal{T}_k$.

Let the elements of the matrix $A = (a_{ij})$ be certain stochastic variables. Denote by $\mathbb{E}[A]$ a matrix obtained from A by the replacement of its elements by their expectations under the condition that $\mathbb{E}[\varepsilon] = \varepsilon$. It is easy to check that for any matrices A and B the inequalities are valid

$$\mathbb{E}\|A\| \geqslant \|\mathbb{E}[A]\|, \qquad \mathbb{E}\|A \otimes B\| \geqslant \mathbb{E}\|\mathbb{E}[A] \otimes B\| \geqslant \|\mathbb{E}[A] \otimes \mathbb{E}[B]\|.$$

4.1. Lower estimates of the mean time of the network. Suppose, for any $i = 1, \ldots, n$, stochastic variables $\tau_{i1}, \tau_{i2}, \ldots$ are equally distributed and independent. Then the following statements hold.

Lemma 1. *If* $\|\mathbb{E}[\mathcal{T}_1]\| > \mu(\mathbb{E}[\mathcal{T}_1])$, *then the following inequality is true*

$$\mathbb{E}\|A_\nu\| \geqslant \alpha p(1-p^m) - \beta p \left(\frac{1-p^m}{1-p} - m p^m \right) + \gamma \left(\frac{p}{1-p} \right), \tag{4}$$

where $m = \lfloor \alpha/\beta \rfloor$ *and*

$$\alpha = \mathbb{E}\|A(1)\| - \mu(\mathbb{E}[\mathcal{T}_1]), \quad \beta = \|\mathbb{E}[\mathcal{T}_1]\| - \mu(\mathbb{E}[\mathcal{T}_1]), \quad \gamma = \|\mathbb{E}[\mathcal{T}_1]\|.$$

Proof. Consider the mathematical expectation

$$\mathbb{E}\|A_\nu\| = (1-p) \sum_{k=1}^{\infty} \mathbb{E}\|A_k\| p^k = (1-p) \left(\sum_{k=1}^{m} \mathbb{E}\|A_k\| p^k + \sum_{k=m+1}^{\infty} \mathbb{E}\|A_k\| p^k \right).$$

Let

$$\|A_k\| = \|A(k) \otimes \ldots \otimes A(1)\| \geqslant \|(\mathcal{T}_k \otimes \ldots \otimes \mathcal{T}_2) \otimes A(1)\|.$$

Passing to the expectation, we obtain

$$\mathbb{E}\|A_k\| \geqslant \mathbb{E}\|\mathbb{E}(\mathcal{T}_k + \ldots + \mathcal{T}_2) \otimes A(1)\| = \mathbb{E}\|((k-1)\mathbb{E}[\mathcal{T}_1]) \otimes A(1)\|$$
$$\geqslant \mu((k-1)\mathbb{E}[\mathcal{T}_1]) \otimes \mathbb{E}\|A(1)\| = (k-1)\mu(\mathbb{E}[\mathcal{T}_1]) + \mathbb{E}\|A(1)\|.$$

Now for the first sum we have

$$\sum_{k=1}^{m} \mathbb{E}\|A_k\|p^k \geqslant \mu(\mathbb{E}[\mathcal{T}_1]) \sum_{k=1}^{m}(k-1)p^k + \mathbb{E}\|A(1)\| \sum_{k=1}^{m} p^k.$$

In respect to $A_k \geqslant \mathcal{T}_k \otimes \ldots \otimes \mathcal{T}_1 = \mathcal{T}_k + \ldots + \mathcal{T}_1$ we obtain

$$\mathbb{E}\|A_k\| \geqslant \|\mathbb{E}(\mathcal{T}_k + \ldots + \mathcal{T}_1)\| = \|k\mathbb{E}[\mathcal{T}_1]\| = k\|\mathbb{E}[\mathcal{T}_1]\|.$$

Then for the second sum we have the inequality

$$\sum_{k=m+1}^{\infty} \mathbb{E}\|A_k\|p^k \geqslant \|\mathbb{E}[\mathcal{T}_1]\| \sum_{k=m+1}^{\infty} kp^k.$$

Uniting the inequalities obtained and using the above notation α, β, and γ, we find

$$\mathbb{E}\|A_\nu\| \geqslant (1-p)\left(\alpha\sum_{k=1}^{m} p^k - \beta\sum_{k=1}^{m} kp^k + \gamma\sum_{k=1}^{\infty} kp^k\right) = \alpha p(1-p^m) - \beta p\left(\frac{1-p^m}{1-p} - mp^m\right) + \gamma\left(\frac{p}{1-p}\right). \tag{5}$$

Obviously, the right side of (5) takes the greatest value for the same values of m that provide the maximum of the function

$$\Phi(m) = \alpha\sum_{k=1}^{m} p^k - \beta\sum_{k=1}^{m} kp^k = \sum_{k=1}^{m}(\alpha - k\beta)p^k.$$

Note that $\alpha \geqslant \beta > 0$. Then it is clear that the function $\Phi(m)$ is an increasing function while $(\alpha - m\beta) \geqslant 0$ and takes the maximal value for $m = \lfloor \alpha/\beta \rfloor$. □

Lemma 2. *If $\|\mathbb{E}[\mathcal{T}_1]\| = \mu(\mathbb{E}[\mathcal{T}_1])$, then the following inequality holds*

$$\mathbb{E}\|A_\nu\| \geqslant p\left(\alpha + \frac{\gamma}{1-p}\right). \tag{6}$$

Proof. Consider inequality (5). Under the condition $\beta = \|\mathbb{E}[\mathcal{T}_1]\| - \mu(\mathbb{E}[\mathcal{T}_1]) = 0$, the right side of the inequality takes the maximal value for $m = \infty$. After the calculation of the corresponding sums we obtain the inequality (6). □

Note that the condition of the lemma is satisfied if and only if for all $i = 1, \ldots, n$ the values $\mathbb{E}[\tau_{i1}]$ are equal.

4.3. Upper estimates for a mean time of a trouble-free network.

Lemma 3. *The following inequality is true*

$$\mathbb{E}\|A_\nu\| \leqslant \frac{p}{1-p}\mathbb{E}\|A(1)\|. \tag{7}$$

Proof. Taking into account that

$$\|A_\nu\| \leqslant \|A(\nu)\| \otimes \ldots \otimes \|A(1)\| = \|A(\nu)\| + \ldots + \|A(1)\|,$$

and $\mathbb{E}[\nu] = p/(1-p)$, let us use the Vald identity

$$\mathbb{E}\|A_\nu\| \leqslant \mathbb{E}\sum_{k=1}^{\nu}\|A(k)\| = \frac{p}{1-p}\mathbb{E}\|A(1)\|. \quad \square$$

Lemma 4. *Under the condition that $\mathbb{D}\tau_{i1} < \infty$ for all $i = 1, \ldots, n$, the inequality*

$$\mathbb{E}\|A_\nu\| \leqslant p\left(\frac{1}{1-p} + r\right)\mathbb{E}\|\mathcal{T}_1\| + r(1-p)C(p)\sqrt{\mathbb{D}\|\mathcal{T}_1\|} \tag{8}$$

holds, where r is a length of the maximal path in the network graph and $C(p)$ is calculated by the formula

$$C(p) = \frac{p}{4q}\left(1 + \frac{\sqrt{\pi}(1-2q)}{2\sqrt{pq}}(1 - \operatorname{erf}\sqrt{q})\right) + M(p),$$

$$q = -\frac{\ln p}{2}, \quad \operatorname{erf}(x) = \frac{2}{\sqrt{\pi}} \int_0^x \exp(-t^2)dt,$$
$$M(p) = \max_k \left(\frac{k-1}{\sqrt{2k-1}} p^k \right).$$

Proof. We apply the estimate obtained in [5]:

$$\mathbb{E}\|A_k\| \leqslant (k+r)\mathbb{E}\|\mathcal{T}_1\| + r\frac{k-1}{\sqrt{2k-1}}\sqrt{\mathbb{D}\|\mathcal{T}_1\|}.$$

Consider the inequality

$$\mathbb{E}\|A_\nu\| = (1-p)\sum_{k=1}^{\infty} \mathbb{E}\|A_k\|p^k \leqslant (1-p)\left(\mathbb{E}\|\mathcal{T}_1\|\sum_{k=1}^{\infty}(k+r)p^k + r\sqrt{\mathbb{D}\|\mathcal{T}_1\|}\sum_{k=1}^{\infty}\frac{k-1}{\sqrt{2k-1}}p^k\right).$$

Compute the first sum in the right side of the inequality:

$$\sum_{k=1}^{\infty}(k+r)p^k = \frac{p}{1-p}\left(\frac{1}{1-p} + r\right).$$

It is not difficult to verify that for the sum of the second series the following estimate holds

$$\sum_{k=1}^{\infty} \frac{k-1}{\sqrt{2k-1}}p^k \leqslant \int_1^{\infty} \frac{x-1}{\sqrt{2x-1}}p^x dx + \frac{m-1}{\sqrt{2m-1}}p^m,$$

where m is chosen such that the second addend in the right side represents the maximal by its value term of the considered series.

It is obvious that for m to be obtained it is sufficient to compute the root of the derivative of the integrand function by the formula

$$x_0 = \frac{3\ln p - \sqrt{\ln^2 p - 6\ln p + 1} - 1}{4\ln p}$$

and then compare the values of the function nearest to the root integers, situated to the right and to the left of the root.

Whereas

$$\int_1^{\infty} \frac{x-1}{\sqrt{2x-1}}p^x dx = \frac{p}{4q}\left(1 + \frac{\sqrt{\pi}(1-2q)}{2\sqrt{pq}}(1 - \operatorname{erf}\sqrt{q})\right),$$

where $q = -\ln p/2$, we arrive at the inequality (8). □

Below we give the results of computations of estimates for the mean time of work for certain values of p for the network, shown in the above figure. The durations of requirement services in the nodes $i = 1,2,3,4$ are independent and have exponential distributions with parameters λ_i.

In Tab. 1 the results are represented under the condition that $\lambda_i = 1$ for all $i = 1,2,3,4$, in Tab. 2 the results are represented for the case, then $\lambda_1 = \lambda_4 = 1$, $\lambda_2 = 2$, and $\lambda_3 = 3$. The tables contain also the estimates for a mean time of the systems work and the bounds of corresponding confidence intervals on the level 0,95, computed on the basis of the imitational modelling of the system by 10000 of independent realizations.

Table 1

p	Estimate (4)	Results of simulation	Estimate (7)	Estimate (8)
0.05	0.178	0.181 ∓ 0.007	0.184	0.452
0.10	0.361	0.385 ∓ 0.013	0.389	0.808
0.20	0.750	0.793 ∓ 0.021	0.875	1.589
0.50	2.250	2.488 ∓ 0.069	3.500	4.776
0.80	6.000	7.729 ∓ 0.298	14.000	12.743
0.90	11.250	14.888 ∓ 0.401	31.500	23.730
0.95	21.375	27.812 ∓ 0.753	66.500	44.834

Table 2

p	Estimate (6)	Results of modelling	Estimate (7)	Estimate (8)
0.05	0.133	0.137 ∓ 0.011	0.139	0.362
0.10	0.267	0.250 ∓ 0.018	0.293	0.639
0.20	0.544	0.523 ∓ 0.029	0.658	1.249
0.50	1.155	1.811 ∓ 0.068	2.633	3.744
0.80	4.416	5.452 ∓ 0.186	10.533	9.936
0.90	9.247	10.770 ∓ 0.360	23.700	18.427
0.95	19.134	20.611 ∓ 0.705	50.033	34.712

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