J. DIFFERENTIAL GEOMETRY
103 (2016) 377-424

ESTIMATES OF THE MEAN FIELD EQUATIONS WITH INTEGER SINGULAR SOURCES: NON-SIMPLE BLOWUP

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Abstract

Let M be a compact Riemann surface, $\alpha_j > -1$, and $h(x)$ a positive C^2 function of M. In this paper, we consider the following mean field equation:

$$\Delta u(x)+\rho\left(\frac{h(x)e^{u(x)}}{\int_M h(x)e^{u(x)}}-\frac{1}{|M|}\right)=4\pi\sum_{j=1}^{d}\alpha_j\left(\delta_{q_j}-\frac{1}{|M|}\right) \text{ in } M.$$

We prove that for $\alpha_j \in \mathbb{N}$ and any $\rho > \rho_0$, the equation has one solution at least if the Euler characteristic $\chi(M) \leq 0$, where $\rho_0 = \max_M (2K - \Delta \ln h + N^*)$, K is the Gaussian curvature, and $N^* = 4\pi\sum_{j=1}^{d}\alpha_j$. This result was proved in [**10**] when $\alpha_j = 0$. Our proof relies on the bubbling analysis if one of the blowup points is at the vortex q_j. In the case where $\alpha_j \notin \mathbb{N}$, the sharp estimate of solutions near q_j has been obtained in [**11**]. However, if $\alpha_j \in \mathbb{N}$, then the phenomena of non-simple blowup might occur. One of our contributions in part 1 is to obtain the sharp estimate for the non-simple blowup phenomena.

1. Introduction

In this paper, we consider the following mean field equation of Liouville type:

$$(1.1)\qquad \begin{cases} \Delta u(x)+\rho\frac{h(x)e^{u(x)}}{\int_M h(x)e^{u(x)}dx}=4\pi\sum_{j=1}^{d}\alpha_j\delta_{q_j} \text{ in } \Omega \\ u(x)=0,\ x\in\partial\Omega, \end{cases}$$

where Ω is a bounded smooth domain in $\mathbb{R}^2$, $\alpha_j > -1$, δ_{q_j} is the Dirac measure at q_j, and $\rho \in \mathbb{R}^+$, or

(1.2)
$$\Delta u(x)+\rho\left(\frac{h(x)e^{u(x)}}{\int_M h(x)e^{u(x)}dx}-\frac{1}{|M|}\right)=4\pi\sum_{j=1}^{d}\alpha_j\left(\delta_{q_j}-\frac{1}{|M|}\right) \text{ in } M,$$

Received 6/20/2014.

where (M,g) is a compact Riemann surface and $|M|$ is the area. Here, Δ stands for the Beltrami–Laplacian operator on (M,g). Throughout the paper, we always assume $h(x)$ to be a positive C^2 function either on $\bar{\Omega}$ or on M and $u|_{\partial\Omega}=0$ for (1.1) or $\int_M u dx = 0$ for (1.2), respectively.

Equations (1.1) and (1.2) have arisen in many different areas in mathematics and physics. For example, we consider the following singular Liouville equation:

$$\Delta u + e^u = \rho\delta_0 \text{ on } T, \tag{1.3}$$

where T is a flat torus. By integration, equation (1.3) becomes a mean field type equation

$$\Delta u + \rho\left(\frac{e^u}{\int_T e^u} - \frac{1}{|T|}\right) = \rho\left(\delta_0 - \frac{1}{|T|}\right) \text{ on } T.$$

In geometry, equation (1.3) comes form a prescribed curvature problem. In general, for a compact Riemann surface (M,g) with constant Gaussian curvature, we may consider the following equation:

$$\Delta w(x) + h(x)e^{w(x)} - 2k = 4\pi\sum_{j=1}^{d}\alpha_j\delta_{q_j}, \tag{1.4}$$

where k is the constant Gaussian curvature of the given metric g and $h(x)$ is a positive function on M. For any solution $w(x)$ to (1.4), equation (1.4) is equivalent to saying that the new metric $\tilde{g} := e^v g$ (where $v = w - \ln 2$) has Gaussian curvature $\tilde{k}(x) = h(x)$ outside those $q_j's$. By integrating the equation, the function $w(x)$ satisfies the equation (1.2) with $\rho = 2k + 4\pi\sum_{j=1}^{d}\alpha_j$. Thus, equation (1.2) can be viewed as a generalization of (1.4). Since equation (1.4) has singular source at q_j, the conformal metric $e^v g$ is degenerate at q_j and is called a metric on M with conic singularity at those $q_j's$. In particular, when $M = \mathbb{S}^2$ and $\alpha_j = 0\ \forall j$, equation (1.4) is related to the well-known Nirenberg problem.

For equation (1.3), there is another application to the complex Monge–Ampère equation

$$\det\left(\frac{\partial^2 w}{\partial z_i\partial\bar{z}_j}\right)_{i,j=1}^{d} = e^{-w} \text{ on } (T\backslash\{0\})^d, \tag{1.5}$$

the dth Cartesian product of $T\backslash\{0\}$. For any solution u to equation (1.3), the function $w(z_1,\dots,z_d) = -\sum_{i=1}^{d}u(z_i) + d\log 4$ satisfies (1.5) with a logarithmic singularity along the normal crossing divisor $D = T^d\backslash(T\backslash\{0\})^d$. In particular, bubbling solutions to (1.3) will give some examples of bubbling solutions to the complex Monge–Ampère equation (1.5). Those examples might be useful in the study of geometry related to the degenerate complex Monge–Ampère equations.

In physics, equation (1.2) can be derived from the mean field limit of point vortices of the Euler flow, as studied by Caglioti et al. [**13**, **14**], and Chanillo and Kiessling [**8**]. Recently, it has drawn a lot of attention due to its application to many physics models, including the Chern–Simons–Higgs theory, see Jackiw and Weinberg [**18**], and the electroweak theory, (see Ambjorn and Olesen [**1**]). In the electroweak theory of Glashow, Salam, and Weinberg, Ambjorn and Olesen found that periodic vortices could be realized as solutions of a self-dual Bogomol'nyi type equation, which can be further reduced to

$$\begin{cases} \Delta u + 4g^2 e^u + g^2 e^w = 4\pi \sum_{\ell=1}^m n_\ell \delta_{p_\ell} \text{ in } T, \\ \Delta w - 2g^2 e^u - \frac{g^2}{2\cos^2\theta}\left(e^w - \varphi_0^2\right) = 0, \end{cases} \tag{1.6}$$

where φ_0, θ, g are constants. By integration, we have

$$4g^2 \int_T e^u = \frac{4\pi N - g^2 \varphi_0^2 |T|}{\sin^2\theta}$$

and

$$g^2 \int_T e^w = \frac{g^2\varphi_0^2 |T| - 4\pi\cos^2\theta N}{\sin^2\theta},$$

where $N = \sum_{\ell=1}^m n_\ell$. The necessary condition for solvability of (1.6) is that N must satisfy

$$g^2\varphi_0^2 < \frac{4\pi N}{|T|} < \frac{g^2\varphi_0^2}{\cos^2\theta}. \tag{1.7}$$

The conjecture proposed in [**29**] is to ask whether (1.7) is also sufficient for the solvability of (1.6) or not. This conjecture has been partially proved in [**12**] by applying the degree theory for the mean field equation.

Theorem A. *Assume*

$$\frac{4\pi N - g^2\varphi_0^2 |T|}{\sin^2\theta} \notin 8\pi\mathbb{N}.$$

*Then (*1.7*) is a necessary and sufficient condition for the existence of a self-dual vortex solution of (*1.6*).*

However, the conjecture is still open when $\frac{4\pi N - g^2\varphi_0^2|T|}{\sin^2\theta} \in 8\pi\mathbb{N}$ (the critical case) and the study of the equation (1.2) at $\alpha_j \in \mathbb{N}$ and $\rho \in 8\pi\mathbb{N}$ might be useful to the conjecture in this critical case.

Since the RHS of equation (1.1) or (1.2) contains some singular terms, in order to eliminate the singularity, we introduce the Green function $G(x,y)$ on M:

$$\begin{cases} \Delta G(x,y) = -\delta_y(x) + 1 \text{ on } M \\ \int_M G(x,y)\,dx = 0 \end{cases}, \tag{1.8}$$

where we assume $|M| = 1$. Also, we set $\gamma(x,y) = G(x,y) + \frac{1}{2\pi}\ln|x-y|$, the regular part of $G(x,y)$. Let $\gamma(x) = \gamma(x,x)$. Then $\gamma(x)$ is well defined in M and $\gamma(x) \equiv$*constant* if g is the standard metric of constant curvature. In terms of the Green function, a solution $u(x)$ of (1.2) can be written as

$$u(x) = w(x) + \ln \int_M h e^u dx + u_0(x),$$

where

$$u_0(x) = -4\pi \sum_{j=1}^{d} \alpha_j G(x, q_j)$$

and we still use $u(x)$ to denote $w(x)$. Then $u(x)$ satisfies

$$\Delta u(x) + \rho \left(h^*(x) e^{u(x)} - 1 \right) = 0 \text{ in } M, \tag{1.9}$$

where

$$h^*(x) = h(x) e^{u_0(x)}. \tag{1.10}$$

Throughout this paper, we shall consider equations (1.2) and (1.9) equivalent, where h^* and h are connected by (1.10).

For the last several decades, equations (1.1) and (1.2) have been extensively studied, we refer [**21**, **22**, **23**, **26**, **27**, **28**, **29**] and references therein for the recent development of this subject. Let $a_j > -1$, and define the critical set Λ by

$$\Lambda = \left\{ 8\pi k + \sum_{j\in A} 8\pi (1+\alpha_j) \,\middle|\, k \in \mathbb{N}^+ \cup \{0\},\ A \subset \{1, \cdots, d\} \right\}.$$

It has been proved that when $\rho \notin \Lambda$, solutions of either (1.1) or (1.2) are uniformly bounded outside of those vortex points $\{q_1, \ldots, q_d\}$. See [**2**] for the case of all $\alpha_j = 0$, and [**3**, **4**] for the general case. Thus, the topological Leray-Schauder degree d_ρ for the equation (1.1) or (1.2) can be well defined. In a series of papers, the counting formula for d_ρ has been proved by Chen and Lin. See [**9**, **10**]. A consequence of the degree-counting formulas is that equation (1.1) or (1.2) has a solution if $\alpha_j \in \mathbb{N}$ and $\rho \notin \Lambda$ and Ω or M are not simply connected. Among others, Chen and Lin proved the following.

Theorem B ([12]). *Let $\alpha_j \in \mathbb{N}\ \forall j$, $\chi(M) \le 0$ and, $h(x)$ a positive C^1 function. If $\rho \notin \Lambda$, then the topological degree $d_\rho > 0$ and equation (1.2) has a solution. Here, $\chi(M) = 2 - 2g$ is the Euler characteristic number of M.*

So, (1.1) or (1.2) have been understood well if $\rho \notin \Lambda$. In this series of papers, we want to extend Theorem A and Theorem B to cover the case with the parameter ρ in $8\pi\mathbb{N}$. Among others, we prove the following theorem.

Theorem 1. *Let $\alpha_j \in \mathbb{N}\ \forall j$, $\chi(M) \leq 0$, and $h(x)$ a positive C^2 function. Then there exists $\rho_0 > 0$ such that for any $\rho > \rho_0$ equation (1.2) has a solution.*

Theorem 1 has been proved in [**10**] for $\alpha_i = 0\ \forall i$. Our method to prove Theorem 1 is to show the following results.

Theorem 2. *Let $\alpha_j > -1$, and let ρ_0 be defined by*

$$\rho_0 = \max_M \left(2K - \Delta \ln h + N^*\right), \tag{1.11}$$

where K is the Gaussian curvature of (M, g) and $N^* = 4\pi \sum_{j=1}^d \alpha_j$. If u_k is a sequence of solutions to (1.2) with

$$\lim_{k\to\infty} \rho_k = \rho_\infty \in \Lambda,\ \rho_\infty > \rho_0,$$

and

$$\rho_k > \rho_\infty \quad \text{for large } k,$$

then u_k is uniformly bounded in $C^2_{loc}\left(M\backslash\{q_1,\ldots,q_d\}\right)$.

This apriori bound was established in [**3**, **4**] when all α_j are not positive integers. When one of α_j is a positive integer, some additional difficulties arise. One of them is that *the phenomenon of non-simple blowup* might happen. In the literature, there are no sharp estimates for bubbling solutions near a non-simple blowup point. The main contribution of this article is to prove such sharp estimates near a non-simple blowup point.

We prove the apriori bound by contradiction. Suppose there is a sequence of bubbling solutions u_k of (1.2) with ρ_k and $\lim_{k\to\infty} \rho_k = \rho_\infty \in \Lambda$ and blowup at $\{p_1, \ldots, p_m\}$. The sharp estimate of u_k near their blowup points has been done for $p_j \notin \{q_1,\ldots,q_d\}$, or $p_j \in \{q_1,\ldots,q_d\}$ with $\alpha_j \notin \mathbb{N}$ in [**9**] and [**11**], respectively. However, the analysis is more complicated when $p_j \in \{q_1,\ldots,q_d\}$ and α_j is a positive integer.

For each p_j, we choose $r_0 > 0$ such that in $B_{2r_0}(p_j)\setminus\{p_j\}$, u_k has no blowup points. Let

$$\alpha(p_j) = \begin{cases} 0 & \text{if } \ p_j \notin \{q_1,\ldots,q_d\} \\ \alpha_j & \text{if } \ p_j \in \{q_1,\ldots,q_d\} \end{cases}. \tag{1.12}$$

We put

$$\rho_{k,p_j} = \rho_k \int_{B_{r_0}(p_j)} h^* e^{u_k} dx \text{ and } \rho_{\infty,p_j} = \lim_{k\to\infty} \rho_{k,p_j} = 8\pi\left(1+\alpha(p_j)\right), \tag{1.13}$$

$$u_k(p_{k,j}) = \max_{B_{r_0}(p_j)} u_k(x) = \lambda_{k,p_j}, \tag{1.14}$$

where $p_{k,j}$ is the local maximum point of u_k near p_j. There are two fundamental questions that will be addressed in the present paper and the second paper of this series:

(i) Are the heights of the bubbles at different blowup points comparable to each other?

(ii) What is the asymptotic formula of $\rho_{k,q} - \rho_{\infty,q}$ in terms of the height of the bubble at q?

Here, q is one of the vortex points with $\alpha(q) \in \mathbb{N}$ and $\rho_{k,q}$, $\rho_{\infty,q}$ are defined by (1.13). We shall answer the first question completely in this paper. Hereafter, the notation $A_k = O(B_k)$ for any two sequences of numbers means that there exists $C > 0$, independent of k such that $|A_k| \leq C|B_k|$. Similarly, $A_k = o(B_k)$ means that $\frac{A_k}{B_k} \to 0$ as $k \to +\infty$.

Our first main result is the following theorem.

Theorem 3. *Let $\alpha_j > -1$, and let $h(x)$ be a C^1 positive function on M. Suppose u_k is a sequence of blowup solutions to (1.9) and $p_1, \ldots, p_m$ are the blowup points. Then*

(i)

$$|\lambda_{k,i} - \lambda_{k,j}| = O(1) \ \ \forall i \neq j. \tag{1.15}$$

(ii)

$$u_k(x) = -\lambda_{k,i} + O(1) \ \ \forall x \in \partial B_{r_0}(p_i), \tag{1.16}$$

where $O(1)$ is independent of k.

The crucial step of Theorem 3 is to prove (1.16); then (1.15) follows immediately. When $\alpha(p_j) = 0$ or $\alpha(p_j) = \alpha_j \notin \mathbb{N}$, (1.16) is a consequence of simple blowup property. *Simple blowup property* means that u_k can be locally well controlled by an entire solution of its limiting problem. More precisely, let $v_k(y)$ be

$$v_k(y) = u_k\left(\varepsilon_{k,p_j} y + p_j\right) - \lambda_{k,j} \text{ for } |y| \leq \frac{1}{\varepsilon_{k,p_j}}, \text{ where } \varepsilon_{k,j} = e^{-\frac{\lambda_{k,j}}{2\left(1+\alpha\left(p_j\right)\right)}}.$$

Then after scaling, a subsequence of v_k would converge to U in $C^2_{loc}\left(\mathbb{R}^2\right)$, where U is an entire solution to

$$\begin{cases} \Delta U + |y|^{2\alpha} e^U = 0 \text{ in } \mathbb{R}^2 \\ \max U = 0 \end{cases}. \tag{1.17}$$

In [**24**], Parajapat and Tarantello have completely classified all solutions of (1.17), that is,

$$U(y; a) = -2\ln\left(1 + \left|y^{1+\alpha} - a\right|^2\right) \tag{1.18}$$

for some $a \in \mathbb{C}$, where $y^{1+\alpha}$ is the $(1+\alpha)$-th power of complex number $y = y_1 + iy_2$. Clearly,

$$a = \lim_{k \to +\infty} \left(\frac{p_{k,j} - p_j}{\varepsilon_{k,p_j}}\right)^{1+\alpha}. \tag{1.19}$$

In particular, for $\alpha = 0$ or $\alpha \notin \mathbb{N}$, we have

$$\frac{p_{k,j} - p_j}{\varepsilon_{k,p_j}} \to 0.$$

That is,

$$a = 0.$$

We say that u_k satisfies the simple blowup property at p_j if

$$|v_k - U(y)| \le C \text{ for } |y| \le \frac{r_0}{\varepsilon_{k,p_j}}$$

for some positive C independent of k and y. When $\alpha(p_j)$ is not a positive integer, the simple blowup property for p_j has been proved by Y.Y. Li in [**19**] for $\alpha(p_j) = 0$ and by Bartolucci, Chen, Lin, and Tarantello in [**3**] for $0 \neq \alpha_j \notin \mathbb{N}$. Obviously, this simple blowup property implies (1.16).

When a blowup point p_j is one of the vortex points, that is, $p_j = q_j$. The simple blowup property may not be true if $\alpha(q_j) = \alpha_j \in \mathbb{N}$. Two cases may occur if $\alpha_j \in \mathbb{N}$.

Case 1: $|p_{k,j} - q_j| = O\left(\varepsilon_{k,q_j}\right)$, $\varepsilon_{k,q_j} = e^{-\frac{\lambda_{k,j}}{2\left(1+\alpha_j\right)}}$.

For $\alpha_j \in \mathbb{N}$, in general, $a \neq 0$, and then $U(z;a)$ is no longer radially symmetric. The non-symmetry of $U(z;a)$ would cause a lot of troubles in the bubbling analysis of u_k. Even so, we still could prove that u_k is *simply bubbling* at q_j for Case 1, that is,

$$|v_k(y) - U(y;a)| \le C \text{ for } |y| \le \frac{r_0}{\varepsilon_{k,q_j}}. \tag{1.20}$$

Inequality (1.20) implies (1.16) for Case 1. For a proof of (1.20), see Appendix A.

Case 2: $\lim_{k\to+\infty} \frac{\left|p_{k,j}-q_j\right|}{\varepsilon_{k,q_j}} = +\infty$

In this case, we see that u_k is *not* simply blowing up at q_j. The method for this case would be different from Case 1. This is a new phenomenon that might occur only at the case when $\alpha_j \in \mathbb{N}$. However, this phenomenon also appears in the study of the $SU(3)$ Toda system. Studying this non-simple blowup phenomenon for the scalar equation should be very useful for the system case. In Case 2, we could also prove the estimate (1.16). We briefly discuss it here. Let

$$|p_{k,j} - q_j| = \delta_{k,j}. \tag{1.21}$$

After scaling by

$$\hat{u}_k(y) = u_k(\delta_{k,j} y + q_j) + 2(1+\alpha_j)\ln \delta_{k,j},$$

$\hat{u}_k(y)$ would blow up at $\left\{e_1, e_2, \dots, e_{1+\alpha_j}\right\}$ with $e_{\ell+1} = q_j + e^{i\frac{2\pi\ell}{1+\alpha_j}}$. Let

$$\hat{\mu}_{k,j} = \lambda_{k,j} + 2(1+\alpha_j)\ln \delta_{k,j} \tag{1.22}$$

and

$$\sigma_{k,j} = e^{-\frac{\hat{\mu}_{k,j}}{2}}. \tag{1.23}$$

Then we can prove that

$$\hat{u}_k(y) = -\hat{\mu}_{k,j} - \sum_{\ell=1}^{1+\alpha_j} 4 \ln |y - e_\ell| + O(1) \tag{1.24}$$

uniformly for all $y \in B_{\frac{1}{\delta_{k,j}}}(0) \setminus \cup_{\ell=1}^{1+\alpha_j} B_{r_0}(e_\ell)$. This implies that (1.16) holds in Case 2. See Theorem 8 and (2.16).

Next, we turn to the question (ii), that is $\rho_{k,j} - \rho_{\infty,j}$. Let $\rho_{k,j}$ be the local mass defined by (1.13) at p_j. By Theorem 3, we have

$$\rho_k - \rho_\infty = \sum_{i=1}^{m} [\rho_{k,i} - 8\pi(1 + \alpha(p_i))] + O\left(e^{-\lambda_k}\right), \tag{1.25}$$

where

$$\lambda_k = \max_j \lambda_{k,j}$$

and

$$\rho_\infty = 8\pi \sum_{i=1}^{m} (1 + \alpha(p_i))$$

and $\alpha(p_i)$ is defined in (1.12). Hence, our second question is how to find asymptotic formulas of $\rho_{k,i} - 8\pi(1 + \alpha(p_i))$.

When $\alpha(p) = 0$ (i.e., $p \notin \{q_1, \ldots, q_d\}$), there is a function $Q(x)$ (see definition in Section 2) such that $\nabla Q(p) = 0$. With this property, Chen and Lin proved:

Theorem C ([**9**]). *Let (u_k, ρ_k) be a sequence of solutions of (1.9) that blows up at $\{p_1, \ldots, p_m\}$. Suppose $\alpha(p) = 0$. Then we have*

$$\rho_{k,p} - 8\pi = \frac{16\pi}{\rho_\infty h_0(p)} (\Delta \ln h(p) - N^* + \rho_\infty - 2K(p)) \varepsilon_{k,p}^2 |\ln \varepsilon_{k,p}| + O\left(\varepsilon_{k,p}^2\right),$$

where $K(x)$ denotes the Gaussian curvature and $N^* = 4\pi \sum_{j=1}^{d} \alpha_j$.

When one of the vortex points is a blowup point—say, $p = q$ and $\alpha(q) \notin \mathbb{N}$—$\nabla Q(p)$ may not be 0. With the help of $a = 0$ in (1.18), Chen and Lin also proved:

Theorem D ([**11**]). *Let (u_k, ρ_k) be a sequence of solutions of (1.9) that blows up at $\{p_1, \ldots, p_m\}$. Suppose $\alpha(p) = \alpha(q) \notin \mathbb{N} \cup \{0\}$. Then we have*

$$\begin{aligned}\rho_{k,q} - 8\pi\big(1 + \alpha(q)\big) = d\big(q, \alpha(q)\big) \big(\Delta \ln h(q) - N^* \\ + \rho_\infty - 2K(q)\big)\varepsilon_{k,q}^2 + o(1)\varepsilon_{k,q}^2,\end{aligned}$$

where $d(q, \alpha(q))$ is a positive constant depending on q and $\alpha(q)$.

For the case $p = q$ and $\alpha(q) \in \mathbb{N}$, again, we also have two cases that need to be considered, simple blowup and non-simple blowup.

Case 1 (Simple blowup):

In this case, a and $\nabla Q(q)$ both may not be 0. The situation here is more complicated than before, and we need to solve an associated linearized problem. Even so, we could also have following sharp estimate.

Theorem 4. *Let (u_k, ρ_k) be a sequence of solutions of (1.9) that blows up at $\{p_1, \dots, p_m\}$. Suppose $p_i = q_i$ and (1.19) holds. Then we have*

$$\rho_{k,i} - 8\pi(1+\alpha_i) = F_1(a;\alpha_i)(\Delta \ln h(q_i) - N^* + \rho_\infty - 2K(q_i))\varepsilon_{k,i}^2 + o(1)\varepsilon_{k,i}^2,$$

where $F_1(a;\alpha_i)$ is a positive constant depending on a and α_i.

Because the proof is technical and also related to existence of some linearized problem, we will give the proof of Theorem 4 in the second paper of this series.

Case 2 (Non-simple blowup):

The sharp estimate for non-simple blowup is the main concern in the present paper, and to obtain the sharp estimate of $\rho_{k,i} - 8\pi(1+\alpha(p_i))$, different estimates in different regions are needed. More precisely, we use the simple blowup property in each bubbling region, $B_{r_0}(e_\ell)$, $\ell = 1, \cdots, (1+\alpha(q))$, and the estimate (1.24) outside the bubbling regions, $B_{\frac{1}{\delta_{k,i}}}(0) \setminus \cup_{\ell=1}^{1+\alpha(q)} B_{r_0}(e_\ell)$. Then we have the following.

Theorem 5. *Let Case 2 hold for $p_i = q_i$. Then*

$$\begin{aligned}\rho_{k,i} - 8\pi(1+\alpha_i) = \frac{32(1+\alpha_i)\pi}{\rho_\infty h_i(q_i)} \big(&\Delta \ln h(q_i) - N^* \\ &+ \rho_\infty - 2K(q_i)\big)\delta_{k,i}^2\sigma_{k,i}^2 |\ln \sigma_{k,i}| + O\left(\delta_{k,i}^2\sigma_{k,i}^2\right)\end{aligned}$$

where $\delta_{k,i}$ and $\sigma_{k,i}$ are defined by (1.21) and (1.23).

By Lemma 9 in Section 4, we have

$$\delta_{k,i}^2 = C\hat{\mu}_{k,i} e^{-\hat{\mu}_{k,i}}(1+o(1)), \text{ for some } C > 0,$$

and thus

$$2\ln \delta_{k,i} = -\hat{\mu}_{k,i} + O(\ln \hat{\mu}_{k,i}). \tag{1.26}$$

Recall (1.22) that

$$\hat{\mu}_{k,i} = \lambda_{k,i} + 2(1+\alpha_i)\ln \delta_{k,i}.$$

Then by (1.26), we have

$$\hat{\mu}_{k,i} = \frac{\lambda_{k,i}}{(2+\alpha_i)}(1+o(1)). \tag{1.27}$$

From (1.23) and (1.27), we have the following corollary.

Corollary 1.

$$O\left(\delta_{k,i}^2\sigma_{k,i}^2\left|\ln\sigma_{k,i}\right|\right)=O\left(\hat{\mu}_{k,i}^2e^{-2\hat{\mu}_{k,i}}\right)=O\left(\lambda_{k,i}^2e^{-\frac{2}{2+\alpha_i}\lambda_{k,i}}\right).$$

This estimate is new and is obtained through delicate application of the Pohozaev identity. Compared with the case of simple blowup (see Theorem 4), the order of non-simple blowup is relatively much smaller than that of simple blowup.

Actually, Theorem 2 is an application of Theorem 4 and Theorem 5. We explain it as follows. Let ρ_0 be the number defined in (1.11). Suppose (u_k,ρ_k) is a sequence of bubbling solutions with $\rho_k\to\rho\in 8\pi\mathbb{N}$. Let p be any blowup point of u_k. When $p\neq q$ (i.e., $\alpha(p)=0$), from Theorem B, we have $\rho_{k,p}-8\pi>0$ provided $\rho>\rho_0$. When $p=q_i$ (i.e. $\alpha(p)=\alpha_i\in\mathbb{N}$), from Theorem 4 and Theorem 5, we also have $\rho_{k,q_i}-8\pi(1+\alpha_i)>0$ provided $\rho>\rho_0$. Thus, from (1.25), we have

$$\rho_k-\rho>0$$

for any $\rho>\rho_0$ and $\rho\in 8\pi\mathbb{N}$. For any $\rho\in 8\pi\mathbb{N}$ and $\rho>\rho_0$, we choose $\rho_k<\rho$ and $\rho_k\to\rho$ as $k\to\infty$. By Theorem B, there is a sequence of solutions u_k of (1.2) with ρ_k for each k. Then by the above results, u_k is uniformly bounded in $C_{loc}^2\left(M\backslash\{q_1,\ldots,q_d\}\right)$. Therefore, after passing limit, u_k converges to a solution u_∞ of (1.2) with $\rho\in 8\pi\mathbb{N}$ and Theorem 1 and Theorem 2 follows.

For (1.1) with the Dirichlet problem, we also have the following theorem.

Theorem 6. *Let $\alpha_j\in\mathbb{N}\ \forall j$, Ω be a non-simply connected domain in $\mathbb{R}^2$, and let $h(x)$ be a positive C^2 function on Ω. Suppose that*

$$\Delta\ln h(x)>N^*.$$

Then equation (1.1) always possesses a solution for all $\rho>0$.

The organization of this paper is as follows: In Section 2, we introduce those notations and definitions stated in our main theorems and discuss the property (1.16) for both cases. In Section 3, we prove Theorem 8, which implies (1.16) for the non-simple blowup case. In Section 4, we will prove Theorem 5. In Appendix A, we give a proof of Theorem 7, which implies (1.16) for the simple blowup case.

2. Preliminary

2.1. Definitions and Notations. In this section, we shall introduce some notation that is stated in our main theorems. Recall that, in terms of the Green function, equation (1.2) is equivalent to the following equation:

$$\Delta u(x)+\rho\left(h^*(x)e^{u(x)}-1\right)=0 \text{ in } M, \tag{2.1}$$

where

$$h^*(x) = h(x)\,e^{u_0(x)}. \tag{2.2}$$

Note that near q_i, $h^*(x)$ has the form in a local coordinate

$$h^*(x) = h_i(x)\,|x-q_i|^{2\alpha_i} \text{ in } |x-q_i| \le r_0 \text{ for some small } r_0 > 0,$$

where

$$h_i(x) = h(x)\,e^{-4\pi\alpha_i\gamma(x,q_i)}e^{-\sum_{j\neq i}^{d}4\pi\alpha_j G(x,q_j)} > 0 \text{ in } |x-q_i| \le r_0. \tag{2.3}$$

For the simplicity of notation, we write $h_i(x) \equiv h^*(x)$ in a neighborhood of p if $p \notin \{q_1, \ldots, q_d\}$. Then we have

$$\nabla \ln h_i(p_i) = \begin{cases} \nabla \ln h(p_i) - \sum_{j=1}^{d} 4\pi\alpha_j \nabla G(p_i, q_j) \text{ if } p_i \notin \{q_1, \ldots, q_d\} \\ \nabla \ln h(q_i) - \sum_{j\neq i}^{d} 4\pi\alpha_j \nabla G(q_i, q_j) - 4\pi\alpha_i \nabla\gamma(p_i) \text{ if } p_i = q_i \end{cases}$$

and

$$\Delta \ln h_i(x) = \Delta \ln h(x) - \sum_{j=1}^{d} 4\pi\alpha_j.$$

Let (u_k, ρ_k) be a sequence of bubbling solutions of (2.1) that blows up at $\{p_1, \ldots, p_m\}$. Then it is known that

$$\rho_k h(x)\,e^{u_k(x)} \to \sum_{i=1}^{m} 8\pi\,(1+\alpha(p_j))\,\delta_{p_i},$$

where $\alpha(p_i)$ is defined in (1.12). This fact follows from work by Brezis and Merle [**2**], Li and Shafrir [**20**], and Bartolucci and Tarantello [**4**]. Furthermore, p_i can be determined a priori. Let $Q_i(x)$ defined by

$$Q_i(x) = \ln h_i(x) + 8\pi\,(1+\alpha(p_i))\,\gamma(x, p_i) + \sum_{j\neq i}^{m} \rho_{\infty,j} G(x, p_j).$$

Then we have

$$\nabla Q_i(p_i) = 0 \text{ provided that } \alpha(p_i) = 0, \text{ i.e., } p_i \notin \{q_1, \ldots, q_d\}. \tag{2.4}$$

See [**9**] for the proof. Note that at $p_i = q_i$, $\nabla Q_i(q_i)$ may not be 0. When $\alpha_i \in \mathbb{N}$, we will see later that this would cause a lot of additional difficulties in the bubbling analysis.

The estimate $\rho_{k,i} - \rho_{\infty,i}$ is our major concern in this paper. Theorem 4 and Theorem 5 say that $\rho_{k,i} - \rho_{\infty,i}$ can be expressed by some local terms. Since it is local in principle, for simplicity we assume that M has a flat metric near a neighborhood of each blowup point. For the general case, our method presented here can be modified easily as in [**9**, **10**].

2.2. Simple blowup. In order to compare heights of any two bubbles of u_k, the first step is to prove (1.16) near each blowup point p. As discussed before, if $p \notin \{q_1, \ldots, q_d\}$ or $p = q_i$ with $\alpha(q_i) \notin \mathbb{N}$, then (1.16) was proved in [**9**, **19**], and [**3**]. In this section, we may assume $p = q = 0$ and $\alpha = \alpha(0) \in \mathbb{N}$. Without loss of generality, we can always assume that $u_k(x)$ satisfies

$$\begin{cases} \Delta u_k(x) + \rho_k h_0(x) |x|^{2\alpha} e^{u_k(x)} = 0 \text{ in } B_1(0) \\ \left| u_k(x) - u_k\left(x'\right) \right| \leq c \text{ for } |x| = \left| x' \right| = 1 \\ \rho_{k,0} := \int_{B_1(0)} \rho_k h_0(x) |x|^{2\alpha} e^{u_k(x)} \to 8\pi(1+\alpha) \\ 0 \text{ is the only blowup point for } u_k(x) \text{ in } B_1(0) \end{cases}, \tag{2.5}$$

where c is a constant and h_0 is defined in (2.3).

Set

$$\lambda_{k,0} = u_k(p_{k,0}) = \max_{x \in B_1(0)} u_k(x), \text{ and } \varepsilon_k = e^{-\frac{\lambda_{k,0}}{2(1+\alpha)}}.$$

For Case 1, we assume

$$\lim_{k\to\infty} \left(\frac{p_{k,0}}{\varepsilon_k} \right)^{1+\alpha} = a \in \mathbb{R}^2. \tag{2.6}$$

Under (2.6), we can show that u_k has a simple blowup at 0; that is, the following estimate holds.

Theorem 7. *Let u_k be a solution of (2.5). Suppose that (2.6) holds true. Then there exists $r_0 > 0$ and $C > 0$ such that*

$$|u_k(x) - U_k(x)| \leq C \text{ for all } x \in B_{r_0}(0), \tag{2.7}$$

where

$$U_k(x) = \lambda_{k,0} - 2\ln\left(1 + \frac{\rho_k h_0(0)}{8(1+\alpha)^2} e^{\lambda_{k,0}} \left| x^{1+\alpha} - p_{k,0}^{1+\alpha} \right|^2 \right).$$

The proof will be given in Appendix A. Obviously, on the boundary $|x| = r_0$, (2.7) implies (1.16).

2.3. Non-simple blowup. Next, we consider the case when (2.6) fails, that is,

$$\lim_{k\to\infty} \frac{|p_{k,0}|}{\varepsilon_k} = \infty. \tag{2.8}$$

In this case, 0 is no longer a simple blowup point. In fact, we shall prove u_k has $\alpha + 1$ local maximum points. Let $u_k(x)$ satisfy (2.5) and set

$$\delta_k = |p_{k,0}|.$$

Define

$$\hat{u}_k(y) = u_k(\delta_k y) + 2(1+\alpha)\ln\delta_k \text{ for } |y| \leq \frac{1}{\delta_k}. \tag{2.9}$$

Then

$$\left\{\begin{array}{l}\Delta \hat{u}_k(y)+\rho_k|y|^{2\alpha} h_0(\delta_k y) e^{\hat{u}_k(y)}=0 \text{ in } B_{\frac{1}{\delta_k}}(0) \\ \left|\hat{u}_k(y)-\hat{u}_k\left(y^{\prime}\right)\right| \leq C \text{ for } y, y^{\prime} \in \partial B_{\frac{1}{\delta_k}}(0)\end{array}\right., \tag{2.10}$$

and by (2.8) we have

$$\hat{u}_k\left(\frac{p_{k,0}}{|p_{k,0}|}\right)=\hat{\mu}_k=\lambda_{k,0}+2(1+\alpha)\ln \delta_k \rightarrow \infty. \tag{2.11}$$

Hence $e_1=\lim_{k\to\infty}\frac{p_{k,0}}{|p_{k,0}|}$ is a blowup point of $\hat{u}_k$. By the Brezis–Merle theorem, $\hat{u}_k$ blows up at a finite set $S=\{e_1,\ldots,e_{1+n}\}$. Moreover, from Green's representation formula, we have

$$\left|\hat{u}_k(y)-\hat{u}_k\left(y^{\prime}\right)\right| \leq C \text{ for } y, y^{\prime} \in \partial B_1(e_\ell),\ 1 \leq \ell \leq 1+n. \tag{2.12}$$

See Lemma 1 in Section 3 for the proof. By (2.12), we conclude that $\hat{u}_k$ is simply bubbling at each e_ℓ. In Section 3, we will prove that $\hat{u}_k$ does not blow up at 0. Thus, by (2.4), $\{e_1,\ldots,e_{1+n}\}$ satisfies

$$2\alpha\frac{e_\ell}{|e_\ell|^2}=\sum_{j\neq\ell}^{1+n}\frac{e_i-e_\ell}{|e_i-e_\ell|^2} \text{ for } \ell=1,\ldots,(1+n), \tag{2.13}$$

and we have the following important estimate for non-simple blowup.

Theorem 8. *Let $\hat{u}_k$ be defined in (2.9). Then $n=\alpha$, and*

$$\hat{u}_k(y)=-\hat{\mu}_k-\sum_{\ell=1}^{1+\alpha}4\ln|y-e_\ell|+O(1) \ \textit{for} \ |y|\leq\frac{1}{\delta_k}. \tag{2.14}$$

By using potential analysis, we could prove the estimate

$$\hat{u}_k(y)=-\hat{\mu}_k-\frac{\rho_{k,0}}{2\pi}\ln|y|+O(1)$$

for $\ln\frac{1}{\delta_k}<|y|<\frac{1}{\delta_k}$. See Lemma 4. However, the crucial step to obtain estimate (2.14) is to prove

$$|\rho_{k,0}-8\pi(1+\alpha)|=O\left(\left(\ln\frac{1}{\delta_k}\right)^{-1}\right),$$

which is an application of the Pohozaev identity. See Lemma 6. The proof of Theorem 8 will be given in the Section 3.

By Theorem 8, on $\partial B_{\frac{1}{\delta_k}}(0)$ we have

$$\hat{u}_k(y)=-\hat{\mu}_k+4(1+\alpha)\ln\delta_k+O(1) \text{ for } |y|=\frac{1}{\delta_k}, \tag{2.15}$$

and by transferring back to $u_k(x)$, we conclude that for $|x|=1$

$$\begin{aligned} u_k(x) &= \hat{u}_k(y)-2(1+\alpha)\ln\delta_k \\ &= -\lambda_{k,0}+O(1). \end{aligned} \tag{2.16}$$

This proves (1.16) for $p=q$ where u_k is non-simple blowup.

3. Proof of Theorem 8

In this section, we are going to prove Theorem 8. Recall that $u_k(x)$ satisfies (2.5) and

$$\rho_{k,0}=\int_{B_1(0)}\rho_k|x|^{2\alpha}h_0(x)e^{u_k(x)}dx\to\rho_{\infty,0}=8\pi(1+\alpha).\tag{3.1}$$

Let $\hat{u}_k(y)=u_k(\delta_k y)+2(1+\alpha)\log\delta_k$ for $|y|\le\frac{1}{\delta_k}$, where δ_k is given by (1.21). Then

$$\begin{cases}\Delta\hat{u}_k(y)+\rho_k|y|^{2\alpha}h_0(\delta_k y)e^{\hat{u}_k(y)}=0\text{ in }B_{\frac{1}{\delta_k}}(0)\\ \left|\hat{u}_k(y)-\hat{u}_k\left(y'\right)\right|\le C_1\text{ for }y,\,y'\in\partial B_{\frac{1}{\delta_k}}(0)\end{cases}.$$

In order to prove Theorem 8, we need several lemmas borrow the ideas from [**3**].

Notice that $e_1=\lim_{k\to\infty}\frac{p_{k,0}}{|p_{k,0}|}$ is a blowup point of $\hat{u}_k$. Applying results of Brezis and Merle [**2**] or Bartolucci and Tarantello [**4**], since $\int_{B_{\frac{1}{\delta_k}}(0)}\rho_k|y|^{2\alpha}h_0(\delta_k y)e^{\hat{u}_k(y)}\le C$, there exists a finite blowup set $S=\{e_1,e_2,\dots,e_{1+n}\}$ and $\hat{u}_k\to-\infty$ uniformly on any compact subset of $\mathbb{R}^2\backslash S$ and

$$\rho_k|y|^{2\alpha}h_0(\delta_k y)e^{\hat{u}_k(y)}\to\sum_{\ell=1}^{1+n}m_\ell\delta_{e_\ell},$$

where

$$m_\ell=\lim_{k\to\infty}m_{k,\ell}=\lim_{k\to\infty}\int_{B_1(e_\ell)}\rho_k|y|^{2\alpha}h_0(\delta_k y)e^{\hat{u}_k(y)}.$$

In order to determine m_ℓ, we have to prove the bounded oscillation of $\hat{u}_k$ near each e_ℓ.

Lemma 1.

$$\left|\hat{u}_k(z)-\hat{u}_k\left(z'\right)\right|\le C\tag{3.2}$$

for $z,\,z'\in\partial B_{r_0}(e_\ell)$, and $e_\ell\in S$.

Proof. Let $r_0>0$ such that e_ℓ is the only blowup point of $\hat{u}_k(z)$ in $B_{4r_0}(e_\ell)$. By Green's formula, for any $z\in B_{\frac{1}{\delta_k}}(0)$, we have

$$\hat{u}_k(z)=\int_{B_{\frac{1}{\delta_k}}(0)}\rho_k|y|^{2\alpha}h_0(\delta_k y)e^{\hat{u}_k(y)}G_k(y,z)\,dy+\phi_k(z)+2(1+\alpha)\ln\delta_k+d_k$$

where $G_k(y,z)$ is the Dirichlet Green function in $B_{\frac{1}{\delta_k}}(0)$ and $\phi_k(y)$ is the harmonic function with $\phi_k|_{\partial B_{\frac{1}{\delta_k}}}=u_k(\delta_k y)-d_k$, $d_k=\frac{1}{2\pi}\int_{\partial B_1}u_k d\sigma$.

More precisely,
(3.3)

$$G_k(x,y) = -\frac{1}{2\pi}\ln|x-y| + \frac{1}{2\pi}\ln\left|\delta_k|y|x - \frac{1}{\delta_k}\frac{y}{|y|}\right| \text{ for } x,y \in B_{\frac{1}{\delta_k}}(0).$$

For $z,\ z^{'} \in \partial B_{r_0}(e_\ell)$, we have

$$\left|\phi_k(z) - \phi_k\left(z^{'}\right)\right| \le \sup_{y_1,y_2\in\partial B_{\frac{1}{\delta_k}}(0)} |\phi_k(y_1) - \phi_k(y_2)| \le C$$

and

$$\begin{aligned}\hat{u}_k(z) - \hat{u}_k\left(z^{'}\right) &= \int_{B_{\frac{1}{\delta_k}}(0)} \rho_k|y|^{2\alpha} h_0(\delta_k y) e^{\hat{u}_k(y)}\left(G_k(y,z) - G_k\left(y,z^{'}\right)\right)dy \\ &\quad + \phi_k(z) - \phi_k\left(z^{'}\right) \\ &= -\frac{1}{2\pi}\int_{B_{\frac{1}{\delta_k}}(0)} \rho_k|y|^{2\alpha} h_0(\delta_k y) e^{\hat{u}_k(y)} \ln\frac{|y-z|}{|y-z^{'}|}dy + O(1).\end{aligned}$$

For $y \in \left\{|y-e_\ell| < \frac{r_0}{2}\right\} \cup \{|y-e_\ell| > 2r_0\}$, we have $\left|\ln\frac{|y-z|}{|y-z^{'}|}\right| \le c_1$, and this implies that

$$\int_{\left\{|y-e_\ell|<\frac{r_0}{2}\right\}\cup\{|y-e_\ell|>2r_0\}} \rho_k|y|^{2\alpha} h_0(\delta_k y) e^{\hat{u}_k(y)} \ln\frac{|y-z|}{|y-z^{'}|}dy = O(1).$$

On the other hand, for $y \in \left\{\frac{r_0}{2} \le |y-e_\ell| \le 2r_0\right\}$, we have $\hat{u}_k(y) \to -\infty$ uniformly as $k \to \infty$. Thus, we can conclude that

$$\left|\hat{u}_k(z) - \hat{u}_k\left(z^{'}\right)\right| \le C \text{ for } z,\ z^{'} \in \partial B_{r_0}(e_\ell).$$

q.e.d.

From the property of (3.2), Li [**19**] and Bartolucci and Tarantello [**4**] proved that $m_\ell = 8\pi$ if $e_\ell \ne 0$ and $m_\ell = 8\pi(1+\alpha)$ if $e_\ell = 0$, respectively. Therefore, we have the following Corollary 3.2.

Corollary 2. *$0 \notin S$.*

Proof. Suppose the contrary holds. Then we have

$$\lim_{k\to\infty}\int_{B_{r_0}(0)} \rho_k|y|^{2\alpha} h_0(\delta_k y) e^{\hat{u}_k(y)}dy = 8\pi(1+\alpha). \tag{3.4}$$

Since e_1 is also a blowup point of $\hat{u}_k(y)$, by (3.4), we have

$$\begin{aligned} 8\pi(1+\alpha) &= \lim_{k\to\infty}\int_{B_1(0)} \rho_k |x|^{2\alpha} h_0(x) e^{u_k(x)} dx \\ &= \lim_{k\to\infty}\int_{B_{\frac{1}{\delta_k}}(0)} \rho_k |y|^{2\alpha} h_0(\delta_k y) e^{\hat{u}_k(y)} dy \\ &\geq \lim_{k\to\infty}\left(\int_{B_{r_0}(0)} \rho_k |y|^{2\alpha} h_0(\delta_k y) e^{\hat{u}_k(y)} dy\right. \\ &\left. + \int_{B_{r_0}(e_1)} \rho_k |y|^{2\alpha} h_0(\delta_k y) e^{\hat{u}_k(y)} dy\right) \\ &= 8\pi(1+\alpha) + 8\pi. \end{aligned}$$

This leads a contradiction to (3.1). Thus, 0 cannot be a blowup point of $\hat{u}_k$. q.e.d.

Now, we prove that the locations of e_ℓ are the partitions of the unit circle.

Lemma 2.

$$n = \alpha$$

and

$$e_{\ell+1} = \exp\left(i\frac{2\pi}{1+\alpha}\ell\right),\ \ell = 0,\ldots,\alpha.$$

Proof. From Lemma 1, $\hat{u}_k$ is simply bubbling at each e_ℓ. By (2.4), e_ℓ satisfies

$$2\alpha\frac{1}{e_\ell} = 4\sum_{j\neq\ell}^{1+n}\frac{1}{e_\ell - e_j} \text{ for } \ell = 1,\ldots,(1+n),$$

and hence

$$\alpha = 2\sum_{j\neq\ell}^{1+n}\frac{e_\ell}{e_\ell - e_j}.$$

Then

$$(1+n)\alpha = \sum_{j=1}^{1+n}\alpha = 2\sum_{j=1}^{1+n}\sum_{j\neq\ell}^{1+n}\frac{e_\ell}{e_\ell - e_j} = 2\frac{(1+n)n}{2}.$$

Hence $n = \alpha$, that is, $\hat{u}_k$ blows up at $S = \{e_1, \cdots, e_{1+\alpha}\}$. To solve e_j, we let $I = \{e_1, \ldots, e_{1+\alpha}\}$, $I_\ell = I\setminus\{e_\ell\}$ and $I_{\ell,j} = I\setminus\{e_\ell, e_j\}$. We introduce the following notation:

$$\binom{I}{k} = \sum_{j_1<\ldots<j_k} e_{j_1}\ldots e_{j_k}, \text{ where } e_{j's} \in I \text{ for } k = 1,\ldots,1+\alpha$$

and

$$\binom{I_\ell}{k} = \sum_{j_1<\cdots<j_k} e_{j_1}\cdots e_{j_k}, \text{ where } e_{j's} \in I_\ell \text{ for } k = 1, \ldots, \alpha.$$

Since

$$\alpha = 2\sum_{j\neq\ell}^{1+\alpha} \frac{e_\ell}{e_\ell - e_j}$$

we have

$$\alpha \prod_{j\in I_\ell} (e_\ell - e_j) = 2e_\ell \sum_{j\neq\ell}^{1+\alpha} \prod_{m\in I_{\ell,j}} (e_\ell - e_m)\,. \tag{3.5}$$

Then expanding (3.5), we have

$$\begin{aligned} &\alpha\left(e_\ell^\alpha + \sum_{k=1}^{\alpha-1} (-1)^k \binom{I_\ell}{k} e_\ell^{\alpha-k} + (-1)^\alpha \binom{I_\ell}{\alpha}\right) \\ &= 2\alpha e_\ell^\alpha + \sum_{k=1}^{\alpha-1} (-1)^k\, 2\,(\alpha-k) \binom{I_\ell}{k} e_\ell^{\alpha-k}. \end{aligned} \tag{3.6}$$

Multiplying e_ℓ on both sides of (3.6), we obtain

$$\alpha e_\ell^{\alpha+1} + \sum_{k=1}^{\alpha-1} (-1)^k (\alpha - 2k) \binom{I_\ell}{k} e_\ell^{\alpha+1-k} + \alpha\,(-1)^{1+\alpha}\, e_1 e_2 \ldots e_{1+\alpha} = 0. \tag{3.7}$$

In particular, (3.7) can be rewritten as

$$\alpha e_\ell^{\alpha+1} + \sum_{k=1}^{\alpha-1} (-1)^k (\alpha-k)(1-k) \binom{I}{k} e_\ell^{\alpha+1-k} + \alpha\,(-1)^{1+\alpha}\, e_1 e_2 \ldots e_{1+\alpha} = 0.$$

Thus, for each e_ℓ, we have

$$e_\ell^{\alpha+1} + \sum_{k=1}^{\alpha-1} (-1)^k \frac{(\alpha-k)(1-k)}{\alpha} \binom{I}{k} e_\ell^{\alpha+1-k} + (-1)^{1+\alpha}\, e_1 e_2 \ldots e_{1+\alpha} = 0.$$

This implies that e_ℓ, $\ell = 1, \ldots, 1+\alpha$ are the solutions of

$$z^{1+\alpha} + \sum_{k=1}^{\alpha-1} (-1)^k \frac{(\alpha-k)(1-k)}{\alpha} \binom{I}{k} z^{1+\alpha-k} + (-1)^{1+\alpha}\, e_1 e_2 \ldots e_{1+\alpha} = 0.$$

Thus,

$$\begin{aligned} (z-e_1)\cdots(z-e_{1+\alpha}) &= z^{1+\alpha} + \sum_{k=1}^{\alpha-1} (-1)^k \frac{(\alpha-k)(1-k)}{\alpha} \binom{I}{k} z^{1+\alpha-k} \\ &\quad + (-1)^{1+\alpha}\, e_1 e_2 \ldots e_{1+\alpha}. \end{aligned} \tag{3.8}$$

On the other hand, by direct expansion, we have
(3.9)
$$(z-e_1)\dots(z-e_{1+\alpha}) = z^{1+\alpha}+\sum_{k=1}^{\alpha-1}(-1)^k\binom{I}{k}z^{1+\alpha-k}+(-1)^{1+\alpha}e_1e_2\dots e_{1+\alpha}.$$

Comparing (3.8) and (3.9), we have
$$\binom{I}{k}=0 \text{ for } k=1,\dots,\alpha-1,$$
and hence
$$(z-e_1)\dots(z-e_{1+\alpha}) = z^{1+\alpha}+(-1)^{1+\alpha}e_1e_2\dots e_{1+\alpha}.$$
Since $e_1=1$, we have
$$(-1)^{1+\alpha}e_1e_2\dots e_{1+\alpha}=-1.$$
Thus, $\{e_1,\dots,e_{1+\alpha}\}$ are solutions of
$$z^{1+\alpha}-1=0,$$
and this implies that
$$e_{\ell+1}=\exp\left(i\frac{2\pi}{1+\alpha}\ell\right),\ \ell=0,\dots,\alpha.$$
q.e.d.

To prove Theorem 8, we need to prove following decay estimate of $\hat{u}_k$.

Lemma 3. *For every small $\theta>0$ there exists $R_\theta>1$ and $k_\theta\in\mathbb{N}$ such that $\forall\,|z|>2R_\theta$ and $k\geq k_\theta$; then we have*
$$\hat{u}_k(z)\leq-\hat{\mu}_k-\left(\frac{\rho_{k,0}}{2\pi}-2\theta\right)\ln|z|+O(1).$$

Proof. By the simple blowup property of $\hat{u}_k$, we have
$$\hat{u}_k(y)=-\hat{\mu}_k+O(1) \text{ for } y\in B_2(0)\setminus\cup_{\ell=1}^{1+\alpha}B_{r_0}(e_\ell).$$
In particular,
$$\hat{u}_k(0)=-\hat{\mu}_k+O(1).$$
By Green's formula, for any $z\in B_{\frac{1}{\delta_k}}(0)\setminus\cup_{\ell=1}^{1+\alpha}B_{r_0}(e_\ell)$, we have
$$\hat{u}_k(z)=\int_{B_{\frac{1}{\delta_k}}(0)}\rho_k|y|^{2\alpha}h_0(\delta_k y)e^{\hat{u}_k(y)}G_k(y,z)\,dy+\phi_k(z)+2(1+\alpha)\ln\delta_k+d_k.$$
Thus,
$$\hat{u}_k(0)=\int_{B_{\frac{1}{\delta_k}}(0)}\rho_k|y|^{2\alpha}h_0(\delta_k y)e_k^{\hat{u}_k(y)}G_k(y,0)\,dy+\phi_k(0)+2(1+\alpha)\ln\delta_k+d_k.$$

Since

$$\left|\phi_k(z)-\phi_k(0)\right| \le \sup_{z,z'\in\partial B_{\frac{1}{\delta_k}}(0)} \left|\phi_k(z)-\phi_k\left(z'\right)\right| \le C_1,$$

we have

$$\begin{aligned}\hat{u}_k(z) &= \int_{B_{\frac{1}{\delta_k}}(0)} \rho_k |y|^{2\alpha} h_0(\delta_k y) e^{\hat{u}_k(y)} \left(G_k(y,z)-G_k(y,0)\right) dy + \hat{u}_k(0) + O(1) \\ &= -\hat{\mu}_k + \int_{B_{\frac{1}{\delta_k}}(0)} \rho_k |y|^{2\alpha} h_0(\delta_k y) e^{\hat{u}_k(y)} \left(G_k(y,z)-G_k(y,0)\right) dy + O(1).\end{aligned}$$

Notice that

$$\begin{aligned}G_k(y,z) &= -\frac{1}{2\pi}\ln|y-z| + \frac{1}{2\pi}\ln\left|\delta_k|y|z-\frac{1}{\delta_k}\frac{y}{|y|}\right| \\ &= -\frac{1}{2\pi}\ln|y-z| + \frac{1}{2\pi}\ln\frac{1}{\delta_k} + \frac{1}{2\pi}\ln\left|\delta_k^2|y|z-\frac{y}{|y|}\right|\end{aligned}$$

and

$$G_k(y,0) = -\frac{1}{2\pi}\ln|y| + \frac{1}{2\pi}\ln\frac{1}{\delta_k}.$$

Since $y \in B_{\frac{1}{\delta_k}}(0)$, we have

$$\frac{1}{2\pi}\ln\left|\delta_k^2|y|z-\frac{y}{|y|}\right| = O(1),$$

and hence

$$\left|G_k(y,z)-G_k(y,0)\right| = \frac{1}{2\pi}\ln\frac{|y|}{|y-z|} + O(1).$$

Thus,

$$\begin{aligned}&\int_{B_{\frac{1}{\delta_k}}(0)} \rho_k |y|^{2\alpha} h_0(\delta_k y) e^{\hat{u}_k(y)} \left(G_k(y,z)-G_k(y,0)\right) dy \\ &= \frac{1}{2\pi}\int_{B_{\frac{1}{\delta_k}}(0)} \rho_k |y|^{2\alpha} h_0(\delta_k y) e^{\hat{u}_k(y)} \ln\frac{|y|}{|y-z|} dy + O(1).\end{aligned}$$

Hence, we have

$$\hat{u}_k(z) = -\hat{\mu}_k + \frac{1}{2\pi}\int_{B_{\frac{1}{\delta_k}}(0)} \left(\ln\frac{|y|}{|y-z|}\right) \rho_k |y|^{2\alpha} h_0(\delta_k y) e^{\hat{u}_k(y)} dy + O(1).$$

For a small $\theta > 0$, we can choose $R_\theta > 1$ and k_θ large such that for $k \ge k_\theta$ we have

$$\frac{1}{2\pi}\int_{|y|\le R_\theta} \rho_k |y|^{2\alpha} h_0(\delta_k y) e^{\hat{u}_k(y)} dy \ge \left(\frac{\rho_{k,0}}{2\pi} - \frac{\theta}{\alpha+2}\right).$$

Taking $|z| > 2R_\theta$ and $k \geq k_\theta$, then we decompose

$$\begin{aligned}
&\frac{1}{2\pi}\int_{B_{\frac{1}{\delta_k}}(0)}\left(\ln\frac{|y|}{|y-z|}\right)\rho_k\,|y|^{2\alpha}\,h_0\left(\delta_k y\right)e^{\hat{u}_k(y)}dy\\
&=\frac{1}{2\pi}\int_{|y|\leq R_\theta}\left(\ln\frac{|y|}{|y-z|}\right)\rho_k\,|y|^{2\alpha}\,h_0\left(\delta_k y\right)e^{\hat{u}_k(y)}dy\\
&+\frac{1}{2\pi}\int_{R_\theta\leq|y|\leq\frac{|z|}{2}}\left(\ln\frac{|y|}{|y-z|}\right)\rho_k\,|y|^{2\alpha}\,h_0\left(\delta_k y\right)e^{\hat{u}_k(y)}dy\\
&+\frac{1}{2\pi}\int_{B_{\frac{|z|}{2}}(z)}\left(\ln\frac{|y|}{|y-z|}\right)\rho_k\,|y|^{2\alpha}\,h_0\left(\delta_k y\right)e^{\hat{u}_k(y)}dy\\
&+\frac{1}{2\pi}\int_{B_k'}\left(\ln\frac{|y|}{|y-z|}\right)\rho_k\,|y|^{2\alpha}\,h_0\left(\delta_k y\right)e^{\hat{u}_k(y)}dy,
\end{aligned}$$

where $B_k' = B_{\frac{1}{\delta_k}}(0)\setminus\left(B_{\frac{|z|}{2}}(0)\cup B_{\frac{|z|}{2}}(z)\right)$. Since $\ln\frac{|y|}{|y-z|} \leq C$ in $\left\{R_\theta\leq|y|\leq\frac{|z|}{2}\right\}\cup B_k'$, we have

$$\frac{1}{2\pi}\int_{R_\theta\leq|y|\leq\frac{|z|}{2}}\left(\ln\frac{|y|}{|y-z|}\right)\rho_k\,|y|^{2\alpha}\,h_0\left(\delta_k y\right)e^{\hat{u}_k(y)}dy = O\left(1\right)$$

and

$$\frac{1}{2\pi}\int_{B_k'}\left(\ln\frac{|y|}{|y-z|}\right)\rho_k\,|y|^{2\alpha}\,h_0\left(\delta_k y\right)e^{\hat{u}_k(y)}dy = O\left(1\right).$$

Next, set $D_\alpha = B_{\frac{|z|}{2}}(z)\cap\left\{|y-z|<|z|^{-(1+\alpha)}\right\}$

$$\begin{aligned}
&\int_{B_{\frac{|z|}{2}}(z)}\left(\ln\frac{|y|}{|y-z|}\right)\rho_k\,|y|^{2\alpha}\,h_0\left(\delta_k y\right)e^{\hat{u}_k(y)}dy\\
&=\int_{D_\alpha}\left(\ln\frac{|y|}{|y-z|}\right)\rho_k\,|y|^{2\alpha}\,h_0\left(\delta_k y\right)e^{\hat{u}_k(y)}dy\\
&+\int_{B_{\frac{|z|}{2}}(z)\setminus D_\alpha}\left(\ln\frac{|y|}{|y-z|}\right)\rho_k\,|y|^{2\alpha}\,h_0\left(\delta_k y\right)e^{\hat{u}_k(y)}dy\\
&\leq\int_{D_\alpha}\left(\ln\frac{1}{|y-z|}\right)\rho_k\,|y|^{2\alpha}\,h_0\left(\delta_k y\right)e^{\hat{u}_k(y)}dy\\
&+(\alpha+2)\int_{B_{\frac{|z|}{2}}(z)}\rho_k\,|y|^{2\alpha}\,h_0\left(\delta_k y\right)e^{\hat{u}_k(y)}dy\ln|z| + O\left(1\right).
\end{aligned}$$

Since $\hat{u}_k(y)$ is a simple blowup in each $B_{r_0}(e_\ell)$, we have $\hat{u}_k(y) < 0$ in $B_{\frac{1}{\delta_k}}(0) \setminus \cup_{\ell=1}^{1+\alpha} B_{r_0}(e_\ell)$. Thus, $\hat{u}_k(y) < 0$ in D_α, and we have

$$\begin{aligned}&\int_{D_\alpha}\left(\ln\frac{1}{|y-z|}\right)\rho_k|y|^{2\alpha}h_0(\delta_k y)e^{\hat{u}_k(y)}dy\\&\le C|z|^{2\alpha}\int_{\left\{|y-z|<|z|^{-(1+\alpha)}\right\}}\left(\ln\frac{1}{|y-z|}\right)dy = O(1).\end{aligned}$$

By the choice of θ and $|z| > 2R_\theta$, we have

$$(\alpha+2)\int_{B_{\frac{|z|}{2}}(z)}\rho_k|y|^{2\alpha}h_0(\delta_k y)e^{\hat{u}_k(y)}dy\ln|z| \le \theta\ln|z|.$$

Hence, for $|z| > 2R_\theta$, we have

$$\begin{aligned}\hat{u}_k(z) &\le -\hat{\mu}_k + \frac{1}{2\pi}\int_{|y|\le R_\theta}\left(\ln\frac{|y|}{|y-z|}\right)\rho_k|y|^{2\alpha}h_0(\delta_k y)e^{\hat{u}_k(y)}dy\\&\quad+\theta\ln|z| + O(1)\\&\le -\hat{\mu}_k + \frac{1}{2\pi}\ln\frac{2R_\theta}{|z|}\int_{|y|\le R_\theta}\rho_k|y|^{2\alpha}h_0(\delta_k y)e^{\hat{u}_k(y)}dy + \theta\ln|z| + O(1)\\&\le -\hat{\mu}_k - \left(\frac{\rho_{k,0}}{2\pi} - 2\theta\right)\ln|z| + O(1).\end{aligned}$$

This completes the proof. q.e.d.

From Lemma 3 and $\rho_{k,0} \to 8\pi(1+\alpha)$, we have

$$\int_{B_{\frac{1}{\delta_k}}(0)}|\ln|y||\,|y|^{2\alpha}h_0(\delta_k y)e^{\hat{u}_k(y)}dy \le C \tag{3.10}$$

and

$$\int_{B_{\frac{1}{\delta_k}}(0)}|y|\,|y|^{2\alpha}h_0(\delta_k y)e^{\hat{u}_k(y)}dy \le C. \tag{3.11}$$

With the help of (3.10), we can refine the decay estimate.

Lemma 4. $\forall y \in B_{\frac{1}{\delta_k}}(0)\setminus B_{\ln\frac{1}{\delta_k}}(0)$, *and so we have*

$$\hat{u}_k(y) = -\hat{\mu}_k - \frac{\rho_{k,0}}{2\pi}\ln|y| + O(1). \tag{3.12}$$

Proof. Let $\tilde{r}_0 > 0$ be a fixed small positive number. Define

$$\tilde{\rho}_{k,0}(z) = \int_{|y|\le\tilde{r}_0|z|}\rho_k|y|^{2\alpha}h_0(\delta_k y)e^{\hat{u}_k(y)}dy.$$

Then by Lemma 3, for $\ln\frac{1}{\delta_k} \leq |z| \leq \frac{1}{\delta_k}$, we have

(3.13)

$$\begin{aligned}
|\tilde{\rho}_{k,0}(z) - \rho_{k,0}| &= \int_{\tilde{r}_0|z|<|y|\leq\frac{1}{\delta_k}} \rho_k |y|^{2\alpha} h_0(\delta_k y) e^{\hat{u}_k(y)} dy \\
&\leq C \int_{\tilde{r}_0 \ln\frac{1}{\delta_k}<|y|\leq\frac{1}{\delta_k}} |y|^{2\alpha} e^{-\hat{\mu}_k - \left(\frac{\rho_{k,0}}{2\pi} - 2\theta\right)\ln|y| + O(1)} dy \\
&= O\left(\left(\ln\frac{1}{\delta_k}\right)^{-2} e^{-\hat{\mu}_k}\right)
\end{aligned}$$

and

(3.14)

$$\int_{\tilde{r}_0|z|<|y|\leq\frac{1}{\delta_k}} \left(\ln\frac{|y|}{|y-z|}\right) \rho_k |y|^{2\alpha} h_0(\delta_k y) e^{\hat{u}_k(y)} dy = O\left(\left(\ln\frac{1}{\delta_k}\right)^{-1}\right).$$

Hence, by (3.10), (3.13), and (3.14), we have

$$\begin{aligned}
\hat{u}_k(z) &= -\hat{\mu}_k + \frac{1}{2\pi}\int_{B_{\frac{1}{\delta_k}}(0)} \left(\ln\frac{|y|}{|y-z|}\right) \rho_k |y|^{2\alpha} h_0(\delta_k y) e^{\hat{u}_k(y)} dy + O(1) \\
&= -\hat{\mu}_k + \frac{1}{2\pi}\int_{|y|\leq\tilde{r}_0|z|} \left(\ln\frac{1}{|y-z|}\right) \rho_k |y|^{2\alpha} h_0(\delta_k y) e^{\hat{u}_k(y)} dy + O(1) \\
&= -\hat{\mu}_k - \frac{1}{2\pi}\tilde{\rho}_{k,0}(z)\ln|z| + \frac{1}{2\pi}\int_{|y|\leq\tilde{r}_0|z|} \left(\ln\frac{|z|}{|y-z|}\right) \rho_k \\
&\quad |y|^{2\alpha} h_0(\delta_k y) e^{\hat{u}_k(y)} dy + O(1) \\
&= -\hat{\mu}_k - \frac{\rho_{k,0}}{2\pi}\ln|z| + O(1).
\end{aligned}$$

q.e.d.

To use the Pohozaev identity, we need to estimate the gradient of $\hat{u}_k$ in $B_{\frac{1}{\delta_k}}(0) \backslash B_{\ln\frac{1}{\delta_k}}(0)$.

Lemma 5. $\forall y \in B_{\frac{1}{\delta_k}}(0) \backslash B_{\ln\frac{1}{\delta_k}}(0)$, *and so we have*

$$\nabla\hat{u}_k(y) = -\frac{\rho_{k,0}}{2\pi}\frac{y}{|y|^2} + O\left(\frac{1}{|y|^2}\right). \tag{3.15}$$

Proof. By Green's formula, we have

$$\nabla \hat{u}_k(z) + \frac{\rho_{k,0}}{2\pi}\frac{z}{|z|^2} = \frac{1}{2\pi}\int_{|y|\le \frac{1}{\delta_k}} \left\{\frac{z}{|z|^2} - \frac{z-y}{|z-y|^2}\right\} |y|^{2\alpha} h_0(\delta_k y) e^{\hat{u}_k(y)} dy$$

$$= \frac{1}{2\pi}\int_{\left\{|y|\le \frac{1}{\delta_k}\right\}\cap\left\{|y-z|\ge \frac{|z|}{2}\right\}} \left\{\frac{z}{|z|^2} - \frac{z-y}{|z-y|^2}\right\} |y|^{2\alpha} h_0(\delta_k y) e^{\hat{u}_k(y)} dy$$

$$+ \frac{1}{2\pi}\int_{\left\{|y-z|\le \frac{|z|}{2}\right\}} \left\{\frac{z}{|z|^2} - \frac{z-y}{|z-y|^2}\right\} |y|^{2\alpha} h_0(\delta_k y) e^{\hat{u}_k(y)} dy$$

First, by mean the value theorem, for any $|z|>1$, we have

$$\left|\frac{z}{|z|^2} - \frac{z-y}{|z-y|^2}\right| \le 4\frac{|y|}{|z|^2}, \quad \forall y \in \left\{|y-z| \ge \frac{|z|}{2}\right\}.$$

On the other hand,

$$\left|\frac{z}{|z|^2} - \frac{z-y}{|z-y|^2}\right| \le \frac{2}{|z-y|}, \quad \forall y \in \left\{|y-z| \le \frac{|z|}{2}\right\}.$$

Hence,

(3.16)

$$\left|\nabla \hat{u}_k(z) + \frac{\rho_{k,0}}{2\pi}\frac{z}{|z|^2}\right| \le \frac{2}{\pi |z|^2}\int_{\left\{|y|\le \frac{1}{\delta_k}\right\}\cap\left\{|y-z|\ge \frac{|z|}{2}\right\}} |y|\,|y|^{2\alpha} h_0(\delta_k y) e^{\hat{u}_k(y)} dy$$
$$+ \frac{1}{\pi}\int_{\left\{|y-z|\le \frac{|z|}{2}\right\}} \frac{1}{|z-y|} |y|^{2\alpha} h_0(\delta_k y) e^{\hat{u}_k(y)} dy.$$

By (3.11), we have

$$\frac{2}{\pi |z|^2}\int_{\left\{|y|\le \frac{1}{\delta_k}\right\}\cap\left\{|y-z|\ge \frac{|z|}{2}\right\}} |y|\,|y|^{2\alpha} h_0(\delta_k y) e^{\hat{u}_k(y)} dy \le \frac{C}{|z|^2}. \tag{3.17}$$

To estimate the second integral, by Lemma 3, we may fix $R_\theta >> 1$ and $k_\theta \in \mathbb{N}$ sufficiently large such that

$$|y|^{2\alpha} h_0(\delta_k y) e^{\hat{u}_k(y)} \le |y|^{-\frac{7}{2}}, \text{ for } |y| \ge R_\theta \text{ and } k \ge k_\theta. \tag{3.18}$$

Since $y \in \left\{|y-z| \le \frac{|z|}{2}\right\}$, this implies that $\frac{|z|}{2} \le |y| \le \frac{3|z|}{2}$. Thus,

(3.19)
$$\int_{\left\{|y-z|\le \frac{|z|}{2}\right\}} \frac{1}{|z-y|} |y|^{2\alpha} h_0(\delta_k y) e^{\hat{u}_k(y)} dy$$
$$\le \int_{\left\{|y-z|\le \frac{|z|}{2}\right\}} \frac{1}{|z-y|} |y|^{-\frac{7}{2}} dy$$
$$\le \frac{C}{|z|^{\frac{7}{2}}}\int_{\left\{|y-z|\le \frac{|z|}{2}\right\}} \frac{1}{|z-y|} dy \le \frac{C'}{|z|^{\frac{5}{2}}}.$$

Combining (3.16), (3.17), and (3.19), we obtain

$$\left|\nabla \hat{u}_k(z) + \frac{\rho_{k,0}}{2\pi}\frac{z}{|z|^2}\right| = O\left(\frac{1}{|z|^2}\right).$$

q.e.d.

From (3.12) and (3.15), for $\delta_k \ln \frac{1}{\delta_k} \le |x| \le 1$, we have

$$u_k(x) = -\lambda_k + \left(\frac{\rho_{k,0}}{2\pi} - 4(1+\alpha)\right)\ln \delta_k - \frac{\rho_{k,0}}{2\pi}\ln|x| + O(1) \tag{3.20}$$

and

$$\nabla u_k(x) = \frac{-\rho_{k,0}}{2\pi}\frac{x}{|x|^2} + O\left(\frac{\delta_k}{|x|^2}\right). \tag{3.21}$$

(3.20) and (3.21) will be used in the boundary terms of the following Pohozaev identity.

Lemma 6.

$$|\rho_{k,0} - 8\pi(1+\alpha)| = O\left(\left(\ln\frac{1}{\delta_k}\right)^{-1}\right).$$

Proof. Apply Pohozaev identity in the region $B_k = B_{\delta_k \ln \frac{1}{\delta_k}}(0)$, to obtain

(3.22)
$$\int_{B_k}\left(2\rho_k|x|^{2\alpha}h_0(x) + \rho_k x\cdot\nabla\left(|x|^{2\alpha}h_0(x)\right)\right)e^{u_k(x)}dx$$
$$= \int_{\partial B_k} r\left[\left(\frac{\partial u_k}{\partial \nu}\right)^2 - \frac{1}{2}|\nabla u_k|^2\right]d\sigma + \int_{\partial B_k} r\rho_k|x|^{2\alpha}h_0(x)e^{u_k}d\sigma.$$

Then, inserting (3.20) and (3.21) into both sides of (3.22), we have

(3.23)
$$\int_{B_k}\left(2\rho_k|x|^{2\alpha}h_0(x) + \rho_k x\cdot\nabla\left(|x|^{2\alpha}h_0(x)\right)\right)e^{u_k(x)}dx$$
$$= \int_{B_k} 2(1+\alpha)\rho_k|x|^{2\alpha}h_0(x)e^{u_k(x)}dx + \int_{B_k}\rho_k(x\cdot\nabla h_0(x))|x|^{2\alpha}e^{u_k(x)}dx$$
$$= 2(1+\alpha)\rho_{k,0} + O(1)\int_{B_1\setminus B_k}|x|^{2\alpha}e^{u_k(x)}dx + \int_{B_k}\rho_k(x\cdot\nabla h_0(x))|x|^{2\alpha}e^{u_k(x)}dx.$$

For the last integral, by the scaling $x = \delta_k y$, we have

$$
\begin{aligned}
&\int_{B_k} \rho_k \left(x \cdot \nabla_x h_0\left(x\right)\right) \left|x\right|^{2\alpha} e^{u_k(x)} dx \\
&= \int_{B_k} \rho_k O\left(\left|x\right|\right) \left|x\right|^{2\alpha} e^{u_k(x)} dx \\
&= \int_{|y| \le \ln \frac{1}{\delta_k}} \rho_k \delta_k O\left(\left|y\right|\right) \left|y\right|^{2\alpha} e^{\hat{u}_k(y)} dy \\
&= \sum_{\ell=1}^{1+\alpha} \int_{B_{r_0}(e_\ell)} \rho_k \delta_k O\left(\left|y\right|\right) \left|y\right|^{2\alpha} e^{\hat{u}_k(y)} dy \\
&+ \int_{B_{R_\theta}(0) \setminus \cup_{\ell=1}^{1+\alpha} B_{r_0}(e_\ell)} \rho_k \delta_k O\left(\left|y\right|\right) \left|y\right|^{2\alpha} e^{\hat{u}_k(y)} dy \\
&+ \int_{R_\theta \le |y| \le \ln \frac{1}{\delta_k}} \rho_k \delta_k O\left(\left|y\right|\right) \left|y\right|^{2\alpha} e^{\hat{u}_k(y)} dy.
\end{aligned}
$$

Since $\left|\hat{u}_k\left(y\right) - \ln \frac{e^{\hat{\mu}_k}}{\left(1 + e^{\hat{\mu}_k} \left|y - e_\ell\right|^2\right)}\right| = O\left(1\right)$ for $\left|y - e_\ell\right| \le r_0$, we have

$$
\int_{B_{r_0}(e_\ell)} \rho_k \delta_k O\left(\left|y\right|\right) \left|y\right|^{2\alpha} e^{\hat{u}_k(y)} dy = \delta_k O\left(1\right) \int_{B_{r_0}(e_\ell)} e^{\hat{u}_k(y)} dy = O\left(\delta_k\right). \tag{3.24}
$$

Again, since $\hat{u}_k\left(y\right) = -\hat{\mu}_k + O\left(1\right)$ in $B_{R_\theta}\left(0\right) \setminus \cup_{\ell=1}^{1+\alpha} B_{r_0}\left(e_\ell\right)$, we have

$$
\int_{B_{R_\theta}(0) \setminus \cup_{\ell=1}^{1+\alpha} B_{r_0}(e_\ell)} \rho_k \delta_k O\left(\left|y\right|\right) \left|y\right|^{2\alpha} e^{\hat{u}_k(y)} dy = \delta_k O\left(e^{-\hat{\mu}_k}\right). \tag{3.25}
$$

Next, by Lemma 3, we have

$$
\begin{aligned}
&\int_{R_\theta \le |y| \le \ln \frac{1}{\delta_k}} \rho_k \delta_k O\left(\left|y\right|\right) \left|y\right|^{2\alpha} e^{\hat{u}_k(y)} dy \\
&\le \rho_k \delta_k \int_{R_\theta \le |y| \le \ln \frac{1}{\delta_k}} O\left(\left|y\right|\right) \left|y\right|^{2\alpha} e^{-\hat{\mu}_k - \left(\frac{\rho_{k,0}}{2\pi} - 2\theta\right) \ln|y| + O(1)} dy \\
&= \delta_k O\left(e^{-\hat{\mu}_k}\right).
\end{aligned} \tag{3.26}
$$

Then by (3.24), (3.25), (3.26), and (3.20), we have

$$\begin{aligned}
&\int_{B_k}\left(2\rho_k|x|^{2\alpha}h_0(x)+\rho_k x\cdot\nabla\left(|x|^{2\alpha}h_0(x)\right)\right)e^{u_k(x)}dx\\
&=2(1+\alpha)\rho_{k,0}+O(1)\int_{B_1\setminus B_k}|x|^{2\alpha}e^{u_k(x)}dx+O(\delta_k)\\
&=2(1+\alpha)\rho_{k,0}+O(1)\int_{B_1\setminus B_k}|x|^{2\alpha}e^{-\hat{\mu}_k+\left(\frac{\rho_{k,0}}{2\pi}-2(1+\alpha)\right)\ln\delta_k}\\
&-\frac{\rho_{k,0}}{2\pi}\ln|x|+O(1)\,dx+O(\delta_k)\\
&=2(1+\alpha)\rho_{k,0}+\left(\ln\frac{1}{\delta_k}\right)^{-\frac{\rho_{k,0}}{2\pi}+2(1+\alpha)}e^{-\hat{\mu}_k}+O(\delta_k)\ \text{ as } k\to\infty.
\end{aligned}$$

For the boundary term, by (3.20) and (3.21), we have

$$\begin{aligned}
&\int_{\partial B_k}r\left[\left(\frac{\partial u_k}{\partial\nu}\right)^2-\frac{1}{2}|\nabla u_k|^2\right]d\sigma+\int_{\partial B_k}r\rho_k|x|^{2\alpha}h_0(x)e^{u_k}d\sigma\\
&=\frac{\rho_{k,0}^2}{4\pi}+O\left(\left(\ln\frac{1}{\delta_k}\right)^{-1}\right)\ \text{ as } k\to\infty.
\end{aligned}\tag{3.27}$$

From (3.23) and (3.27), we have

$$|\rho_{k,0}-8\pi(1+\alpha)|=O\left(\left(\ln\frac{1}{\delta_k}\right)^{-1}\right).$$

q.e.d.

Proof of Theorem 8:
First, we claim that

$$\hat{u}_k(y)=-\hat{\mu}_k-4(1+\alpha)\ln|y|+O(1)\ \text{ for } R\le|y|\le\frac{1}{\delta_k},$$

where R is chosen such that (3.18) holds true. By (3.12) and Lemma 6, we have

$$\hat{u}_k(y)=-\hat{\mu}_k-4(1+\alpha)\ln|y|+O(1)\ \text{ for }\ |y|\ge\ln\frac{1}{\delta_k}.\tag{3.28}$$

Thus, we need to prove (3.28) for $R\le|y|\le\ln\frac{1}{\delta_k}$. By considering

$$f_\pm(y)=\hat{u}_k(y)+\hat{\mu}_k+4(1+\alpha)\ln|y|\mp\left(4c_1-c_1|y|^{-\frac{1}{2}}\right)$$

on $B_{\ln\frac{1}{\delta_k}}(0)\setminus B_R(0)$, we have

$$\begin{aligned}\Delta f_+(y) &= \Delta\hat{u}_k(y) + \frac{1}{4}c_1|y-e_\ell|^{-\frac{5}{2}}\\ &= -\rho_k|y|^{2\alpha}h_0(\delta_k y)e^{\hat{u}_k(y)} + \frac{1}{4}c_1|y-e_\ell|^{-\frac{5}{2}}\\ &\geq -|y|^{-\frac{7}{2}} + \frac{1}{4}c_1\sum_{\ell=1}^{1+\alpha}|y-e_\ell|^{-\frac{5}{2}}.\end{aligned}$$

Thus, by choosing a suitable constant c_1, we have

$$\Delta f_+(y) > 0$$

and

$$f_+(y) < 0 \text{ on } \partial\left(B_{\ln\frac{1}{\delta_k}}(0)\setminus B_R(0)\right).$$

Hence, by the maximum principle, we have

$$\hat{u}_k(y) \leq -\hat{\mu}_k - 4(1+\alpha)\ln|y| + O(1),$$

and, similarly, we also have

$$\hat{u}_k(y) \geq -\mu_k - 4(1+\alpha)\ln|y| + O(1).$$

Thus, we obtain

$$\hat{u}_k(y) = -\hat{\mu}_k - 4(1+\alpha)\ln|y| + O(1) \text{ in } B_{\ln\frac{1}{\delta_k}}(0)\setminus B_R(0). \tag{3.29}$$

By (3.29) and (3.28), for $|y|\geq R$, we have

$$\begin{aligned}\hat{u}_k(y) &= -\hat{\mu}_k - 4(1+\alpha)\ln|y| + O(1)\\ &= -\hat{\mu}_k - \sum_{\ell=1}^{1+\alpha}4\ln|y-e_\ell| + O(1).\end{aligned} \tag{3.30}$$

Again, by considering

$$\hat{f}_\pm(y) = \hat{u}_k(y) + \hat{\mu}_k + \sum_{\ell=1}^{1+\alpha}4\ln|y-e_\ell| \mp \left(4\hat{c}_1 - \hat{c}_1\sum_{\ell=1}^{1+\alpha}|y-e_\ell|^{-\frac{1}{2}}\right)$$

on $B_R(0)\setminus\cup_{\ell=1}^{1+\alpha}B_{r_0}(e_\ell)$, we have

$$\begin{aligned}\Delta\hat{f}_\pm(y) &= \Delta\hat{u}_k(y) \pm \frac{1}{4}\hat{c}_1\sum_{\ell=1}^{1+\alpha}|y-e_\ell|^{-\frac{5}{2}}\\ &= -\rho_k|y|^{2\alpha}h_0(\delta_k y)e^{\hat{u}_k(y)} \pm \frac{1}{4}\hat{c}_1\sum_{\ell=1}^{1+\alpha}|y-e_\ell|^{-\frac{5}{2}}\\ &= -\rho_k|y|^{2\alpha}h_0(\delta_k y)e^{-\mu_k+O_R(1)} \pm \frac{1}{4}\hat{c}_1\sum_{\ell=1}^{1+\alpha}|y-e_\ell|^{-\frac{5}{2}},\end{aligned}$$

where we have used the property that $\hat{u}_k = -\hat{\mu}_k + O(1)$ inside $B_R(0) \setminus \cup_{\ell=1}^{1+\alpha} B_{r_0}(e_\ell)$. Thus, by choosing a suitable constant $\hat{c}_1$, we have

$$\Delta \hat{f}_+(y) > 0$$

and

$$\hat{f}_+(y) < 0 \text{ on } \partial\left(B_R(0) \setminus \cup_{\ell=1}^{1+\alpha} B_{r_0}(e_\ell)\right).$$

Hence, by the maximum principle, we have

$$\hat{u}_k(y) \le -\hat{\mu}_k - \sum_{\ell=1}^{1+\alpha} 4\ln|y-e_\ell| + O(1) \text{ for } y \in B_R(0) \setminus \cup_{\ell=1}^{1+\alpha} B_{r_0}(e_\ell),$$

and, similarly, we also have

$$\hat{u}_k(y) \ge -\mu_k - \sum_{\ell=1}^{1+\alpha} 4\ln|y-e_\ell| + O(1) \text{ for } y \in B_R(0) \setminus \cup_{\ell=1}^{1+\alpha} B_{r_0}(e_\ell).$$

This implies that
(3.31)

$$\hat{u}_k(y) = -\hat{\mu}_k - \sum_{\ell=1}^{1+\alpha} 4\ln|y-e_\ell| + O(1) \text{ for } y \in B_R(0) \setminus \cup_{\ell=1}^{1+\alpha} B_{r_0}(e_\ell).$$

Combining (3.30) and (3.31), we obtain

$$\hat{u}_k(y) = -\hat{\mu}_k - \sum_{\ell=1}^{1+\alpha} 4\ln|y-e_\ell| + O(1) \text{ for } y \in B_{\frac{1}{\delta_k}}(0) \setminus \cup_{\ell=1}^{1+\alpha} B_{r_0}(e_\ell).$$

This completes the proof. q.e.d.

4. Sharp Estimates: Non-simple blowup

In this section, we are going to prove Theorem 5. Recall that u_k is a sequence of bubbling solutions of

$$\Delta u_k + \rho_k (h^*(x) e^{u_k} - 1) = 0 \text{ in } M \tag{4.1}$$

that blows up at $\{p_1, \dots, p_m\}$. Let $\lambda_k = \max_j \lambda_{k,j}$, $\alpha = \max_j \alpha(p_j)$. Then by Theorem 3(i), we have

$$\lambda_{k,j} = \lambda_k + O(1) \ \ \forall j = 1, \dots, m. \tag{4.2}$$

Define

$$\varepsilon_k = e^{-\frac{\lambda_k}{2(1+\alpha)}} \text{ and } \varepsilon_{k,j} = e^{-\frac{\lambda_{k,j}}{2\left(1+\alpha\left(p_j\right)\right)}}.$$

Then

$$\varepsilon_k = \max_j \varepsilon_{k,j}.$$

Set $\omega_k(x)$ to be the error term outside the blowup points defined by

$$\omega_k(x) = u_k(x) - \bar{u}_k - \sum_{j=1}^{m} \rho_{k,j} G(x, p_j) \text{ on } M \setminus \cup_{j=1}^{m} B_{\frac{r_0}{2}}(p_j), \tag{4.3}$$

where $\bar{u}_k$ is the average of u_k, that is,

$$\bar{u}_k = \frac{1}{|M|}\int_M u_k(x)\,dx \tag{4.4}$$

and $\rho_{k,j}$ is the local mass defined in (1.13).

First, we have following error estimate of ω_k.

Lemma 7. *$|\omega_k(x)| + |\nabla\omega_k(x)| = O(\varepsilon_k)$ on $M\backslash \cup_{j=1}^m B_{\frac{r_0}{2}}(p_j)$.*

Proof. By Green's formula and (4.2), we have

$$\begin{aligned} u_k(x) - \bar{u}_k &= \int_M \rho_k\left(h^*(z)\,e^{u_k(z)} - 1\right) G(x,z)\,dz \\ &= \sum_{i=1}^m \rho_{k,i} G(x,p_i) + \sum_{i=1}^m \int_{B_{\frac{r_0}{2}}(p_i)} \rho_k h^*(z)\,e^{u_k(z)}\big(G(x,z) \\ &\quad - G(x,p_i)\big)dz + O\left(e^{-\lambda_k}\right). \end{aligned} \tag{4.5}$$

where $G(x,y)$ is defined in (1.8). Thus, we need to estimate

$$\int_{B_{\frac{r_0}{2}}(p_i)} \rho_k h^*(z)\,e^{u_k(z)}\left(G(x,z) - G(x,p_i)\right)dz.$$

First, for those p_i with $\alpha(p_i) = 0$, this estimate has been done in [**9**]. So we have

$$\int_{B_{\frac{r_0}{2}}(p_i)} \rho_k h^*(z)\,e^{u_k(z)}\left(G(x,z) - G(x,p_i)\right)dz = O\left(\lambda_{k,i}e^{-\lambda_{k,i}}\right) = o(\varepsilon_k). \tag{4.6}$$

For $p_i = q_i$ with $\alpha(p_i) = \alpha_i$, we have two cases we need to disscuss: (i) For Case 1 (simple blowup), by Theorem 7, we have

$$\begin{aligned} &\int_{B_{\frac{r_0}{2}}(p_i)} \rho_k h^*(z)\,e^{u_k(z)}\left(G(x,z) - G(x,q_i)\right)dz \\ &= \int_{B_{\frac{r_0}{2}}(p_i)} |z-q_i|^{2\alpha_i}\,e^{U_{k,i}(z)} O(|z-q_i|)\,dz = O\left(e^{-\frac{\lambda_{k,i}}{2(1+\alpha_i)}}\right) = O(\varepsilon_k), \end{aligned} \tag{4.7}$$

where

$$U_{k,i}(z) = \lambda_{k,i} - 2\ln\left(1 + \frac{\rho_k h_i(q_i)}{8(1+\alpha_i)^2}e^{\lambda_{k,i}}\left|(z-q_i)^{1+\alpha_i} - (p_{k,i}-q_i)^{1+\alpha_i}\right|^2\right).$$

(ii) For Case 2 (non-simple blowup), by scaling $z = \delta_{k,i} y + q_i$, we have

$$\begin{aligned}
&\int_{B_{\frac{r_0}{2}}(p_i)} \rho_k h^* (z) e^{u_k(z)} (G(x,z) - G(x,q_i))\, dz \\
&= \int_{B_{\frac{r_0}{2\delta_{k,i}}}(0)} \rho_k |y|^{2\alpha_i} h_i (y) e^{\hat{u}_k(y)} (G(x, \delta_{k,i} y + q_i) - G(x,q_i))\, dy.
\end{aligned}$$

Since $x \in M \setminus \cup_{j=1}^m B_{\frac{r_0}{2}}(p_j)$, we have

$$G(x, \delta_{k,i} y + q_i) - G(x,q_i) = O(\delta_{k,i} |y|).$$

Thus,

$$\begin{aligned}
&\int_{B_{\frac{r_0}{2\delta_{k,i}}}(0)} \rho_k |y|^{2\alpha_i} e^{\hat{u}_k(y)} (G(x, \delta_{k,i} y + q_i) - G(x,q_i))\, dy \\
&= \int_{B_{\frac{r_0}{2\delta_{k,i}}}(0)} \rho_k |y|^{2\alpha_i} e^{\hat{u}_k(y)} O(\delta_{k,i} |y|)\, dy \\
&= \sum_{\ell=1}^{1+\alpha_i} \int_{B_{r_1}(e_\ell)} \rho_k |y|^{2\alpha_i} e^{\hat{u}_k(y)} O(\delta_{k,i} |y|)\, dy \\
&+ \int_{B_{\frac{r_0}{2\delta_{k,i}}}(0) \setminus \cup_{\ell=1}^{1+\alpha_i} B_{r_1}(e_\ell)} \rho_k |y|^{2\alpha_i} e^{\hat{u}_k(y)} O(\delta_{k,i} |y|)\, dy.
\end{aligned}$$

Since $\hat{u}_k(y)$ is a simple blowup inside each $B_{r_1}(e_\ell)$, we have

$$\left| \hat{u}_k(y) - \hat{U}_{k,\ell}(y) \right| \le C \text{ for } y \in B_{r_1}(e_\ell),\ 1 \le \ell \le 1 + \alpha_i,$$

where

$$\hat{U}_{k,\ell}(y) = \hat{\mu}_k - 2 \log \left(1 + \frac{\rho_k h^* (q_i + \delta_{k,i} e_\ell)}{8} e^{\hat{\mu}_k} |y - e_\ell|^2 \right).$$

Thus, we have

$$\begin{aligned}
(4.8) \qquad &\int_{B_{r_1}(e_\ell)} \rho_k |y|^{2\alpha_i} e^{\hat{u}_k(y)} O(\delta_{k,i} |y|)\, dy \\
&= \int_{B_{r_1}(e_\ell)} \rho_k |y|^{2\alpha_i} e^{\hat{U}_{k,\ell}(y)} O(\delta_{k,i} |y|)\, dy = O\left(\delta_{k,i} e^{\frac{-\hat{\mu}_{k,i}}{2}} \right).
\end{aligned}$$

By Theorem 8, we have

(4.9)

$$\int_{B_{\frac{r_0}{2\delta_{k,i}}}(0)\setminus\cup_{\ell=1}^{1+\alpha_i}B_{r_1}(e_\ell)} \rho_k |y|^{2\alpha_i} e^{\hat{u}_k(y)} O(\delta_{k,i}|y|)\, dy$$

$$= \int_{B_{\frac{r_0}{2\delta_{k,i}}}(0)\setminus\cup_{\ell=1}^{1+\alpha_i}B_{r_1}(e_\ell)} \rho_k |y|^{2\alpha_i} e^{-\hat{\mu}_{k,i}-\sum_{\ell=1}^{1+\alpha_i} 4\ln|y-e_\ell|+O(1)} O(\delta_{k,i}|y|)\, dy$$

$$= O\left(\delta_{k,i} e^{-\hat{\mu}_{k,i}}\right)$$

Hence, by (4.8) and (4.9), we have

$$\int_{B_{\frac{r_0}{2\delta_{k,i}}}(0)} \rho_k |y|^{2\alpha_i} e^{\hat{u}_k(y)} \left(G(x,\delta_{k,i}y+p_i) - G(x,p_i)\right) dy = O\left(\delta_{k,i} e^{\frac{-\hat{\mu}_k}{2}}\right).$$

Since

$$\hat{\mu}_k = \lambda_{k,i} + 2(1+\alpha_i)\ln \delta_{k,i},$$

we have

$$\delta_{k,i} e^{-\frac{\hat{\mu}_k}{2}} = \varepsilon_{k,i}\left(\frac{\varepsilon_{k,i}}{\delta_{k,i}}\right)^{\alpha_i} = o(\varepsilon_k).$$

Therefore, we obatin

$$\int_{B_{\frac{r_0}{2}}(p_i)} \rho_k h^*(z) e^{u_k(z)} \left(G(x,z) - G(x,p_i)\right) dz = o(\varepsilon_k). \tag{4.10}$$

Hence, from (4.5), (4.6), (4.7), and (4.10), we conclude that

$$|\omega_k(x)| = O(\varepsilon_k),$$

and similar for $|\nabla\omega_k(x)|$. q.e.d.

Next, we want to compute the difference of $\rho_{k,i} - 8\pi(1+\alpha_i)$ under the assumption

$$\lim_{k\to+\infty} \frac{|p_k - q|}{\varepsilon_{k,q}} = +\infty.$$

Since this is a local estimate, we may assume $p = q = 0$ for simplicity and adapt the flat metric near the blowup point 0. We also use $\rho_{k,0}$, δ_k,...etc. to denote $\rho_{k,i}$ and $\delta_{k,i}, \ldots$ etc. To compute the difference of $\rho_{k,0} - 8\pi(1+\alpha)$, we will apply the method in [**9**]. Now, we localize the problem as follows:

Define

$$G_k^*(x) = \rho_{k,0}\gamma(x,0) + \sum_{p_j\neq 0} \rho_{k,j} G(x,p_j), \tag{4.11}$$

where $G(x,y)$ is defined by (1.8). Let

$$\tilde{u}_k(x) = u_k(x) - \left(G_k^*(x) - G_k^*(0)\right) \text{ in } B_1(0).$$

In order to eliminate the boundary oscillation, we introduce a harmonic function ϕ_k satisfying

$$\begin{cases} \Delta\phi_k = 0 \text{ in } B_1(0) \\ \phi_k|_{\partial B_1(0)} = \tilde{u}_k - m_k \end{cases},$$

where

$$m_k = \frac{1}{2\pi}\int_{\partial B_1(0)} \tilde{u}_k.$$

By mean value property, we have $\phi_k(0) = 0$. Let

$$\tilde{\tilde{u}}_k = \tilde{u}_k - \phi_k \text{ in } B_1(0).$$

Then

$$\begin{cases} \Delta\tilde{\tilde{u}}_k + \rho_k |x|^{2\alpha} \hat{h}_0(x) e^{\tilde{\tilde{u}}_k} = \left(\rho_k - \sum_{j=1}^m \rho_{k,j}\right) = O\left(e^{-\lambda_k}\right) \text{ in } B_1(0) \\ \tilde{\tilde{u}}_k = \frac{1}{2\pi}\int_{\partial B_1(0)} \tilde{u}_k \text{ on } \partial B_1(0), \end{cases}$$

where

$$\hat{h}_0(x) = h_0(x) e^{\phi_k(x) + G_k^*(x) - G_k^*(0)} \text{ and } \hat{h}_0(0) = h_0(0). \tag{4.12}$$

Notice that

$$\phi_k(x) = \omega_k(x) - \frac{1}{2\pi}\int_{\partial B_1} \omega_k(x)\, d\sigma \text{ for } |x| = 1,$$

where $\omega_k(x)$ is given by (4.3). By Lemma 7 and the maximum principle, we have that

$$|\phi_k(x)| = |\nabla\phi_k(x)| = O(\varepsilon_k) \text{ for } |x| \le 1.$$

Let

$$\hat{u}_k(y) = \tilde{\tilde{u}}_k(\delta_k y) + 2(1+\alpha)\ln\delta_k \text{ in } B_{\frac{1}{\delta_k}}(0).$$

Then $\hat{u}_k(y)$ satisfies

$$\begin{cases} \Delta\hat{u}_k(y) + \rho_k |y|^{2\alpha} \hat{h}_0(\delta_k y) e^{\hat{u}_k(y)} = O\left(e^{-\lambda_k}\right) \text{ in } B_{\frac{1}{\delta_k}}(0) \\ \hat{u}_k(y) = m_k + 2(1+\alpha)\ln\delta_k \text{ on } \partial B_{\frac{1}{\delta_k}}(0) \end{cases}.$$

As we have discussed in Section 2, $\hat{u}_k(y)$ blows up simply at $\{e_1, e_2, \ldots, e_{1+\alpha}\}$ with $e_{\ell+1} = \exp\left(i\frac{2\pi\ell}{1+\alpha}\right)$, $0 \le \ell \le \alpha$. Let $r_0 > 0$. Define

$$\hat{\mu}_{ki} = \max_{B_{r_0}(e_i)} \hat{u}_k(y) = \hat{u}_k(e_{ki})$$

and

$$\rho_{k,0}^i = \int_{B_{r_0}(e_{ki})} \rho_k |y|^{2\alpha} \hat{h}_0(\delta_k y) e^{\hat{u}_k(y)} dy \to 8\pi,\ 1 \le i \le 1+\alpha.$$

Now, we define the error term $\hat{\omega}_k(y)$ outside the bubbling region by

$$\hat{\omega}_k(y) = \hat{u}_k(y) - (m_k + 2(1+\alpha)\ln\delta_k) - \sum_{i=1}^{1+\alpha} \rho_{k,0}^i G_k(e_{ki}, y) \tag{4.13}$$

for $y \in B_{\frac{1}{\delta_k}}(0) \setminus \cup_{i=1}^{1+\alpha} B_{r_0}(e_{ki})$, where $G_k(x,y)$ is defined by (3.3).

Let

$$\hat{U}_{ki}(y) = \hat{\mu}_k - 2\ln\left(1 + c_k e^{\hat{\mu}_k} |y - e_{ki}|^2\right), \ c_k = \frac{\rho_k \hat{h}_0(\delta_k e_{ki})}{8}, \tag{4.14}$$

and

$$\hat{H}_{ki}(y) = \rho_{k,0}^i \gamma_k(y, e_{ki}) + \sum_{j \neq i}^{1+\alpha} \rho_{k,0}^j G_k(y, e_{kj}),$$

where $\gamma_k(x,y)$ is the regular part of $G_k(x,y)$. We also define the error term $\hat{\eta}_{ki}(y)$ inside the bubbling region by

$$\hat{\eta}_{ki}(y) = \hat{u}_k(y) - \hat{U}_{ki}(y) - \left(\hat{H}_{ki}(y) - \hat{H}_{ki}(e_{ki})\right) \text{ in } B_{r_0}(e_{ki}).$$

Then $\hat{\eta}_{ki}$ satisfies

$$\Delta\hat{\eta}_{ki}(y) + \rho_k \hat{h}_0(\delta_k e_{ki}) e^{\hat{U}_{ki}(y)} \hat{D}_k(y, \hat{\eta}_{ki}(y)) = 0 \text{ in } B_{r_0}(e_{ki}), \tag{4.15}$$

where

$$\hat{D}_k(y, \hat{\eta}_{ki}(y)) = e^{\hat{\eta}_{ki}(y) + \hat{Q}_{ki}(y) - \hat{Q}_{ki}(e_{ki})} - 1,$$

$$\hat{Q}_{ki}(y) = 2\alpha \ln|y| + \ln \hat{h}_0(\delta_k y) + \hat{H}_{ki}(y). \tag{4.16}$$

By (4.16), (4.12), and (4.11), we have

$$\begin{aligned} \Delta\hat{Q}_{ki}(e_{ki}) &= \Delta \ln \hat{h}_0(\delta_k e_{ki}) = \delta_k^2 \left[\Delta \ln h_0(\delta_k e_{ki}) + \sum_{j=1}^m \rho_{k,j}\right] \\ &= \delta_k^2 [\Delta \ln h_0(0) + \rho_k] + o\left(\delta_k^2\right). \end{aligned} \tag{4.17}$$

Moreover, since $\hat{u}_k$ is simply bubbling at each e_ℓ, we have

$$|\hat{\mu}_{ki} - \hat{\mu}_{kj}| = O(1).$$

Let

$$\hat{\mu}_k = \max_{1 \le i \le 1+\alpha} \hat{\mu}_{ki}.$$

Then we have the following estimate. See the proof in [**9**].

Lemma 8 ([**9**])**.**

$$\left|\nabla \hat{Q}_{ki}(e_{ki})\right| = O\left(e^{-\hat{\mu}_k}\right), \ \textit{for } 1 \le i \le 1+\alpha, \tag{4.18}$$

$$|\hat{\omega}_k(y)| = O\left(e^{-\hat{\mu}_k}\right) \ \forall y \in B_{\frac{1}{\delta_k}}(0) \setminus \cup_{i=1}^{1+\alpha} B_{r_0}(e_{ki}), \tag{4.19}$$

(4.20)

$$-m_k+2(1+\alpha)\ln\delta_k=\hat{\mu}_k+2\ln\frac{\rho_k\hat{h}_0(\delta_k e_{ki})}{8}-\sum_{j\neq i}^{1+\alpha}4\ln|e_{ki}-e_{kj}|+O\left(\hat{\mu}_k e^{-\hat{\mu}_k}\right),$$

$$\rho^i_{k,0}-8\pi=\frac{16\pi}{\rho_k\hat{h}_0(\delta_k e_{ki})}\Delta\hat{Q}_{ki}(e_{ki})\hat{\mu}_k e^{-\hat{\mu}_k}+O\left(e^{-\hat{\mu}_k}\right), \tag{4.21}$$

$$\begin{aligned}\hat{\eta}_{ki}(y)=&-\frac{8}{\rho_{k\hat{h}_0(\delta_k e_{ki})}}\Delta\hat{Q}_{ki}(e_{ki})e^{-\hat{\mu}_k}\left[\ln\left(e^{\frac{\hat{\mu}_k}{2}}|y-e_{ki}|+2\right)\right]^2\\&+O\left(\ln\left(e^{\frac{\hat{\mu}_k}{2}}|y-e_{ki}|+2\right)\right)e^{-\hat{\mu}_k}\ \ in\ B_{r_0}(e_{ki}).\end{aligned} \tag{4.22}$$

Next, we want to compare the order of δ_k and $e^{-\hat{\mu}_k}$. In fact, we could have the following estimate.

Lemma 9.

$$\delta_k^2=C\hat{\mu}_k e^{-\hat{\mu}_k}+O\left(e^{-\hat{\mu}_k}\right)$$

for some constant $C>0$.

Proof. Without loss of generality, we may assume $e_1=1$. Let $u^*_k(y)=\hat{u}_k(y+1)$. Then

$$\Delta u^*_k(y)+\rho_k|y+1|^{2\alpha}\hat{h}_0(\delta_k(y+1))e^{u^*_k(y)}=0\text{ in }B_{r_0}(0).$$

Let ξ be a constant unit vector and apply the Pohozaev identity; we then obtain

$$\begin{aligned}&\int_{\partial B_{r_0}(0)}(\nu\cdot\nabla u^*_k)(\xi\cdot\nabla u^*_k)-\frac{1}{2}(\nu\cdot\xi)|\nabla u^*_k|^2d\sigma\\&+\int_{\partial B_{r_0}(0)}(\nu\cdot\xi)\rho_k|y+1|^{2\alpha}\hat{h}_0(\delta_k(y+1))e^{u^*_k}d\sigma\\&=\int_{B_{r_0}(0)}\rho_k\hat{h}_0(\delta_k(y+1))\left[\xi\cdot\nabla|y+1|^{2\alpha}\right]e^{u^*_k}dy\\&+\int_{B_{r_0}(0)}\rho_k|y+1|^{2\alpha}\left[\xi\cdot\nabla\hat{h}_0(\delta_k(y+1))\right]e^{u^*_k}dy.\end{aligned} \tag{4.23}$$

For simplicity, we use {l.o.t.} to denote those terms whose order is $O\left(\delta_k\hat{\mu}_k e^{-\hat{\mu}_k}\right)$ after integrating over $B_{r_0}(0)$. By Taylor expansion, we have

$$
\begin{aligned}
&\hat{h}_0\left(\delta_k\left(y+1\right)\right)\left[\xi\cdot\nabla\left|y+1\right|^{2\alpha}\right]\\
&=\hat{h}_0\left(\delta_k\left(y+1\right)\right)\left|y+1\right|^{2\alpha}\left[\xi\cdot\nabla\ln\left|y+1\right|^{2\alpha}\right]\\
&=2\alpha\xi_1\hat{h}_0\left(\delta_k\left(y+1\right)\right)\left|y+1\right|^{2\alpha}+\alpha\hat{h}_0\left(\delta_k\left(y+1\right)\right)\left|y+1\right|^{2\alpha}\\
&\quad\left[\xi\cdot\nabla\left(\ln\left|y+1\right|^{2}-2y_1\right)\right]\\
&=2\alpha\xi_1\hat{h}_0\left(\delta_k\left(y+1\right)\right)\left|y+1\right|^{2\alpha}+\alpha\hat{h}_0\left(\delta_k\left(y+1\right)\right)\left|y+1\right|^{2\alpha}\\
&\quad\left[2\xi_1\left(\frac{y_1+1}{\left|y+1\right|^{2}}-1\right)+2\xi_2\frac{y_2}{\left|y+1\right|^{2}}\right]\\
&=2\alpha\xi_1\hat{h}_0\left(\delta_k\left(y+1\right)\right)\left|y+1\right|^{2\alpha}-2\alpha\xi_1\hat{h}_0\left(\delta_k\left(y+1\right)\right)\left|y+1\right|^{2(\alpha-1)}\left|y\right|^{2}\\
&\quad+\left\{\text{l.o.t.}\right\}.
\end{aligned}
$$

Therefore, we have

$$
\begin{aligned}
&\int_{B_{r_0}(0)}\rho_k\hat{h}_0\left(\delta_k\left(y+1\right)\right)\left[\xi\cdot\nabla\left|y+1\right|^{2\alpha}\right]e^{u_k^*}dy\\
&=2\alpha\xi_1\int_{B_{r_0}(0)}\rho_k\left|y+1\right|^{2\alpha}\hat{h}_0\left(\delta_k\left(y+1\right)\right)e^{u_k^*}dy\\
&-2\alpha\xi_1\int_{B_{r_0}(0)}\rho_k\hat{h}_0\left(\delta_k\left(y+1\right)\right)\left|y+1\right|^{2(\alpha-1)}\left|y\right|^{2}e^{u_k^*}dy+O\left(\delta_k\hat{\mu}_ke^{-\hat{\mu}_k}\right)\\
&=2\alpha\rho_{k,0}^{1}\xi_1-2\alpha\xi_1\int_{B_{r_0}(0)}\rho_k\hat{h}_0\left(\delta_k\left(y+1\right)\right)\left|y+1\right|^{2(\alpha-1)}\left|y\right|^{2}e^{u_k^*}dy\\
&\quad+O\left(\delta_k\hat{\mu}_ke^{-\hat{\mu}_k}\right).
\end{aligned}
$$

Similarly, since

$$
\begin{aligned}
&\left|y+1\right|^{2\alpha}\left[\xi\cdot\nabla\hat{h}_0\left(\delta_k\left(y+1\right)\right)\right]\\
&=\hat{h}_0\left(\delta_k\left(y+1\right)\right)\left|y+1\right|^{2\alpha}\left[\xi\cdot\nabla\ln\hat{h}_0\left(\delta_k\left(y+1\right)\right)\right]\\
&=\hat{h}_0\left(\delta_k\left(y+1\right)\right)\left|y+1\right|^{2\alpha}\left[\xi_1\partial_1\ln\hat{h}_0\left(\delta_k\left(y+1\right)\right)+\xi_2\partial_2\ln\hat{h}_0\left(\delta_k\left(y+1\right)\right)\right]\\
&=\hat{h}_0\left(\delta_k\left(y+1\right)\right)\left|y+1\right|^{2\alpha}\xi_1\\
&\quad\left[\partial_1\ln\hat{h}_0\left(0\right)\delta_k+\partial_{11}\ln\hat{h}_0\left(0\right)\delta_k^2\left(y_1+1\right)+\partial_{12}\hat{h}_0\left(0\right)\delta_k^2y_2\right]\\
&+\hat{h}_0\left(\delta_k\left(y+1\right)\right)\left|y+1\right|^{2\alpha}\xi_2\\
&\quad\left[\partial_2\ln\hat{h}_0\left(0\right)\delta_k+\partial_{22}\ln\hat{h}_0\left(0\right)\delta_k^2y_2+\partial_{12}\ln\hat{h}_0\left(0\right)\delta_k^2\left(y_1+1\right)\right]+\left\{\text{l.o.t.}\right\}\\
&=\hat{h}_0\left(\delta_k\left(y+1\right)\right)\left|y+1\right|^{2\alpha}\left[\xi\cdot\nabla\ln\hat{h}_0\left(0\right)\right]\delta_k\\
&+\hat{h}_0\left(\delta_k\left(y+1\right)\right)\left|y+1\right|^{2\alpha}\left[\xi_1\partial_{11}\ln\hat{h}_0\left(0\right)+\xi_2\partial_{12}\ln\hat{h}_0\left(0\right)\right]\delta_k^2+\left\{\text{l.o.t.}\right\},
\end{aligned}
$$

we have

$$\begin{aligned}
&\int_{B_{r_0}(0)} \rho_k |y+1|^{2\alpha} \left[\xi\cdot\nabla\hat{h}_0(\delta_k(y+1))\right] e^{u_k^*}dy \\
&= \int_{B_{r_0}(0)} \rho_k\hat{h}_0(\delta_k(y+1))|y+1|^{2\alpha} e^{u_k^*}dy \left[\xi_1\partial_{11}\ln\hat{h}_0(0)+\xi_2\partial_{12}\ln\hat{h}_0(0)\right]\delta_k^2 \\
&+ \int_{B_{r_0}(0)} \rho_k\hat{h}_0(\delta_k(y+1))|y+1|^{2\alpha} e^{u_k^*}dy \left[\xi\cdot\nabla\ln\hat{h}_0(0)\right]\delta_k + O\left(\delta_k\hat{\mu}_k e^{-\hat{\mu}_k}\right) \\
&= \int_{B_{r_0}(0)} \rho_k\hat{h}_0(\delta_k(y+1))|y+1|^{2\alpha} e^{u_k^*}dy \left[\xi_1\partial_{11}\ln\hat{h}_0(0)+\xi_2\partial_{12}\ln\hat{h}_0(0)\right]\delta_k^2 \\
&+ \rho_{k,0}^1\xi\cdot\nabla\ln\hat{h}_0(0)\delta_k + O\left(\delta_k\hat{\mu}_k e^{-\hat{\mu}_k}\right).
\end{aligned}$$

Then we obtain

(4.24)

$$\begin{aligned}
&\int_{B_{r_0}(0)} \rho_k\hat{h}_0(\delta_k(y+1))\left[\xi\cdot\nabla|y+1|^{2\alpha}\right] e^{u_k^*}dy \\
&+ \int_{B_{r_0}(0)} \rho_k|y+1|^{2\alpha}\left[\xi\cdot\nabla_y\hat{h}_0(\delta_k(y+1))\right] e^{u_k^*}dy \\
&= -2\alpha\xi_1\int_{B_{r_0}(0)} \rho_k\hat{h}_0(\delta_k(y+1))|y+1|^{2(\alpha-1)}|y|^2 e^{u_k^*}dy \\
&+ \int_{B_{r_0}(0)} \rho_k\hat{h}_0(\delta_k(y+1))|y+1|^{2\alpha} e^{u_k^*}dy \left[\xi_1\partial_{11}\ln\hat{h}_0(0)+\xi_2\partial_{12}\ln\hat{h}_0(0)\right]\delta_k^2 \\
&+ 2\alpha\rho_{k,0}^1\xi_1 + \rho_{k,0}^1\xi\cdot\nabla\ln\hat{h}_0(0)\delta_k + O\left(\delta_k\hat{\mu}_k e^{-\hat{\mu}_k}\right).
\end{aligned}$$

Finally, we consider the boundary term in (4.23). By Theorem 8, we have

$$\int_{\partial B_{r_0}(0)} (\nu\cdot\xi)\rho_k|y+1|^{2\alpha}\hat{h}_0(\delta_k(y+1)) e^{u_k^*}d\sigma = O\left(e^{-\hat{\mu}_k}\right). \tag{4.25}$$

On the other hand, by (4.16), (4.18), and (4.19), we have

$$\begin{aligned}
&\int_{\partial B_{r_0}(0)} (\nu\cdot\nabla u_k^*)(\xi\cdot\nabla u_k^*) - \frac{1}{2}(\nu\cdot\xi)|\nabla u_k^*|^2 d\sigma \\
&= -\rho_{k,0}^1\xi\cdot\nabla\hat{H}_{k1}(e_1) + O(|\hat{\omega}_k|) \\
&= 2\alpha\rho_{k,0}^1\xi_1 + \rho_{k,0}^1\xi\cdot\nabla\ln\hat{h}_0(0)\delta_k + O\left(e^{-\hat{\mu}_k}\right).
\end{aligned} \tag{4.26}$$

Combining (4.24), (4.25), and (4.26), we have

$$\begin{aligned}
&-2\alpha\int_{B_{r_0}(0)} \rho_k\hat{h}_0(0)|y+1|^{2(\alpha-1)}|y|^2 e^{u_k^*}dy\xi_1 \\
&+ \int_{B_{r_0}(0)} \rho_k|y+1|^{2(\alpha-1)} e^{u_k^*}dy\left[\partial_{11}\hat{h}_0(0)\xi_1+\partial_{12}\hat{h}_0(0)\xi_2\right]\delta_k^2 \\
&= O\left(e^{-\hat{\mu}_k}\right) + O\left(\delta_k\hat{\mu}_k e^{-\hat{\mu}_k}\right).
\end{aligned}$$

This implies that

$$\delta_k^2 = C\hat{\mu}_k e^{-\hat{\mu}_k} + O\left(e^{-\hat{\mu}_k}\right) + O\left(\delta_k \hat{\mu}_k e^{-\hat{\mu}_k}\right),$$

and hence

$$\delta_k^2 = C\hat{\mu}_k e^{-\hat{\mu}_k} + O\left(e^{-\hat{\mu}_k}\right).$$

q.e.d.

To compute the difference $\rho_k - 8\pi(1+\alpha)$ accurately, we need to improve the estimate (2.14). Let $\Omega_k = B_{\frac{1}{\delta_k}}(0)$ and $\Omega_{ki} \subset \Omega_k$ such that $\Omega_k = \bigcup_{i=1}^{1+\alpha} \Omega_{ki}$, $\Omega_{ki} \cap \Omega_{kj} = \emptyset$ and $B_{r_0}(e_{ki}) \subset \Omega_{ki}$. Then we have the following estimate.

Corollary 3. *For $y \in \Omega_{ki} \backslash B_{r_0}(e_{ki})$, we have*

$$\hat{u}_k(y) = -\hat{\mu}_k - 2\ln \frac{\rho_k \hat{h}_0(\delta_k e_{ki})}{8} + 4\sum_{i \neq j}^{1+\alpha} \ln|e_{ki} - e_{kj}|$$
$$- 4\sum_{i=1}^{1+\alpha} \ln|y - e_{ki}| + O\left(\hat{\mu}_k e^{-\hat{\mu}_k}\right). \tag{4.27}$$

Proof. By (4.13) and (4.19), we have

$$\hat{u}_k(y) = m_k + 2(1+\alpha)\ln\delta_k + \sum_{i=1}^{1+\alpha} \rho_{k,0}^i G_k(e_{ki}, y) + O\left(e^{-\hat{\mu}_k}\right).$$

From (3.3), we have

$$G_k(e_{ki}, y) = -\frac{1}{2\pi}\ln|y - e_{ki}| + \frac{1}{2\pi}\ln\frac{1}{\delta_k} + \frac{1}{2\pi}\ln\left|\delta_k^2 |y| e_{ki} - \frac{y}{|y|}\right|$$
$$= -\frac{1}{2\pi}\ln|y - e_{ki}| + \frac{1}{2\pi}\ln\frac{1}{\delta_k} + O\left(\delta_k^2\right).$$

By (4.20), (4.21), and Lemma 9, we have

$$\hat{u}_k(y) = -\hat{\mu}_k - 2\ln \frac{\rho_k \hat{h}_0(\delta_k e_{ki})}{8} + 4\sum_{i \neq j}^{1+\alpha} \ln|e_{ki} - e_{kj}|$$
$$- 4\sum_{i=1}^{1+\alpha} \ln|y - e_{ki}| + O\left(\hat{\mu}_k e^{-\hat{\mu}_k}\right).$$

q.e.d.

Now, we are at the stage to prove Theorem 5.

Proof of Theorem 5: Let $\Omega_k' = B_{\frac{1}{\delta_k}}(0) \setminus \cup_{i=1}^{1+\alpha} B_{r_0}(e_{ki})$ and recall $\sigma_k = e^{-\frac{\hat{\mu}_k}{2}}$. Then

(4.28)

$$\begin{aligned}\rho_{k,0} &= \int_{\Omega_k} \rho_k |y|^{2\alpha} \hat{h}_0(\delta_k y) e^{\hat{u}_k(y)} dy \\ &= \sum_{i=1}^{1+\alpha} \int_{B_{r_0}(e_{ki})} \rho_k |y|^{2\alpha} \hat{h}_0(\delta_k y) e^{\hat{u}_k(y)} dy + \int_{\Omega_k'} \rho_k |y|^{2\alpha} \hat{h}_0(\delta_k y) e^{\hat{u}_k(y)} dy,\end{aligned}$$

and

$$\begin{aligned}&\sum_{i=1}^{1+\alpha} \int_{B_{r_0}(e_{ki})} \rho_k |y|^{2\alpha} \hat{h}_0(\delta_k y) e^{\hat{u}_k(y)} dy \\ &= \sum_{i=1}^{1+\alpha} \int_{B_{r_0}(e_{ki})} \rho_k \hat{h}_0(\delta_k e_{ki}) e^{\hat{U}_{ki}(y)} dy \\ &+ \sum_{i=1}^{1+\alpha} \int_{B_{r_0}(e_{ki})} \rho_k \hat{h}_0(\delta_k e_{ki}) e^{\hat{U}_{ki}(y)} \hat{D}_k(y, \hat{\eta}_{ki}(y)) dy \\ &= 8\pi(1+\alpha) - \sum_{i=1}^{1+\alpha} \int_{\mathbb{R}^2 \setminus B_{r_0}(e_{ki})} \rho_k \hat{h}_0(\delta_k e_{ki}) e^{\hat{U}_{ki}(y)} dy \\ &+ \sum_{i=1}^{1+\alpha} \int_{B_{r_0}(e_{ki})} \rho_k \hat{h}_0(\delta_k e_{ki}) e^{\hat{U}_{ki}(y)} \hat{D}_k(y, \hat{\eta}_{ki}(y)) dy.\end{aligned} \tag{4.29}$$

First, by (4.14), we have

$$\begin{aligned}&-\int_{\mathbb{R}^2 \setminus B_{r_0}(e_{ki})} \rho_k \hat{h}_0(\delta_k e_{ki}) e^{\hat{U}_{ki}(y)} dy \\ &= \frac{64}{\rho_k \hat{h}_0(e_{ki})} \int_{\mathbb{R}^2 \setminus B_{r_0}(e_{ki})} \frac{-1}{|y-e_{ki}|^4} dy \sigma_k^2 + O(\sigma_k^4),\end{aligned} \tag{4.30}$$

Let $\hat{\varphi}_{ki}(y) = -1 + \frac{2}{1+c_k e^{\hat{\mu}_k}|y-e_{ki}|^2}$, $c_k = \frac{\rho_k \hat{h}_0(\delta_k e_{ki})}{8}$. Then $\hat{\varphi}_{ki}(y)$ satisfies

$$\Delta \hat{\varphi}_{ki}(y) + \rho_k \hat{h}_0(\delta_k e_{ki}) e^{\hat{U}_{ki}(y)} \hat{\varphi}_{ki}(y) = 0. \tag{4.31}$$

By (4.15), we have

$$\begin{aligned}&\int_{B_{r_0}(e_{ki})} \rho_k \hat{h}_0(\delta_k e_{ki}) e^{\hat{U}_{ki}(y)} \hat{D}_k(y, \hat{\eta}_{ki}(y)) dy \\ &= \int_{B_{r_0}(e_{ki})} \hat{\varphi}_{ki}(y) \Delta \hat{\eta}_{ki}(y) - \hat{\eta}_{ki}(y) \Delta \hat{\varphi}_{ki}(y) dy \\ &- \int_{\partial B_{r_0}(e_{ki})} (1+\hat{\varphi}_{ki}) \frac{\partial}{\partial \nu} \hat{\eta}_{ki} d\sigma + \int_{\partial B_{r_0}(e_{ki})} \hat{\eta}_{ki} \frac{\partial}{\partial \nu} \hat{\varphi}_{ki} d\sigma.\end{aligned}$$

From (4.22), for the boundary term we have

$$-\int_{\partial B_{r_0}(e_{ki})}(1+\hat{\varphi}_{ki})\frac{\partial}{\partial\nu}\hat{\eta}_{ki}d\sigma+\int_{\partial B_{r_0}(e_{ki})}\hat{\eta}_{ki}\frac{\partial}{\partial\nu}\hat{\varphi}_{ki}d\sigma=O\left(|\ln\sigma_k|\,\sigma_k^4\right).$$

Note that

$$\hat{D}_k\left(y,\hat{\eta}_{ki}\left(y\right)\right)-\hat{\eta}_{ki}\left(y\right)=\hat{D}_k\left(y,0\right)+\hat{\eta}_{ki}\left(y\right)\hat{D}_k\left(y,0\right)+O\left(\left|\hat{\eta}_{ki}\left(y\right)\right|^2\right).$$

By (4.31) and (4.22), we get

$$\begin{aligned}&\int_{B_{r_0}(e_{ki})}\hat{\varphi}_{ki}\left(y\right)\Delta\hat{\eta}_{ki}\left(y\right)-\hat{\eta}_{ki}\left(y\right)\Delta\hat{\varphi}_{ki}\left(y\right)dy\\&=-\int_{B_{r_0}(e_{ki})}\rho_k\hat{h}_0\left(\delta_ke_{ki}\right)e^{\hat{U}_{ki}}\hat{\varphi}_{ki}\left(y\right)\left(\hat{D}_k\left(y,\hat{\eta}_{ki}\left(y\right)\right)-\hat{\eta}_{ki}\left(y\right)\right)dy\\&=-\int_{B_{r_0}(e_{ki})}\rho_k\hat{h}_0\left(\delta_ke_{ki}\right)e^{\hat{U}_{ki}}\hat{\varphi}_{ki}\left(y\right)\hat{D}_k\left(y,0\right)dy+O\left(\sigma_k^4\right).\end{aligned}$$

From the definition of $\hat{\varphi}_{ki}\left(y\right)$, we have

(4.32)

$$\begin{aligned}&\int_{B_{r_0}(e_{ki})}-\rho_k\hat{h}_0\left(\delta_ke_{ki}\right)e^{\hat{U}_{ki}}\hat{\varphi}_{ki}\left(y\right)\hat{D}_k\left(y,0\right)dy\\&=\int_{B_{r_0}(e_{ki})}\rho_k\hat{h}_0\left(\delta_ke_i\right)e^{\hat{U}_{ki}}\hat{D}_k\left(y,0\right)dy-2\int_{B_{r_0}(e_{ki})}\frac{\rho_k\hat{h}_0\left(\delta_ke_{ki}\right)e^{\hat{U}_{ki}}\hat{D}_k\left(y,0\right)}{1+c_ke^{\hat{\mu}_k}\left|y-e_{ki}\right|^2}dy.\end{aligned}$$

By Taylor expansion, we obtain

(4.33)

$$\begin{aligned}&\hat{D}_k\left(y,0\right)\\&=e^{\hat{Q}_{ki}(y)-\hat{Q}_{ki}(e_{ki})}-1\\&=\nabla\hat{Q}_{ki}\left(e_{ki}\right)\left(y-e_{ki}\right)+\frac{1}{2}\nabla^2\hat{Q}_{ki}\left(e_{ki}\right)\left(y-e_{ki}\right)^2+\frac{1}{6}\nabla^3\hat{Q}_{ki}\left(e_{ki}\right)\left(y-e_{ki}\right)^3\\&+\frac{1}{2}\left[\nabla\hat{Q}_{ki}\left(e_{ki}\right)\left(y-e_{ki}\right)+\frac{1}{2}\nabla^2\hat{Q}_{ki}\left(e_{ki}\right)\left(y-e_{ki}\right)^2\right]^2\\&+\frac{1}{6}\left[\nabla\hat{Q}_{ki}\left(e_{ki}\right)\left(y-e_{ki}\right)\right]^3+O\left(\left|y-e_{ki}\right|^4\right).\end{aligned}$$

By (4.18), (4.33), and the symmetry, we have

$$\int_{B_{r_0}(e_{ki})}\frac{\rho_k\hat{h}_0\left(\delta_ke_{ki}\right)e^{\hat{U}_{ki}}\hat{D}_k\left(y,0\right)}{1+c_ke^{\hat{\mu}_k}\left|y-e_{ki}\right|^2}dy=O\left(|\ln\sigma_k|\,\sigma_k^4\right).$$

For any $\theta > 0$, we have

$$
\begin{aligned}
&\int_{B_{r_0}(e_{ki})} \rho_k \hat{h}_0 (\delta_k e_{ki}) e^{\hat{U}_{ki}} \hat{D}_k (y,0)\, dy \\
&= \int_{B_\theta(e_{ki})} \rho_k \hat{h}_0 (\delta_k e_{ki}) e^{\hat{U}_{ki}} \hat{D}_k (y,0)\, dy \\
&+ \int_{B_{r_0}(e_{ki})\setminus B_\theta(e_{ki})} \rho_k \hat{h}_0 (\delta_k e_{ki}) e^{\hat{U}_{ki}} \hat{D}_k (y,0)\, dy \\
&= \frac{32\pi}{\rho_k \hat{h}_0 (\delta_k e_{ki})} \Delta \hat{Q}_{ki} (e_{ki}) \sigma_k^2 \left|\ln \sigma_k\right| + \frac{32\pi}{\rho_k \hat{h}_0 (\delta_k e_{ki})} \Delta \hat{Q}_{ki} (e_{ki}) (\ln \theta) \sigma_k^2 \\
&+ \int_{B_{r_0}(e_{ki})\setminus B_\theta(e_{ki})} \rho_k \hat{h}_0 (\delta_k e_{ki}) e^{\hat{U}_{ki}} \hat{D}_k (y,0)\, dy + o_\theta (1) \sigma_k^4 + O \left(\left|\ln \sigma_k\right| \sigma_k^4\right),
\end{aligned}
$$

where $o_\theta (1) \to 0$ as $\theta \to 0$. Moreover, from (4.33) and the symmetry, we get

$$
\begin{aligned}
&\int_{B_{r_0}(e_{ki})\setminus B_\theta(e_{ki})} \rho_k \hat{h}_0 (\delta_k e_{ki}) e^{\hat{U}_{ki}} \hat{D}_k (y,0)\, dy \\
&= \frac{64}{\rho_k \hat{h}_0 (\delta_k e_{ki})} \int_{B_{r_0}(e_{ki})\setminus B_\theta(e_{ki})} \frac{e^{\hat{Q}_{ki}(y)-\hat{Q}_{ki}(e_{ki})} - 1}{\left|y - e_{ki}\right|^4} dy \sigma_k^2 + O \left(\left|\ln \sigma_k\right| \sigma_k^4\right).
\end{aligned}
$$

From (4.16), we have

$$
\begin{aligned}
&\frac{64}{\rho_k \hat{h}_0 (e_{ki})} \int_{B_{r_0}(e_{ki})\setminus B_\theta(e_{ki})} \frac{e^{\hat{Q}_{ki}(y)-\hat{Q}_{ki}(e_{ki})} - 1}{\left|y - e_{ki}\right|^4} dy \sigma_k^2 \\
&= \frac{64 \Pi_{i\neq j}^{1+\alpha} \left|e_{ki} - e_{kj}\right|^4}{\rho_k \hat{h}_0 (\delta_k e_{ki})} \int_{B_{r_0}(e_{ki})\setminus B_\theta(e_{ki})} \frac{\left|y\right|^{2\alpha} e^{\ln \hat{h}_0(\delta_k y) - \ln \hat{h}_0(\delta_k e_{ki})}}{\Pi_{\ell=1}^{1+\alpha} \left|y - e_{k\ell}\right|^4} dy \sigma_k^2 \\
&- \frac{64}{\rho_k \hat{h}_0 (\delta_k e_{ki})} \int_{B_{r_0}(e_{ki})\setminus B_\theta(e_{ki})} \frac{1}{\left|y - e_{ki}\right|^4} dy \sigma_k^2 + O \left(\left|\ln \sigma_k\right| \sigma_k^4\right),
\end{aligned}
$$

and thus

$$\begin{aligned}(4.34)\quad &\int_{B_{r_0}(e_{ki})} \rho_k \hat{h}_0(\delta_k e_{ki}) e^{\hat{U}_{ki}} \hat{D}_k(y,0)\,dy \\ &= \frac{32\pi}{\rho_k \hat{h}_0(\delta_k e_{ki})} \Delta \hat{Q}_{ki}(e_{ki}) \sigma_k^2 |\ln \sigma_k| \\ &- \frac{64}{\rho_k \hat{h}_0(e_{ki})} \int_{B_{r_0}(e_{ki}) \setminus B_\theta(e_{ki})} \frac{1}{|y-e_{ki}|^4} dy \sigma_k^2 \\ &+ \frac{64}{\rho_k \hat{h}_0(\delta_k e_{ki})} \int_{B_{r_0}(e_{ki}) \setminus B_\theta(e_{ki})} \\ &\quad \frac{\Pi_{i\neq j}^{1+\alpha} |e_{ki}-e_{kj}|^4 |y|^{2\alpha} e^{\ln \hat{h}_0(\delta_k y) - \ln \hat{h}_0(\delta_k e_{ki})}}{\Pi_{\ell=1}^{1+\alpha} |y-e_{k\ell}|^4} dy \sigma_k^2 \\ &+ \frac{32\pi}{\rho_k \hat{h}_0(\delta_k e_{ki})} \Delta \hat{Q}_{ki}(e_{ki}) (\ln \theta) \sigma_k^2 + O\left(|\ln \sigma_k| \sigma_k^4\right) + o_\theta(1) \sigma_k^4.\end{aligned}$$

By (4.27), we have

$$\begin{aligned}(4.35)\quad &\int_{\Omega_k'} \rho_k |y|^{2\alpha} \hat{h}_0(\delta_k y) e^{\hat{u}_k(y)} dy \\ &= \sum_{i=1}^{1+\alpha} \int_{\Omega_{ki} \setminus B_{r_0}(e_{ki})} \frac{64 \Pi_{i\neq j}^{1+\alpha} |e_{ki}-e_{kj}|^4}{\rho_k \hat{h}(\delta_k e_{ki})} \frac{|y|^{2\alpha} e^{\ln \hat{h}_0(\delta_k y) - \ln \hat{h}_0(\delta_k e_{ki})}}{\Pi_{\ell=1}^{1+\alpha} |y-e_{k\ell}|^4} dy \sigma_k^2 + O\left(\sigma_k^4\right).\end{aligned}$$

Combining (4.28), (4.29), (4.30), (4.34), and (4.35), we obtain

$$\begin{aligned}&\rho_{k,0} - 8\pi(1+\alpha) \\ &= \sum_{i=1}^{1+\alpha} \left[\begin{array}{l} \frac{64}{\rho_k \hat{h}_0(\delta_k e_{ki})} \int_{\mathbb{R}^2 \setminus B_\theta(e_{ki})} \frac{-1}{|y-e_{ki}|^4} dy \\ + \frac{64}{\rho_k \hat{h}_0(\delta_k e_{ki})} \int_{\Omega_{ki} \setminus B_\theta(e_{ki})} \frac{\Pi_{i\neq j}^{1+\alpha} |e_{ki}-e_{kj}|^4 |y|^{2\alpha} e^{\ln \hat{h}_0(\delta_k y) - \ln \hat{h}_0(\delta_k e_{ki})}}{\Pi_{\ell=1}^{1+\alpha} |y-e_{k\ell}|^4} dy \\ + \frac{32\pi}{\rho_k \hat{h}_0(\delta_k e_{ki})} \Delta \hat{Q}_{ki}(e_{ki}) (\ln \theta) \end{array} \right] \sigma_k^2 \\ &+ \sum_{i=1}^{1+\alpha} \frac{32\pi}{\rho_k \hat{h}_0(\delta_k e_{ki})} \Delta \hat{Q}_{ki}(e_{ki}) \sigma_k^2 |\ln \sigma_k| + O\left(|\ln \sigma_k| \sigma_k^4\right) + o_\theta(1) \sigma_k^4.\end{aligned}$$

Notice that

$$\begin{aligned}(4.36)\quad &e^{\ln \hat{h}_0(\delta_k y) - \ln \hat{h}_0(\delta_k e_{ki})} \\ &= 1 + \nabla \ln \hat{h}_0(\delta_k e_{ki}) \delta_k (y - e_{ki}) + \frac{1}{2} \nabla^2 \ln \hat{h}_0(\delta_k e_{ki}) \delta_k^2 (y-e_{ki})^2 \\ &+ \frac{1}{2} \left[\nabla \ln \hat{h}_0(\delta_k e_{ki}) \delta_k (y-e_{ki}) \right]^2 + O\left(\delta_k^3 |y-e_{ki}|^3\right).\end{aligned}$$

Then by using (4.36) and (4.17) and letting $\theta \to 0$, we have

$$\lim_{\theta\to 0}\sum_{i=1}^{1+\alpha}\left[\begin{array}{l}\frac{64}{\rho_k\hat{h}_0(\delta_k e_{ki})}\int_{\mathbb{R}^2\setminus B_\theta(e_{ki})}\frac{-1}{|y-e_{ki}|^4}dy\\ +\frac{64}{\rho_k\hat{h}_0(\delta_k e_{ki})}\int_{\Omega_{ki}\setminus B_\theta(e_{ki})}\frac{\Pi_{i\neq j}^{1+\alpha}|e_{ki}-e_{kj}|^4|y|^{2\alpha}e^{\ln\hat{h}_0(\delta_k y)-\ln\hat{h}_0(\delta_k e_{ki})}}{\Pi_{\ell=1}^{1+\alpha}|y-e_{k\ell}|^4}dy\\ +\frac{32\pi}{\rho_k\hat{h}_0(\delta_k e_{ki})}\Delta\hat{Q}_{ki}(e_{ki})(\ln\theta)\end{array}\right]\sigma_k^2$$

$$=\lim_{\theta\to 0}\frac{64}{\rho_k\hat{h}_0(\delta_k e_{ki})}$$

$$\left[\sum_{i=1}^{1+\alpha}\int_{\mathbb{R}^2\setminus B_\theta(e_{ki})}\frac{-1}{|y-e_{ki}|^4}dy+\int_{\mathbb{R}^2\setminus\cup_{i=1}^{1+\alpha}B_\theta(e_{ki})}\frac{\Pi_{i\neq j}^{1+\alpha}|e_{ki}-e_{kj}|^4|y|^{2\alpha}}{\Pi_{\ell=1}^{1+\alpha}|y-e_{k\ell}|^4}dy\right]\sigma_k^2$$

$$+O\left(\delta_k^2\sigma_k^2\right).$$

Recall that $e_\ell = e^{i\frac{2\pi(\ell-1)}{1+\alpha}}$, and by direct computation we have

$$\lim_{\theta\to 0}\left[\sum_{i=1}^{1+\alpha}\int_{\mathbb{R}^2\setminus B_\theta(e_i)}\frac{-1}{|y-e_i|^4}dy+\int_{\mathbb{R}^2\setminus\cup_{i=1}^{1+\alpha}B_\theta(e_i)}\frac{\Pi_{i\neq j}^{1+\alpha}|e_i-e_j|^4|y|^{2\alpha}}{\Pi_{\ell=1}^{1+\alpha}|y-e_\ell|^4}dy\right]=0.$$

Therefore, we obtain

$$\rho_{k,0}-8\pi(1+\alpha)=\frac{32(1+\alpha)\pi}{\rho_k h_0(0)}\Delta\hat{Q}_{ki}(e_{ki})\sigma_k^2|\ln\sigma_k|+O\left(|\ln\sigma_k|\sigma_k^4+\delta_k^2\sigma_k^2\right).$$

Then by Lemma 9 and (4.17), we get

$$\rho_{k,0}-8\pi(1+\alpha)=\frac{32(1+\alpha)\pi}{\rho_k h_0(0)}(\Delta\ln h_0(0)+\rho_\infty)\delta_k^2\sigma_k^2|\ln\sigma_k|+O\left(\delta_k^2\sigma_k^2\right).$$

This completes the proof. q.e.d.

Appendix A. Simple blowup

Here, we are going to prove Theorem 7. Let us recall that $u_k(x)$ satisfies

$$\text{(A.1)}\qquad\left\{\begin{array}{l}\Delta u_k(x)+\rho_k h_0(x)|x|^{2\alpha}e^{u_k(x)}=0\text{ in }B_1(0)\\ \left|u_k(x)-u_k\left(x'\right)\right|\leq c\text{ for }|x|=\left|x'\right|=1\\ \int_{B_1(0)}\rho_k h_0(x)|x|^{2\alpha}e^{u_k(x)}dx\to\rho_{\infty,0}=8\pi(1+\alpha)\\ 0\text{ is the only blowup point for }u_k(x)\text{ in }B_1(0)\end{array}\right..$$

Let $v_k(z)=u_k(\varepsilon_k z)-\lambda_k$. Then

$$\text{(A.2)}\qquad\left\{\begin{array}{l}\Delta v_k(z)+\rho_k h_0(\varepsilon_k z)|z|^{2\alpha}e^{v_k(z)}=0\text{ in }B_{\frac{1}{\varepsilon_k}}(0)\\ \left|v_k(z)-v_k\left(z'\right)\right|\leq c\text{ for }|z|=\left|z'\right|=\frac{1}{\varepsilon_k}\\ v_k(z)\leq 0\end{array}\right..$$

Define

$$\rho_{k,0}=\int_{B_1(0)}\rho_k h_0(x)|x|^{2\alpha}e^{u_k(x)}dx.$$

Proof of Theorem 7: Set $V_k(z) = -2\ln\left(1 + \frac{\rho_{\infty,0}h_0(0)}{8(1+\alpha)^2}\left|z^{1+\alpha} - a_k\right|^2\right)$. Then $V(z)$ satisfies

$$\text{(A.3)}\qquad \begin{cases} \Delta V_k(z) + \rho_{\infty,0}h_0(0)\,|z|^{2\alpha} e^{V(z)} = 0 \text{ in } \mathbb{R}^2 \\ V_k\left(a_k^{\frac{1}{1+\alpha}}\right) = 0 \\ \int_{\mathbb{R}^2} \rho_{\infty,0}h_0(0)\,|z|^{2\alpha} e^{V_k(z)}dz = 8\pi(1+\alpha) \end{cases}.$$

Notice that

$$|u_k(x) - U_k(z)| \le C \text{ in } B_1(0) \text{ if and only if } |v_k(z) - V_k(z)| \le C \text{ in } B_{\frac{1}{\varepsilon_k}}(0).$$

Now we divide the proof into several steps as follows:

Step 1: $v_k(z) \to V(z)$ in $C^2_{loc}(\mathbb{R}^2)$, where $V(z) = \lim_k V_k(z) = -2\ln\left(1 + \frac{\rho_{\infty,0}h_0(0)}{8(1+\alpha)^2}\left|z^{1+\alpha} - a\right|^2\right)$.

Since $v_k(z) \le 0$ and $v_k\left(a_k^{\frac{1}{1+\alpha}}\right) = u_k(p_k) - \lambda_k = 0$, by applying results of Brezis and Merle [**2**] or Bartolucci and Tarantello [**4**], we conclude that $v_k(z)$ is convergent in $C^2_{loc}(\mathbb{R}^2)$ to $v_\infty(z)$ and $v_\infty(z)$ satisfies

$$\text{(A.4)}\qquad \begin{cases} \Delta v_\infty(z) + \rho_{\infty,0}h_0(0)\,|z|^{2\alpha} e^{v_\infty(z)} = 0 \text{ in } \mathbb{R}^2 \\ \int_{\mathbb{R}^2} \rho_{\infty,0}h(0)\,|z|^{2\alpha} e^{v_\infty(z)}dz = 8\pi(1+\alpha) \end{cases}.$$

By the classification of entire solution of (A.4), we have

$$v_\infty(z) = \ln\frac{\mu}{\left(1 + \frac{\rho_{\infty,0}h_0(0)\mu}{8(1+\alpha)^2}\left|z^{1+\alpha} - b\right|^2\right)^2}$$

for some $\mu > 0$ and $b \in \mathbb{C}$. Next, we have to determine μ and b. Notice that $a_k^{\frac{1}{1+\alpha}}$ is the maximum point of v_k and $a_k \to a$ with $|a| < \infty$. Since $v_k \to v_\infty$ in $C^2_{loc}(\mathbb{R}^2)$, $a^{\frac{1}{1+\alpha}}$ is the maximum point of v_∞ and thus $b = a$. Furthermore, since $v_k\left(a_k^{\frac{1}{1+\alpha}}\right) = 0$, we have $v_\infty\left(a^{\frac{1}{1+\alpha}}\right) = 0$. This implies that $\mu = 1$. That is, $v_\infty(z) = V(z)$. This completes the proof of Step 1.

By Step 1, we have

$$\text{(A.5)}\qquad v_k(0) = O(1).$$

Then by Green's formula and (A.5), we have

(A.6)

$$v_k(z) = \frac{1}{2\pi}\int_{|y|\le\frac{1}{\varepsilon_k}}\left(\ln\frac{|y|}{|z-y|}\right)\rho_k\,|y|^{2\alpha} h_0(\varepsilon_k y)\,e^{v_k(y)}dy + O(1).$$

Step 2: $|v_k(z) + 4(1+\alpha)\ln|z|| \le C$ for $\ln\frac{1}{\varepsilon_k} \le |z| \le \frac{1}{\varepsilon_k}$.

Note that when $|z|$ is sufficient large, $V_k(z)$ is controlled by $-4(1+\alpha)\ln|z|$. Thus,

$$|v_k(z) - V_k(z)| \le C \text{ if and only if } |v_k(z) + 4(1+\alpha)\ln|z|| \le C$$

as $|z|$ is large enough. Notice that

(A.7)

$$\begin{aligned} |v_k(z) + 4(1+\alpha)\ln|z|| &= \left| v_k(z) + \frac{\rho_{k,0}}{2\pi}\ln|z| + \left(4(1+\alpha) - \frac{\rho_{k,0}}{2\pi}\right)\ln|z| \right| \\ &\le \left| v_k(z) + \frac{\rho_{k,0}}{2\pi}\ln|z| \right| + \left| \left(4(1+\alpha) - \frac{\rho_{k,0}}{2\pi}\right)\ln|z| \right|. \end{aligned}$$

Thus, we have to estimate $\left|v_k(z) + \frac{\rho_{k,0}}{2\pi}\ln|z|\right|$ and $\left|\left(4(1+\alpha) - \frac{\rho_{k,0}}{2\pi}\right)\ln|z|\right|$ for $\ln\frac{1}{\varepsilon_k} \le |z| \le \frac{1}{\varepsilon_k}$.

From (A.6) and the same argument as Lemma 3, we have the following.

Estimate 1: For any $\theta > 0$, there exist $R_\theta > 1$ and $k_\theta \in \mathbb{N}$ such that for $|z| \ge 2R_\theta$ and $k \ge k_\theta$, we have

$$v_k(z) \le -\left(\frac{\rho_{k,0}}{2\pi} - 2\theta\right)\ln|z| + O(1). \tag{A.8}$$

From (A.8) and $\rho_{k,0} = 8\pi(1+\alpha) + o(1)$, it follows that

$$\int_{|y|\le\frac{1}{\varepsilon_k}} |\ln|y||\,|y|^{2\alpha} h_0(\varepsilon_k y)\, e^{v_k(y)} dy \le C_1 \tag{A.9}$$

and

$$\int_{|y|\le\frac{1}{\varepsilon_k}} |y|\,|y|^{2\alpha} h_0(\varepsilon_k y)\, e^{v_k(y)} dy \le C_2. \tag{A.10}$$

By a similar argument as in Lemma 4 and Lemma5, we have the following.

Estimate 2: For $\ln\frac{1}{\varepsilon_k} \le |z| \le \frac{1}{\varepsilon_k}$, we have

$$v_k(z) = -\frac{\rho_{k,0}}{2\pi}\ln|z| + O(1) \tag{A.11}$$

Estimate 3: For $\ln\frac{1}{\varepsilon_k} \le |z| \le \frac{1}{\varepsilon_k}$, we have

$$\left|\nabla v_k(z) + \frac{\rho_{k,0}}{2\pi}\frac{z}{|z|^2}\right| \le \frac{C}{|z|^2}. \tag{A.12}$$

By (A.11) and (A.12), we have, for $\varepsilon_k \ln\frac{1}{\varepsilon_k} \le |x| \le 1$,

$$u_k(x) = -\frac{\rho_{k,0}}{2\pi}\ln|x| + \left(\frac{\rho_{k,0}}{2\pi} - 2(1+\alpha)\right)\ln\varepsilon_k + O(1) \tag{A.13}$$

and

$$\nabla u_k(x) = -\frac{\rho_{k,0}}{2\pi}\frac{x}{|x|^2} + O\left(\frac{\varepsilon_k}{|x|^2}\right). \tag{A.14}$$

Estimate 4: $|\rho_{k,0} - 8\pi(1+\alpha)| = O\left(\left(\ln \frac{1}{\varepsilon_k}\right)^{-1}\right)$.

Applying the Pohozaev identity in the region $B_k = \left\{|x| \le \varepsilon_k \ln \frac{1}{\varepsilon_k}\right\}$, we obtain

$$\begin{aligned}\text{(A.15)}\quad &\int_{B_k} \left(2\rho_k |x|^{2\alpha} h_0(x) + \rho_k x \cdot \nabla\left(|x|^{2\alpha} h_0(x)\right)\right) e^{u_k(x)} dx \\ &= \int_{\partial B_k} r\left[\left(\frac{\partial}{\partial \nu} u_k\right)^2 - \frac{1}{2}|\nabla u_k|^2 + \rho_k |x|^{2\alpha} h_0(x) e^{u_k(x)}\right] d\sigma,\end{aligned}$$

where $r = |x|$. Substituting (A.13) and (A.14) into both sides of (A.15), we find that

$$\begin{aligned}&\int_{\partial B_k} r\left[\left(\frac{\partial}{\partial \nu} u_k\right)^2 - \frac{1}{2}|\nabla u_k|^2 + \rho_k |x|^{2\alpha} h_0(x) e^{u_k(x)}\right] d\sigma \\ &= \frac{\rho_{k,0}^2}{4\pi} + O\left(\left(\ln \frac{1}{\varepsilon_k}\right)^{-1}\right) \text{ as } k \to \infty\end{aligned}$$

and

$$\begin{aligned}&\int_{B_k} \left(2\rho_k |x|^{2\alpha} h_0(x) + \rho_k x \cdot \nabla\left(|x|^{2\alpha} h_0(x)\right)\right) e^{u_k(x)} dx \\ &= 2(1+\alpha)\rho_{k,0} + O(1)\int_{B_1(0)\setminus B_k} |x|^{2\alpha} e^{u_k(x)} dx \\ &+ \int_{B_k} \rho_k (x \cdot \nabla h_0(x)) |x|^{2\alpha} e^{u_k(x)} dx \\ &= 2(1+\alpha)\rho_{k,0} + O(1)\left(\ln \frac{1}{\varepsilon_k}\right)^{-\frac{\rho_{k,0}}{2\pi}+2(1+\alpha)} + O\left(\varepsilon_k \ln \frac{1}{\varepsilon_k}\right) \text{ as } k \to \infty.\end{aligned}$$

Therefore,

$$\text{(A.16)}\qquad \rho_{k,0} = 8\pi(1+\alpha) + O\left(\left(\ln \frac{1}{\varepsilon_k}\right)^{-1}\right).$$

Then Step 2 follows from (A.7), (A.11), and (A.16).

Step 3: $|v_k(z) + 4(1+\alpha)\ln|z|| \le C$ for $R \le |z| \le \ln \frac{1}{\varepsilon_k}$, where R is a fixed but large number.

We choose $R > 1$ such that $|z|^{2\alpha} h_0(\varepsilon_k z) e^{v_k(z)} \le \frac{1}{|z|^{\frac{7}{2}}}$ for $|z| \ge R$. Then we construct two functions $w_\pm(z)$ as follows:

$$w_\pm(z) = -4(1+\alpha)\ln|z| \pm \left(c_1 - c_1 |z|^{-\frac{1}{2}}\right)$$

with a suitable choice of c_1. Then

$$\Delta w_+(z) = -\frac{1}{4}c_1 |z|^{-\frac{5}{2}}$$

$$\Delta w_-(z) = \frac{1}{4}c_1 |z|^{-\frac{5}{2}}$$

for $|z| > R$. By considering $(w_+(z) - v_k(z))$, we have

$$\Delta (w_+(z) - v_k(z)) = -\frac{1}{4}c_1 |z|^{-\frac{5}{2}} + \rho_k |z|^{2\alpha} h_0(\varepsilon_k z) e^{v_k(z)} \leq 0$$

for $|z| > R$. Now, choose $c_1 > 0$ such that $w_+(z) \geq v_k(z)$ on $|z| = R$ and $|z| = \ln \frac{1}{\varepsilon_k}$. Thus, by the maximum principle, we have $v_k(z) \leq w_+(z)$ for $R \leq |z| \leq \ln \frac{1}{\varepsilon_k}$. Similarly, we also have $v_k(z) \geq w_-(z)$ for $R \leq |z| \leq \ln \frac{1}{\varepsilon_k}$.

From Step 1 to Step 3, we complete the proof of Theorem 7. q.e.d.

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