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ESTIMATES FOR THE PRINCIPAL SPECTRUM POINT FOR CERTAIN TIME-DEPENDENT PARABOLIC OPERATORS

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Abstract. Non-autonomous parabolic equations are discussed. The periodic case is considered first and an estimate for the principal periodic-parabolic eigenvalue is obtained by relating the original problem to the elliptic one obtained by time-averaging. It is then shown that an analogous bound may be obtained for the principal spectrum point in the almost periodic case. These results have applications to the stability of nonlinear systems and hence, for example, to permanence for biological systems.

1. Introduction

We shall examine a class of time-dependent parabolic problems. In order to clarify the purpose of the analysis, we discuss first a relatively simple case with time periodicity (see [9]). The basic question concerns the principal eigenvalue (p.e.v.) for the partial differential equation

$$-\omega\partial_t v + \mu\Delta v + h(x,t)v = \lambda v, \tag{1.1}$$

where Δ is the Laplacian, ω is the frequency and $h(x,\cdot)$ has period unity. Here the domain Ω is bounded and suitable boundary conditions are imposed on v at $\partial\Omega$, the boundary of Ω (the details are given in section 2). The p.e.v. is the unique eigenvalue of equation (1.1) corresponding to non-negative v. The problem is important from the point of view of applications. The most obvious of these is to extend the analysis of stability for reaction-diffusion equations from the class of autonomous equations, which have been much studied, to periodic problems, which are more realistic in some contexts, for example in biological applications. In particular, the sign of λ determines the stability of the origin, and hence whether or not persistence (that is, the long-time survival of the species) holds for the scalar problem

$$\omega\partial_t z = zm(x,t,z) + \mu\Delta z, \tag{1.2}$$

where $h(x,t) = m(x,t,0)$. This is discussed at length in [9] and applications to systems are covered in [2]. The importance of eigenvalue estimates for the autonomous problem corresponding to equation (1.1) is well known and there has been considerable research directed towards this problem both theoretically and numerically. A key point is that the operator is formally self-adjoint and powerful

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variational techniques are available. Of course equation (1.1) is not self-adjoint and this rules out one of the main approaches. Estimates for λ are available under certain assumptions, e.g. μ small, but the theory (see [9]) is much less developed. We present in section 2 a general estimate, with the Laplacian replaced by an arbitrary (not necessarily self-adjoint) elliptic operator, which appears to be new; this is that $\lambda \geq \lambda^*$ where λ^* is the p.e.v. for the corresponding problem for the time average of h. This has a surprising implication: the addition of temporal variation (with time average zero at every spatial point) tends to destabilise an equilibrium under all circumstances. For the biological problem this means that spatio-temporal variation **always** favours persistence (in the above sense).

In section 3 we generalise (1.1) in a different direction by examining the case when $h(x, \cdot)$ is almost periodic. Here λ must be interpreted as the principal spectrum point (or upper Lyapunov exponent), which is defined in detail later. Almost-periodic time variation is also of interest in the biological context, (see [3] for ordinary differential equation models and [14] for a model involving partial differential equations). The result is clearly analogous to that for the previous case; the Lyapunov exponent satisfies $\lambda \geq \lambda^*$ and again time variation favours persistence.

2. A bound for the principal eigenvalue in the periodic case

The assumptions we shall make are similar to those in [9] with small adjustments. The most important change is in the sign of the eigenvalue, which is here chosen to fit in with the natural notation of the next section. Let

$$Lv = \sum_{i,\,j=1}^{n} \partial_i (a_{i\,j} \partial_j v) + \sum_{j=1}^{n} b_j \, \partial_j v,$$

where the $a_{i\,j}$ and b_j are functions of x only. Assume throughout this section that the following hold for some $\theta > 0$:

(C1) $\Omega \subset \mathbb{R}^n$ is bounded and open with $\partial\Omega \in C^{2+\theta}$.

(C2) Let $\partial/\partial n$ denote differentiation along the outward normal. The boundary conditions are one of the following types:
  - (a) $v = 0$ on $\partial\Omega$ (Dirichlet boundary condition),
  - (b) $\partial v/\partial n + b_0(x)v = 0$ on $\partial\Omega$, where $b_0 \in C^{1+\theta}$ and $b_0 \geq 0$ $(x \in \partial\Omega)$ (regular oblique derivative boundary condition).

(C3) $a_{i\,j} \in C^{1+\theta}(\bar{\Omega}),\ b_j \in C^{\theta}(\bar{\Omega})$. Also $h \in F$ where $F = \{w \in C^{\theta,\theta/2}(\bar{\Omega} \times \mathbb{R}) :\ w$ 1-periodic in $t\}$.

(C4) We may assume that $a_{i\,j} = a_{j\,i}$ $(x \in \bar{\Omega})$ without loss of generality. Suppose also that $-L$ is uniformly elliptic, that is, there exists $\delta > 0$ such that

$$\sum_{i,\,j=1}^{n} a_{i\,j} \xi_i \xi_j \geq \delta |\xi|^2 \ (\xi \in \mathbb{R}^n,\ x \in \bar{\Omega}).$$

Define the time average for $h : \bar{\Omega} \times \mathbb{R} \to \mathbb{R}$ as follows:

$$\hat{h}(x) = \int_0^1 h(x,t) dt,$$

and take $h(x,t) = \hat{h}(x) + H(x,t)$, so that $\hat{H}(x) = 0$ $(x \in \bar{\Omega})$. Consider the principal eigenvalue problem for the elliptic equation

$$Lu + \hat{h}u = \lambda^* u, \tag{2.1}$$

and for the periodic-parabolic equation

$$-\omega\partial_t v + Lv + hv = \lambda v, \tag{2.2}$$

under one of the boundary conditions (C2).

Theorem 2.1. *The inequality $\lambda \geq \lambda^*$ holds, with equality if and only if H is independent of x.*

We need a simple preliminary result.

Lemma 2.2. *With $f = (f_1, ..., f_n)^T$, let $f : [0,1] \to \mathbb{R}^n$ be continuous. Then for each x,*

$$\sum_{i,\,j=1}^{n} a_{i\,j} \int_0^1 f_i(t)dt \int_0^1 f_j(t)dt \leq \sum_{i,\,j=1}^{n} a_{i\,j} \int_0^1 f_i(t)f_j(t)dt. \tag{2.3}$$

Proof. By (C4), there is an orthogonal matrix L such that $A = L^T DL$ where D is diagonal with entries $d_i > 0$. With x, y column vectors in $\mathbb{R}^n$, set $y = Lx$. Then

$$\sum_{i,\,j=1}^{n} x_i a_{i\,j} x_j = \sum_{i=1}^{n} d_i y_i^2. \tag{2.4}$$

Put first $x_i = \int_0^1 f_i(t)dt$ and $y = Lx = L\int_0^1 f(t)dt = \int_0^1 Lf(t)dt$. Then take $x_i = f_i(t)$ and $y(t) = Lx(t)$. Using (2.4), one sees that (2.3) becomes

$$\sum_{i=1}^{n} d_i \left(\int_0^1 y_i(t)dt \right)^2 \leq \sum_{i=1}^{n} d_i \int_0^1 y_i^2(t)dt.$$

The result follows from Schwarz's inequality. □

Proof of Theorem 2.1. Let

$$w(x) = \exp\left(\int_0^1 \ln v(x,t)dt \right). \tag{2.5}$$

For boundary conditions (C2)(b) this definition is valid for $x \in \bar{\Omega}$. For (C2)(a), suppose (2.5) holds for $x \in \Omega$ and extend w to $\bar{\Omega}$ by defining it to be zero on $\partial\Omega$. Differentiation of (2.5) gives

$$\frac{\partial_j w}{w} = \int_0^1 \frac{\partial_j v}{v} dt. \tag{2.6}$$

Multiplication of each equation by $a_{i\,j}$, summation over j and differentiation with respect to x_i yields the relation

$$-\frac{\partial_i w}{w^2} \sum_{j=1}^{n} a_{i\,j}\partial_j w + \frac{1}{w}\partial_i \sum_{j=1}^{n} a_{i\,j}\partial_j w$$
$$= -\int_0^1 \frac{\partial_i v}{v^2} \sum_{j=1}^{n} a_{i\,j}\partial_j v dt + \int_0^1 \frac{1}{v}\partial_i \left(\sum_{j=1}^{n} a_{i\,j}\partial_j v \right) dt.$$

The first term of this equation may be expressed in terms of v by using (2.6). Now sum over i and rearrange to obtain

$$\begin{aligned}\frac{1}{w}\sum_{i,j=1}^{n}\partial_i(a_{ij}\partial_j w) &= \sum_{i,j=1}^{n}\int_0^1\frac{\partial_i v}{v}dt\,a_{ij}\int_0^1\frac{\partial_j v}{v}dt\\ &\qquad -\sum_{i,j=1}^{n}\int_0^1\frac{\partial_i v}{v}a_{ij}\frac{\partial_j v}{v}dt+\int_0^1\frac{1}{v}\sum_{i,j=1}^{n}\partial_i(a_{ij}\partial_j v)dt\\ &\qquad \le \int_0^1\frac{1}{v}\sum_{i,j=1}^{n}\partial_i(a_{ij}\partial_j v)dt\end{aligned}$$

by Lemma 2.2 with $f_i = v^{-1}\partial_i v$. Therefore, from the differential equation (2.2) for v,

$$\frac{1}{w}\sum_{i,j=1}^{n}\partial_i(a_{ij}\partial_j w) \le \int_0^1\frac{1}{v}\left(\omega\partial_t v-\sum_{i=1}^{n}b_i\partial_i v-hv+\lambda v\right)dt.$$

From periodicity, the first term on the right-hand side is zero. Thus, using (2.6) again and rearranging, we obtain

$$Lw \le -(\hat{h}-\lambda)w. \tag{2.7}$$

We note for future reference that strict inequality holds for some x unless, for each i and $x \in \bar{\Omega}$, $v^{-1}\partial_i v$ is independent of t.

We first prove that $\lambda \ge \lambda^*$. For if $\lambda < \lambda^*$, since $w > 0$ $(x \in \Omega)$, from (2.7) there is a function m with $m(x) < 0$ $(x \in \Omega)$ such that

$$Lw + (\hat{h}-\lambda^*)w = m(x). \tag{2.8}$$

From (2.1), zero is the principal eigenvalue of the operator $L + \hat{h} - \lambda^*$. It follows (see [1], Proof of Theorem 4.3 and Theorem 3.2 for example) that zero is also the principal eigenvalue of the adjoint operator with corresponding eigenfunction u', say, which is also strictly positive in Ω. From a standard Fredholm alternative argument, equation (2.8) has a solution if and only if $\int_\Omega m(x)u'(x)dx = 0$. However, this is impossible as $u' > 0$ and $m < 0$ on Ω. This contradiction proves that $\lambda \ge \lambda^*$.

Finally, consider the case of equality, that is, $\lambda = \lambda^*$. Clearly this holds if $H(x,t)$ is a function of t only, say g, where we recall the definition $h(x,t) = \hat{h}(x) + H(x,t)$. To see this one only needs to substitute

$$v(x,t) = u(x)\exp\left(\frac{1}{\omega}\int_0^t g(s)ds\right)$$

into equation (2.2) and use equation (2.1). On the other hand, it was noted above that strict inequality holds in (2.7) for some x (and hence $\lambda < \lambda^*$ by a minor amendment of the argument in the last paragraph) unless for each i and $x \in \Omega$, $v^{-1}\partial_i v$ is independent of t, say $v^{-1}\partial_i v = F_i(x)$ for some F. Then $\nabla \ln v = F$ and $\nabla\partial \ln v/\partial t = 0$. Thus $v = X(x)T(t)$ for some smooth X, T (where X is positive and T is periodic and positive). From equation (2.2), for all $x \in \Omega$ and $t \in \mathbb{R}$

$$-\omega\frac{T'}{T} + \frac{LX}{X} + h = \lambda.$$

Therefore h is of the form asserted. □

3. A bound for the principal spectrum point in the almost periodic case

In this section, we consider the spectrum of the almost periodic parabolic problem

$$\begin{cases} \partial_t v = \mu \Delta v + h(x,t) v & (x \in \Omega), \\ \\ Bv = 0 & (x \in \partial\Omega), \end{cases} \tag{3.1}$$

where Ω satisfies (C1), $Bv = 0$ denotes one of the boundary conditions (C2) (a) or (b), and $h \in C^{\theta,\theta/2}(\bar{\Omega} \times \mathbb{R})$ and is uniformly almost periodic in t. To do so, we first give a brief review of the basic theory of almost periodic functions and construct a skew-product semiflow associated with (3.1). Then we introduce the definition of the dynamic spectrum and present the continuous separation property of (3.1). Finally we provide a bound for the principal spectrum point of (3.1) (Theorem 3.14).

It should be pointed out that, as in the previous section, the results in this section also hold when the Laplacian in (3.1) is replaced by an arbitrary elliptic operator. But for simplicity in notation, we consider the relatively simple equation (3.1) only. It should also be pointed out that if h is actually periodic, then Theorem 3.14 implies Theorem 2.1, but Theorem 2.1 can be proved directly as is done.

3.1. **Almost periodic functions.**

Definition 3.1.

1) A function $f \in C(\mathbb{R}, \mathbb{R}^m)$ is said to be almost periodic if for any $\epsilon > 0$, the set

$$T(f,\epsilon) = \{\tau : |f(t+\tau) - f(t)| < \epsilon \ \forall \ t \in \mathbb{R}\}$$

is a relatively dense subset of $\mathbb{R}$, that is, there is positive number L such that $[a, a+L] \cap T(f,\epsilon) \neq \emptyset \ \forall \ a \in \mathbb{R}$, where $|\cdot|$ is the usual Euclidean norm in $\mathbb{R}^m$. For further discussion see [6].

2) Let $D \subset \mathbb{R}^n$. A function $f \in C(D \times \mathbb{R}, \mathbb{R}^m)$: $(x,t) \mapsto f(x,t)$ is said to be uniformly almost periodic in t if f is almost periodic in t for each $x \in D$, and for any compact set $E \subset D$, f is uniformly continuous on $E \times \mathbb{R}$.

3) Let $D \subset \mathbb{R}^n$ and $f \in C(D \times \mathbb{R}, \mathbb{R}^m)$ be uniformly almost periodic in t. Then $H(f) = cl\{f \cdot \tau : \tau \in \mathbb{R}\}$ is called the hull of f, where $f \cdot \tau(x,t) = f(x, t+\tau)$ and the closure is taken in the compact open topology.

Lemma 3.2.

1) *Let $f \in C(\mathbb{R}, \mathbb{R})$ be almost periodic. Then $\lim_{T\to\infty} \frac{1}{T}\int_0^T g(t)dt$ exists and is independent of $g \in H(f)$.*

2) *Let $D \subset \mathbb{R}^n$ and $f \in C(D \times \mathbb{R}, \mathbb{R}^m)$: $(x,t) \mapsto f(x,t)$ be uniformly almost periodic in t. Then $H(f)$ is compact in the compact open topology.*

3) *Let $D \subset \mathbb{R}^n$ and $f_k \in C(D \times \mathbb{R}, \mathbb{R}^m) : (x,t) \mapsto f_k(x,t)$ $(k \in \mathbb{Z}^+)$ be a family of uniformly almost periodic functions in t and $\lim_k f_k(x,t) = f_0(x,t)$ uniformly for $t \in \mathbb{R}$ and x in compact subsets of D. Then $f_0(x,t)$ is uniformly almost periodic in t.*

Proof. See [6]. □

3.2. **Skew-product semiflow.** Consider (3.1). Let $X \subset L^p(\Omega)$ $(p > n)$ be a fractional power space of $-\Delta : \mathcal{D} \to L^p(\Omega)$ satisfying $X \hookrightarrow C^\nu(\bar{\Omega})$ for some ν with $1 < \nu < 2$, where $\mathcal{D} = \{v \in H^{2,p}(\Omega) | Bv = 0 \text{ on } \partial\Omega\}$ (see [8]). Let $\|\cdot\|$ be the norm of X. Then (3.1) generates a skew-product semiflow:

$$\Pi_t : X \times H(h) \to X \times H(h), \quad t \geq 0, \tag{3.2}$$

$$\Pi_t(v_0, k) = (v(\cdot, t; v_0, k), k \cdot t),$$

where $v(x, t; v_0, k)$ is the solution of

$$\begin{cases} \partial_t v = \mu \Delta v + k(x,t)v & (x \in \Omega), \\ Bv = 0 & (x \in \partial\Omega), \end{cases} \tag{3.3}$$

with $v(x, 0; v_0, k) = v_0(x)$ $(k \in H(h))$.

Let $X_+ = \{v \in X | v(x) \geq 0 \text{ for } x \in \Omega\}$. Note that $\operatorname{Int} X_+ \neq \emptyset$. Hence X_+ defines a strong ordering on X as follows:

$$v_1 \leq v_2 \quad \text{iff} \quad v_1(x) \leq v_2(x) \quad \text{for} \quad \text{all} \quad x \in \Omega,$$

$$v_1 < v_2 \quad \text{iff} \quad v_1 \leq v_2 \quad \text{but} \quad v_1 \neq v_2,$$

$$v_1 \ll v_2 \quad \text{iff} \quad v_2 - v_1 \in Int X_+.$$

Lemma 3.3. *The skew-product semiflow $\{\Pi_t\}_{t \in \mathbb{R}^+}$ in (3.2) is strongly monotone in the sense that $\Pi_t(v_0, k) \gg 0$ for any $t > 0$, $k \in H(h)$, and $v_0 \in X_+$.*

Proof. This follows from the strong maximum principle for parabolic equations ([7]). ∎

3.3. **Dynamic spectrum and Lyapunov exponent.** For given $\sigma \in \mathbb{R}$, define

$$\Pi_t^\sigma : X \times H(h) \to X \times H(h), \quad t \geq 0, \tag{3.4}$$

$$\Pi_t^\sigma(v_0, k) = (\Phi_\sigma(t, k)v_0, k \cdot t),$$

where $\Phi_\sigma(t, k)v_0 = e^{-\sigma t} v(\cdot, t; v_0, k)$. We say that $\Phi_\sigma(t, k)v_0$ has a *negative continuation* if the solution $v(\cdot, t; v_0, k)$ of (3.3) has a backward extension for all $t < 0$. Define

$$\mathcal{B}_\sigma = \{(v_0, k) \in X \times H(h) \ : \ \Phi_\sigma(t,k)v_0 \text{ has a negative continuation and } \sup_{t \in \mathbb{R}} \|\Phi_\sigma(t,k)v_0\| < \infty\}. \tag{3.5}$$

For a given operator $P : X \to X$, let $R(P)$ denote the range of P.

Definition 3.4. Given $\sigma \in \mathbb{R}$, the linear skew-product semiflow (3.4) is said to have an exponential dichotomy (ED) if there exist $\beta > 0$, $C > 0$, and continuous projections $P(k) : X \to X$ $(k \in H(h))$ such that for any $k \in H(h)$ the following holds:

1) $\Phi_\sigma(t, k)P(k) = P(k \cdot t)\Phi_\sigma(t, k)$ for $t \in \mathbb{R}^+$;
2) $\Phi_\sigma(t, k)|_{R(P(k))} : R(P(k)) \to R(P(k \cdot t))$ is an isomorphism for $t \in \mathbb{R}^+$ (hence $\Phi_\sigma(-t, k) := \Phi_\sigma^{-1}(t, k \cdot (-t)) : R(P(k)) \to R(P(k \cdot (-t))$ is well defined for $t \in \mathbb{R}^+$);

3)

$$\|\Phi_\sigma(t,k)(I-P(k))\| \leq Ce^{-\beta t} \quad \text{for} \quad t \in \mathbb{R}^+,$$

and

$$\|\Phi_\sigma(t,k)P(k)\| \leq Ce^{\beta t} \quad \text{for} \quad t \in \mathbb{R}^-.$$

Definition 3.5. Given $\sigma \in \mathbb{R}$, (3.4) is said to be weakly hyperbolic if $\mathcal{B}_\sigma = \{0\} \times H(h)$.

Definition 3.6. The set $\Sigma = \{\sigma \in \mathbb{R} : (3.4) \text{ admits no ED}\}$ is called the Sacker-Sell or the dynamic spectrum of (3.2) or (3.1).

Lemma 3.7. *Given $\sigma \in \mathbb{R}$,* (3.4) *admits ED if and only if it is weakly hyperbolic.*

Proof. See [12]. □

Lemma 3.8. Σ *has one of the following forms:*

1) $\Sigma = \emptyset$*;*
2) $\Sigma = \bigcup_{i=1}^{\infty}[a_i, b_i]$*;*
3) $\Sigma = (-\infty, b_\infty]$*;*
4) $\Sigma = (-\infty, b_\infty] \cup (\bigcup_{i=1}^{k}[a_i, b_i])$ *for some integer k,*

where the intervals are nonempty and nonoverlapping and $\{a_i\}$, $\{b_i\}$ are decreasing sequences of real numbers with $a_i \leq b_i$ (an interval degenerates to a point if $a_i = b_i$).

Proof. See [4] or [10]. □

Definition 3.9. For given $k \in H(h)$, define

$$\lambda(k) = \limsup_{t\to\infty} \frac{\ln \|\Phi(t,k)\|}{t} \tag{3.6}$$

where $\Phi(t,k) = \Phi_0(t,k)$. The number

$$\lambda = \sup_{k\in H(h)} \lambda(k) \tag{3.7}$$

is called the upper Lyapunov exponent of (3.2) or (3.1).

Lemma 3.10. $\sup \Sigma = \lambda$*.*

Proof. See [13]. □

We remark that if Σ is of form 2) or 4) in Lemma 3.8, then by invariant manifold theory ([5]) the spectrum interval $[a_1, b_1]$ determines the stability of the zero solution of (3.1). We may call $[a_1, b_1]$ the *principal spectrum interval* (*principal spectrum point* if $a_1 = b_1$) of (3.1). If h is actually periodic ($h(x, t+T) = h(x,t)$), then Σ is of form 2) in Lemma 3.8 and $a_i = b_i$ for each $i = 1, 2, \cdots$. Moreover, $\lambda = a_1$ is the principal eigenvalue of the periodic parabolic equation

$$\begin{cases} -\partial_t v + \mu\Delta v + h(x,t)v = \lambda v & (x \in \Omega), \\ Bv = 0 & (x \in \partial\Omega), \\ v(x,0) = v(x,T). \end{cases} \tag{3.8}$$

We shall prove that in the almost periodic case Σ is of form 2) or 4) in Lemma 3.8, and $a_1 = b_1$ (Theorem 3.14).

3.4. **Continuous separation.**

Definition 3.11. The skew-product semiflow (3.2) is said to admit a continuous separation if there are subspaces $\{X_1(k)\}_{k\in H(h)}$, $\{X_2(k)\}_{k\in H(h)}$ with the following properties:

1) $X = X_1(k) \oplus X_2(k)$ $(k \in H(h))$ and $X_1(k)$, $X_2(k)$ vary continuously in $k \in H(h)$;
2) $X_1(k) = span\{v(k)\}$, where $v(k) \in \operatorname{Int} X_+$ and $\|v(k)\| = 1$ $(k \in H(h))$;
3) $X_2(k) \cap X_+ = \{0\}$ $(k \in H(h))$;
4) for any $t > 0$ and $k \in H(h)$, $\Phi(t,k)X_1(k) = X_1(k\cdot t)$ and $\Phi(t,k)X_2(k) \subset X_2(k\cdot t)$;
5) there are $M > 0$ and $\delta > 0$ such that for any $k \in H(h)$ and $w \in X_2(k)$ with $\|w\| = 1$,
$$\|\Phi(t,k)w\| \le Me^{-\delta t}\|\Phi(t,k)v(k)\| \quad (t > 0).$$

Note that intuitively, a continuous separation of (3.2) means that for any $k \in H(h)$, there is a time dependent split of X, $X = X_1(k\cdot t) \oplus X_2(k\cdot t)$, which is invariant (Definition 3.11 4)) and exponentially separated (Definition 3.11 5)) under (3.3). Moreover, X_1 is spanned by a positive vector (Definition 3.11 2)) and X_2 does not contain a positive vector (Definition 3.11 3)). It then implies that the stability of the zero solution of (3.1) is determined by the 'one dimensional flow' $\Pi_t : X_1(k) \times \{k\} \to X_1(k\cdot t) \times \{k\cdot t\}$, and the Lyapunov exponent $\lambda = \sup_{k\in H(h)} \limsup_{t\to\infty} \|\Phi(t,k)v(k)\|/t$. Clearly, when $h(x,t)$ is time independent or periodic, by the Krein-Rutman theorem (see Theorems 7.1 and 7.2 in [9]), (3.2) admits a continuous separation with $X_1(h)$ being the eigenspace corresponding to the principal eigenvalue. In general, we have the following extension of the Krein-Rutman theorem.

Lemma 3.12. *The skew-product semiflow* (3.2) *admits a continuous separation.*

Proof. See [11] or [13]. ☐

Observe that $\{\|v(k)\|_2 : k \in H(h)\}$ is bounded away from zero, where $\|\cdot\|_2$ denotes the norm in $L^2(\Omega)$. Let

$$\tilde v(\cdot,t;k) = \frac{v(k\cdot t)}{\|v(k\cdot t)\|_2} \quad (k \in H(h)) \tag{3.9}$$

and $\tilde v(k) = \tilde v(\cdot,0;k)$. We have

Lemma 3.13. *For any $k \in H(h)$, the following holds:*

1) $\tilde v(x,t;k)$ *is differentiable in t.*
2) $\tilde v(x,t;k) \in C(\bar\Omega \times \mathbb{R},\mathbb{R})$ *and* $\nabla\tilde v(x,t;k) \in C(\bar\Omega\times\mathbb{R},\mathbb{R}^n)$ *are uniformly almost periodic in t.*
3) $\Delta\tilde v(x,t;k) \in C(\Omega\times\mathbb{R},\mathbb{R})$ *is uniformly almost periodic in t.*

Proof. 1) Note that $\tilde v(x,t;k)$ can also be expressed as
$$\tilde v(x,t;k) = \frac{v(x,t;v(k),k)}{\|v(\cdot,t;v(k),k)\|_2}.$$
1) follows from the differentiability of $v(x,t;v(k),k)$ and $\|v(\cdot,t;v(k),k)\|_2$ in t.

2) First, it is clear that $\tilde v(x,t;k) \in C(\bar\Omega\times\mathbb{R},\mathbb{R})$ and $\nabla\tilde v(x,t;k) \in C(\bar\Omega\times\mathbb{R},\mathbb{R}^n)$. Next, by Lemma 3.12, $v(k\cdot t) \in C(\bar\Omega\times\mathbb{R},\mathbb{R})$, $\nabla v(k\cdot t) \in C(\bar\Omega\times\mathbb{R},\mathbb{R}^n)$, and

$\|v(k\cdot t)\|_2 \in C(\mathbb{R},\mathbb{R})$ are uniformly almost periodic in t. Hence $\tilde{v}(x,t;k)$ and $\nabla\tilde{v}(x,t;k)$ are uniformly almost periodic in t for $x\in\bar{\Omega}$.

3) Similarly, it is clear that $\Delta\tilde{v}(x,t;k)\in C(\Omega\times\mathbb{R},\mathbb{R})$. Let

$$s(t,k)=\|v(\cdot,t;\tilde{v}(k),k)\|_2.$$

Then $v(x,t;\tilde{v}(k),k)=s(t,k)\tilde{v}(x,t;k)$. By 1), $s(t,k)$ is differentiable in t. Hence

$$\begin{aligned} s_t(t,k)\tilde{v}(x,t;k)+s(t,k)\tilde{v}_t(x,t;k) &= \mu s(t,k)\Delta\tilde{v}(x,t;k) \\ &\quad +k(x,t)s(t,k)\tilde{v}(x,t;k). \end{aligned} \tag{3.10}$$

It then follows from $\int_\Omega \tilde{v}^2(x,t;k)dx=1$ that

$$s_t(t,k)=\tilde{k}(t,k)s(t,k) \tag{3.11}$$

where

$$\begin{aligned} \tilde{k}(t,k) &= \mu\int_{\partial\Omega}\tilde{v}(x,t;k)\frac{\partial\tilde{v}}{\partial n}(x,t;k)dx-\mu\int_\Omega|\nabla\tilde{v}(x,t;k)|^2dx \\ &\quad +\int_\Omega k(x,t)|\tilde{v}(x,t;k)|^2dx. \end{aligned} \tag{3.12}$$

By 2), $\tilde{k}(t,k)$ is almost periodic in t and hence is bounded. By (3.10) and (3.11),

$$\tilde{v}_t(x,t;k)=\mu\Delta\tilde{v}(x,t;k)+k(x,t)\tilde{v}(x,t;k)-\tilde{k}(t,k)\tilde{v}(x,t;k). \tag{3.13}$$

Then by a priori estimates for parabolic equations and the boundedness of $\tilde{v}(\cdot,t;k)$,

$$\lim_{h\to 0}\frac{\tilde{v}_{x_i}(x+he_j,t;k)-\tilde{v}_{x_i}(x,t;k)}{h}=\tilde{v}_{x_ix_j}(x,t;k)$$

uniformly for $t\in\mathbb{R}$ and x in compact subset sets of Ω, where e_j is jth unit vector of $\mathbb{R}^n$. By Lemma 3.2 and 2), $\Delta\tilde{v}$ is uniformly almost periodic in t. □

3.5. **A bound for the principal spectrum point.** Consider (3.1) and its average companion

$$\begin{cases} \partial_t u=\mu\Delta u+\hat{h}(x)u & (x\in\Omega), \\ Bu=0 & (x\in\partial\Omega), \end{cases} \tag{3.14}$$

where $\hat{h}(x)=\lim_{t\to\infty}\frac{1}{t}\int_0^t h(x,\tau)d\tau$. Let Σ and λ be the spectrum and upper Lyapunov exponent of (3.2), respectively, and λ^* be the principal eigenvalue of

$$\begin{cases} \mu\Delta u+\hat{h}(x)u=\lambda^* u & (x\in\Omega), \\ Bu=0 & (x\in\partial\Omega). \end{cases} \tag{3.15}$$

Then we have the following main results.

Theorem 3.14.

1) Σ *is of form* 2) *or* 4) *in Lemma 3.8 and* $a_1=b_1$ *(hence* $\lambda=a_1$*).*

2) $\lambda\ge\lambda^*$.

Proof. 1) By Definitions 3.9 and 3.11 and Lemma 3.12,

$$\lambda=\sup_{k\in H(h)}\limsup_{t\to\infty}\frac{1}{t}\ln\|\Phi(t,k)\tilde{v}(k)\|.$$

Observe that $\Phi(t,k)\tilde{v}(k)=v(\cdot,t;\tilde{v}(k),k)=s(t,k)\tilde{v}(\cdot,t;k)$, where $s(t,k)$ satisfies (3.11). By (3.12), $\tilde{k}(t,k)$ is almost periodic in t and $\lim_{t\to\infty}\frac{1}{t}\int_0^t\tilde{k}(\tau,k)d\tau$

exists and is independent of $k \in H(h)$. Therefore $\lim_{t\to\infty} \frac{1}{t} \ln \|\Phi(t,k)\tilde{v}(k)\| = \lim_{t\to\infty} \frac{1}{t} \int_0^t \tilde{k}(\tau,k)d\tau$ exists and is independent of $k \in H(h)$. It then follows from Lemma 3.12 that (3.4) is weakly hyperbolic for any $\sigma \in (\lambda-\delta,\lambda)$. Now by Lemma 3.7, $\sigma \notin \Sigma$ for any $\sigma \in (\lambda-\delta,\lambda)$. Hence by Lemma 3.8, Σ is of the form 2) or 4) in that lemma and, by Lemma 3.10,

$$\lambda = a_1 = b_1 = \lim_{t\to\infty} \frac{1}{t} \ln \|\Phi(t,k)\tilde{v}(k)\|.$$

2) Let $\phi(x) = \lim_{t\to\infty} \exp(\frac{1}{t} \int_0^t \ln \tilde{v}(x,\tau;h)d\tau)$. By Lemma 3.13,

$$\nabla\phi(x) = \lim_{t\to\infty} \left[\exp\left(\frac{1}{t}\int_0^t \ln \tilde{v}(x,\tau;h)d\tau\right) \cdot \frac{1}{t}\int_0^t \frac{\nabla\tilde{v}(x,\tau;h)}{\tilde{v}(x,\tau;h)}d\tau\right],$$

$$\begin{aligned}(3.16)\qquad \Delta\phi = \lim_{t\to\infty} &\left[\exp\left(\frac{1}{t}\int_0^t \ln \tilde{v}(x,\tau;h)d\tau\right) \cdot \left|\frac{1}{t}\int_0^t \frac{\nabla\tilde{v}(x,\tau;h)}{\tilde{v}(x,\tau;h)}d\tau\right|^2\right. \\ &+ \left.\exp\left(\frac{1}{t}\int_0^t \ln \tilde{v}(x,\tau;h)d\tau\right) \cdot \frac{1}{t}\int_0^t \left(\frac{\Delta\tilde{v}(x,\tau;h)}{\tilde{v}(x,\tau;h)} - \left|\frac{\nabla\tilde{v}(x,\tau;h)}{\tilde{v}(x,\tau;h)}\right|^2\right)d\tau\right]\end{aligned}$$

for $x \in \Omega$, and

$$B\phi = 0 \quad \text{for} \quad x \in \partial\Omega.$$

By Schwarz's inequality

$$\left|\frac{1}{t}\int_0^t \frac{\nabla\tilde{v}(x,\tau;h)}{\tilde{v}(x,\tau;h)}d\tau\right|^2 \le \frac{1}{t}\int_0^t \left|\frac{\nabla\tilde{v}(x,\tau;h)}{\tilde{v}(x,\tau;h)}\right|^2 d\tau.$$

It follows from (3.13) and (3.16) that

$$\begin{cases} \mu\Delta\phi \le -(\hat{h}-\lambda)\phi & (x \in \Omega), \\ B\phi = 0 & (x \in \partial\Omega). \end{cases}$$

Assertion 2) then follows from the arguments in section 2. □

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