Estimates for the kernel and continuity properties of pseudo-differential operators

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0. Introduction

This work studies the continuity of pseudo-differential operators in Hörmander's class $\mathcal{L}^m_{\varrho,\delta}$ (cf. [11]) in several function spaces, including L^p spaces, Hardy spaces, weak L^1 and BMO. The basic assumption throughout the paper is that $0<\varrho\leqq 1$, $0\leqq\delta<1$, in particular, all our results are valid for $\delta\geqq\varrho$. The stress is on sharp conditions over the order and type of the operators.

Our point of view is that in many spaces continuity should follow from the functional calculus and simple computations, once L^2 estimates and suitable estimates for the kernel are known. Thus, we prove three different types of estimates for kernels of pseudo-differential operators: pointwise, integral and "dyadic integral" in § 1, § 2 and § 5 respectively; the first two types extend [14, p. 1053] and [1, p. 75], the last one may be new. Then we combine these estimates with the L^2-continuity results proved in [13] to obtain (L^1, weak L^1), (L^∞, BMO), (L^p, L^q) and (H^p, L^p) continuity conditions that extend or improve results due to C. Fefferman [9], L. Hörmander [11] and J. Alvarez and M. Milman [1] (most results are classic for $\delta<\varrho$ or $\delta\leqq\varrho$). We also prove a pointwise estimate for the sharp maximal function $(Lf)^\#$ in terms of the generalized Hardy—Littlewood maximal function $M_p f$ for some pseudo-differential operators L extending [2, p. 424]. It is well-known that these pointwise estimates give weighted L^p estimates for L.

When $\varrho=1$, pseudo-differential operators of non-positive order are associated to standard kernels, i.e., they are generalized Calderón—Zygmund operators. However, when $\varrho<1$, in order to obtain the best continuity properties, one is led to consider kernels that blow up at the diagonal faster than standard kernels. It is then natural to ask to what extent properties valid for operators associated to stan-

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dard kernels, remain true for more general kernels. In that direction we exhibit, following a construction of Hörmander's [12], an operator whose distribution kernel grows at the diagonal slightly faster than a standard kernel, satisfies $T1=T^*1=0$ and the weak boundedness property, but is unbounded in L^2. This shows that the so called $T1$ theorem of G. David and J. L. Journé [7] cannot hold for kernels which are slightly worse than standard ones.

Since we do not aim at achieving minimal smoothness assumptions we just consider smooth symbols, but inspection of most proofs would show the precise number of derivatives that is being used. The functions considered, unless otherwise indicated, are complex functions defined on $\mathbf{R}^n$. The characteristic function of a subset $A\subset\mathbf{R}^n$ will be denoted by χ_A. The letter C will denote positive constants that may vary at different occurrences. We use Δ to denote both the diagonal in $\mathbf{R}^n\times\mathbf{R}^n$ and the Laplace operator in $\mathbf{R}^n$; the meaning will be clear from context. The Lebesgue measure of a measurable set $A\subset\mathbf{R}^n$ is written as $|A|$. Finally, $[x]$ will indicate the integral part of a real number x, i.e., the largest integer $\leqq x$.

Following [11], a symbol in $S^m_{\varrho,\delta}$ will be a smooth function $p(x,\xi)$ defined on $\mathbf{R}^n\times\mathbf{R}^n$, satisfying the estimates

$$|D_x^\alpha D_\xi^\beta p(x,\xi)| \leqq C_{\alpha\beta}(1+|\xi|)^{m-\varrho|\beta|+\delta|\alpha|}.$$

We will always assume that $0<\varrho\leqq 1$, $0\leqq\delta<1$, even though some times the condition $\varrho>0$ will suffice, like in some parts of Theorem 1.1 below. As usual, $\mathscr{L}^m_{\varrho,\delta}$ will denote the class of operators with symbol in $S^m_{\varrho,\delta}$. A few times we will mention pseudo-differential operators defined by amplitudes $a(x,y,\xi)$. Most of the results also hold for these operators, although sometimes appropriate changes in the order will be needed for $\delta>\varrho$. The organization of the paper is as follows:

§ 1. Pointwise estimates for the distribution kernel;
§ 2. Integral estimates for the distribution kernel;
§ 3. (H^1, L^1), $(L^1$, weak $L^1)$, (L^∞, BMO), (L^p, L^q)-continuity;
§ 4. Pointwise estimates for the sharp maximal function;
§ 5. Continuity on Hardy spaces;
§ 6. A counterexample to a $T1$ theorem.

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1. Pointwise estimates for the distribution kernel

Theorem 1.1. *Let* $L\in\mathscr{L}^m_{\varrho,\delta}$, $0<\varrho\leqq 1$, $0\leqq\delta<1$, *be a pseudo-differential operator in* $\mathbf{R}^n$ *with symbol* $p(x,\xi)$ *and let* $k(x,y)$ *be the distribution kernel of L defined by the oscillatory integral*

$$k(x,y)=(2\pi)^{-n}\int e^{i(x-y)\cdot\xi}p(x,\xi)\,d\xi.$$

a) *(Pseudo-local property.) The distribution k is smooth outside the diagonal. Moreover, given* $\alpha,\beta\in Z^n_+$ *there exists* $N_0\in Z_+$ *such that for each* $N\geqq N_0$,

$$\sup_{x\neq y}|x-y|^N|D^\alpha_x D^\beta_y k(x,y)|<\infty.$$

b) *Suppose that p has compact support in* ξ *uniformly with respect to x. Then k is smooth, and given* $\alpha,\beta\in Z^n_+$, $N\in Z_+$ *there is* $C>0$ *such that*

$$|D^\alpha_x D^\beta_y k(x,y)|\leqq C(1+|x-y|)^{-N}.$$

c) *Suppose that* $m+M+n<0$ *for some* $M\in Z_+$. *Then k is a bounded continuous function with bounded continuous derivatives of order* $\leqq M$.

d) *Suppose that* $m+M+n=0$, *for some* $M\in Z_+$. *Then there exists a constant* $C>0$ *such that*

$$\sup_{|\alpha+\beta|=M}|D^\alpha_x D^\beta_y k(x,y)|\leqq C\big|\log|x-y|\big|,\quad x\neq y.$$

e) *Suppose that* $m+M+n>0$ *for some* $M\in Z_+$. *Then there exists a positive constant C such that*

$$\sup_{|\alpha+\beta|=M}|D^\alpha_x D^\beta_y k(x,y)|\leqq C|x-y|^{-(m+M+n)/\varrho},\quad x\neq y.$$

Proof. The first three statements are part of the classical theory and their proof relies on repeated integration by parts. Statements d) and e) will be now proved simultaneously. We will first consider the case $0<\varrho<1$. Observe that if $k(x,y)$ is the kernel of a pseudo-differential operator in $\mathscr{L}^m_{\varrho,\delta}$, it follows that $D^\alpha_x D^\beta_y k(x,y)$ is the kernel of a pseudo-differential operator of order $\leqq m+|\alpha|+|\beta|$. Hence, it is enough to prove d) and e) when $\alpha=\beta=0$.

Let $\varphi\in C^\infty_c(\mathbf{R})$, $\varphi\geqq 0$ supported in the interval $[1/2,1]$, $\int\varphi(t)dt=1$ and set $\varphi(\xi,t)=\varphi(|\xi|^{1-\varrho}-t)$, for $\xi\in\mathbf{R}^n$. According to a) and b), it is enough to estimate $k(x,y)$ for $|x-y|<1$ assuming that $p(x,\xi)$ vanishes identically for $|\xi|\leqq 1$. Thus,

$$k(x,y)=(2\pi)^{-n}\int_1^\infty\int_{\mathbf{R}^n}e^{i(x-y)\cdot\xi}p(x,\xi)\varphi(\xi,t)\,d\xi\,dt. \tag{1.1}$$

Let

$$k(x,y,t)=(2\pi)^{-n}\int_{\mathbf{R}^n}e^{i(x-y)\cdot\xi}p(x,\xi)\varphi(\xi,t)\,d\xi. \tag{1.2}$$

Given $\beta\in Z_+^n$ we have

$$(1.3)\qquad (x-y)^\beta k(x,y,t)=\sum_{\alpha\le\beta}C_{\alpha\beta}\int_{\mathbf{R}^n}e^{i(x-y)\cdot\xi}D_\xi^\alpha p(x,\xi)D_\xi^{\beta-\alpha}\varphi(\xi,t)\,d\xi.$$

Since $\varphi(\xi,t)\in S^0_{\varrho,0}$ as a function of ξ uniformly on t, $|\xi|$ is comparable with $t^{1/(1-\varrho)}$ on the support of φ and the volume of the support can be estimated by $Ct^{n/(1-\varrho)-1}$, we obtain from (1.3)

$$|(x-y)^\beta k(x,y,t)|\le Ct^{[(m+n-\varrho|\beta|)/(1-\varrho)]-1}.$$

Set $N=[(m+n)/\varrho]+1$ (the smallest integer $>(m+n)/\varrho$).

Adding those estimates for $\beta=0$ and $|\beta|=N$ we get

$$(1.4)\qquad |k(x,y,t)|\le C\frac{t^{[(m+n)/(1-\varrho)]-1}}{1+t^{\varrho N/(1-\varrho)}|x-y|^N}.$$

Hence, (1.4) shows that $k(x,y,t)$ is integrable as a function of t for $x\ne y$. Moreover, according to (1.1) and (1.2),

$$(1.5)\qquad |k(x,y)|\le\begin{cases}C|x-y|^{-(m+n)/\varrho} & \text{if } m+n>0\\ C\big|\log|x-y|\big| & \text{if } m+n=0.\end{cases}$$

This completes the proof of d) and e) when $\varrho<1$. The proof for $\varrho=1$ is analogous, using the decomposition

$$(1.6)\qquad k(x,y)=(2\pi)^{-n}\int_0^\infty\int_{\mathbf{R}^n}e^{i(x-y)\cdot\xi}p(x,\xi)\psi(\xi/t)\,d\xi\,dt/t,$$

where $0\le\psi\in C_c^\infty(\mathbf{R})$ is supported in $[1/2,2]$ and has integral equal to 1 with respect to the measure dt/t.

Remarks. a) Very precise representations of $k(x,y)$ can be found in [4, p. 54], [16, p. 59] when the symbol is an asymptotic sum of homogeneous terms of decreasing order.

b) When $m=0$, $|\alpha+\beta|\le1$, $\varrho=1$, the estimates in e) reflect the well-known fact that that operators in $\mathcal{L}^0_{1,\delta}$ are associated to standard kernels in the sense of Coifman and Meyer [6, p. 78].

c) Estimates (1.5) for $\varrho=1$ appear in [14, p. 40].

d) When $m=-n(1-\varrho)$, $|\alpha+\beta|=1$, (1.5) yields

$$(1.7)\qquad |\nabla k(x,y)|\le\frac{C}{|x-y|^{n+1/\varrho}}.$$

Operators with kernels satisfying (1.7) have been studied in [3], [1], [2]. They are non-convolution generalizations of some weakly strongly singular operators considered by C. Fefferman [8, p. 21].

e) Theorem 1.1 remains true when the operator is defined by an amplitude instead of a symbol.

f) The mean value theorem combined with Theorem 1.1 implies the following estimate that we note for further reference.

g) When $\varrho=1$ and $\delta=0$ pointwise estimates have been obtained by A. Laptev [17].

(1.8)

$$|k(x, y)-k(z, y)|+|k(y, x)-k(y, z)| \leqq \frac{|x-z|}{|y-z|^{(m+n+1)/\varrho}} \quad \text{if} \quad 2|x-z| \leqq |y-z|.$$

We show now that the estimates of Theorem 1.1 are sharp.

Theorem 1.2. *Suppose that* $m+n\geqq 0$ *and* $0<\varrho\leqq 1$. *Then there exists an operator* $L\in\mathscr{L}^m_{\varrho,0}$ *such that for each* $\varepsilon>0$

$$\sup_{x\neq y} |x-y|^{(m+n-\varepsilon)/\varrho}|k(x, y)| = \infty.$$

Proof. We will first conclude the result assuming that the following statement is true:

Given $\varepsilon>0$ $\exists L_\varepsilon\in\mathscr{L}^m_{\varrho,\delta}$ *such that*

$$\sup_{x\neq y} |x-y|^{(m+n-\varepsilon)/\varrho}|k_\varepsilon(x, y)| = \infty, \tag{1.9}$$

where k_ε *denotes the distribution kernel of the operator* L_ε.

Now, for $\varepsilon>0$ and $N\in Z_+$ fixed, set

$$F^N_\varepsilon = \{a\in S^m_{\varrho,\delta}\colon \sup_{x\neq y} |x-y|^{(m+n-\varepsilon)/\varrho}|k(x, y)| \leqq N\}.$$

This is a closed subset of $S^m_{\varrho,\delta}$. Moreover, F^N_ε has empty interior. In fact, if a_ε denotes the symbol of the operator L_ε satisfying (1.9), given $a\in F^N_\varepsilon$, $a+ta_\varepsilon\notin F^N_\varepsilon$, for every $t>0$. Thus, Baire's theorem implies that $\cup F^N_{1/j}$ is a proper subset of $S^m_{\varrho,\delta}$. To complete the proof, it remains to show (1.9). If this statement were not true for some positive ε, the closed graph theorem would imply that the map

$$S^m_{\varrho,\delta}\ni a \mapsto |x-y|^{(m+n-\varepsilon)/\varrho}k(x, y)\in L^\infty(\mathbf{R}^n\times\mathbf{R}^n\setminus\Delta)$$

would be continuous. In that case, given $M>0$, there has to exist a continuous seminorm $|||\cdot|||$ in $S^m_{\varrho,\delta}$ such that

$$a\in S^m_{\varrho,\delta},\quad |||a||| \leqq 1 \Rightarrow \sup_{x\neq y} |x-y|^{(m+n-\varepsilon)/\varrho}|k(x, y)| \leqq M. \tag{1.10}$$

We will exhibit a symbol in $S^m_{\varrho,\delta}$ that violates (1.10). In the construction below we assume that $\varrho<1$. The case $\varrho=1$ is similar.

It is convenient to use the notation $\|\xi\|=\max(|\xi_1|, \dots, |\xi_n|)$ for a vector $\xi\in\mathbf{R}^n$. Consider a function, $0\leqq\varphi(x)\in\mathcal{S}(\mathbf{R}^n)$, such that $\varphi(x)\geqq 1$ if $|x|\leqq 1$ and its Fourier transform $\hat{\varphi}(\xi)$ is supported in the cube $\|\xi\|<1/2$. Set

$$a(\xi)=\sum_{p=1}^{\infty} a_p(\xi) \tag{1.11}$$

with

$$a_p(\xi)=p^{m/(1-\varrho)}\sum_{\|k\|=p} e^{i\gamma(p,k)}\hat{\varphi}(p^{-\varrho/(1-\varrho)}\xi-k), \tag{1.12}$$

where k runs over the lattice points of $\mathbf{R}^n$ and $\gamma(p, k)$ denotes real numbers to be chosen later. It is easy to check that $a_p(\xi)$ is supported in the cubic annulus $(p-1/2)p^{\varrho/(1-\varrho)}\leqq\|\xi\|\leqq(p+1/2)p^{\varrho/(1-\varrho)}$. Furthermore, there are $(2p+1)^n-(2p-1)^n$ terms in the sum (1.12) corresponding to functions with disjoint supports. Thus, for every multi-index α we have the estimate

$$|D_\xi^\alpha a(\xi)| \leqq C_\alpha(1+|\xi|)^{m-\varrho\alpha}, \quad \xi\in\mathbf{R}^n,$$

which shows that $a\in S^m_{\varrho,0}$. Moreover, the constants C_α do not depend on $\gamma(p, k)$ so we may assume without loss of generality that $|||a|||\leqq 1$ for all choices of $\gamma(p, k)$. A simple computation shows that off the diagonal, the distribution kernel of the convolution operator associated to the symbol a is given by

$$k(x, y)=\sum_{p=1}^{\infty} p^{(m+n\varrho)/(1-\varrho)}$$

$$\times\sum_{\|k\|=p}\exp\left[ip^{\varrho/(1-\varrho)}k\cdot(x-y)+i\gamma(p, k)\right]\varphi\left(p^{\varrho/(1-\varrho)}(x-y)\right).$$

Taking $x_0\neq y_0\in\mathbf{R}^n$ and defining $\gamma(p, k)=-p^{\varrho/(1-\varrho)}k\cdot(x_0-y_0)$ we obtain

$$k(x_0, y_0)=\sum_{p=1}^{\infty} p^{(m+n\varrho)/(1-\varrho)}\sum_{\|k\|=p}\varphi\left(p^{\varrho/(1-\varrho)}(x-y)\right). \tag{1.13}$$

Let N be the smallest integer such that $p^{\varrho/(1-\varrho)}|x_0-y_0|\geqq 1$ (in particular, $N^{\varrho/(1-\varrho)}\geqq |x_0-y_0|^{-1}$). Then (1.13) and the choice of φ imply

$$k(x_0, y_0)\geqq\sum_{p=1}^{N-1} p^{(m+n\varrho)/(1-\varrho)}[(2p+1)^n-(2p-1)^n]$$

$$\geqq C\sum_{p=1}^{N-1} p^{[(m+n\varrho)/(1-\varrho)]+n-1}\geqq CN^{(m+n)/(1-\varrho)}\geqq C|x_0-y_0|^{-(m+n)/\varrho}.$$

Therefore, $|x_0-y_0|^{(m+n-\varepsilon)/\varrho}k(x_0, y_0)\geqq C|x_0-y_0|^{-\varepsilon}$. This violates (1.10) if x_0 and y_0 are taken close enough.

2. Integral estimates for the distribution kernel

Theorem 2.1. *Let* $L\in\mathcal{L}^m_{\varrho,\delta}$, $0<\varrho\leqq 1$, $0\leqq\delta<1$, *with kernel* $k(x,y)$ *and set* $\lambda=\max\{0,(\delta-\varrho)/2\}$.

a) *If* $m\leqq -n[(1-\varrho)/2+\lambda]$, *then*

$$(2.1)\qquad \begin{aligned} &\sup_{|y-z|\leqq\sigma}\int_{|x-z|>2\sigma^\varrho}|k(x,y)-k(x,z)|\,dx\leqq C,\quad \sigma<1;\\ &\sup_{|y-z|\leqq\sigma}\int_{|x-z|>2\sigma}|k(x,y)-k(x,z)|\,dx\leqq C,\quad \sigma>1.\end{aligned}$$

b) *If* $m\leqq -n(1-\varrho)/2$, *then*

$$(2.2)\qquad \begin{aligned} &\sup_{|y-z|\leqq\sigma}\int_{|x-z|>2\sigma^\varrho}|k(y,x)-k(z,x)|\,dx\leqq C,\quad \sigma<1;\\ &\sup_{|y-z|\leqq\sigma}\int_{|x-z|>2\sigma}|k(y,x)-k(z,x)|\,dx\leqq C,\quad \sigma>1.\end{aligned}$$

Proof. The proof is an adaptation of [14, p. 1053] (see also [15, p. 272]). According to Theorem 1.1 b), there is no loss of generality in assuming that the symbol $p(x,\xi)$ of L vanishes for $|\xi|\leqq 1$. Consider a function $0\leqq\psi\in C_c^\infty(\mathbf{R})$ supported in the interval $[1/2, 1]$ such that

$$(2.3)\qquad \int_0^\infty \psi(t^{-1})t^{-1}\,dt=\int_1^2\psi(t^{-1})t^{-1}\,dt=1.$$

Let

$$k(x,y,t)=(2\pi)^{-n}\int e^{i(x-y)\cdot\xi}p(x,\xi)\psi(|\xi|/t)\,d\xi,$$

so

$$(2.4)\qquad k(x,y)=\int_0^\infty k(x,y,t)\,dt/t=\int_1^\infty k(x,y,t)\,dt/t.$$

We will prove (2.1). Let us first take $0<\sigma<1$. Let $N>n/2$ be a natural number. Then,

$$(2.5)\qquad \int_{|x-z|>2\sigma^\varrho}|k(x,y,t)-k(x,z,t)|\,dx$$

$$\leqq\Big[\int(1+t^{2\varrho}|x-z|^2)^N|k(x,y,t)-k(x,z,t)|^2\,dx\Big]^{1/2}\Big[\int(1+t^{2\varrho}|x-z|^2)^{-N}\,dx\Big]^{1/2}$$

$$\leqq C\Big[\int(1+t^{2\varrho}|x-z|^2)^N|k(x,y,t)-k(x,z,t)|^2\,dx\Big]^{1/2}t^{-\varrho n/2}.$$

Let us estimate the integral in (2.5). Given a multi-index α, $|\alpha|\leqq N$, we have

$$t^{\varrho|\alpha|}(x-z)^\alpha\int(e^{i(x-y)\cdot\xi}-e^{i(x-z)\cdot\xi})p(x,\xi)\psi(|\xi|/t)\,d\xi$$

$$=t^{\varrho|\alpha|}(x-z)^\alpha\int e^{i(x-z)\cdot\xi}(e^{i(z-y)\cdot\xi}-1)p(x,\xi)\psi(|\xi|/t)\,d\xi$$

$$=\sum_{\beta\leqq\alpha}C_{\alpha\beta}t^{\varrho|\alpha|}\int e^{i(x-z)\cdot\xi}D_\xi^\beta[(e^{i(z-y)\cdot\xi}-1)p(x,\xi)]t^{-|\alpha-\beta|}(D_\xi^{\alpha-\beta}\psi)(|\xi|/t)\,d\xi.$$

Now,

$$|e^{i(z-y)\cdot\xi}-1|\leqq|y-z|\,|\xi|\leqq t\sigma$$

on the support of $\psi(|\xi|/t)$ for $|y-z|<\sigma$. On the other hand, if $|\gamma|>0$,

$$|D_\xi^\gamma e^{i(z-y)\cdot\xi}| \leqq |y-z|^{|\gamma|} \leqq t^{-|\gamma|}(t\sigma)^{|\gamma|} \leqq C|\xi|^{-|\gamma|}(t\sigma)^{|\gamma|}$$

for $|y-z|<\sigma$ if $|\xi|\sim t$.

Let us now suppose that $t\sigma<1$. The above estimates show that if $\chi\in C_c^\infty(\mathbf{R}_+)$ is equal to 1 on the support of ψ (so $\psi=\chi\psi$), the set of functions

$$\{|\xi|^{n(1-\varrho)/2+\varrho|\beta|}D_\xi^\beta[(e^{i(z-y)\cdot\xi}-1)p(x+z,\xi)]\chi(|\xi|/t):\ |y-z|<\sigma,\ z\in\mathbf{R}^n\}$$

is a bounded subset of $S_{\varrho,\delta}^{m+n(1-\varrho)/2}$, with bounds $\leqq Ct\sigma$, for each $0\leqq\beta\leqq\alpha$. At this point we recall

Theorem 2.2. ([13], p. 766.) *Assume that* $m\leqq -n\max\{0,(\delta-\varrho)/2\}=-n\lambda$, $0<\varrho\leqq 1$, $0\leqq\delta<1$ *and let* $a(x,\xi)\in S_{\varrho,\delta}^m$ *verify*

$$|D_x^\alpha D_\xi^\beta a(x,\xi)| \leqq M(1+|\xi|)^{m+\delta|\alpha|-\varrho|\beta|}$$

for $|\alpha|, |\beta|\leqq[n/2]+1$. *Then, the pseudo-differential operator with symbol a is bounded in* L^2 *with a norm proportional to the best bound M.*

Since $m+n(1-\varrho)/2\leqq -n\lambda$, the family of symbols defined above gives rise to operators bounded on L^2, with norm $\leqq Ct\sigma$.

Thus the integral in (2.5) can be estimated by

$$Ct\sigma\sum_{\beta\leqq\alpha,|\alpha|\leqq N} C_{\alpha\beta}\|t^{\varrho|\alpha|}|\xi|^{-n(1-\varrho)/2-\varrho|\beta|}t^{-|\alpha-\beta|}(D^{\alpha-\beta}\psi)(|\xi|/t)\|_{L^2} \leqq Ct^{\varrho n/2}t\sigma.$$

This means that we may estimate (2.5) by $Ct\sigma$, provided that $t\sigma\leqq 1$. Let us see next what estimate can be obtained without the restriction $t\sigma\leqq 1$. Since $|x-z|>2\sigma^\varrho$ and $|y-z|<\sigma$ imply that $|x-y|>\sigma^\varrho$ for $\sigma<1$, we have

$$\int_{|x-z|>2\sigma^\varrho}|k(x,y,t)|\,dx$$

$$\leqq \Big[\int_{\mathbf{R}^n}(t^{2\varrho}|x-y|^2)^N|k(x,y,t)|^2\,dx\Big]^{1/2}\Big[\int_{|x-y|>\sigma^\varrho}(t^{2\varrho}|x-y|^2)^{-N}\,dx\Big]^{1/2}$$

$$\leqq C\Big[\int(t^{2\varrho}|x-y|^2)^N|k(x,y,t)|^2\,dx\Big]^{1/2}t^{-\varrho N}\sigma^{\varrho(n/2-N)}.$$

Let $|\alpha|=N$. Then,

$$t^{\varrho|\alpha|}(x-y)^\alpha\int e^{i(x-y)\cdot\xi}p(x,\xi)\psi(|\xi|/t)\,d\xi$$

$$=\sum_{\beta\leqq\alpha}C_{\alpha\beta}t^{\varrho|\alpha|}\int e^{i(x-y)\cdot\xi}D_\xi^\beta p(x,\xi)t^{-|\alpha-\beta|}(D_\xi^{\alpha-\beta}\psi)(|\xi|/t)\,d\xi.$$

Now, for each $\beta\leqq\alpha$, $y\in\mathbf{R}^n$, the L^2-norm of the function of the x-variable

$$\int e^{i(x-y)\cdot\xi}D_\xi^\beta p(x,\xi)t^{-|\alpha-\beta|}(D^{\alpha-\beta}\psi)(|\xi|/t)\,dx$$

is the same as the L^2-norm of

$$\int e^{ix\cdot\xi}D_\xi^\beta p(x+y,\xi)t^{-|\alpha-\beta|}(D_\xi^{\alpha-\beta}\psi)(|\xi|/t)\,d\xi.$$

On the other hand, $\{|\xi|^{n(1-\varrho)/2+\varrho|\beta|} D_\xi^\beta p(x+y, \xi)\colon y\in\mathbf{R}^n\}$ is a bounded subset of $S_{\varrho,\delta}^{m+n(1-\varrho)/2}$ for each $\beta\leqq\alpha$, so Theorem 2.2 implies that the corresponding pseudo-differential operators are uniformly bounded in L^2. Thus, the first factor above can be estimated by

$$C\sum_{\beta\leqq\alpha,\,|\alpha|=N} t^{\varrho|\alpha|}\, t^{-n(1-\varrho)/2-\varrho|\beta|}\, t^{-\varrho|\alpha-\beta|}\, t^{n/2} = Ct^{\varrho n/2}.$$

This proves that

$$\int_{|x-z|>2\sigma^\varrho} |k(x, y, t)|\, dx \leqq C(t\sigma)^{\varrho(n/2-N)}. \tag{2.6}$$

In the same way, it can be proved that

$$\int_{|x-z|>2\sigma^\varrho} |k(x, z, t)|\, dx \leqq C(t\sigma)^{\varrho(n/2-N)}. \tag{2.7}$$

Finally, using these estimates and (2.4), we get that if $0<\sigma<1$ and $|x-z|<\sigma$,

$$\int_{|x-z|>2\sigma^\varrho} |k(x, y)-k(x, z)|\, dx \leqq C\Big[\int_1^{\sigma^{-1}} t\sigma+\int_{\sigma^{-1}}^\infty (t\sigma)^{\varrho(n/2-N)}\Big]\, dt/t \leqq C.$$

To prove (2.1) when $\sigma\geqq 1$ it will suffice to show that

$$\sup_{|y-z|<\sigma} \int_{|x-z|>\sigma} |k(x, y, t)-k(x, z, t)|\, dx \leqq C(t\sigma)^{\varrho(n/2-N)}$$

with $N>n/2$. This can be checked along the line of proof of (2.6) and (2.7).

Let us now prove (2.2). It is an adjoint version of (2.1) and its proof is somewhat simpler. The fact that a symbol does not depend on y (as an amplitude would), accounts for the less restrictive conditions on the order (for $\delta>\varrho$). We will only outline the main steps in the proof. We may write

$$\begin{aligned} &(2\pi)^n\big(k(y, x, t)-k(z, x, t)\big) \\ &= \int e^{-i(x-y)\cdot\xi}\big(p(y, \xi)-p(z, \xi)\big)\psi(|\xi|/t)\, d\xi+\int e^{-ix\cdot\xi}(e^{iy\cdot\xi}-e^{iz\cdot\xi})p(z, \xi)\psi(|\xi|/t)\, d\xi \\ &= f(x-y, y, z, t)+g(x, y, z, t), \end{aligned} \tag{2.8}$$

and we wish to estimate

$$\int_{|x-z|>2\sigma^\varrho} |f(x-y, y, z, t)|\, dx \quad\text{and}\quad \int_{|x-z|>2\sigma^\varrho} |g(x, y, z, t)|\, dx,$$

when $|y-z|<\sigma$ and $\sigma<1$. We have

$$\int |g(x, y, z, t)|\, dx \leqq Ct^{-\varrho n/2}\Big[\int |g(x, y, z, t)|^2(1+t^{2\varrho}|x|^2)^N\, dx\Big]^{1/2}. \tag{2.9}$$

The integrand on the right-hand side of (2.9) is a sum of terms of the form

$$C\,|(t^\varrho x)^\alpha g(x, y, z, t)|^2$$

with $|\alpha|\leqq N$. Since g is the Fourier transform in the first variable of $G(\xi, y, z, t)=$

$(e^{iy\cdot\xi}-e^{iz\cdot\xi})p(z,\xi)\psi(|\xi|/t)$, we obtain, by the Plancherel theorem, assuming that $t\sigma<1$,

(2.10)
$$\int |g(x,y,z,t)|\,dx \leqq Ct^{-\varrho n/2}\sum_{|\alpha|\leqq N} t^{\varrho|\alpha|}\|D_\xi^\alpha G(\cdot,y,z,t)\|_{L^2}$$
$$\leqq Ct^{-\varrho n/2}\sum_{|\alpha|\leqq N,\,\beta\leqq\alpha}\|D_\xi^\beta[(e^{iy\cdot\xi}-e^{iz\cdot\xi})p(z,\xi)]D_\xi^{\alpha-\beta}[\psi(|\xi|/t)]\|_{L^2}$$
$$\leqq Ct\sigma t^{n(1-\varrho)/2+m}\leqq Ct\sigma,\quad t\sigma<1,\quad t\geqq 1,$$

where we have used the estimates:

(2.11)
$$|D_\xi^\beta[(e^{iy\cdot\xi}-e^{iz\cdot\xi})p(z,\xi)]|\leqq Ct\sigma t^{m-\varrho|\beta|}\ \text{ if }\ |y-z|\leqq\sigma,\ t\sigma<1,\ 1/2\leqq|\xi|/t\leqq 1,\ |\beta|\leqq N,$$
$$|D_\xi^{\alpha-\beta}[\psi(|\xi|/t)]|\leqq C_{\alpha\beta}t^{|\beta|-|\alpha|},\quad \beta\leqq\alpha,\quad |\operatorname{supp}\psi(|\xi|/t)|\leqq Ct^{n/2}.$$

A similar argument shows that for all $t\geqq 1$, and $\sigma<1$

(2.12)
$$\int_{|x-z|>2\sigma^\varrho}|g(x,y,z,t)|\,dx\leqq C(t\sigma)^{(n/2-N)\varrho}.$$

Furthermore, estimates analogous to (2.10) and (2.12) are valid for $f(x-y,y,z,t)$ (observe that $\int_{|x-z|>2\sigma^\varrho}|f(x-y,y,z,t)|\,dx=\int_{|x+y-z|>2\sigma^\varrho}|f(x,y,z,t)|\,dx\leqq \leqq\int_{|x|>\sigma^\varrho}|f(x,y,z,t)|\,dx$ for $\sigma<1$). They are proved in the same way, with (2.11) replaced by

$$\sup_{t/2\leqq|\xi|\leqq t}|D_\xi^\beta[p(y,\xi)-p(z,\xi)]|\leqq Ct^{m+\delta-\varrho|\beta|}|y-z|\leqq Ct\sigma t^{m-\varrho|\beta|},$$
$$|y-z|\leqq\sigma,\quad t\geqq 1,\quad |\beta|\leqq N.$$

Hence, we obtain from these estimates

$$\int_1^\infty\int_{|x-z|>2\sigma^\varrho}\big(|g(x,y,z,t)|+|f(x-y,y,z,t)|\big)\,dx\,dt/t\leqq C,$$

which implies (2.2) for $\sigma<1$ in view of (2.4). We leave to the reader the proof of the case $\sigma>1$. This completes the proof of Theorem 2.1.

Remarks. a) Theorem 2.1 extends [1, p. 75].

b) According to (1.8), if $m=-n(1-\varrho)$ then the kernel of an operator in $\mathscr{L}^m_{\varrho,\delta}$, $0<\varrho\leqq 1$, $0\leqq\delta<1$, satisfies

(2.13)
$$|k(x,y)-k(z,y)|+|k(y,x)-k(y,z)|\leqq C\frac{|x-z|}{|y-z|^{n+1/\varrho}},\quad 2|x-z|<|y-z|.$$

An easy computation shows that (2.13) implies (2.1) and (2.2). For $\delta>\varrho$, Theorem 2.1 asserts that it is enough to take the larger $m=-n(1+\delta-2\varrho)/2$ to obtain (2.1) and (2.2). However, as $\delta\to 1$, the latter value of m tends precisely to $-n(1-\varrho)$.

This is natural because the value of δ is irrelevant concerning pointwise estimates for the kernel.

c) We will use (2.1) to prove (L^1, weak L^1) and (H^1, L^1) continuity. On the other hand, (2.2) will be one of the ingredients in proving (L^∞, BMO) continuity.

d) If the operator L is defined in terms of an amplitude $p(x, y, \xi)$, then a variation of the proof of Theorem 2.1 shows that both (2.1) and (2.2) hold if $m \leqq -n[(1-\varrho)/2+2\lambda]$. The more restrictive conditions on the order are due to the fact that one has to replace Theorem 2.2 by [13, Thm. 2].

e) Inspection of the proof of Theorem 2.1 shows that the constant C in (2.1) depends on the size of the derivatives of the symbol $D_x^\alpha D_\xi^\beta p(x, \xi)$ for $|\alpha| \leqq N$, $|\beta| \leqq 2N$, and the constant C in (2.2) on the size of $D_\xi^\beta p(x, \xi)$, $|\beta| \leqq N$, where $N = [n/2]+1$.

3. (H^1, L^1), $(L^1$, weak $L^1)$, (L^∞, BMO), (L^p, L^q)-continuity

As before, we set $\lambda = \max\{0, (\delta-\varrho)/2\}$.

Lemma 3.1. *Given* $L \in \mathscr{L}^m_{\varrho,\delta}$, $0<\varrho\leqq 1$, $0 \leqq \delta < 1$, $m \leqq -n[(1-\varrho)/2+\lambda]$, *$L$ is continuous from L^2 into $L^{2/\varrho}$ and from $L^{2/(2-\varrho)}$ into L^2.*

Proof. According to Theorem 2.2 and the calculus of pseudo-differential operators, the compositions with the Bessel potential of order $n(1-\varrho)/2$, $J^{n(1-\varrho)/2}L$ and $LJ^{n(1-\varrho)/2}$ are bounded in L^2 $(J=(I-\Delta)^{1/2})$. By the Hardy—Littlewood—Sobolev estimates, we know that $J^{-(n-\varrho)/2}$ maps L^2 (resp. $L^{2/(2-\varrho)}$) into $L^{2/\varrho}$ (resp. L^2). Thus the lemma follows by composition.

Theorem 3.2. *Let* $L \in \mathscr{L}^m_{\varrho,\delta}$, $0<\varrho\leqq 1$, $0 \leqq \delta < 1$, $m \leqq -n[(1-\varrho)/2+\lambda]$. *Then, L and L^* map continuously*

a) *the Hardy space H^1 into L^1;*

b) *L^1 into weak L^1;*

c) *L^∞ into BMO.*

Proof. To prove a) for L it suffices to show that there exists a constant $C>0$ such that for each $(1, \infty)$ atom a, we have $\|La\|_{L^1} \leqq C$. Let us recall that a $(1, \infty)$ atom a is a measurable function satisfying the following conditions for some ball $B = B(z, \sigma)$:

$$\operatorname{supp} a \subset B, \quad \|a\|_{L^\infty} \leqq |B|^{-1}, \quad \int a(x)\,dx = 0.$$

Now, let a be a $(1, \infty)$ atom and assume that $\sigma<1$. Set $B' = B(z, 2\sigma^\varrho)$ and $A = \mathbf{R}^n \setminus B'$. Thus,

$$\int |La|\,dx \leqq \int_{B'} |La|\,dx + \int_A |La|\,dx = I_1 + I_2. \tag{3.1}$$

According to Lemma 3.1, L is of type $(2/(2-\varrho), 2)$. This takes care of I_1. In fact,

$$I_1 \leqq C\sigma^{n\varrho/2}\|La\|_{L^2} \leqq C\sigma^{n\varrho/2}\|a\|_{L^{2/(2-\varrho)}}.$$

Using the above conditions on the function a, we can conclude that $I_1 \leqq C$.

Since the mean of a is zero, we can write

$$I_2 \leqq \int_A \int_B |k(x, y)-k(x, z)|\,|a(y)|\,dy\,dx \leqq \sup_{|y-z|<\sigma} \int_A |k(x, y)-k(x, z)|\,dx.$$

Since the kernel of L satisfies (2.1) we also obtain that $I_2 \leqq C$. When $\sigma \geqq 1$, we use (3.1) with $B' = B(z, 2\sigma)$ and $A = \mathbf{R}^n \setminus B'$. To estimate the first term one uses that L is of type $(2, 2)$ and for the second one the second inequality of (2.1) applies.

The only properties that were used to prove a) for L concerned the type of L and (2.1). In view of Lemma 3.1 and Theorem 2.1, these properties are shared by L^*, so L^* satisfies a). We may conclude by duality that c) is also true but a direct proof is easy to get by. Given $f \in L_c^\infty$ and $B = B(z, \sigma)$, let us first decompose f as

$$f = f\chi_{B'} + f\chi_A = f_1 + f_2,$$

where $B' = B(z, 2\sigma^\varrho)$, $A = \mathbf{R}^n \setminus B'$ and $\chi_{B'}$ (resp. χ_A) indicates the characteristic function of B' (resp. A). Let $b = Lf_2(z)$ (notice that the pseudo-local property implies that Lf_2 is smooth in B' and b is well defined). Then,

$$|B|^{-1} \int_B |Lf - b|\,dx \leqq |B|^{-1} \int_B |Lf_1|\,dx + |B|^{-1} \int_B |Lf_2 - b|\,dx \leqq I_1 + I_2.$$

Since L is of type $(2, 2/\varrho)$ we have

$$I_1 \leqq |B|^{-\varrho/2}\|Lf_1\|_{L^{2/\varrho}} \leqq C|B|^{-\varrho/2}\|f_1\|_{L^2} \leqq C\|f\|_{L^\infty}.$$

On the other hand,

$$I_2 \leqq |B|^{-1} \int_B \int_A |k(x, y)-k(z, y)|\,|f(y)|\,dy\,dx \leqq C\|f\|_{L^\infty},$$

where we have used that k satisfies (2.2). The same estimate is valid when $\sigma > 1$ (in this case one takes $B' = B(z, 2\sigma)$). Hence,

$$\sup_B \inf_b \frac{1}{|B|} \int_B |Lf(x) - b|\,dx \leqq C\|f\|_{L^\infty}, \quad f \in L_c^\infty,$$

where the infimum is taken over all complex numbers. This inequality implies that L verifies c). The same proof works for L^*.

Finally, to prove b) we recall

Theorem 3.3. ([2, p. 414].) *Let T: $C_c^\infty(\mathbf{R}^n) \to \mathscr{D}'$ be a linear and continuous operator associated to a kernel $k(x, y)$. Suppose that $k(x, y)$ satisfies (2.1) and* *that T is of type $(2, 2)$ and $(q, 2)$, with $1/q=1/2+\beta/n$ for some $(1-\varrho)n/2\leqq\beta<<n/2$. Then, T maps continuously L^1 into weak L^1.*

Then, Theorem 2.1 and Lemma 3.1 show that L as well as L^* satisfy the hypotheses of Theorem 3.3.

Theorem 3.4. *Given $L\in\mathscr{L}^m_{\varrho,\delta}$, $0<\varrho\leqq 1$, $0\leqq\delta<1$, $1<p<\infty$, L is continuous from L^p into itself, provided that*

$$m \leqq -n\left[(1-\varrho)\left|\frac{1}{p}-\frac{1}{2}\right|+\lambda\right]. \tag{3.2}$$

Proof. The proof uses complex interpolation between (H^1, L^1) and (L^2, L^2) for $1<p<2$ and between (L^2, L^2) and (L^∞, BMO) for $2<p<\infty$. In fact, L is of type $(2, 2)$ if $m\leqq -n\lambda$ by Theorem 2.2. On the other hand, according to Theorem 3.2, L will be bounded from H^1 into L^1 and from L^∞ into BMO if $m\leqq -n[(1-\varrho)/2+\lambda]$. Then, (cf. [10, p. 156]) L will be of type (p, p) for $1/p=(1-t)/q+t/2$, $0<t<1$, with $q=1$ or $q=\infty$ if

$$m \leqq -n\lambda t-n[(1-\varrho)/2+\lambda](1-t),$$

that is to say if m satisfies (3.2).

Theorem 3.5. *Let $L\in\mathscr{L}^m_{\varrho,\delta}$, $0<\varrho\leqq 1$, $0\leqq\delta<1$, $1<p\leqq q<\infty$. Then, L is of type (p, q) in the following cases:*

a) *if $1<p\leqq 2\leqq q$ and*

$$m \leqq -n\left(\frac{1}{p}-\frac{1}{q}+\lambda\right); \tag{3.3}$$

b) *if $2\leqq p\leqq q$ and*

$$m \leqq -n\left(\frac{1}{p}-\frac{1}{q}+(1-\varrho)\left(\frac{1}{2}-\frac{1}{p}\right)+\lambda\right); \tag{3.4}$$

c) *if $p\leqq q\leqq 2$ and*

$$m \leqq -n\left(\frac{1}{p}-\frac{1}{q}+(1-\varrho)\left(\frac{1}{q}-\frac{1}{2}\right)+\lambda\right). \tag{3.5}$$

Proof. Let p, q, m as in a) and set $m_1=-n(1/p-1/2)$, $m_2=-n(1/2-1/q)$. Then, $m\leqq m_1+m_2-n\lambda$ and the Hardy—Littlewood—Sobolev estimates imply that J^{m_1} is of type $(p, 2)$ and J^{m_2} is of type $(2, q)$. Set

$$L = J^{m_2}(J^{-m_2}LJ^{-m_1})J^{m_1}.$$

The calculus of pseudo-differential operators shows that the term between parentheses is a pseudo-differential operator in $L^{-n\lambda}_{\varrho,\delta}$, bounded in L^2 by Theorem 2.2. This proves a).

Consider now p, q, m as in b) and set $m'=-n(1/p-1/q)$ so $J^{m'}$ is of type (p, q). Then, $J^{-m'}L$ has order $m-m'\leqq -n(1/2-1/p+\lambda)$ and Theorem 3.4 implies that it is of type (p,p). Thus, $L=J^{m'}(J^{-m'}L)$ is of type (p,q). Finally to show c) one sets $m'=-n(1/p-1/q)$ and writes $L=(LJ^{-m'})J^{m'}$ to obtain the desired conclusion.

Remarks. a) Theorems 3.2 and 3.4 extend [9, p. 414] to the case $\delta\geqq\varrho$.

b) Theorem 3.5 extends the results of [11, p. 162] in two ways. First by including the case $\delta\geqq\varrho$. Second, by allowing equality in (3.4) and (3.5). This answers a question posed in [11, p. 163].

c) Observe that (3.4) reduces to (3.3) for $p=2$ and (3.5) reduces to (3.3) for $q=2$. Also, Theorem 3.5 specializes to Theorem 3.4 as $p=q$.

4. Pointwise estimates for the sharp maximal functions

The sharp maximal function $f^\#$ of C. Fefferman and E. M. Stein, (cf. [10. p. 153]) is defined as follows,

$$f^\#(x) = \sup_B \inf_b \frac{1}{|B|} \int_B |f(y)-b|\, dy$$

where the infimum is taken over all complex numbers $b\in\mathbf{C}$ and the supremum is taken over all balls $B\subset\mathbf{R}^n$ that contain the point x (taking balls centered at x decreases $f^\#$ by a fixed ratio depending only on the dimension n). The condition $f^\#\in L^\infty$ characterizes BMO. On the other hand, knowing that $f\in L^{p_0}$ for some $1\leqq p_0<\infty$, $f\in L^p$ if and only if $f^\#\in L^p$, $p_0\leqq p<\infty$ ([10, p. 153]).

Given $1\leqq p<\infty$, $M_pf(x)$ will denote the function $M(|f|^p)^{1/p}(x)$, where M stands for the Hardy—Littlewood maximal function.

As before, $\lambda=\max\{(\delta-\varrho)/2, 0\}$.

Theorem 4.1. *Given* $L\in\mathscr{L}^m_{\varrho,\delta}$, $0<\varrho\leqq 1$, $0\leqq\delta<1$ *and given* $p>1$, *there exist* $C_p>0$ *such that*

(4.1) $$(Lf)^\#(x) = C_pM_pf(x), \quad f\in C_c^\infty(\mathbf{R}^n),$$

provided that

$$0<\varrho\leqq\frac{1}{2}\left(1-\frac{2n\lambda}{n+2}\right) \quad \text{or} \quad \varrho=1,$$

and

$$m\leqq -n(1-\varrho)-\mu,$$

where

$$2\mu = 1+n(\varrho+\lambda)-\sqrt{[1+n(\varrho+\lambda)]^2-4n\lambda}.$$

(Notice that $\mu=0$ *if* $\lambda=0$.*)*

Proof. Since $M_p f(x)$ is an increasing function of p, it suffices to prove (4.1) when $1<p\leqq 2$. Let $B=B(z,\sigma)$ be such that $|x-z|<\sigma$ and let us suppose that $\sigma<1$. Given $f\in C_c^\infty(\mathbf{R}^n)$, we decompose f as

$$f = f\chi_{B'}+f\chi_A = f_1+f_2,$$

where $B'=B(z, 2\sigma^\theta)$, $A=\mathbf{R}^n\setminus B'$, and $0<\theta\leqq 1$ will be chosen conveniently. We have

$$\frac{1}{|B|}\int_B |Lf(y)-Lf_2(z)|\,dy$$

$$\leqq \frac{1}{|B|}\int_B |Lf_1(y)|\,dy+\frac{1}{|B|}\int_B |Lf_2(y)-Lf_2(z)|\,dy = I_1+I_2.$$

Let us assume that for some $q\geqq p>1$, L is of type (p, q). Thus,

$$I_1 \leqq \left[\frac{1}{|B|}\int_B |Lf_1|^q\,dy\right]^{1/q} \leqq C|B|^{-1/q}\|f_1\|_{L^p} \leqq C|B|^{\theta/p-1/q}M_p f(x).$$

On the other hand, if $|y-z|<\sigma$,

$$|Lf_2(y)-Lf_2(z)| \leqq \int_A |k(y, w)-k(z, w)|\,|f(w)|\,dw.$$

According to (1.8)

$$|k(y, w)-k(z, w)| \leqq C\frac{|y-z|}{|w-z|^{n+\alpha}},$$

when $|y-z|<\sigma<1$, $|w-z|>2\sigma^\theta$, where $\alpha=[m+n(1-\varrho)+1]/\varrho$. Thus,

$$I_2 \leqq C\sum_{j=1}^\infty \frac{\sigma}{(2^{j+1}\sigma^\theta)^{n+\alpha}}\int_{B(z,2^{j+1}\sigma^\theta)} |f(w)|\,dw \leqq C\sigma^{1-\theta\alpha}\sum_{j=1}^\infty 2^{-j\alpha}Mf(x).$$

So, in order to obtain (4.1), the parameters θ, q, α, have to satisfy the conditions:

(4.2) $$\theta \geqq \frac{p}{q};$$

(4.3) $$\alpha > 0;$$

(4.4) $$\theta \leqq \min\{1, 1/\alpha\}.$$

Moreover, L has to be of type (p, q), and the order m must satisfy $m+n+1>0$.

If $\varrho=1$, then $\lambda=\mu=0$. Then Theorem 3.4 implies that all required conditions are satisfied if we take $m\leqq 0$, $\theta=\alpha=1$ and $q=p$. Let us assume now that $\varrho<1$. According to Theorem 3.5, L will be of type (p, q) if

(4.5) $$1<p\leqq 2\leqq q \quad\text{and}\quad m\leqq -n\left(\frac{1}{p}-\frac{1}{q}+\lambda\right).$$

To avoid working with negative numbers let $\beta=-m$ and let us suppose that $\beta<n(1-\varrho)+1$ is such that for each $1<p\leqq 2$ there exists $q\geqq 2$ satisfying (4.5) and

$$\frac{p}{q} \leqq \frac{1}{\alpha}. \tag{4.6}$$

This would allow us to find $0<\theta\leqq 1$ satisfying all the other conditions. Hence, in view of (4.5) and (4.6), we have

$$\beta \geqq n\left(\frac{1}{q}-\frac{1}{p}+\lambda\right), \quad \frac{p}{q} \leqq \frac{\varrho}{1+n(1-\varrho)-\beta}.$$

Assume that $q\to q_0$ as $p\to 1$. Thus,

$$1+\lambda-\beta/n \leqq \frac{1}{q_0} \leqq \frac{\varrho}{1+n(1-\varrho)-\beta}.$$

The smallest $\beta<n(1-\varrho)+1$ for which the above inequality can occur is

$$\beta_0 = n(1-\varrho) \quad \text{if} \quad \lambda = 0;$$

$$\beta_\lambda = n(1-\varrho)+\mu \quad \text{if} \quad \lambda > 0.$$

It is clear that $\mu=0$ if $\lambda=0$. Now taking $m\leqq-\beta_\lambda=-n(1-\varrho)-\mu$, $\theta=\varrho/(1-\mu)$, $q=p/\theta$, it is easy to check that $0<\theta\leqq 1$, $\varrho\alpha=m+n(1-\varrho)+1>0$ and $q\geqq 2$ provided that $0<\varrho\leqq[1-2n\lambda/(n+2)]/2$. Hence, conditions (4.2), (4.3) and (4.4) are satisfied and this implies (4.1) for $\sigma<1$. The case $\sigma>1$ can be handled in a similar way using the decomposition

$$f = f\chi_{B'}+f\chi_A$$

with $B'=B(z, 2\sigma)$ and $A=\mathbf{R}^n\setminus B'$. We leave details to the reader.

Remarks. a) With the same technique one can extend Theorem 4.1 to the range $[1-2n\lambda/(n+2)]/2<\varrho<1$. However, one has to rely in this case on part c) of Theorem 3.5 instead of using a) of the same theorem. This makes the conditions rather cumbersome and we have no indication that the restriction that the method imposes on the order of the pseudo-differential operator is sharp, so we do not state this extension explicitly.

b) For $\delta\leqq\varrho\leqq 1/2$, Theorem 4.1 states that $(Lf)^\#\leqq C_p M_p f(x)$ for all $p>1$ if $m\leqq -n(1-\varrho)$. On the other hand S. Chanillo and A. Torchinsky proved that $(Lf)^\#(x)\leqq CM_2 f(x)$ if $m\leqq -n(1-\varrho)/2$ ([5]) and asked whether $p=2$ was the smallest possible value of p for that order. This question remains open.

5. Continuity on Hardy spaces

We start by proving "dyadic" estimates of the type considered in [2]. We use the notation

$$C_j(z, r) = \{x\in\mathbf{R}^n\colon\ 2^j r < |x-z| < 2^{j+1}r\},\quad j = 1, 2, \ldots .$$

The meaning of λ is the same as in the previous sections.

Theorem 5.1. *Let* $L\in\mathscr{L}^m_{\varrho,\delta}$, $0<\varrho\leqq 1$, $0\leqq\delta<1$, *with kernel* $k(x, y)$. *Then,*

a) *If* $m\leqq -n[(1-\varrho)/2+\lambda]$, $0<\theta\leqq 1$, $j=1, 2, \ldots$,

$$(5.1)\qquad \begin{aligned} &\sup_{|y-z|<\sigma}\int_{C_j(z,\sigma^\theta)}|k(x, y)-k(x, z)|\,dx \leqq C2^{-j/\varrho}\sigma^{1-\theta/\varrho}\quad \text{if}\quad \sigma<1;\\ &\sup_{|y-z|<\sigma}\int_{C_j(z,\sigma)}|k(x, y)-k(x, z)|\,dx \leqq C2^{-j}\quad \text{if}\quad \sigma\geqq 1.\end{aligned}$$

b) *If* $m\leqq -n(1-\varrho)/2$, $0<\theta\leqq 1$, $j=1, 2, \ldots$,

$$(5.2)\qquad \begin{aligned} &\sup_{|y-z|<\sigma}\int_{C_j(z,\sigma^\theta)}|k(y, x)-k(z, x)|\,dx \leqq C2^{-j/\varrho}\sigma^{1-\theta/\varrho}\quad \text{if}\quad \sigma<1;\\ &\sup_{|y-z|<\sigma}\int_{C_j(z,\sigma)}|k(y, x)-k(z, x)|\,dx \leqq C2^{-j}\quad \text{if}\quad \sigma\geqq 1.\end{aligned}$$

Proof. The proof is similar to that of Theorem 2.1. With the same notation, we start estimating for $\sigma<1$, $|y-z|<\sigma$ and $0<\theta\leqq 1$,

$$(5.3)\qquad \int_{C_j(z,\sigma^\theta)}|k(x, y, t)-k(x, z, t)|\,dx$$

$$\leqq \Big[\int(1+t^{2\varrho}|x-z|^2)^N|k(x, y, t)-k(x, z, t)|^2\,dx\Big]^{1/2}\Big[\int_{C_j(z,\sigma^\theta)}(1+t^{2\varrho}|x-z|^2)^{-N}dx\Big]^{1/2},$$

where $N>n/2$ is a natural number to be determined later. We know from the proof of Theorem 2.1 that the first factor on the right-hand side of (5.3) is dominated by

$$C\sigma t\, t^{\varrho n/2}\quad\text{if}\quad \sigma t\leqq 1.$$

To estimate the second factor consider the function

$$(5.4)\qquad F(r) = \Big[\int_r^{2r}(1+s^2)^{-N}s^{n-1}\,ds\Big]^{1/2},\quad 0<r<\infty.$$

It is clear that F is smooth, $F(r)\sim r^{n/2}$ as $r\to 0$ and $F(r)\sim r^{n/2-N}$ as $r\to\infty$. It is easy to check that the second factor on the right-hand side of (5.3) is dominated by

$$Ct^{-\varrho n/2}F(t^\varrho 2^j\sigma^\theta),$$

which implies that

$$(5.5)\qquad \int_{C_j(z,\sigma^\theta)}|k(x, y, t)-k(x, z, t)|\,dx\leqq Ct\sigma F(t^\varrho 2^j\sigma^\theta),\quad t\sigma\leqq 1.$$

A similar computation, as in the proof of Theorem 2.1, shows that

$$\int_{C_j(z,\sigma^\theta)} |k(x,y,t)|+|k(x,z,t)|\,dx \leqq C(t^\varrho 2^j\sigma^\theta)^{n/2-N}. \tag{5.6}$$

Hence, (2.4), (5.5) and (5.6) yield

(5.7)
$$\int_{C_j(z,\sigma^\theta)} |k(x,y)-k(x,z)|\,dx \leqq C\Big[\int_0^{\sigma^{-1}} t\sigma F(t^\varrho 2^j\sigma^\theta)+\int_{\sigma^{-1}}^\infty (t^\varrho 2^j\sigma^\theta)^{n/2-N}\Big]\,dt/t.$$

Now we choose N so that $\varrho(N-n/2)>1$, which makes $\int F(t^\varrho)dt<\infty$ in view of (5.4). Then (5.7) implies

$$\int_{C_j(z,\sigma^\theta)} |k(x,y)-k(x,z)|\,dx$$
$$\leqq C(2^{-j/\varrho}\sigma^{1-\theta/\varrho}+2^{j(n/2-N)}\sigma^{(1-\theta/\varrho)\varrho(N-n/2)}) \leqq C2^{-j/\varrho}\sigma^{1-\theta/\varrho},\quad 0<\sigma<1,$$

by the choice of N. This proves (5.1) for $\sigma<1$. Inequality (5.1) for $\sigma\geqq 1$ as well as (5.2) can be proved along the same lines and will be left to the reader.

The next theorem deals with (H^p, L^p) continuity for some range of $p\leqq 1$ and extends results of [9].

Theorem 5.2. *Let* $L\in\mathscr{L}^m_{\varrho,\delta}$, $0<\varrho\leqq 1$, $0\leqq\delta<1$. *Assume that*

$$m\leqq -\beta-n\lambda$$

for some $n(1-\varrho)/2\leqq\beta<n/2$ *and set*

$$\frac{1}{p_0}=\frac{1}{2}+\frac{\beta(1/\varrho+n/2)}{n(1/\varrho-1+\beta)} \tag{5.8}$$

(it is understood that for $\varrho=1$, $p_0=n/(n+1)$ *even when* $\beta=0$*). Then L maps continuously H^p into L^p for* $p_0\leqq p\leqq 1$, *when* $0<\varrho<1$, *and for* $p_0<p\leqq 1$, *when* $\varrho=1$.

Proof. The proof relies on

Theorem 5.3. [2, p. 412.] *Let* T: $C_c^\infty(\mathbf{R}^n)\to\mathscr{D}'$ *be a linear operator associated to a kernel* $k(x,y)$ *satisfying* (5.1). *Moreover, suppose that T extends to operators of type* $(2,2)$ *and* $(r,2)$ *with* $1/r=1/2+\beta/n$, $n(1-\varrho)/2\leqq\beta<n/2$, $0<\varrho\leqq 1$. *Then T maps continuously H^p into L^p for* $p_0\leqq p\leqq 1$, *when* $0<\varrho<1$, *and for* $p_0<p\leqq 1$, *when* $\varrho=1$.

We now check that the hypotheses of Theorem 5.3 are satisfied. Indeed, since $\beta\geqq n(1-\varrho)/2$, it follows from Theorem 5.1 that $k(x,y)$ satisfies (5.1). Next, Theorem 3.5 and the fact that $m\leqq-\beta-n\lambda$ imply that L is of type $(r,2)$ if $1/r=1/2+\beta/n$ and certainly of type $(2,2)$. The proof is complete.

Theorem 5.4. *Let* $L\in\mathcal{L}^m_{\varrho,\delta}$, $0<\varrho\leqq 1$, $0\leqq\delta<1$. *Assume that*

$$m \leqq -\beta - n\lambda$$

for some $n(1-\varrho)/2\leqq\beta<n/2$ *and* $L^*(1)=0$ *in the sense of BMO. Then L maps continuously* H^p *into itself for* $p_0<p\leqq 1$ *where* p_0 *is given by* (5.8).

Proof. According to Theorem 3.1 L^* maps L^∞ into BMO, so the condition $L^*(1)=0$ is well defined. Furthermore, L is of types $(2, 2)$ and $(r, 2)$ with $1/r=1/2+\beta/n$, and its kernel satisfies (5.1). Applying [2, Thm. 3.5] we get the result.

Remarks. a) The same kind of analysis shows that if we change in Theorem 5.4 condition $L^*(1)=0$ to $L(1)=0$ (in the sense of BMO), we will obtain that L maps BMO into itself.

b) Theorems 5.3 and 5.4 were known for $\varrho=1$ when the pseudo-differential operators involved are associated to standard kernels.

6. A counterexample to a $T1$ theorem

In this section we show that operators with kernels verifying estimates slightly worse than those of standard kernels do not verify a $T1$ theorem, i.e., its L^2-continuity is not related to the conditions $T1$ and $T^*1\in BMO$.

Lemma 6.1. *Let* $T\colon C_c^\infty(\mathbf{R}^n)\to\mathcal{D}'$ *be a linear and continuous operator whose distribution kernel is locally integrable outside the diagonal and satisfies*

$$(6.1)\qquad |k(x,y)-k(z,y)| \leqq C\frac{|x-z|}{|y-z|^{n+\mu}} \quad \text{if} \quad 2|x-z|<|y-z|$$

for some $\mu>0$. *Then, given* $f\in\mathcal{C}^\infty\cap L^\infty$, Tf *is well defined as a linear, continuous functional on*

$$\mathcal{D}_0 = \left\{g\in C_c^\infty(\mathbf{R}^n)\colon \int g(x)\,dx = 0\right\}.$$

Proof. It follows [7, p. 372]. Let us fix $f\in\mathcal{C}^\infty\cap L^\infty$ and $z\in\mathbf{R}^n$. Given a ball $B(z,\sigma)$ with $\sigma\geqq 1$, let us consider a test function $\varphi\in C_c^\infty(\mathbf{R}^n)$, such that $0\leqq\varphi\leqq 1$, $\varphi=1$ on a neighborhood of $\overline{B(z, 2\sigma)}$.

The adjoint T^* is continuous and linear from $C_c^\infty(\mathbf{R}^n)$ into $\mathcal{D}'$ and associated to the kernel $\overline{k(y,x)}$. Thus $\langle T(f\varphi), g\rangle=\langle T^*g, f\varphi\rangle$ is well defined for any test function g if we denote by $\langle\, ,\rangle$ the duality between test functions and distributions. Suppose that $g\in\mathcal{D}_0$ and $\operatorname{supp} g\subset B(z,\sigma)$, $\sigma\geqq 1$. Then, for $|y-z|>2\sigma$, we have

$$T^*g(y) = \int \overline{k(x,y)}g(x)\,dx = \int \left(\overline{k(x,y)-k(z,y)}\right)g(x)\,dx.$$

Thus,

$$|T^*g(y)| \leqq C\frac{\sigma^{n+1}}{|y-z|^{n+\mu}}\|g\|_{L^\infty},$$

which implies

$$|\langle T^*g, f(1-\varphi)\rangle| \leqq C\|f\|_{L^\infty}\|g\|_{L^\infty}\sigma^{n+1}\int_{|y-z|>2\sigma}|y-z|^{-n-\mu}\,dy = C\|f\|_{L^\infty}\|g\|_{L^\infty}\sigma^{n+1-\mu}.$$

Hence, we may define

$$\langle Tf, g\rangle \stackrel{\text{def}}{=} \langle T(f\varphi), g\rangle + \int T^*g(y)f(y)(1-\varphi(y))\,dy, \quad g\in C_c^\infty(B(z,\sigma)).$$

It is easy to check that this definition of Tf is independent of φ as long as $\varphi=1$ on a neighborhood of $\overline{B(z, 2\sigma)}$, and it agrees with the original definition of Tf if f is compactly supported. Since $\sigma\geqq 1$ is arbitrary, Tf defines an element of $\mathcal{D}_0'$.

One technical ingredient in the $T1$ theorem of David and Journé is the so-called weak boundedness property, *WBP*. Given an operator $T\colon C_c^\infty(\mathbf{R}^n)\to\mathcal{D}'$, linear and continuous, T has the *WBP* if:

For each bounded subset $\mathcal{B}\subset C_c^\infty(\mathbf{R}^n)$, there exists a constant $C>0$ such that, for any $a\in\mathbf{R}^n$, $t>0$, $\varphi, \psi\in\mathcal{B}$,

$$\left|\left\langle T\left(\varphi\left(\frac{\cdot-a}{t}\right)\right), \psi\left(\frac{\cdot-a}{t}\right)\right\rangle\right| \leqq Ct^n.$$

It is simple to verify that a pseudo-differential operator will satisfy the *WBP* if its symbol is bounded.

Theorem 6.2. *For each $\varepsilon>0$, there exists a linear and continuous operator $T\colon C_c^\infty(\mathbf{R}^n)\to\mathcal{D}'$ with a distribution kernel $k(x,y)$, smooth outside the diagonal and verifying*

$$(6.2)\qquad |\nabla k(x,y)| \leqq \frac{C}{|x-y|^{n+1+\varepsilon}}, \quad x\neq y,$$

such that:

i) *T has the WBP;*
ii) *$T1=T^*1=0$;*
iii) *T is unbounded in L^2.*

Proof. We follow closely Hörmander's construction [12, p. 530]. Let us denote $\|\xi\|=\max(|\xi_1|, \ldots, |\xi_n|)$, $\xi\in\mathbf{R}^n$, and let $e=(1,0,\ldots,0)\in\mathbf{R}^n$ be the first vector in the canonical basis. Consider a function $0\leqq\varphi\in C_c^\infty(\mathbf{R}^n)$, supported in the cube $\|\xi\|<1/2$, such that $\varphi=1$ if $\|\xi\|\leqq 1/4$, and set, for $0<\varrho<\delta<1$ fixed,

$$(6.3)\quad p_j(x,\xi) = \sum_{\|k\|<2^{j(\delta-\varrho)}}\exp[-i2^{j\varrho}k\cdot x]\varphi(2^{-j\varrho}\xi-2^{j(1-\varrho)}e-k), \quad j=1,2,\ldots,$$

where k runs over the lattice points of $\mathbf{R}^n$. For any fixed $j \in \mathbf{Z}_+$, all terms in (6.3) have disjoint supports and the number of terms is $(2[2^{j(\delta-\varrho)}]+1)^n \geqq (2^{j(\delta-\varrho)}-1)^n$. The ξ-projection of the support of $p_j(x, \xi)$ is contained in a cube centered at $2^j e$ with side $l_j \leqq \frac{3}{2} 2^{j\delta}$. Hence, $\xi \sim 2^j$ on the support of p_j and there exists j_0 depending on δ such that for $j, l \geqq j_0$ the supports of p_j and p_l are disjoint if $j \neq l$. Thus,

$$p(x, \xi) = \sum_{j=j_0}^{\infty} p_j(x, \xi) \tag{6.4}$$

is a symbol in $S^0_{\varrho,\delta}$. We will check that if T is the pseudo-differential operator with symbol given by (6.4), (6.3), and if $0<\varrho<\delta<1$ and j_0 are chosen conveniently, T has all required properties. First, Theorem 1.1 e) with $M=1$ shows that (6.2) holds for $|x-y| \leqq 1$ if $(n+1)/\varrho < n+1+\varepsilon$, and a) of the same theorem guarantees that (6.2) holds for $|x-y|>1$. So (6.2) is achieved by fixing ϱ close enough to 1. Furthermore, T has the WBP property because $p(x, \xi)$ is bounded.

Now, $T1$ is defined in the sense of distributions and equal to the function $p(x, 0)$. Hence, if we increase j_0 so that $2^{j_0(1-\varrho)}-2^{j_0(\delta-\varrho)}>1/2$, all terms in (6.3) will vanish at $\xi=0$ for $j \geqq j_0$ and we get $T1=0$. On the other hand, T^*1 is also defined in the sense of distributions and a simple computation gives

$$T^*1(y) = \sum_{j=j_0}^{\infty} \sum_{\|k\|<2^{j(\delta-\varrho)}} \varphi(-2^{j(1-\varrho)}e) \exp[-i2^{j\varrho}k \cdot y].$$

By our previous choice of j_0, we have for $j \geqq j_0$, $\|2^{j(1-\varrho)}e\| = 2^{j(1-\varrho)} \geqq 2^{j_0(1-\varrho)} > 1/2$ so $T^*1=0$.

Finally, we recall Hörmander's argument to show that T is unbounded in L^2. Take $0 \neq \psi \in \mathscr{S}(\mathbf{R}^n)$, with $\hat{\psi}$ compactly supported in the cube $\|\xi\|<1/4$ and set

$$\hat{u}_j = \sum_{\|k\|<2^{j(\delta-\varrho)}} \hat{\psi}(\xi - 2^j e - 2^{j\varrho}k), \tag{6.5}$$

so, in particular, all terms in (6.5) have disjoint supports. Then,

$$p(x, \xi)\hat{u}_j(\xi) = \sum_{\|k\|<2^{j(\delta-\varrho)}} \exp[-i2^{j\varrho}k \cdot x] \hat{\psi}(\xi - 2^j e - 2^{j\varrho}k),$$

where we have used that $\varphi(2^{-j\varrho}\xi)=1$ on the support of $\hat{\psi}$. Thus,

$$Tu_j(x) = \exp[i2^j x \cdot e] \sum_{\|k\|<2^{j(\delta-\varrho)}} \psi(x),$$

so

$$\|Tu_j\|_{L^2} \geqq (2^{j(\delta-\varrho)}-1)^n \|\psi\|_{L^2}. \tag{6.6}$$

On the other hand, $\|u_j\|_{L^2} = C\|\hat{u}_j\|_{L^2} \leqq (2^{j(\delta-\varrho)+1}+1)^{n/2} \|\psi\|_{L^2}$ which in view of (6.6) shows that T is unbounded in L^2.

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