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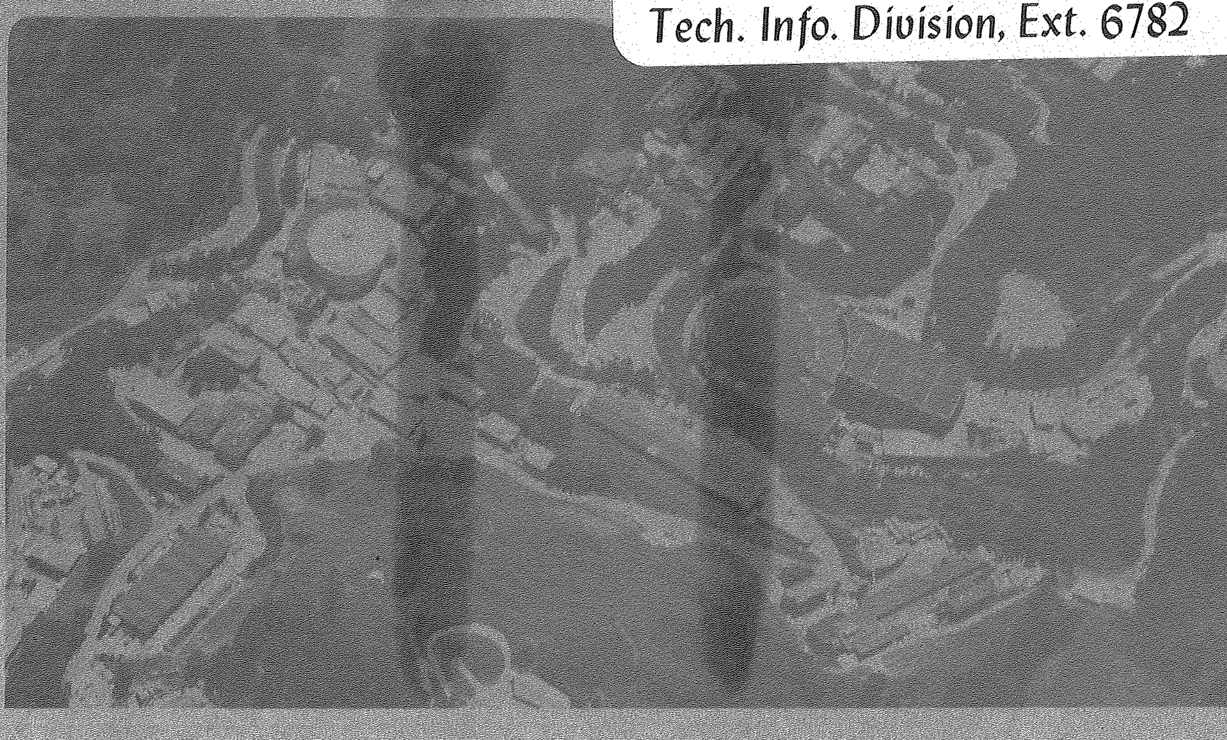

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ABSTRACT

We apply a vortex method to the analysis of the inertial range in fully developed, spatially periodic turbulence. We find that the highly stretched vorticity collects itself into a body of decreasing volume; the numerical results are compatible with the conjecture of Mandelbrot and Frisch et al. that this body has Hausdorff dimension ~2.5. We find an inertial range spectrum of the form $E(k) \sim k^{-(1+\beta)}$, with $\beta = 0.84 \pm 0.03$, a value compatible with experimental data. The calculations suggest that the solutions of Euler's equations in three dimensions blow up in a finite time.

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§1 INTRODUCTION

The inertial range has been the object of many experimental and theoretical investigations since its existence was postulated by Kolmogorov. Kolmogorov's theory and its successors, based mainly on scaling arguments, can be made to fit the experimental data (for reviews, see [8],[16],[23]). However, no analysis of the properties of the inertial range based on the equations of fluid flow has been successfully performed to date. The goal of the present paper is to fill this gap at least in part by computing properties of the inertial range numerically with a vortex method.

Vortex methods are natural candidates for the analysis of flow at high Reynolds number because they place no lower bound on the magnitude of resolvable scales (see e.g.,[7],[9], and the review paper [19]). Our method does not differ in any essential way from earlier vortex methods. We use some labor saving devices but make sure the results are independent of these devices. Numerical innovations are introduced mainly in the analysis of the outcome of the calculations.

The energy E as a function of scale r (= "the structure function") has, in the inertial range, the well known form

$$E(r) = \text{constant} \times r^{\beta} \quad , \qquad \beta = \text{constant} \tag{1}$$

β is the inertial range exponent. The corresponding energy spectrum can be found by Fourier transformation,

$$E(k) = \text{constant} \times k^{-(1+\beta)} \quad ,$$

where k is the wave number. Kolmogorov's original theory predicted $\beta = 2/3$; it is now believed that $\beta > 2/3$; we find $\beta = 0.84$ with a standard deviation of 0.03. This value is compatible with recent experimental values (see the discussion in [12]).

We find that the vorticity stretches unevenly, and that the highly stretched vorticity collects itself into a body with a shrinking volume. The numerical results are compatible with the conjectures of Mandelbrot [20],[21] and Frisch et al [12] that this body approximates a set with Hausdorff dimension ~ 2.5. From the point of view of applications, this may be our most important result (see [10]).

Our calculations also suggest that if the flow can be described by Euler's equation, the vorticity becomes infinite in a finite time for all but special initial data. The time at which the blow-up occurs can be estimated, and the estimates are apparently consistent with recent estimates of Morf et al [24] for the Green-Taylor problem. A stronger statement cannot be made because we could not afford to solve the Green-Taylor problem and had to be content with simpler initial data.

Finally, the calculations support the assumptions in vorticity-based closures for the Navier-Stokes equations introduced by the author [8] and by P. Bernard [3],[4].

§2 THE EQUATIONS OF MOTION

We shall be solving the incompressible Euler equations in vorticity form,

$$\partial_t \underline{\xi} + (\underline{u}\cdot\underline{\nabla})\underline{\xi} - (\underline{\xi}\cdot\underline{\nabla})\underline{u} = 0 \quad , \tag{2a}$$

$$\underline{\xi} = \text{curl}\ \underline{u} \ , \quad \text{div}\ \underline{u} = 0 \ , \tag{2b,c}$$

where $\underline{u}$ is the velocity, $\underline{\xi}$ the vorticity, t the time, and $\underline{\nabla}$ the differentiation vector. These equations are to be solved in a unit cube with periodic boundary conditions. Note that the dissipation term of the Navier-Stokes has been dropped. We are interested in the inertial range of turbulence, i.e., in the limiting behavior of solutions of the Navier-Stokes equation as the viscosity tends to zero and the scale tends to zero, in that order. The first limit is assumed to have been reached. The mathematical question which must be answered is whether the solutions of the Navier-Stokes equations tend, as the viscosity tends to zero, to the solutions of Euler's equations (2) fast enough for the inertial range properties to be the same (for example, convergence in L_1 will do, see [6]). The appropriate theorems in three-dimensional space have not been proved. However, the analogous theorems have been proved in two dimensions [1], and in one dimension for Burgers' equation even when neither the data nor the solution are smooth [17]. Numerically, viscosity is easily added to the algorithms below (see [7] or [9]) and does not modify any of the results. We shall therefore be content with equations (2). In a related problem involving boundaries [10] we shall use the full Navier-Stokes equations.

We assume that at time $t=0$ the vorticity field can be written as a sum of N closed vortex filaments; the circulation of the i-th filament is Γ_i. We shall be interested mostly in those asymptotic properties of the solutions which are independent of the data, and there is no need to describe here in what sense the filaments approximate the initial data. Let $\underline{r}(t)$ denote a point on one of the filaments. The motion of that point as determined by equations (2) can be found from the Biot-Savart law ([18])

$$\frac{d\underline{r}}{dt} = -\frac{1}{4\pi}\sum_{i=1}^{N}\int_{\substack{i\text{-th}\\ \text{filament}}} \frac{\underline{a}\times d\underline{s}}{a^3} \tag{3}$$

where $\underline{s} = \underline{s}(r')$ is the unit tangent to the i-th filament at $\underline{r}'$, $ds = ds(r')$ is the arc length along a filament, $d\underline{s} = \underline{s}\,ds$, $\underline{r}'$ is a coordinate vector along a filament, $\underline{a} = \underline{r} - \underline{r}'$, and a is the length of $\underline{a}$. Equation (3) will be our main equation.

§3 A THREE-DIMENSIONAL VORTEX METHOD

We shall approximate equations (3) by the simplest possible vortex method; for more accurate methods, see [2],[9],[13],[19]. We shall have a comment on the reasons for the success of this method in the concluding section.

Place N_i points on the i-th vortex; let their coordinates be

$$\underline{r}_j^i = (x_j^i, y_j^i, z_j^i)\ , \qquad j = 1,\dots,N_i\ , \qquad i = 1,\dots,N\ .$$

Denote the segment $\overline{r_j^i, r_{j'}^i}$, $j' = (j+1)\bmod N_i$, by S_j^i. Assume all the segments have lengths less than a predetermined small number h. Let $\underline{r}_j^{i,n} = \underline{r}_j^i(nk)$, where n is an integer, k is a time step, and $t = nk$ is the time. The simplest approximation to (2) is

$$r_j^{i,n+1} - r_j^{i,n} = -\frac{k}{4\pi}\sum_{k=1}^{N}\sum_{\ell=1}^{N_i}\frac{\underline{a}\times\Delta\underline{s}}{\phi(a)}\ , \tag{4}$$

where

$$\underline{\Delta s} = \underline{r}^k_{\ell'} - \underline{r}^k_{\ell} \quad ,$$

$$\underline{a} = \tfrac{1}{2}(\underline{r}^k_{\ell'} + \underline{r}^k_{\ell}) - \underline{r}^j_i \quad , \qquad \ell' = (\ell + 1)(\text{mod } N_k) \quad ,$$

$$a = \text{length of } \underline{a} \quad ,$$

and

$$\phi(a) = \begin{cases} a^3 & \text{if} \quad a \geq h \\ h^2 a & \text{if} \quad a < h \quad . \end{cases}$$

The introduction of $\phi(a)$ has the effect of smoothing the self-induction of the vortex lines and the interaction of lines close to each other. More elaborate smoothing was tried without affecting the results. The time step is chosen so that

$$\max_{i,j} \left| \underline{r}^{j,n+1}_i - \underline{r}^{j,n}_i \right| \leq Kk \quad , \tag{5}$$

where K is a moderate constant.

As the flow evolves, the vortex lines become stretched. If a segment S^j_i becomes longer than h, it is broken up into two segments, each of half the original length. If need be, the procedure is repeated. The new points are found by linear interpolation. Here again, a more elaborate procedure is unnecessary.

The stretching of the vortex lines is carefully tracked. Each segment S^j_i is assigned a tag q^j_i. If S^j_i is broken up into two segments, the tag assigned to each has twice the value of q^j_i for the unbroken segment. In this way, the stretching (and therefore change in cross section) of each vortex element during the evolution of the flow is remembered.

If a is large, the interaction of the corresponding segments is small. We set that interaction to zero if $a \geq R$, R constant. This is equivalent to setting $1/\phi = 0$ in (4) if $a \geq R$. We shall have to verify that the results obtained are independent of R for R large enough.

In the spatially periodic initial value problem, we construct the initial vortices as integral lines of the vorticity field

$$\begin{aligned} \xi &= (\xi_1, \xi_2, \xi_3) \quad , \\ \xi_1 &= \sin(2\pi x)\cos(2\pi y)\sin(2\pi z) \quad , \\ \xi_2 &= -\cos(2\pi x)\sin(2\pi y)\sin(2\pi z) \quad , \\ \xi_3 &= -2\cos(2\pi x)\cos(2\pi y)\cos(2\pi z) \quad . \end{aligned} \tag{6}$$

where $\underline{r} = (x,y,z)$ is the coordinate vector. This vorticity field coincides, except for the scaling of $\underline{r}$, with the initial vorticity field of the Green-Taylor problem [25]. The circulations of the vortices are $\Gamma_i = 1$ for all i. The tag q_j^i assigned to each segment is inversely proportional to the length of the vorticity vector given by (6) near that segment. We used few vortex filaments (usually between 1 and 5).

In a periodic flow, each point is affected not only by each segment but also by an infinite set of images of that segment. However, if the long range cut-off R satisfies $R \leq \frac{1}{2}$, in a periodic box of side 1, only one of these interactions is non-zero.

The algorithm has been carefully checked by using it to solve simple problems whose solution is known (e.g., a circular vortex or a helical vortex moving at constant speed).

§4 INTERMITTENCY

It has long been known that highly stretched vorticity occupies a volume much smaller than the total volume available to the flow; this phenomenon is known as "intermittency". Mandelbrot [20],[21] suggested that the highly stretched vorticity collect itself into a body of non-integer Hausdorff dimension D, $D<3$ (For definitions, see [15],[24]; the heuristics of such objects, including graphics, can be found in [21].) Mandelbrot [20] and Frisch et al [12] have presented heuristic theories which relate D to the exponent β in (1); their relation is

$$\beta = 2/3 + (3-D)/3 \quad . \tag{7}$$

Experimental measurements of β suggested to them a value $D \sim 2.5$.

There is a class of objects whose Hausdorff dimension is easily ascertainable. Consider an object with some finite volume. Divide the object into N smaller objects of equal volumes, and throw out all but $M \geqslant 2$ of these smaller objects. Perform the same operations on the remaining volume, and keep on repeating the process. The remainder has Hausdorff dimension $\log M/\log \sqrt[3]{N} = 3 \log M/\log N$ ([15],[21]).

Consider the vorticity distribution produced at a fixed time t by the algorithm of the preceding section. Our goal is to characterize the volume occupied by highly stretched vorticity. Let a,b be positive numbers, $b>1$. Consider the set of segments S_i^j such that

$$ab^{n-1} \leqslant q_i^j \leqslant ab^n \quad . \tag{8}$$

Call this set B_n. Assume that in some remote past all the S_i^j had the same cross section 1. By conservation of volume, the volume of each segment is now proportional to $1/q_i^j$, if one assumes that they have approximately equal lengths. The volume occupied by the segments in B_n is

$$\Delta V_n \cong h \sum_{B_n} \frac{1}{q_i^j} \quad .$$

Let $V_n = \sum_{i=n}^{\infty} \Delta V_i$. (The sum is finite since we have a finite number of segments.)

Imagine that all the segments in the B_m, $m = n, n+1, \ldots$, are compressed and are now temporarily in B_n. The volume they occupy is $\sim V_n$. This volume could be divided among M_n segments, each with volume approximately equal to h/ab^{n-1}, $M_n \sim V_n ab^{n-1}/h$. Now stretch anew all the segments which do not belong to B_n. Their volume could be distributed among M_{n+1} pieces of equal volume $\sim h/ab^{n-1}$, where $M_{n+1} \sim V_{n+1} ab^{n-1}/h$. Now stretch anew all the segments which do not belong to B_{n+1} and evaluate M_{n+2}. In a finite number of steps, this restretching rebuilds the vorticity distribution at the given fixed time. At each step of the restretching, we start with M_i objects of equal volume, $i \geq n$, and, if we are interested only in the highly stretched vorticity, we leave all but M_{i+1} behind. Define the number $D_i = 3 \log M_{i+1} / \log M_i$. The number $(3-D_i)$ is a measure of the reduction in the volume occupied by stretched vorticity. If the D_i are all equal to a number D, the stretched vorticity will end up in an object of Hausdorff dimension D.

In particular runs, the number of segments is finite, and thus we can have at best an approximation to an object of non-integer Hausdorff dimension. This is consistent with the discussion in the section on blow-up where it will be shown that the time at which a singularity can first appear in the solution is never reached in our runs.

We allow the computed vorticity to contain at most N_{max} segments. When this limit is reached, we restart the calculation. However, we do not wish to lose whatever convergence to an asymptotic distribution of the q_j^i had occurred. To prevent the loss, we proceed as follows: the cube is divided into 8 cubes of side 1/2. Seven of these are thrown out. The remaining vortex lines are reconnected, and the new connecting segments are assigned appropriate q's (usually, $q = a$ from (8)). The smaller cube is then stretched to fill the original cube. This operation should reduce the number of segments by approximately a factor of 4, without disturbing unduly the distribution of the q's.

As the segments stretch, the ΔV_i's which correspond to smaller values of q become zero, and the corresponding D_i's are $D_i = 3$. The parameter a in (7) should be chosen so that these empty sets do not interfere with the calculation. There are a number of easy ways to do that (visual inspection is one); we found heuristically that the following strategy was effective: at t=0 set a=1. Double a each time $\Delta V_1 < \Delta V_2 < \Delta V_3$. The excuse for this strategy is that the ΔV_i provides a rough measure of the energy in the corresponding scales, and if that energy is increasing we are not in the inertial range.

We picked N_{max} between 300 and 500. With such values of N_{max}, it turns out that if b in (8) is larger than 2, there are too few D_i's for a trend to be discernible; if $b < 1.6$, the number of B_i's increases so much that the fluctuations in the D_i become too large.

In the runs we made, there were on the average about 20 segments per B_i, and thus a fluctuation of about $\sqrt{20}$ can be expected in their distribution, and thus in M_i, and therefore a fluctuation of about $1/\sqrt{20} \sim 20\%$ is the value of the D_i. In Table I we present values of D_i obtained in successive steps, with $N_{max} = 400$, $K = 1$ (which determines k through (5)), $b = 1.8$, $R = 0.3$. The initial conditions consist of three vortex lines passing through three points picked at random. We start the table with step 20 which allows the distribution of the q's to have reached some equilibrium. After step 27, several steps are omitted because the calculation was restarted and there were too few segments for the results to be meaningful. If M_i or M_{i+1} is zero, D_i is not defined. We omit the last defined value of D_i which is greatly affected by the void that follows it.

The distribution of the D_i's is not detectably affected by the value of R as long as $R \geqslant 0.3$, by the value of K if $K \leqslant 1.2$, by N_{max} if $N_{max} \geqslant 300$, by h if $h \leqslant 0.15$, or by the initial conditions. If the computed values of D_i are viewed as independent estimates of a fixed number D, then the usual estimation method yields $D = 2.55$ with a standard deviation of 0.05. Intermittency is clearly present. We can conclude that the vorticity collects itself into a body which approximates an object with Hausdorff dimension ~ 2.5 only if the D_i are indeed estimates of a fixed quantity. An inspection of Table I shows that the data are consistent with such a conclusion, and thus with the Mandelbrot conjecture.

It is not clear to me what the relationship is between the present calculation and recent work regarding the Hausdorff dimension of the singular set of the solutions of the Navier-Stokes equation [11],[24]. However, this singular set should be contained in the body of stretched vorticity. It is interesting to note that Foias and Temam [11] obtained 2.5 as an upper bound for the Hausdorff dimension of the singular set.

TABLE I. Values of D_i, $N_{max} = 400$, $b = 1.8$, $K = 1$, $h = 0.1$, $R = 0.3$.

Step 20	Step 21	Step 22	Step 23
2.67	2.57	2.60	2.65
2.28	2.46	2.63	2.75
2.66	2.79	2.89	2.87
1.18	1.88	2.02	2.27
	1.90	2.21	1.69
			2.66
Step 24	**Step 25**	**Step 26**	**Step 27**
2.74	2.84	2.87	2.64
2.80	2.83	2.86	2.58
2.85	2.53	2.61	2.84
2.48	2.47	2.54	2.30
2.29	2.80	2.86	2.31
2.72	1.76	2.05	
2.45	2.54	2.41	
Step 30	**Step 31**	**Step 32**	**Step 33**
2.28	2.58	2.74	2.78
2.92	2.95	2.97	2.98
2.16	1.97	2.25	2.48
2.72	2.74	2.52	2.40
2.45	2.60	2.73	2.73
	2.09	2.36	2.47
		1.52	2.93
			1.93

The Hausdorff dimension of the stretched vorticity set plays a major role in a numerical method used to solve problems involving inhomogeneous turbulence [10].

§5 ENERGY AND VORTICITY SPECTRA

We now attempt to ascertain the exponent β in equation (1). It is of course out of the question to perform a Fourier transform on our velocity field and evaluate $E(k)$ directly. An alternate physical space argument for determining β (see [8],[12]) involves the notion of energy per scale of energy containing eddy, which is too vague to be quantifiable. However, the analogous notion of mean square vorticity per scale of vorticity containing eddy is sufficiently well defined. The vorticity spectrum $Z(k)$ is related to $E(k)$ by

$$Z(k) = \text{constant} \times k^2 E(k) \quad ,$$

while the vorticity structure function (mean square vorticity as a function of scale) is given by

$$Z(r) = \text{constant} \times r^{\beta-2} \tag{9}$$

(see e.g., [8]). We shall evaluate $Z(r)$.

We assume that the vortex filaments remain tube-like, i.e., two of their dimensions shrink as they are stretched, and the ratio of these dimensions remains bounded by some moderate number. The energy transfer to higher wave numbers is a reflection of the vortex stretching process. The cross section of a vortex segment S_j^i is inversely proportional to its length ℓ_j^i multiplied by its tag q_j^i, as a consequence of the conservation of volume. The corresponding length scale is $L_j^i = 1/\sqrt{\ell_j^i\, q_j^i}$. If $\Gamma_i = 1$, and at some remote past all the q's were equal to 1, then by conservation of circulation the vorticity in S_j^i is $\ell_j^i q_j^i$, and the vorticity squared integrated over the segment is approximately equal to $(\ell_j^i)^2 q_j^i$.

Let c,d be two positive numbers, $d > 1$. Consider the segments such that

$$cd^{n-1} \leqslant (L_j^i)^{-1} \leqslant cd^n \quad , \qquad n \text{ integer} \;.$$

Call this set of segments C_n. The vorticity in C_n is

$$Z_n = \sum_{C_n} (\ell_j^i)^2 q_j^i \quad .$$

Note that we explicitly use the ℓ_j^i in the construction of this section, while in the analogous construction of the previous section we were content with the (correct) assumption that the ℓ_j^i were O(h). The reason for the change is that numbers $\ell_j^i q_j^i$ are marginally more smoothly distributed than the numbers q_j^i alone, not enough to make a difference in the earlier calculation but enough to make a difference here. If Z_n, Z_{n+1} are not zero, and $Z(r) = \text{constant} \times r^{-\gamma}$, then the following quantity is an estimate of γ:

$$\gamma_n = \frac{\log(Z_{n+1}/Z_n)}{\log d} \tag{10}$$

It is easy to see that an average of γ_n's is also an estimate of γ. The corresponding estimates of β can be deduced from the relation $\beta = 2 - \gamma$ which follows from (9).

The Z_n are noisy quantities and the estimates $\beta_n = 2-\gamma_n$ are sensitive functions of the Z_n; some statistical analysis has to be performed. At each time step, we exclude the estimate β_n which corresponds to the highest n (because it cannot be associated with an equilibrium). We average the other β_n's and consider the outcomes at different time steps to be independent estimates of β. We exclude those β which result from averaging a list of values of β_n which contain fewer than N_β entries (typically, $N_\beta = 6$ or 8), because such lists tend to be dominated by non-equilibrium conditions at the end of the range of n's. We also avoid averaging β_n's from ranges of n where many of the Z_n's are zero.

The same problems arise with the choices of c and d as arose with the choices of a and b in the last section. We double c if the list of β_n's at a given time step is too long or if the ΔV's are increasing at the beginning of the list. We pick c so that we have a list of sensible length to average (between 6 and 30 entries). With N_{max} in the neighborhood of 300-400, a reasonable choice of d is $d = 1.2$ or $d = 1.25$. The results are insensitive to the choices of these parameters as well as to the other parameters already discussed in the preceding section. The same restarting procedure was used as in the preceding section.

In Table II we exhibit part of a typical list of Z_n's and the corresponding β_n's. In Table III we exhibit part of a list of estimates of β from successive time steps. We made runs with a variety of initial data and numerical parameters. Four runs with $K = 1$, $h = 0.1$, $N_{max} = 350$, $R = 0.3$, $N_\beta = 6$, $d = 1.2$ but differing initial conditions yielded an average value of β equal to 0.845 with a standard deviation of 0.03. Other runs were compatible with this result, i.e., were within one standard deviation of this computed mean. A typical run contained about 200 steps; at the end

TABLE II. Typical vlaues of Z_n and β_n at a fixed time, d = 1.25.

Z_n	β_n
4.957	undefined
4.972	1.97
4.41	2.54
11.73	-2.39
12.34	1.78
17.90	0.33
5.19	7.55
11.62	-1.61
72.20	-6.18
125.06	-0.46
257.18	-1.23
19.70	13.51 (omitted in averaging)
0	undefined

TABLE III. Successive estimates of β at different time steps.

Step	Estimate of β
18	--
19	1.37
20	1.02
21	0.88
22	0.50
23	0.17
24	0.74
25	0.43
26	0.59
27	0.42
28	0.20
. . . .	. . . .
59	0.63
60	1.12
61	0.99
. . . .	. . . .
80	1.04
81	0.54
82	1.69
83	1.50
84	1.68

of such a run, some segments had been stretched by a factor of $\sim 10^{10}$, not counting the stretching due to the restarting procedure. Runs could not be continued ad infinitum because of the problems with overflow in the computer.

The value of β is compatible with the experimental estimate $\beta = 0.82$ from which Frisch et al deduced that $D \simeq 2.5$, in particular with equation (7). It is significantly higher than the Kolmogorov result $\beta = 0.66$.

Note that the constnuction of the present section is compatible with the coherent structure model of the inertial range (see [6],[8]). In [16], Kraichnan argued that such a model is unlikely because it would require a core structure for each vortex with an unphysical cusp. However, once it is realized that the core structure does not have to be constant along a vortex line and that parts of a filament can be stretched more than others, this difficulty disappears.

§6 FINITE TIME BLOW-UP FOR EULER'S EQUATIONS

We now turn to the question: does the amount of vorticity in our cube become infinite in a finite time when the flow is described by Euler's equations? The question is of significance in turbulence theory (see e.g. [6]). No analytical answer is available. An affirmative answer based on a tentative turbulence model has been given by Brauer et al [5]. Morf et al [22] have given an affirmative answer for the Green-Taylor initial data (6), on the basis of a numerical analytical continuation method; they predicted a blow-up time $t_* \sim 5.2/4\pi^2$ in our units. (The factor $4\pi^2$ appears because of the difference in non-dimensionalization between their paper and the present paper.)

It is, of course, impossible to watch the vorticity until it becomes infinite in a numerical calculation, since the amount of labor also becomes infinite. We proceed indirectly: assume that the details of the interaction of vortex segments at a substantial distance from each other do not affect the issue of blow-up. Start the calculation at time t=0 with a time step k. When the number of segments exceeds N_{max}, delete all but a corner of the cube, reconnect the vortex lines, stretch the corner into the cube and restart the calculation, as was done above. Since the cube has periodic boundary conditions, the omitted parts of the cube are replaced by a periodic continuation of the corner which is retained.

Can the restarted calculation be viewed as a continuation of the earlier part of the calculation? The restarting process focuses attention on a corner of the original cube. We assume that the replacement of the missing parts of the cube by a periodic continuation of the retained part has little effect on the overall stretching process. The stretching of the corner into the whole cube doubles the length scale in the corner. Let the energy in the cube before the restarting process be E_0, and let E_1 be the energy in the cube after the restarting process. Because of intermittency and numerical fluctuations, there is no simple relationship between E_1 and E_0. A typical velocity in the original cube is $\sqrt{E_0}$, and in the reconstituted tube, $\sqrt{E_1}$. The time scale must now be reduced by a factor $2\sqrt{E_0/E_1}$ (as can easily be seen from the fact that a vortex filament must now travel twice as far to reach a given configuration, and has E_0/E_1 less energy for doing so). On the new time scale, the restarted calculation can be viewed as a continuation of the old calculation, with attention focused on a part of the original domain. If the time is advanced by k in the restarted calculation, real time is advanced by $k/2\sqrt{E_0/E_1}$; after the next restarting process, real time will be advanced by $k/(4\sqrt{E_0/E_1}\sqrt{E_0'/E_1'})$, where E_0', E_1' are the corresponding energies at the beginning and the end of the next restarting procedure; this process can be continued ad infinitum. The sequence of new starts focuses attention on ever decreasing scales of motion.

Suppose the sum of the real time increments (sum of the k's divided by the appropriate scaling factors) converges to a time t_*. Then the conclusion must be that the at time t_* the vorticity in the original cube becomes infinite.

The energies E_0, E_1 for use in the scaling can be estimated by the formula for the energy of a vortex system (see e.g. [18])

$$E = \sum_{i=j}^{N} \sum_{j=1}^{N} \Gamma_i \Gamma_j \int \frac{d\underline{s} \cdot d\underline{s}'}{r} \qquad (11)$$

where as before, $\underline{s}$ is the unit normal to a vortex, ds is the arc length along a vortex, $d\underline{s} = \underline{s}\, ds$, r is the distance between the locations of ds and ds', Γ_i is the circulation of the i-th vortex line, and N is the number of vortex lines. The integral in (11) must be approximated and a short range cut-off for r must be used in a manner similar to the manner in which equations (3) were approximated. We omit the details. The resulting values of E_0 and E_1 are quite approximate (in particular, formula (11) does not take into account the changes in the cores of the vortices, which are important in earlier arguments); numerical experiment shows that the ratio E_0/E_1 is computed with an error of about 10%, which is adequate for the purpose of the present qualitative argument. Only the vorticity in the cube is taken into account, since energy of translation does not affect the stretching process.

TABLE IV.

Number of steps between restarts	Scaling factors $2\sqrt{E_0/E_1}$
12	1.79
2	2.00
8	2.03
8	1.75
4	0.49
1	3.95
6	3.18
10	3.43
11	4.48
10	2.30
5	1.27
⋮	⋮

In Table IV, we display the number of steps between restarts in a run with $h = 0.1$, $K = 1$, $N_{max} = 350$, $R = 0.3$, and initial data consisting of three vortex filaments placed at random. The number of steps is irregular (mainly because of intermittency) but exhibits no increasing or decreasing trend. We also display the corresponding factors $2\sqrt{E_0/E_1}$. The series of real time increments is seen to converge roughly like $\sum_{i=1}^{\infty} (1/2)^i$. This rate of convergence is compatible with equation (1) if $\beta \sim 1$.

In fact, the series of real time increments converged for all initial data we picked, except for initial data which consisted of a single straight line, a single circle or a single helix under conditions where the periodic images did not affect the motion and even then only if round-off errors were sharply controlled. The limit time t_* is, of course, dependent on the initial conditions; in particular, if the initial vortices all have the same circulation Γ, one can see from (3) that t_* is proportional to $1/\Gamma$.

For a fixed set of initial data, t_* was found to be quite independent of the numerical parameters, in particular of N_{max} (which controls the frequency of restarts). For example, with three vortices passing through the points (0.47, 0.51, 0.53), (0.97, 0.52, 0.47), (0.46, 0.96, 0.43), and following integral lines of the Green-Taylor vorticity field (6), with $\Gamma_1 = \Gamma_2 = \Gamma_3 = 1$, we found $4\pi^2 t_* = 1.74$ with $N_{max} = 350$, $4\pi^2 t_* = 1.79$ with $N_{max} = 400$, $4\pi t_* = 1.73$ with $N_{max} = 450$.

In order to compare our results with the results of Morf et al, we have to approximate the Green-Taylor data by vortex filaments. This requires a substantial number of filaments, more than we can afford. However, we can make the following very approximate argument: the mean length of the vorticity vector in the Green-Taylor data (6) is 0.77.

If those data were approximated by vortex filaments, the sum of their circulations would be approximated by 0.77. In the calculation just described, the sum of the circulations is 3. Since t_* is inversely proportional to the circulation, $4\pi^2 t_*$ for the Green-Taylor problem should be roughly $1.75 \times 3/0.77 = 6.82$, a value quite comparable with the value $4\pi^2 t_* = 5.2$ of Morf et al. The trouble with this argument is that it is very unreliable; with different initial data and similar reasoning we have obtained values of $4\pi^2 t_*$ for the Green-Taylor problem which varied between 4 and 8; all these values are, however, of the same order of magnitude as the value of Morf et al, lending credence both to their conclusion and to ours.

§7. CONCLUSIONS

We have used a vortex method to calculate properties of the inertial range in turbulence. We have in particular approximately verified well known conjectures regarding these properties; these conjectures will be used elsewhere in the process of solving problems with inhomogeneous turbulence [10].

It is interesting to note that the computed results are insensitive to numerical parametres, and in particular to the value of the long range cut-off R. Related conclusions have been reached in a two-dimensional problem solved by a vortex method [14]. This fact is consistent with one's intuitive notion that phenomena as ubiquitous as the power law inertial spectrum, and which have been found experimentally in a wide variety of physical situations, should be independent of the very fine details in the equations of motion. Similar conclusions have been reached in the theory of critical phenomena (see e.g. [23]). The independence of R for R large enough is also consistent with the widespread belief that the energy cascade is fairly localized in space (see e.g. [16]) and also with the assumption that vorticity patches at some distance from each are statistically independent. This assumption is crucial to coarse-grained vorticity closures for inhomogeneous turbulence ([3],[4],[8]).

A listing of the program used in the calculations of this paper is available from the author. The calculations were performed on a VAX computer at the Lawrence Berkeley Laboratory.

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