ESTIMATES OF THE BERGMAN KERNEL FUNCTION ON PSEUDOCONVEX DOMAINS WITH COMPARABLE LEVI FORM

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Abstract. Let Ω be a smoothly bounded pseudoconvex domain in $\mathbb{C}^n$ and let $z^0 \in b\Omega$ be a point of finite type. We also assume that the Levi form of $b\Omega$ is comparable in a neighborhood of z^0. Then we get precise estimates of the Bergman kernel function, $K_\Omega(z,w)$, and its derivatives in a neighborhood of z^0

1. Introduction

The purpose of this paper is to give precise estimates of the Bergman kernel function $K_\Omega(z,w)$ and its derivatives near the boundary of a smooth pseudoconvex domain Ω of finite type with comparable Levi-form.

For strongly pseudoconvex domain Ω in $\mathbb{C}^n$, the boundary of a suitable ball locally approximates $b\Omega$ near the point $z^0 \in b\Omega$ in question and this approximation is often the first step taken when analyzing the Bergman kernel function on Ω [9,10,12]. When Ω is weakly pseudoconvex domain of finite type, different approaches should be applied according to the local geometry of $b\Omega$ [3, 6, 7, 11, 14, 16, 17]. In the rest of this paper, we let Ω be a smoothly bounded pseudoconvex domain in $\mathbb{C}^n$ with smooth defining function r, i.e., $\Omega = \{z \in \mathbb{C}^n : r(z) < 0\}$, and let $K_\Omega(z,w)$ be the corresponding Bergman kernel function.

Let $\lambda_1(z), \dots, \lambda_{n-1}(z)$ be the eigenvalues of the Levi-form, $\partial\overline{\partial}r$, near a point $z^0 \in b\Omega$. We say Ω has comparable Levi-form near z^0 if there are a constant $c > 0$ and a neighborhood U of z^0 such that

$$\lambda_k(z) \geq c \cdot \sum_{i=1}^{n-1} \lambda_i(z), \quad k = 1, 2, \dots, n-1, \quad z \in U. \tag{1.1}$$

For example, let $r(z) = 2\text{Re}z_3 + (|z_1|^2 + |z_2|^2)^2$ be a defining function for a domain Ω in $\mathbb{C}^3$ near the origin. Then the Levi-form of $b\Omega$ satisfies (1.1) near the origin, and hence Ω has a comparable Levi-form near the origin.

Let $z^0 \in b\Omega$ be a point of finite type m in the sense of D'Angelo [8] and assume that the Levi-form is comparable near z^0. In this case, the author analyzed the local geometry of $b\Omega$ near z^0 and estimated the Bergman kernel function $K(z,z)$

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on the diagonal [7]. In this paper, we estimate $K_\Omega(z,w)$ and its derivatives on and off the diagonal near z^0.

For each z' near z^0, we will construct a biholomorphism $\Phi_{z'}:\mathbb{C}^n\longrightarrow\mathbb{C}^n$, $\Phi_{z'}^{-1}(\Omega)=\Omega_{z'}$ depending on z', but with holomorphic Jacobian uniformly nonsingular in a fixed neighborhood U of z^0. Because of the transformation formula for the Bergman kernel function, we state our estimates on $\Omega_{z'}=\Phi_{z'}^{-1}(\Omega)$, that is, with respect to special coordinates $\zeta=\Phi_{z'}^{-1}(z)$ defined for each reference point $z'\in U$. For $z^1,z^2\in\Omega$ near z^0, set $\zeta^i=\Phi_{z'}^{-1}(z^i)$, $i=1,2$, $z'=\pi(z^1)$ and $\Omega_{z'}=\Phi_{z'}^{-1}(\Omega)$, where π is the projection onto $b\Omega$. In the rest of this paper we let α, β be multi-indices and let $\alpha'=(\alpha_1,\dots,\alpha_{n-1},0)$, $\alpha''=(0,\alpha_2,\dots,\alpha_{n-1},0)$, etc.

Theorem 1.1. *Let Ω be a smoothly bounded pseudoconvex domain in $\mathbb{C}^n$ and $z^0\in b\Omega$ be a point of finite type. Assume that Ω has a comparable Levi-form in a neighborhood of z^0. Then there is a neighborhood V of z^0 such that for all n-indices α, β and $z^1,z^2\in V$, there exists a constant $C_{\alpha,\beta}$ such that*

$$|D^\alpha_{\zeta^1}\overline{D}^\beta_{\zeta^2}K_{\Omega_{z'}}(\zeta^1,\zeta^2)|\le C_{\alpha,\beta}\sum_{l=2}^m A_l(z')^{(2(n-1)+|\alpha'+\beta'|)/l}.\delta^{-2-\alpha_n-\beta_n-(2(n-1)+|\alpha'+\beta'|)/l},$$

where $\delta=(|\rho(\zeta^1)|+|\rho(\zeta^2)|+|\zeta^1_n-\zeta^2_n|+\sum_{j=1}^{n-1}\sum_{l=2}^m A_l(z')|\zeta^1_j-\zeta^2_j|^l)$, and where $\zeta^i=(\zeta^i_1,\dots,\zeta^i_n)$, $i=1,2$, are the coordinates defined by $\Phi_{z'}$ and the functions $A_l(z')$ are explicit functions, given by certain derivatives of r, and $\rho=r\circ\Phi_{z'}$.

Remark 1.2. In [7], the author estimated the Bergman kernel function $K(z,z)$ on the diagonal:

$$K(z,z)\approx\sum_{l=2}^m A_l(z')^{2(n-1)/l}\cdot|r(z)|^{-2-(2(n-1)/l)}, \tag{1.2}$$

and this is the case when $z^1=z^2$, and $\alpha,\beta=0$, in Theorem 1.1.

There is a close relation between the estimates of the Bergman kernel function and the existence of the peak functions on the domains in question. A point $z^0\in b\Omega$ is a peak point if there is a function $f\in A(\Omega)$ such that $f(z^0)=1$, and $|f(z)|<1$ for $z\in\overline{\Omega}-\{z^0\}$. Here $A(\Omega)$ denotes the set of functions which are holomorphic on Ω. In [4, 5], the author proposed a method, which is a modification of Fornaess and McNeal's method [13], to construct a peak function for the domains in $\mathbb{C}^n$ when the optimal estimates of the Bergman kernel function are known. Namely, for each neighborhood V of $z^0\in b\Omega$ we construct a regular bumping family of pseudoconvex domains outside V, and use Bishop's $\frac14-\frac34$ method on bumped domains, and we obtain the following theorem.

Theorem 1.3. *Let Ω and $z^0\in b\Omega$ be as in Theorem 1.1. Then for each small neighborhood $\widetilde{V}$ of z^0, there is a Hölder continuous peak function which peaks at z^0 and extends holomorphically up to $\overline{\Omega}\setminus\widetilde{V}$.*

The existence of peak functions for $A(\Omega)$ implies that Ω is complete in the Carathéodory metric. Since the Carathéodory metric is smaller than the Kobayashi metric and the Bergman metric, we obtain the following corollary as an immediate application of Theorem 1.3.

Corollary 1.4. *Let Ω and z^0 be as in Theorem 1.1. Then Ω is complete in a neighborhood of z^0 in the Kobayashi, Bergman and Carathéodory metrics.*

2. Special coordinates and polydiscs.

Let Ω, $z^0 \in b\Omega$ and U be as in Section 1. In this section we want to show that about each point z' in U, there is a special coordinates ζ about z' and a polydisc of maximal size on which the function $r(z)$ changes by no more than some prescribed small number $\delta > 0$.

We may assume that there are coordinate functions $z_1, ..., z_n$ such that $|\frac{\partial r}{\partial z_n}(z)| \geq c > 0$ for all $z \in U$. We first take the following special coordinates which reflects the local geometry of $b\Omega$ near $z^0 \in b\Omega$ ([7, Proposition 2.1]).

Proposition 2.1. *For each $z' \in U$ and positive integer m, there is a biholomorphism $\Phi_{z'} : \mathbb{C}^n \longrightarrow \mathbb{C}^n$, $\Phi_{z'}^{-1}(z') = 0$, such that*

$$\rho(\zeta) := r(\Phi_{z'}(\zeta)) = r(z') + Re\zeta_n + \sum_{\substack{j+k\leq m \\ j,k\geq 1}} a_{jk}(z')\zeta_1^j\overline{\zeta}_1^k + \sum_{\substack{|\alpha'+\beta'|\leq m \\ |\alpha'|,|\beta'|\geq 1 \\ 1\leq|\alpha''+\beta''|\leq m}} b_{\alpha'\beta'}(z')\zeta^{\alpha'}\overline{\zeta}^{\beta'} + \mathcal{O}(|\zeta'|^{m+1} + |\zeta||\zeta_n|). \tag{2.1}$$

We now show how to define a polydisc around z' in ζ-coordinates. Set

$$\begin{aligned} A_{l_1}(z') &= max\{|a_{jk}(z')| : j + k = l_1\} \\ A_{l_2}(z') &= max\{|b_{\alpha'\beta'}(z')| : |\alpha' + \beta'| = l_2\}, \quad 2 \leq l_1, l_2 \leq m. \end{aligned} \tag{2.2}$$

For each $\delta > 0$, we define $\tau(z', \delta)$ by :

$$\tau(z', \delta) = min\{(\delta/A_{l_i}(z'))^{\frac{1}{l_i}} : 2 \leq l_1, l_2 \leq m\}. \tag{2.3}$$

If we assume that the type at $z^0 \in b\Omega$ is m then it follows that $|a_{jk}(z')| + |b_{\alpha'\beta'}(z')| \geq c > 0$ for some $j + k = m$, or $|\alpha' + \beta'| = m$, for all $z' \in U$, provided U is sufficiently small. This gives us the inequality :

$$\delta^{\frac{1}{2}} \lesssim \tau(z', \delta) \lesssim \delta^{\frac{1}{m}}, \quad z' \in U, \tag{2.4}$$

and if $\delta' < \delta''$, then

$$(\delta'/\delta'')^{\frac{1}{2}}\tau(z', \delta'') \lesssim \tau(z', \delta') \lesssim (\delta'/\delta'')^{\frac{1}{m}}\tau(z', \delta''). \tag{2.5}$$

The following lemma shows that $|a_{jk}(z')|$ terms are major terms to define $\tau(z', \delta)$ in (2.3). One can refer a proof in [7,Lemma 2.4].

Lemma 2.2. *There is $c_0 > 0$ (independent of z' and $\delta > 0$) such that*

$$|a_{jk}| \geq c_0 \cdot \delta\tau^{-j-k} \quad \text{for some } j+k \leq m.$$

By virtue of (2.2) we then have

$$A_{l_1}(z') \approx \delta\tau^{-l_1}, \quad \text{for some } \; 2 \leq l_1 \leq m. \tag{2.6}$$

Set $\tau(z', \delta) = \tau$ for a convenience, and define

$$\begin{aligned} &R_\delta(z') = \{\zeta \in \mathbb{C}^n : |\zeta_k| \leq \tau, \; 1 \leq k \leq n-1, \; |\zeta_n| \leq \delta\}, \quad \text{and} \\ &Q_\delta(z') = \{\Phi_{z'}(\zeta) : \zeta \in R_\delta(z')\}. \end{aligned}$$

Then by virtue of the definition of $\tau(z', \delta)$, it follows that $R_{c\delta}(z')$ is contained in $\Omega_{z'} = \Phi_{z'}^{-1}(\Omega)$ for a fixed constant $c > 0$ (independent of z' and $\delta > 0$).

In [7], the author constructed the following family of bounded plurisubharmonic weight functions with essentially maximal Hessian in a thin strip near the boundary of Ω. For $\epsilon > 0$, we let $\Omega_\epsilon = \{z : r(z) < \epsilon\}$ and set

$$S(\epsilon) = \{z : -\epsilon < r(z) < \epsilon\}.$$

Theorem 2.2. *For all small $\delta > 0$, there is a plurisubharmonic function $g_\delta \in C^\infty(\Omega_\delta)$ with the following properties,*

(i) $|g_\delta(z)| \leq 1$, $z \in U \cap \Omega_\delta$.

(ii) *For all $L = \sum_{j=1}^n b_j L_j$ at $z \in U \cap S(\delta)$,*

$$\partial\overline{\partial} g_\delta(z)(L, \overline{L}) \approx \tau^{-2} \sum_{k=1}^{n-1} |b_k|^2 + \delta^{-2}|b_n|^2,$$

(iii) *If $\Phi_{z'}$ is the map associated with a given $z' \in U \cap S(\delta)$, then for all $\zeta \in R_\delta(z')$ with $|\rho(\zeta)| < \delta$,*

$$|\partial^\alpha \overline{\partial}^\beta (g_\delta \circ \Phi_{z'})(\zeta)| \lesssim C_{\alpha,\beta} \delta^{-\alpha_n - \beta_n} \tau^{-|\alpha' + \beta'|}.$$

For $z' \in U \cap \Omega$ and $\delta > 0$, we define a biholomorphism (dilation map) by

$$D_{z'}^\delta(\zeta) := (\tau^{-1}\zeta_1, \dots, \tau^{-1}\zeta_{n-1}, \delta^{-1}\zeta_n) := (w_1, \dots, w_n). \tag{2.7}$$

3. Subelliptic estimates for $\overline{\partial}$ in dilated coordinates.

Let Ω, z_0 and U be as in Section 1. In this section, we want to get uniform subelliptic estimates (independent of δ and z') for $\overline{\partial}$-equation in dilated coordinates $w = D_{z'}^\delta(\zeta)$ defined in (2.7). Set

$$\rho_{z'}^\delta(w) = \delta^{-1}\left(\rho \circ (D_{z'}^\delta)^{-1}(w)\right) \quad \text{and} \quad \Omega_{z'}^\delta = \{w \in \mathbb{C}^n : \rho_{z'}^\delta(w) < 0\}. \tag{3.1}$$

Note that $D^{\delta}_{z'}(R_\delta(z')) = P(0;1) = \{w \in \mathbb{C}^n : |w_i| < 1\} := W$ and $|D^\alpha \rho_\delta(w)| \le C_\alpha$, independent of z' and δ. Set

$$b_j(\zeta) = (\partial\rho/\partial\zeta_n)^{-1}\partial\rho/\partial\zeta_j, \quad 1 \le j \le n-1.$$

In special coordinates, we can write

$$L_j = \frac{\partial}{\partial\zeta_j} - b_j(\zeta)\frac{\partial}{\partial\zeta_n}, \quad 1 \le j \le n-1, \quad \text{and} \quad L_n = \frac{\partial}{\partial\zeta_n},$$

and they form a local frame of $\mathbb{C}T^{1,0}(U)$. In terms of dilated coordinates, set

$$\begin{aligned} L^\delta_j &= \tau(dD^\delta_{z'})_* L_j = \frac{\partial}{\partial w_j} - b_j \circ (D^\delta_{z'})(w)\delta^{-1}\tau\frac{\partial}{\partial w_n}, \quad 1 \le j \le n-1, \quad \text{and} \\ L^\delta_n &= \delta(dD^\delta_{z'})_* L_n = \frac{\partial}{\partial w_n}. \end{aligned} \tag{3.2}$$

Then they form a local frame of $\mathbb{C}T^{1,0}(W)$ in dilated coordinates.

Let $\mathcal{D}^{0,1}(W)$ denote the (0,1)-forms u, $u = \sum_{k=1}^n u_k d\bar{z}_k$, with components in $C^\infty_0(W)$ such that $\sum_{k=1}^n \frac{\partial r}{\partial z_k} u_k = 0$ on $W \cap b\Omega^\delta_{z'}$. Then a subelliptic estimate of order $\epsilon > 0$ holds in W if

$$|||u|||^2_\epsilon \le CQ(u,u) \quad \forall u \in \mathcal{D}^{0,1}(W), \tag{3.3}$$

where $|||\cdot|||_\epsilon$ denotes the tangential Sobolev norm of order ϵ on forms and $Q(u,u) = \|\bar\partial u\|^2 + \|\vartheta u\|^2 + \|u\|^2$ and where ϑ is the formal adjoint of $\bar\partial$.

Let $z' \in U \cap \overline{\Omega}$ and $\delta > 0$ be fixed for a moment. Note that the neighborhood $W = D^\delta_{\zeta'}(R_\delta(z'))$ actually depends on z'. We will show that (3.3) holds independent of z' and δ, and this is a key ingredient to prove Theorem 1.1.

By virtue of Theorem 2.2, there is a family of plurisubharmonic functions $\{g_{\delta\rho}\}_{\rho>0}$ satisfying the properties (i), (ii), (iii) of Theorem 2.2. Set

$$S^\delta_{z'}(\rho) = \{w \in W : -\rho < \rho^\delta_{z'}(w) < \rho\}, \quad \Omega^\delta_{z'}(\rho) = \{w \in \mathbb{C}^n : \rho^\delta_{z'}(w) < \rho\}.$$

Theorem 3.1. *For each small $\rho > 0$, there exists a C^∞ plurisubharmonic function λ_ρ defined on $\Omega^\delta_{z'}(\rho)$ such that*

(i) $|\lambda_\rho| \le 1$ *in* $W \cap \Omega^\delta_{z'}(\rho)$.

(ii) *For all* $L^\delta = \sum_{j=1}^n d_j L^\delta_j$ *at* $w \in W \cap S^\delta_{z'}(\rho)$,

$$\partial\bar\partial\lambda_\rho(w)(L^\delta, \overline{L}^\delta) \approx \tau(z',\delta)^2\tau(z',\rho\delta)^{-2}\sum_{j=1}^{n-1}|d_j|^2 + \rho^{-2}|d_n|^2.$$

(iii) *For all* $w \in W \cap \Omega^\delta_{z'}(\rho)$ *and for each* α, β, *there is* $C_{\alpha,\beta}$ *such that*

$$|\partial^\alpha\bar\partial^\beta\lambda_\rho(w)| \le C_{\alpha,\beta}\rho^{-\alpha_n-\beta_n}\tau(z',\delta)^{|\alpha'+\beta'|}\tau(z',\rho\delta)^{-|\alpha'+\beta'|}.$$

Proof. Let $\{g_{\delta\rho}\}_{\rho>0}$ be the family of plurisubharmonic functions satisfying the properties (i), (ii), (iii) of Theorem 2.2. Set $\lambda_\rho = g_{\delta\rho} \circ (D^\delta_{z'})^{-1}$, where $D^\delta_{z'}$ is the dilation map defined in (2.7). It is clear that λ_ρ is plurisubharmonic and $|\lambda_\rho| \le 1$ in $W \cap \Omega^\delta_{z'}(\rho)$.

Let $L^\delta = \sum_{j=1}^n d_j L^\delta_j$ where L^δ_j is defined in (3.2), $1 \le j \le n$. By functoriality and by the property (ii) of Theorem 2.2, it follows that

$$\begin{aligned}
\partial\overline{\partial}\lambda_\rho(w)(L^\delta, \overline{L}^\delta) &= \partial\overline{\partial} g_{\delta\rho}(\zeta)(d(D^\delta_{z'})^{-1}L^\delta, d(D^\delta_{z'})^{-1}\overline{L}^\delta) \\
&= \partial\overline{\partial} g_{\delta\rho}(\zeta)\left(\tau\sum_{j=1}^{n-1} d_j L_j + \delta d_n L_n, \tau\sum_{j=1}^{n-1}\overline{d}_j\overline{L}_j + \delta\overline{d}_n\overline{L}_n\right) \\
&\approx \sum_{j=1}^{n-1}\tau(z',\delta)^2|d_j|^2\tau(z',\delta\rho)^{-2} + \delta^2|d_n|^2(\delta\rho)^{-2} \\
&= \tau(z',\delta)^2\tau(z',\delta\rho)^{-2}\sum_{j=1}^{n-1}|d_j|^2 + \rho^{-2}|d_n|^2.
\end{aligned}$$

Note that these estimates are independent of z' and δ because the estimates in Theorem 2.2 are independent of z' and δ. This proves (ii). Property (iii) follows from chain rule and the property (iii) of Theorem 2.2. □

Note that the estimates and the constants in (2.4), (2.5) and Theorem 3.1 are independent of δ and z'. Using this fact and Theorem 3.1, we show the following subelliptic estimates of $\overline{\partial}$ equation which is an essential ingredient to get derivative estimates for $K_\Omega(z,w)$. For $0 < b \le 1$, we set

$$P_b = \{w \in W : |w_i| < b\}.$$

Corollary 3.2. *There exist a small constant $b > 0$ (independent of z' and δ) and a constant $C_1 > 0$ so that*

$$|||u|||^2_{1/m} \le C_1 Q(u,u), \quad \forall u \in \mathcal{D}^{0,1}(P_b). \tag{3.4}$$

Proof. By virtue of the relations in (2.4) and (2.5), it follows that

$$\tau(z',\delta)^2\tau(z',\rho\delta)^{-2} \gtrsim (\delta\rho/\delta)^{-\frac{1}{m}} = \rho^{-\frac{1}{m}}. \tag{3.5}$$

By Theorem 3.1 and (3.5), there is a small $b > 0$ such that for each $0 < \rho \le b$, there is a C^∞ plurisubharmonic function λ_ρ, $|\lambda_\rho| \le 1$, satisfying

$$\partial\overline{\partial}\lambda_\rho(w)(L^\delta, \overline{L}^\delta) \gtrsim \rho^{-\frac{1}{m}}|L^\delta|^2, \tag{3.6}$$

for all $w \in W \cap S^\delta_{z'}(\rho)$. Here the estimate in (3.6) is independent of δ and ρ. Note that the existence of the family of plurisubharmonic weight functions, $\{\lambda_\rho\}_{\rho>0}$ satisfying (3.6), is a sufficient condition for the subelliptic estimates for $\overline{\partial}$ of order $1/m$ by the theorem of Catlin [1]. Therefore (3.4) holds for (0,1)-forms on P_b provided b is sufficiently small. □

We also need an estimate on the $\overline{\partial}$-Neumann operator, N_δ, for the domain $\Omega^\delta_{z'} = \{w \in \mathbb{C}^n; \rho^\delta_{z'}(w) < 0\}$, where $\rho^\delta_{z'}$ is defined in (3.1). Using the weighted estimates for $\overline{\partial}$ of Hörmander, Catlin proved that for a smooth bounded pseudoconvex domain Ω in $\mathbb{C}^n$, and any $\lambda \in C^2(\overline{\Omega})$ with $-1 \le \lambda \le 1$,

$$(3.7) \quad \int_\Omega \sum_{j,k=1}^n \frac{\partial^2 \lambda}{\partial z_j \partial \overline{z}_k} u_j \overline{u}_k dV \le 36(\|\overline{\partial} u\|^2 + \|\overline{\partial}^* u\|^2), \quad u \in Dom(\overline{\partial}) \cap Dom(\overline{\partial}^*).$$

Let us fix $b > 0$ so that (3.4) holds on P_b. For each $\delta > 0$, set $\lambda^\delta(w) = g_{b\delta} \circ (D^\delta_{z'})^{-1}(w)$. By virtue of Theorem 3.1 and (3.7), we then have a constant $C_2 > 0$ (independent of z' and δ) so that

$$(3.8) \qquad |L|^2 \le C_2 \partial\overline{\partial}\lambda^\delta(L, \overline{L})(w), \quad w \in P_b.$$

Combining (3.7) and (3.8), we obtain the following relation:

$$(3.9) \qquad \int_{\Omega^\delta_{z'} \cap P_b} |u|^2 \le C_3(\|\overline{\partial}_\delta u\|^2 + \|\overline{\partial}^*_\delta u\|^2), \quad u \in Dom(\overline{\partial}_\delta) \cap Dom(\overline{\partial}^*_\delta),$$

for an independent constant $C_3 > 0$, where $\overline{\partial}_\delta$ and $\overline{\partial}^*_\delta$ refers to operators on $\Omega^\delta_{z'}$.

Now let $g \in L^2_{(0,1)}(\Omega^\delta_{z'})$ and $supp g \subset P_b$. Then $N_\delta g \in Dom(\overline{\partial}_\delta) \cap Dom(\overline{\partial}^*_\delta)$ and by (3.9), there is a unform constant C_4 so that

$$(3.10) \qquad \|N_\delta g\|_{\Omega^\delta_{z'} \cap P_b} \le C_4 \|g\|_{\Omega^\delta_{z'}}, \quad g \in L^2_{(0,1)}(\Omega^\delta_{z'}), \quad supp g \subset P_b.$$

Note that if P^δ is the Bergman projection operator on $\Omega^\delta_{z'}$, then we have the relation, $P^\delta = I - \overline{\partial}^*_\delta N_\delta \overline{\partial}_\delta$, and the Bergman kernel function can be written as :

$$(3.11) \qquad K_{\Omega^\delta_{z'}}(z, w) = P^\delta \phi_w(z),$$

where $\phi_w(z)$ is a polyradial function with center at w, and $\int \phi_w(z) dV = 1$.

The following theorem will be used to show derivative estimates for the kernel function. The proof of the theorem is based on some ideas of Kerzman [15] on the smooth extension of the kernel function, and on McNeal [16] and the author's [6] work on the derivative estimates of the kernel function. We use the relation (3.11) and the estimates in (3.10) as well as the crucial estimates in (3.4), the subelliptic estimates for $\overline{\partial}$-Neumann problem. One can refer a detailed proof in [6, 15, 16].

Theorem 3.3. *For K_1, $K_2 \subset\subset \mathbb{C}^n$ with $K_1 \cap K_2 = \emptyset$, and α, β any n-indices, there exists a constant $C_{\alpha,\beta}$ such that, for small $\delta > 0$,*

$$|D^\alpha_z \overline{D}^\beta_w K_{\Omega^\delta_{z'}}(z, w)| \le C_{\alpha,\beta}, \quad (z, w) \in (K_1 \cap P_b \cap \Omega^\delta_{z'}) \times (K_2 \cap P_b \cap \Omega^\delta_{z'}),$$

where $P_b = \{w \in \mathbb{C}^n : |w_i| < b\}$.

4. Estimates of the Bergman kernel function.

In this section we prove Theorem 1.1 and Theorem 1.3 of Section 1.

Proof of Theorem 1.1. Let $z^1, z^2 \in U$ and set $z' = \pi(z^1)$, where π is the projection onto $b\Omega$. Let $\Phi_{z'}$ be the map associated with z' as defined in Proposition 2.1 and set $\zeta^i = \Phi_{z'}^{-1}(z^i)$, $i = 1, 2$. Let $\zeta^1 = (\zeta_1^1, \zeta_2^1, \dots, \zeta_n^1) \in V$ be a point whose closest point in $b\Omega_{z'}$ is 0 in special coordinates, and $\zeta^2 = (\zeta_1^2, \zeta_2^2, \dots \zeta_n^2) \in V$. Set, for b_0 to be determined,

$$\delta = b_0^{-1}\left(|\rho(\zeta^1)| + |\rho(\zeta^2)| + |\zeta_n^1 - \zeta_n^2| + \sum_{j=1}^{n-1}\sum_{l_1=2}^{m} A_{l_1}(z')|\zeta_j^1 - \zeta_j^2|^{l_1}\right),$$

and denote $D_{z'}^{\delta}(\zeta^i)$ by w^i, $i = 1, 2$ where A_{l_1} and $D_{z'}^{\delta}$ are defined as in (2.2) and (2.7) respectively. We claim that $\zeta^1, \zeta^2 \in P(0; C_0 b_0 \delta) = \{\zeta \in \mathbb{C}^n : |\zeta_j| \le \tau(z', C_0 b_0 \delta),\ 1 \le j \le n-1,\ |\zeta_n| \le C_0 b_0 \delta\}$ for some $C_0 > 0$. Since $|\zeta^1| \approx |\rho(\zeta^1)|$, it follows that

$$|\zeta^1| \lesssim |\rho(\zeta^1)| \lesssim b_0\delta,$$

and hence for each $j = 1, \dots, n-1$, it follows, from the definition of δ and $\tau(z', \delta)$, that

$$\begin{aligned} |\zeta_n^2| + \sum_{l_1=2}^{m} A_{l_1}(z')|\zeta_j^2|^{l_1} &\lesssim |\zeta_n^2 - \zeta_n^1| + |\zeta_n^1| \\ &+ \sum_{l_1=2}^{m} A_{l_1}(z')|\zeta_j^2 - \zeta_j^1|^{l_1} + \sum_{l_1=2}^{m} A_{l_1}(z')|\zeta_j^1|^{l_1} \lesssim b_0\delta. \end{aligned}$$

So $|\zeta_n^2| \lesssim b_0\delta$ and $A_{l_1}(z')|\zeta_j^2|^{l_1} \lesssim b_0\delta$ for $l_1 = 2, \dots, m$. By virtue of the important relation in (2.8), it follows that

$$|\zeta_j^2| \lesssim \tau(z', b_0\delta), \quad j = 1, \dots, n-1.$$

Therefore we have $\zeta^1, \zeta^2 \in P_\zeta(0; C_0 b_0 \delta)$, for some $C_0 > 0$, and hence if b_0 is sufficiently small, then $w^1, w^2 \in D_{z'}^{\delta}(P_{z'}(0; C_1 b_0 \delta)) \subset P_b$, where $b > 0$ is the number as in Corollary 3.2. Note that for the special δ, we have, in dilated coordinates, that

$$0 < b_0 \le \left(|\rho_\delta(w^1)| + |\rho_\delta(w^2)| + |w_n^1 - w_n^2| + \sum_{j=2}^{n-1}\sum_{l_1=2}^{m} A_{l_1}(z')|w_j^1 - w_j^2|^{l_1}\right).$$

If $|w^1 - w^2| < a_0$ for a_0 small enough, then

$$|\rho_\delta(w^1)| + |\rho_\delta(w^2)| > \frac{b_0}{2},$$

and the continuity of ρ_δ together the fact that $|D^\alpha \rho_\delta(w)| \le C_\alpha$ in P_b, independent of δ, give us

$$|\rho_\delta(w^1)| > \frac{b_0}{5}$$

provided a_0 is sufficiently small. Therefore the ball centered at w^1 of radius $a_0 \le \frac{b_0}{5}$ lies in $\Omega^\delta_{z'}$ and contains w^2. Let us fix a_0 (independent of z' and δ). If we set $K_1 = \{w^1\}$ and $K_2 = \{w \in \mathbb{C}^n; w \in P(0;b),\ |w^1 - w| = a_0\}$ in Theorem 3.3, we have for $w \in K_2$,

$$|D^\alpha_{w^1}\overline{D}^\beta_w K_{\Omega^\delta_{z'}}(w^1, w)| \le C_{\alpha,\beta},$$

and hence by the maximum modulus theorem,

$$|D^\alpha_{w^1}\overline{D}^\beta_{w^2} K_{\Omega^\delta_{z'}}(w^1, w^2)| \le C_{\alpha,\beta}. \tag{4.1}$$

If, instead, $|w^1 - w^2| \ge a_0$, Theorem 3.3 immediately applies. Thus (4.1) holds in all the cases. By the chain rule and the transformation formula for the Bergman kernel, one has

$$|D^\alpha_{\zeta^1}\overline{D}^\beta_{\zeta^2} K_{\Omega_{z'}}(\zeta^1, \zeta^2)| \le C_{\alpha,\beta}\delta^{-2-\alpha_n-\beta_n} \cdot \tau(z', \delta)^{-2n+2-|\alpha'+\beta'|}. \tag{4.2}$$

By virtue of (2.6), one obtains that

$$\tau(z', \delta)^{-2n+2-|\alpha'-\beta'|} \approx \sum_{l_1=2}^{m} A_{l_1}(z')^{(2n-2+|\alpha'+\beta'|)/l_1} \cdot \delta^{(-2n+2-|\alpha'+\beta'|)/l_1}. \tag{4.3}$$

Thus we get Theorem 1.1 combining (4.2) and (4.3). □

In [13], Fornaess and McNeal proposed a method to construct a peak function in $\mathbb{C}^n$. In their method, we need precise estimates of the Bergman kernel function and its derivatives, together Hölder estimates for $\overline{\partial}$-equation. Later the author [4, 5] proposed a method which uses a bumping family of pseudoconvex domains. In this method, we do not need the Hölder estimates for $\overline{\partial}$-equation and hence can be applied to wide class of domains in $\mathbb{C}^n$.

Note that (1.1) is an open condition and hence it holds for the bumping family of pseudoconvex domains. Therefore we have the same kinds of estimates for the Bergman kernel function on bumped domain $\widetilde{\Omega}$ which touches $b\Omega$ only at z^0. We show briefly how the peaking can be constructed. For a detailed proof, one can refer [4, 5, 13].

Let N be the interior normal to the boundary of $\widetilde{\Omega}$ at z^0. Then we technically choose a sequence of points $\{q_n\}$ converging to z^0 and set

$$h_n(z) = K(z, q_n)/K(q_n, q_n),$$

and then set

$$f(z) = (1-c)\sum_{n=0}^{\infty} c^n h_n(z),$$

where the constant c is chosen appropriately so that the sequence converges uniformly on compact sets. Since $z^0 \in b\widetilde{\Omega}$ is a point of finite type, the subelliptic estimate for $\overline{\partial}$ equation holds near z^0 on bumped pseudoconvex domain $\widetilde{\Omega}$ by the Theorem of Catlin [1]. Then the estimates for the Bergman kernel function and its derivatives show that the function $f(z)$ is the required Hölder continuous peaking function which peaks at z^0 and analytic on $\overline{\Omega} \setminus V$ where V is a small neighborhood of z_0. This proves Theorem 1.3.

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