As an application we are able to describe the spreading basic sequences in JT and its spreading models.

Recall that a basic sequence is *spreading* if it is equivalent to all of its subsequences. The theory of spreading models can be found in [3].

COROLLARY 7. *Let $\{x_l\}_{l=1}^{\infty}$ be a normalized basic sequence in JT. Then $\{x_l\}_{l=1}^{\infty}$ has a subsequence which is equivalent to either the summing basis for J or to the unit vector basis of l_2. In particular, these two spaces are the only spreading models of JT and every normalized basic sequence in JT admits a spreading subsequence.*

Theorem 5 and the corollary apply to the space $(J \oplus J \oplus \ldots)_{l_2}$ since the latter is a subspace of JT. Thus the above results improve those previously proved by the authors in [6].

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Estimates of Fourier transforms in Sobolev spaces

by

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Abstract. We investigate the Fourier transforms of functions in the Sobolev spaces $W_1^{r_1,\ldots,r_n}$. It is proved that for any function $f \in W_1^{r_1,\ldots,r_n}$ the Fourier transform $\widehat{f}$ belongs to the Lorentz space $L^{n/r,1}$, where $r = n(\sum_{j=1}^n 1/r_j)^{-1} \le n$. Furthermore, we derive from this result that for any mixed derivative $D^s f \quad (f \in C_0^\infty,\ s = (s_1, \ldots, s_n))$ the weighted norm $\|(D^s f)^\wedge\|_{L^1(w)}$ $(w(\xi) = |\xi|^{-n})$ can be estimated by the sum of L^1-norms of all pure derivatives of the same order. This gives an answer to a question posed by A. Pełczyński and M. Wojciechowski.

1. Introduction. For any function $f \in L^1(\mathbb{R}^n)$ its Fourier transform is the function $\widehat{f}$ defined by

$$\widehat{f}(\xi) = \int_{\mathbb{R}^n} f(x)e^{-2\pi i x\cdot\xi}\, dx, \quad \xi \in \mathbb{R}^n.$$

For the Fourier transform of a function $f \in L^p(\mathbb{R}^n)$, $1 \le p \le 2$ (see [13], Ch. 1), we have the following classical inequalities ([2], Ch. 1):

- *the Hausdorff–Young inequality*

$$\|\widehat{f}\|_{p'} \le \|f\|_p, \quad 1 \le p \le 2,\ \frac{1}{p} + \frac{1}{p'} = 1; \tag{1}$$

- *the Hardy–Littlewood–Paley inequality*

$$\Big(\int_{\mathbb{R}^n} |\xi|^{n(p-2)}|\widehat{f}(\xi)|^p\, d\xi\Big)^{1/p} \le c\|f\|_p, \quad 1 < p \le 2. \tag{2}$$

It is well known that (2) is not true for $p = 1$, $n \ge 1$. On the other hand, by Hardy's inequality we know that for any $f \in H^1(\mathbb{R}^n)$,

$$\int_{\mathbb{R}^n} \frac{|\widehat{f}(\xi)|}{|\xi|^n}\, d\xi \le c\|f\|_{H^1}. \tag{3}$$

Furthermore, the inequality (2) can be strengthened in terms of rearrangements.

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Let f be a measurable function on $\mathbb{R}^n$ such that $|\{x : |f(x)| > y\}| < \infty$ for all $y > 0$. The nonincreasing rearrangement of f is defined to be the function f^* that is nonincreasing on $(0,\infty)$ and equimeasurable with $|f(x)|$. We denote by $L^{p,r}(\mathbb{R}^n)$ $(1 \le p, r < \infty)$ the Lorentz space of all functions f, measurable on $\mathbb{R}^n$, for which

$$\|f\|_{p,r} = \left\{ \int_0^\infty [t^{1/p} f^*(t)]^r \frac{dt}{t} \right\}^{1/r} < \infty.$$

Note that $L^{p,r} \subset L^{p,s}$ for $r < s$; in particular, $L^{p,r} \subset L^{p,p} \equiv L^p$ for $r \le p$ (see [1], p. 217).

Suppose $f \in L^{p,r}$ $(1 < p < 2,\ 1 \le r < \infty)$. Then $f \in L^1 + L^2$ and, hence, $\widehat{f}$ is defined. Furthermore (see [6], [7], [12]),

$$\|\widehat{f}\|_{p',r} \le c\|f\|_{p,r} \quad (1/p + 1/p' = 1). \tag{4}$$

This inequality is an $L^{p,r}$-version of the inequality of Hardy and Littlewood

$$\left(\int_0^\infty t^{p-2}\widehat{f}^*(t)^p\,dt\right)^{1/p} \le c\|f\|_p, \quad 1 < p \le 2, \tag{5}$$

which we obtain by setting $r = p$. In view of the Hardy–Littlewood inequality ([1], p. 43), (5) gives a refinement of the inequality (2).

Now let $1 \le p < \infty$ and $r \in \mathbb{N}$. The Sobolev space W_p^r consists of those functions f in $L^p(\mathbb{R}^n)$ for which all distributional derivatives $D^s f$ $(s = (s_1, \ldots, s_n))$ of order $|s| = s_1 + \ldots + s_n \le r$ belong to $L^p(\mathbb{R}^n)$.

A. Pełczyński and M. Wojciechowski [10] obtained the following result as a consequence of an embedding theorem.

THEOREM A. *Let $f \in W_1^r(\mathbb{R}^n)$ $(n \ge 2,\ r \in \mathbb{N})$. Then*

$$\int_{\mathbb{R}^n} |\widehat{f}(\xi)| \cdot |\xi|^{r-n}\,d\xi \le c \sum_{|s|=r} \|D^s f\|_1. \tag{6}$$

The periodic analogue of Theorem A was discovered by Bourgain [4], [5].

Theorem A is equivalent to the following statement: for any derivative $D^k f$ $(k = (k_1, \ldots, k_n),\ |k| = r)$ the following analogue of the Hardy inequality (3) holds:

$$\int_{\mathbb{R}^n} \frac{|(D^k f(\xi))^\wedge|}{|\xi|^n}\,d\xi \le c \sum_{|s|=r} \|D^s f\|_1. \tag{7}$$

It is well known that the L^p-norm of any mixed derivative can be estimated by the sum of the L^p-norms of all non-mixed derivatives of the same order if and only if $1 < p < \infty$ (see [3]). Nevertheless we prove that the right-hand side of the inequality (7) can be replaced by the sum of all pure derivatives of the same order. This gives an answer to a question posed in [10].

2. Main results. Let $1 \le p < \infty$ and $r_1, \ldots, r_n \in \mathbb{N}$. We denote by $W_p^{r_1,\ldots,r_n}$ the anisotropic Sobolev space of all functions $f \in L^p(\mathbb{R}^n)$ for which every distributional partial derivative

$$D_j^{r_j} f \equiv \partial^{r_j} f / \partial x_j^{r_j} \in L^p(\mathbb{R}^n) \quad (j = 1, \ldots, n)$$

exists. If a function f is defined on $\mathbb{R}^n$ and $k \in \mathbb{N}$, then we set

$$\Delta_j^k(h) f(x) = \sum_{\nu=0}^k (-1)^{k-\nu} \binom{k}{\nu} f(x + \nu h e_j),$$

where e_j denotes the jth coordinate vector.

The following theorem was proved in [8].

THEOREM B. *Let $r_1, \ldots, r_n \in \mathbb{N}$ $(n \ge 2)$, $r = n(\sum_{j=1}^n 1/r_j)^{-1}$, $0 < 1 - 1/p < r/n$ and*

$$\alpha_j = r_j \left[1 - \frac{n}{r}\left(1 - \frac{1}{p}\right)\right] \quad (j = 1, \ldots, n).$$

Then for every function $f \in W_1^{r_1,\ldots,r_n}(\mathbb{R}^n)$,

$$\sum_{j=1}^n \int_0^\infty h^{-\alpha_j} \|\Delta_j^{r_j}(h) f\|_p \frac{dh}{h} \le c \sum_{j=1}^n \|D_j^{r_j} f\|_1.$$

This inequality implies the embedding

$$W_1^{r_1,\ldots,r_n} \subset B_{p_1}^{\alpha_1,\ldots,\alpha_n}$$

into the Besov space (see [3] for the definition).

THEOREM 1. *Let $f \in W_1^{r_1,\ldots,r_n}(\mathbb{R}^n)$ $(n \ge 2)$ and $r = n(\sum_{j=1}^n 1/r_j)^{-1} \le n$. Then*

$$\|\widehat{f}\|_{n/r,1} \le c \cdot \sum_{j=1}^n \|D_j^{r_j} f\|_1. \tag{8}$$

Proof. We estimate $\widehat{f}^*(t)$ for fixed $t > 0$. Let E be the set of measure t such that

$$|\widehat{f}(\xi)| \ge \widehat{f}^*(t) \quad \text{for any } \xi \in E. \tag{9}$$

Set $s_k = r/(nr_k)$; then $\sum_{k=1}^n s_k = 1$. Let

$$A_k = \{\xi : |\xi_k| \ge t^{s_k}/2\}, \quad k = 1, \ldots, n.$$

Since $|(\bigcup_{k=1}^n A_k)^c| = |\bigcap_{k=1}^n A_k^c| = t/2^n$, we have $|E \cap \bigcup_{k=1}^n A_k| \ge t/2$. Thus there exists $k = k(t)$ such that $|E \cap A_k| \ge t/(2n)$. Let $Q = E \cap A_k$. Next, let $h > 0$ and $\varphi_h^{(k)}(x) = \Delta_k^{r_k}(h)f(x)$. Then $\widehat{\varphi}_h^{(k)}(\xi) = \widehat{f}(\xi)\sigma(h\xi_k)$, where $\sigma(u) = (e^{2\pi iu} - 1)^{r_k}$. Set $\tau = r_k t^{-s_k}$. We show that for any $\xi \in Q$,

$$\frac{1}{\tau}\int_0^\tau |\sigma(h\xi_k)|\,dh > \frac{1}{2}. \tag{10}$$

Indeed, we have (for $u \in \mathbb{R}$)

$$|\sigma(u)| \ge (1 - \cos 2\pi u)^{r_k} \ge 1 - r_k \cos 2\pi u.$$

Thus, if $|\lambda| \ge t^{s_k}/2$, then

$$\frac{1}{\tau}\int_0^\tau |\sigma(\lambda h)|\,dh \ge \frac{1}{\tau}\int_0^\tau (1 - r_k\cos 2\pi\lambda h)\,dh = 1 - \frac{r_k \sin 2\pi\lambda\tau}{2\pi\lambda\tau} \ge 1 - \frac{r_k}{2\pi|\lambda|\tau} \ge 1 - \frac{1}{\pi}.$$

Now we have, using (9) and (10),

$$\frac{1}{\tau}\int_0^\tau dh \int_Q |\widehat{\varphi}_h^{(k)}(\xi)|\,d\xi = \frac{1}{\tau}\int_0^\tau dh \int_Q |\widehat{f}(\xi)| \cdot |\sigma(h\xi_k)|\,d\xi \ge \frac{1}{2}|Q|\widehat{f}^*(t) \ge \frac{t}{4n}\widehat{f}^*(t).$$

On the other hand, by the Hausdorff–Young inequality (1), for any $p \in (1,2)$,

$$\int_Q |\widehat{\varphi}_h^{(k)}(\xi)|\,d\xi \le |Q|^{1/p}\Big(\int_Q |\widehat{\varphi}_h^{(k)}(\xi)|^{p'}\,d\xi\Big)^{1/p'} \le |Q|^{1/p}\|\varphi_h^{(k)}\|_p \le t^{1/p}\|\varphi_h^{(k)}\|_p.$$

Therefore,

$$\widehat{f}^*(t) \le ct^{1/p-1}\omega_k(r_k t^{-s_k}),$$

where

$$\omega_k(\delta) = \frac{1}{\delta}\int_0^\delta \|\varphi_h^{(k)}\|_p\,dh.$$

By Theorem B, if $1 - 1/p < r/n$, then

$$\sum_{k=1}^n \int_0^\infty h^{-\alpha_k}\|\varphi_h^{(k)}\|_p \frac{dh}{h} \le c\sum_{k=1}^n \|D_k^{r_k} f\|_1, \quad \alpha_k = r_k\Big[1 - \frac{n}{r}\Big(1 - \frac{1}{p}\Big)\Big].$$

Hence,

$$\int_0^\infty t^{r/n-1}\widehat{f}^*(t)\,dt \le c\sum_{k=1}^n \int_0^\infty t^{r/n+1/p-2}\omega_k(t^{-s_k})\,dt \le c'\sum_{k=1}^n \int_0^\infty z^{-\alpha_k-1}\omega_k(z)\,dz \le c''\sum_{k=1}^n \int_0^\infty h^{-\alpha_k-1}\|\varphi_h^{(k)}\|_p\,dh \le B\sum_{k=1}^n \|D_k^{r_k} f\|_1.$$

The proof is completed.

Using the Hardy–Littlewood inequality ([1], p. 43), we obtain the following

COROLLARY 1. *Let $f \in W_p^{r_1,\dots,r_n}$ $(n \ge 2)$ and $r = n(\sum_{j=1}^n 1/r_j)^{-1} < n$. Then for every nonnegative measurable function $w(\xi)$ on $\mathbb{R}^n$ with $w^*(t) = t^{r/n-1}$ $(t > 0)$,*

$$\int_{\mathbb{R}^n} |\widehat{f}(\xi)|w(\xi)\,d\xi \le c\sum_{j=1}^n \|D_j^{r_j} f\|_1. \tag{11}$$

Remark 1. In the limiting case $r = n$ $(n \ge 2)$ Theorem 1 states that for every function $f \in W_1^{r_1,\dots,r_n}$,

$$\|\widehat{f}\|_1 \le c\sum_{j=1}^n \|D_j^{r_j} f\|_1.$$

It is a refinement of the Sobolev theorem, which asserts that in this case every function $f \in W_1^{r_1,\dots,r_n}$ is equivalent to a bounded continuous function on $\mathbb{R}^n$ (see [3], Ch. 3).

Remark 2. The following embedding theorem holds.

THEOREM C. *Let $r_1,\dots,r_n \in \mathbb{N}$ $(n \ge 2)$ and $r = n(\sum_{j=1}^n 1/r_j)^{-1} < n$. Then for any function $f \in W_1^{r_1,\dots,r_n}$,*

$$\|f\|_{n/(n-r),1} \le c\sum_{j=1}^n \|D_j^{r_j} f\|_1.$$

This theorem was proved in [8] (the case $r_1 = \dots = r_n = 1$ was considered in [11]). If $n/(n-r) < 2$, then inequality (8) can be derived from Theorem C and inequality (4).

THEOREM 2. *Let $f \in W_1^{r_1,\dots,r_n}$ $(n \ge 2)$ and $r = n(\sum_{j=1}^n 1/r_j)^{-1}$. Then*

$$\int_{\mathbb{R}^n} |\widehat{f}(\xi)|\Big(\sum_{j=1}^n |\xi_j|^{r_j/r}\Big)^{r-n} d\xi \le c\sum_{j=1}^n \|D_j^{r_j} f\|_1. \tag{12}$$

Proof. It is easy to see that in the case $r < n$ the function

$$w(\xi) = \Big(\sum_{j=1}^n |\xi_j|^{r_j/r}\Big)^{r-n}$$

has nonincreasing rearrangement equivalent to $h(t) = t^{r/n-1}$. Therefore in this case Theorem 2 follows from Corollary 1. To consider the general case we put $s_j = r/(nr_j)$ $(\sum_{j=1}^n s_j = 1)$. Let

$$P_\nu = \{\xi : |\xi_j| \le 2^{\nu s_j}\} \quad (\nu = 0,1,\dots),$$
$$D_0 = P_0, \quad D_\nu = P_\nu - P_{\nu-1} \quad (\nu \ge 1).$$

Next, $D_\nu = \bigcup_{j=1}^n D_\nu^{(j)}$, where
$$D_\nu^{(j)} = \{\xi \in D_\nu : 2^{(\nu-1)s_j} < |\lambda_j| \le 2^{\nu s_j}\} \quad (\nu \ge 1).$$
Let
$$\varphi_h^{(j)}(x) = \Delta_j^{r_j}(h)f(x) \quad \text{and} \quad \delta_\nu^{(j)} = r_j 2^{-\nu s_j}.$$
Reasoning as in the proof of Theorem 1, we have
$$\frac{1}{\delta_\nu^{(j)}} \int_0^{\delta_\nu^{(j)}} dh \int_{D_\nu^{(j)}} |\widehat{\varphi}_h^{(j)}(\xi)|\, d\xi \ge \frac{1}{2} \int_{D_\nu^{(j)}} |\widehat{f}(\xi)|\, d\xi$$
and for any $p \in (1,2)$,
$$\int_{D_\nu^{(j)}} |\widehat{\varphi}_h^{(j)}(\xi)|\, d\xi \le 2^{\nu/p} \|\varphi_h^{(j)}\|_p.$$
Therefore,
$$\int_{D_\nu^{(j)}} |\widehat{f}(\xi) w(\xi)|\, d\xi \le c_1 2^{\nu(r/n-1)} \sum_{j=1}^n \int_{D_\nu^{(j)}} |\widehat{f}(\xi)|\, d\xi$$
$$\le 2c_1 2^{\nu(1/p+r/n-1)} \sum_{j=1}^n \omega_j(\delta_\nu^{(j)}),$$
where
$$\omega_j(\delta) = \frac{1}{\delta} \int_0^\delta \|\varphi_h^{(j)}\|_p\, dh, \quad \nu \ge 1.$$
In the same way as in Theorem 1 we get
$$\sum_{\nu=1}^\infty \int_{D_\nu} |\widehat{f}(\xi)| w(\xi)\, d\xi \le c \sum_{j=1}^n \|D_j^{r_j} f\|_1.$$
Similarly we estimate the integral
$$\int_{D_0} |\widehat{f}(\xi)| w(\xi)\, d\xi,$$
setting $Q_\nu = \{\xi : |\xi| \le 2^{-\nu s_j}\}$ $(\nu = 0, 1, \ldots)$, $E_\nu = Q_\nu - Q_{\nu-1}$ $(\nu \ge 1)$. Further reasoning is the same as above, and this completes the proof.

Now consider the isotropic case $r_1 = \ldots = r_n = r$ $(n \ge 2)$. In this case inequality (12) assumes the form

$$\int_{\mathbb{R}^n} |\widehat{f}(\xi)| \cdot |\xi|^{r-n}\, d\xi \le c \sum_{j=1}^n \|D_j^r f\|_1. \tag{13}$$

As opposed to inequality (6), the right-hand side of (13) contains only the norms of pure derivatives.

Let $f \in W_1^r \subset W_1^{r,\ldots,r}$. For any $s = (s_1, \ldots, s_n)$ with $|s| = r$ we have
$$|(D^s f)^\wedge(\xi)| = (2\pi)^r |\widehat{f}(\xi)| \prod_{j=1}^n |\xi_j|^{s_j} \le (2\pi)^r |\xi|^r |\widehat{f}(\xi)|.$$
Thus, we obtain

THEOREM 3. *Let $f \in W_1^r(\mathbb{R}^n)$ $(n \ge 2,\ r \in \mathbb{N})$. Then*
$$\sum_{|s|=r} \int_{\mathbb{R}^n} \frac{|(D^s f)^\wedge(\xi)|}{|\xi|^n}\, d\xi \le c \sum_{j=1}^n \|D_j^r f\|_1.$$

As is well known, the L^1-norms of mixed derivatives cannot be estimated by the sum of the L^1-norms of directional derivatives of the same order (see [3], Ch. 3).

Remark 3. Theorems 1–3 are true for B-valued functions, where B is a Banach space with non-trivial Fourier type.

In the proofs of Theorems 1–3 we have used only Theorem B and the Hausdorff–Young inequality (for some $p \in (1,2)$). One can easily check that the proof of Theorem B remains valid for B-valued functions for an arbitrary Banach space B. Thus in order to verify Remark 3 it remains to use the definition of the non-trivial Fourier type ([9]).

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Amenability and the second dual of a Banach algebra

by

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Abstract. Amenability and the Arens product are studied. Using the Arens product, derivations from $\mathcal{A}$ are extended to derivations from $\mathcal{A}^{**}$. This is used to show directly that $\mathcal{A}^{**}$ amenable implies $\mathcal{A}$ amenable.

1. Introduction and preliminaries. The study of cohomological properties of $\mathcal{A}^{**}$ in relation to those of $\mathcal{A}$ goes back to B. E. Johnson's seminal article [9]. Recently, Ghahramani, Loy and Willis [4] have studied the amenability and weak amenability of $\mathcal{A}$ in relation to the same properties for $\mathcal{A}^{**}$, with an emphasis on the Banach algebra $L^1(\mathcal{G})$. One of their result is that the amenability of $\mathcal{A}^{**}$ implies the amenability of A: this result was originally proved in [5] by other methods, but has not been published.

In this article, we show how Arens' construction of a product on the second dual of a Banach algebra enables us to extend derivations from $\mathcal{A}$ into a bimodule $\mathcal{X}$ to derivations from $\mathcal{A}^{**}$ into $\mathcal{X}^{**}$, answering a question raised in [9]. This is then used, along with a criterion for amenability which does not involve duals, to give a simple proof that $\mathcal{A}^{**}$ amenable implies $\mathcal{A}$ amenable.

For basic definitions, the reader is referred to [2]. Let $\mathcal{A}$ be a Banach algebra. Then the second dual of $\mathcal{A}$ can also be made into a Banach algebra, using either the *Arens product* or the *reversed Arens product*. For clarity and completeness, we recall precisely a few definitions related to the Arens product, and regroup properties we shall need in a lemma. The reader who wishes to return to the original is referred to [1].

Let X, Y and Z be Banach spaces and let $m : X \times Y \to Z$ be a bounded bilinear map. Let $x \in X$, $x' \in X^*$ and $x'' \in X^{**}$, where X^* is the Banach space dual of X, with similar notations for Y and Z. From m, we can construct a map $m^{***} : X^{**} \times Y^{**} \to Z^{**}$ in the following manner. For $x \in X$, $x' \in X^*$, $x'' \in X^{**}$, and so on, we have maps:

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