Estimates on Green functions and Poisson kernels for symmetric stable processes

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Mathematics Subject Classification (1991): 60J30, 60J45, 60J75, 31C99

1. Introduction

One of the most basic and most important subfamily of Lévy processes is symmetric stable processes. A symmetric α-stable process X on $\mathbf{R}^n$ is a Lévy process whose transition density $p(t, x - y)$ relative to the Lebesgue measure is uniquely determined by its Fourier transform $\int_{\mathbf{R}^n} e^{ix \cdot \xi} p(t, x) dx = e^{-t|\xi|^\alpha}$. Here α must be in the interval (0, 2]. When $\alpha = 2$, we get a Brownian motion running with a time clock twice as fast as the standard one. Brownian motion plays a central role in modern probability theory and has numerous important applications in other scientific areas as well as in many other branches of mathematics. Thus it has been intensively studied. In this paper, symmetric stable processes are referred to the case when $0 < \alpha < 2$, unless otherwise specified. In the last few years there has been an explosive growth in the study of physical and economic systems that can be successfully modeled with the use of stable processes. Stable processes are now widely used in physics, operations research, queuing theory, mathematical finance and risk estimation. In some physics literatures, symmetric α-stable processes are called Lévy flights, and they have been applied to a wide range of very complex physics issues, such as turbulent diffusion, vortex dynamics, anomalous diffusion in rotating flows, and molecular spectral fluctuations. In mathematical finance, stable processes can be used to model stock returns in incomplete market. For these and more applications of stable processes, please see the interesting book [14] by Janicki and Weron and the references therein and the recent article [15] by Klafter, Shlesinger and Zuomofen. In order to make precise predictions about natural phenomena and to better cope with these

widespread applications, there is a need to study the fine properties of symmetric
stable processes, just as for the Brownian motion case. Although a lot is known
about symmetric stable processes and their potential theory (see [1, 2, 3, 4, 9,
10, 12, 14, 16, 17, 20, 21] and the references therein), little is known about the
counterparts to some of the deep results for Brownian motion, such as sharp esti-
mates on Green functions and Poisson kernels of bounded domains. In the special
case of balls, the explicit formulae for the Poisson kernels and Green functions
for symmetric stable processes are known. The formula for the Poisson kernel
of balls were obtained by M. Riesz and the formula for Green function of balls
were obtained by Blumenthal, Getoor and Ray in [3]. Prior to that, Kac, Polland,
Spitzer, Widom, Kesten and Kinney had obtained some results for 1-dimensional
case (cf. [3]).

Unlike the generator Δ of Brownian motion whose time clock is twice as
fast as the standard one, the generator of a symmetric α-stable process with
$0 < \alpha < 2$ is the fractional Laplacian $-(-\Delta)^{\alpha/2}$ which is non-local. Also a
symmetric stable process is a process with discontinuous sample paths and having
heavy tails, while Brownian motion has continuous sample paths and exponential
decay tails. The transition density function $p(t, x-y)$ for discontinuous symmetric
α-stable process X is approximately $c|x - y|^{-(n+\alpha)}$ when $|x - y|$ is large. So X_t
has infinite variance and when $0 < \alpha \leq 1$, $|X_t|$ even has infinite mean. All these
indicate the significant difference between Brownian motion and symmetric α-
stable processes.

In this paper, we will address some of these problems. More specifically, we
will derive precise estimates on Green functions and Poisson kernels of X in
bounded $C^{1,1}$-smooth domains D. That D is $C^{1,1}$ means that for every $z \in \partial D$,
there exists a $r > 0$ such that $B(z, r) \cap \partial D$ is the graph of a function whose
first derivatives are Lipschitz. These estimates are very useful in studying other
properties of symmetric stable processes. As examples of applications of these
estimates, we prove that the 3G Theorem holds for symmetric α-stable processes
on bounded $C^{1,1}$ domains and that the conditional lifetimes for the symmetric
α-stable processes in a bounded $C^{1,1}$ domain are uniformly bounded.

To state the main results of this paper, let X be a symmetric α-stable process
on $\mathbf{R}^n$ with $n \geq 2$ and $0 < \alpha < 2$. The process X is transient and we are going
to use G to denote the potential of X. We know that the Green function of X is
given by

$$(1.1) \quad G(x, y) = \int_0^\infty p(t, x, y) dy = 2^{-\alpha} \pi^{-\frac{n}{2}} \Gamma\left(\frac{n-\alpha}{2}\right) \Gamma\left(\frac{\alpha}{2}\right)^{-1} |x - y|^{\alpha-n}$$

(see, for example, [2]). Here Γ is the Gamma function defined by $\Gamma(\lambda) = \int_0^\infty t^{\lambda-1} e^{-t} dt$ for $\lambda > 0$. For a domain D in $\mathbf{R}^n$, let $\tau_D = \inf\{t > 0 : X_t \notin D\}$.
Adjoin a cemetery point ∂ to D and set

$$X_t^D(\omega) = \begin{cases} X_t(\omega), & \text{if } t < \tau_D, \\ \partial, & \text{if } t \geq \tau_D. \end{cases}$$

X^D is called the subprocess of the symmetric α–stable process X killed upon leaving D, or simply the symmetric α–stable process in D. It is well known that there is a continuous symmetric function $G_D(\cdot, \cdot)$ defined on $D \times D$ except along the diagonal such that for any Borel measurable function $f \geq 0$,

$$E_x \left[\int_0^{\tau_D} f(X_s) ds \right] = \int_D G_D(x, y) f(y) dy.$$

G_D is called the Green function of X in D. Note that G_D has the following scaling property: for $a > 0$,

(1.2) $$G_D(x, y) = a^{\alpha - n} G_{D/a}(x/a, y/a), \quad x, y \in D.$$

The main results of of this paper are summarized as follows.

Theorem 1.1. *Suppose that D is a bounded $C^{1,1}$ domain in $\mathbf{R}^n$. Let $\delta(x) = d(x, \partial D)$ be the Euclidean distance between x and ∂D. Then there exists a $C = C(D, \alpha) > 0$ such that for $x, y \in D$,*

$$G_D(x, y) \leq C \frac{1}{|x - y|^{n-\alpha}}, \tag{1.3}$$

$$G_D(x, y) \leq C \frac{\delta(x)^{\alpha/2}}{|x - y|^{n-\alpha/2}}, \tag{1.4}$$

$$G_D(x, y) \leq C \frac{\delta(x)^{\alpha/2} \delta(y)^{\alpha/2}}{|x - y|^n}, \tag{1.5}$$

$$G_D(x, y) \leq C \frac{\delta(x)^{\alpha/2}}{\delta(y)^{\alpha/2}} \frac{1}{|x - y|^{n-\alpha}}. \tag{1.6}$$

Now that since G_D is a symmetric function, so (1.6) tells that

(1.7) $$G_D(x, y) \leq C \frac{\delta(y)^{\alpha/2}}{\delta(x)^{\alpha/2}} \frac{1}{|x - y|^{n-\alpha}}.$$

Theorem 1.1 is proved by using inversion with respect to spheres along with the explicit formulae for Green functions and exit distributions of X on balls. For the now classical upper bounds for Green functions of Brownian motion, one can see Zhao [23] and the references therein. Comparing with the upper bound estimates above, the following lower bounds on the Green functions are much more difficult to prove. The proof involves some very detailed hard analysis. These lower bounds, in a sense, are generalizations of the results of Zhao [23] for Brownian motions to the discontinuous symmetric stable processes.

Theorem 1.2. *Suppose that D is a bounded $C^{1,1}$ domain in $\mathbf{R}^n$. Then there exists a $C = C(D, \alpha) > 0$ such that for $x, y \in D$,*

$$G_D(x, y) \geq \frac{C}{|x - y|^{n-\alpha}}, \quad \text{if } |x - y| \leq \max \left\{ \frac{\delta(x)}{2}, \frac{\delta(y)}{2} \right\} \tag{1.8}$$

$$G_D(x, y) \geq C \frac{\delta(x)^{\alpha/2} \delta(y)^{\alpha/2}}{|x - y|^n}, \quad \text{if } |x - y| > \max \left\{ \frac{\delta(x)}{2}, \frac{\delta(y)}{2} \right\}. \tag{1.9}$$

The above lower and upper bounds provide precise information about the Green functions. Summarizing them up, we have

Corollary 1.3. *Therefore there is a constant* $C = C(D, \alpha) > 1$ *such that*

$$C^{-1} \min \left\{ \frac{1}{|x - y|^{n-\alpha}}, \frac{\delta(x)^{\alpha/2} \delta(y)^{\alpha/2}}{|x - y|^n} \right\} \le G_D(x, y)$$

$$\le C \min \left\{ \frac{1}{|x - y|^{n-\alpha}}, \frac{\delta(x)^{\alpha/2} \delta(y)^{\alpha/2}}{|x - y|^n} \right\}.$$

In the Brownian motion case, the Poisson kernel in a bounded $C^{1,1}$ domain is the normal derivative of the Green function. In the case of an α–symmetric process, $0 < \alpha < 2$, this kind of relationship can not be expected to hold. For $0 < \alpha < 2$, the symmetric α-stable process has discontinuous sample paths and therefore the exit distribution of X_{τ_D} under P_x does not concentrate on the boundary ∂D. In fact, we have the following

Theorem 1.4. *For every bounded domain D in* $\mathbf{R}^n$ *satisfying uniform exterior cone condition, there is a function* $K_D(x, z)$ *defined on* $D \times D^c$ *such that*

$$E_x[\varphi(X_{\tau_D})] = A(n, \alpha) \int_{D^c} K_D(x, z) \varphi(z) dz, \quad x \in D$$

for every $\varphi \ge 0$ *on* D^c, *where*

(1.10)
$$A(n, \alpha) = \frac{\alpha 2^{\alpha-1} \Gamma(\frac{\alpha+n}{2})}{\pi^{n/2} \Gamma(1 - \frac{\alpha}{2})}.$$

Furthermore

$$K_D(x, z) = A(n, \alpha) \int_D \frac{G_D(x, y)}{|y - z|^{n+\alpha}} dy, \quad x \in D, \ y \in D^c.$$

Recall that a domain D in $\mathbf{R}^n$ is said to satisfy the uniform exterior cone condition if there exist constants $\eta > 0$, $r > 0$ and a cone $\mathscr{C} = \{x = (x_1, \ldots, x_n) \in \mathbf{R}^n : 0 < x_n, (x_1^2 + \cdots + x_{n-1}^2)^{1/2} < \eta x_n\}$ such that for every $z \in \partial D$, there is a cone $\mathscr{C}_z$ with vertex z, isometric to $\mathscr{C}$ and satisfying $\mathscr{C}_z \cap B(z, r) \subset D^c$. It is well known that bounded $C^{1,1}$ domains satisfy the uniform exterior cone condition, therefore the above theorem holds in particular for bounded $C^{1,1}$ domains. In principle, by using Theorem 1.4 and the bounds for the Green functions, one could get two–sided bounds on the Poisson kernels. However, it turns out to be a pretty challenging task.

Theorem 1.5. *Suppose that D is a bounded* $C^{1,1}$ *domain in* $\mathbf{R}^n$. *Then there exists a* $C = C(D, \alpha) > 1$ *such that for* $x \in D$ *and* $z \in \overline{D}^c$,

$$\frac{\delta(x)^{\alpha/2}}{C \, \delta(z)^{\alpha/2} (1 + \delta(z))^{\alpha/2}} \frac{1}{|x - z|^n} \le K_D(x, z)$$

$$\le \frac{C \, \delta(x)^{\alpha/2}}{\delta(z)^{\alpha/2} (1 + \delta(z))^{\alpha/2}} \frac{1}{|x - z|^n}.$$

For the corresponding results for Brownian motion, see Zhao [22]. Using the bounds above, we have

Theorem 1.6 ((3G Theorem)). *Suppose that D is a bounded $C^{1,1}$ domain in $\mathbf{R}^n$. Then there exists a $C = C(D, \alpha) > 0$ such that*

$$\frac{G_D(x,y)G_D(y,w)}{G_D(x,w)} \leq C \frac{|x-w|^{n-\alpha}}{|x-y|^{n-\alpha}|y-w|^{n-\alpha}},$$
$$x, y, w \in D. \tag{1.11}$$

$$\frac{G_D(x,y)K_D(y,z)}{K_D(x,z)} \leq C \frac{|x-z|^{n-\alpha}}{|x-y|^{n-\alpha}|y-z|^{n-\alpha}},$$
$$x, y \in D, \ z \in \overline{D}^c. \tag{1.12}$$

The estimates above are very useful and have a lot of applications. As an example of these applications, we are going to prove that the conditional lifetimes for X in a bounded $C^{1,1}$ domain are uniformly bounded. As another application of our estimates we are also going to give a simple proof of the boundary Harnack principle for nonnegative functions which are harmonic in a bounded $C^{1,1}$ domain D. Recently, the boundary Harnack principle for nonnegative functions which are harmonic in a bounded Lipschitz domain D was proved by Bogdan [4]. To state these results, we first we need some definitions.

Definition 1.1. Let D be a domain in $\mathbf{R}^n$. A locally integrable function f defined on $\mathbf{R}^n$ satisfying the condition $\int_{\{|x|>1\}} |f(x)||x|^{-(n+\alpha)}dx < \infty$ is said to be

1) $(-\Delta)^{\alpha/2}$–*superharmonic in D* if f is lower semicontinuous in D and for each $x \in D$ and each ball $B(x,r)$ with $\overline{B(x,r)} \subset D$,

$$f(x) \geq E_x f(X(\tau_{B(x,r)})).$$

2) $(-\Delta)^{\alpha/2}$–*harmonic in D* if f is continuous in D and for each $x \in D$ and each ball $B(x,r)$ with $\overline{B(x,r)} \subset D$,

$$f(x) = E_x f(X(\tau_{B(x,r)})).$$

Theorem 1.7 (Boundary Harnack Principle). *Suppose that D is a bounded $C^{1,1}$ domain in $\mathbf{R}^n$, V is an open set of $\mathbf{R}^n$ and K is a compact subset of V. Then there is a constant $C = C(D, V, K, \alpha) > 0$ such that for any two $(-\Delta)^{\alpha/2}$– harmonic functions u, v in D which are strictly positive and bounded on $V \cap D$, and vanish on $V \cap D^c$, we have*

$$\frac{u(x)}{v(x)} \leq C \frac{u(y)}{v(y)}, \quad x, y \in K \cap D.$$

For fixed $y \in D$ and $z \in \overline{D}^c$, it is easy to check that $\{G_D(X_t^D, y), \mathscr{F}_t^D\}_{t \geq 0}$ and $\{K_D(X_t^D, z), \mathscr{F}_t^D\}_{t \geq 0}$ are nonnegative P_x-supermartingales for each $x \in D$. So if we extend $G_D(\cdot, y)$ and $K_D(\cdot, z)$ to be zero outside D, then $G_D(\cdot, y)$ and $K_D(\cdot, z)$ are $(-\Delta)^{\alpha/2}$–superharmonic in D.

We can take the h-transform of X^D, where $h(x)$ is taken to be $G_D(x, y)$ or $K_D(x, z)$. For each $y \in D$, the $G(\cdot, y)$-transformed process of X^D is called the y-conditioned symmetric stable process whose state space is $(D \setminus \{y\}) \cup \{\partial\}$, where ∂ is the added cemetery point. The lifetime of the conditional process is $\zeta = \tau_{D \setminus \{y\}}$. We continue to use X_t^D to denote the generic random variable of the conditional process, but use P_x^y and E_x^y to denote its probability and expectation respectively. For $z \in \overline{D}^c$, the $K(\cdot, z)$-transformed process is called the z–conditioned symmetric stable process whose state space is $D \cup \{\partial\}$ and its lifetime is $\zeta = \tau_D$. Its probability and expectation are denoted as P_x^z and E_x^z respectively. Using Theorem 1.6, one can show easily that any bounded $C^{1,1}$ domain is a Cranston-McConnell domain for symmetric stable process. That is,

Theorem 1.8 (Conditional Lifetimes). *Suppose that D is a bounded $C^{1,1}$ domain in $\mathbf{R}^n$. Then*

$$\sup_{z \in \overline{D}^c, x \in D} E_x^z(\tau_D) < \infty \quad and \quad \sup_{y \in D, x \in D \setminus \{y\}} E_x^y(\tau_{D \setminus \{y\}}) < \infty.$$

In [5], we apply the estimates in Theorems 1.1-1.2 and Theorem 1.5 to show that logarithmic Sobolev inequality and intrinsic ultracontractivity hold for symmetric α-stable processes in bounded $C^{1,1}$ domains. We then use these to establish the conditional gauge theorem for symmetric α-stable processes.

The rest of the paper is organized as follows. In Sect. 2 we prove the upper bound estimates Theorem 1.1 for Green function G_D. Due to the length of its argument, the proof for Theorem 1.2 is postponed until Sect. 6. Theorem 1.4 and 1.5 are proved in Sect. 3. The boundary Harnack principle is proved in Sect. 4, the $3G$ Theorem and conditional lifetime Theorem are proved in Sect. 5.

In the sequel we use ω_n to denote the surface area of the unit ball in $\mathbf{R}^n$. That is, $\omega_n = 2\pi^{n/2}\Gamma(\frac{n}{2})^{-1}$. In the proofs of this paper, constants c and C, which do not change their dependence, may change their values from line to line. The Lebesgue measure of a Borel measurable set A will be denoted as $|A|$. For a bounded domain D in $\mathbf{R}^n$, we use d_D to denote the diameter of D.

Although the main results of this paper are stated for bounded $C^{1,1}$ domains only, the assumption about the connectedness of D is not really needed in the proof. All our proofs will go through if D has a finite number of components D_i such that each D_i is bounded $C^{1,1}$ and that $\overline{D_i}$ and $\overline{D_j}$ are disjoint for $i \neq j$.

2. Upper bound estimates for Green functions

It is easy to see from the strong Markov property of X that for each $y \in D$ and $z \in \overline{D}^c$, $G_D(\cdot, y)$ and $K_D(\cdot, z)$ are superharmonic functions in D. If f and g are superharmonic, then so is $f \wedge g$. If f is superharmonic in D and $f \geq 0$ on D^c, then $f \geq 0$ on $\mathbf{R}^n$ by the limit theorem for supermartingales.

The following proposition follows immediately from the scaling property of the symmetric stable processes, so we will omit the proof.

Proposition 2.1. *Let D be a domain in $\mathbf{R}^n$ and $a > 0$ be a constant. Then*

$$G_D(x, y) = a^{\alpha - n} G_{D/a}(x/a, y/a), \quad x, y \in D.$$

Lemma 2.2. *There exists a constant $C = C(n, \alpha) > 0$ such that for any ball $B \subset \mathbf{R}^n$ we have*

$$G_B(x, y) \leq C \frac{\delta_B(x)^{\alpha/2} \delta_B(y)^{\alpha/2}}{|x - y|^n}, \quad x, y \in B,$$

where G_B is the Green function of B and $\delta_B(x) = d(x, \partial B)$.

Proof. We first consider the case that $B = B(0, 1)$. It follows from [3] that

$$G_B(x, y) = 2^{-\alpha} \pi^{-n/2} \Gamma\left(\frac{\alpha}{2}\right)^{-2} \Gamma\left(\frac{n}{2}\right) \int_0^z (u + 1)^{-n/2} u^{\alpha/2 - 1} du \, |x - y|^{\alpha - n}$$

where $z = (1 - |x|^2)(1 - |y|^2)|x - y|^{-2}$. Note that

$$\int_0^z (u + 1)^{-n/2} u^{\alpha/2 - 1} du$$

$$= z^{\alpha/2} \int_0^1 (1 + vz)^{-n/2} v^{\alpha/2 - 1} dv$$

$$= (1 - |x|)^{\alpha/2} (1 - |y|)^{\alpha/2} (1 + |x|)^{\alpha/2} (1 + |y|)^{\alpha/2} |x - y|^{-\alpha} \cdot$$

$$\cdot \int_0^1 (1 + vz)^{-n/2} v^{\alpha/2 - 1} dv$$

$$\leq \left(\int_0^1 (1 + vz)^{-n/2} v^{\alpha/2 - 1} dv\right) 2^{\alpha} \delta_B(x)^{\alpha/2} \delta_B(y)^{\alpha/2} |x - y|^{-\alpha}.$$

Since

$$\int_0^1 (1 + vz)^{-n/2} v^{\alpha/2 - 1} dv < \int_0^1 v^{\alpha/2 - 1} dv < \infty$$

the assertion of Lemma 2.2 holds for $B = B(0, 1)$. For $B = B(0, r)$ with radius r, by Proposition 2.1, $G_{B(0,r)}(x, y) = r^{-n+\alpha} G_{B(0,1)}(\frac{x}{r}, \frac{y}{r})$, we have

$$G_{B(0,r)} \leq C \frac{\delta_B(x)^{\alpha/2} \delta_B(y)^{\alpha/2}}{|x - y|^n}, \quad \forall x, y \in B.$$

$\square$

Lemma 2.3. *There exists $C = C(n, \alpha) > 0$ such that for any ball $B \subset \mathbf{R}^n$ we have*

$$G_B(x, y) \le C \, \frac{\delta_B(x)^{\alpha/2}}{\delta_B(y)^{\alpha/2}|x - y|^{n-\alpha}}, \quad x, y \in B.$$

Proof. Let $G(x, y)$ be the Green function of the whole space. Clearly

$$G_B(x, y) \le G(x, y).$$

Combining this with Lemma 2.2 and formula (1.1), we have

$$
\begin{aligned}
G_B(x, y) &\le G(x, y) \wedge \left(C \frac{\delta_B(x)^{\alpha/2} \delta_B(y)^{\alpha/2}}{|x - y|^n} \right) \\
&\le \frac{C \delta_B(x)^{\alpha/2}}{\delta_B(y)^{\alpha/2}|x - y|^{n-\alpha}} \left(\frac{\delta_B(y)^{\alpha/2}}{\delta_B(x)^{\alpha/2}} \wedge \frac{\delta_B(y)^{\alpha}}{|x - y|^{\alpha}} \right).
\end{aligned}
$$

If $\delta_B(y) > 2\delta_B(x)$, then $|x - y| \ge \delta_B(y) - \delta_B(x) > \frac{1}{2}\delta_B(y)$ and so

$$\frac{\delta_B(y)^{\alpha/2}}{\delta_B(x)^{\alpha/2}} \wedge \frac{\delta_B(y)^{\alpha}}{|x - y|^{\alpha}} \le \frac{\delta_B(y)^{\alpha}}{|x - y|^{\alpha}} \le 2^{\alpha}.$$

If $\delta_B(y) \le 2\delta_B(x)$, then

$$\frac{\delta_B(y)^{\alpha/2}}{\delta_B(x)^{\alpha/2}} \wedge \frac{\delta_B(y)^{\alpha}}{|x - y|^{\alpha}} \le \frac{\delta_B(y)^{\alpha/2}}{\delta_B(x)^{\alpha/2}} \le 2^{\alpha/2}.$$

Lemma 2.3 is thus proved. $\square$

Lemma 2.4. *There exists a constant $C = C(n, \alpha) > 0$ such that for any ball $B \subset \mathbf{R}^n$ we have*

$$G_B(x, y) \le C \, \frac{\delta_B(x)^{\alpha/2}}{|x - y|^{n-\alpha/2}}, \quad x, y \in B.$$

Proof. Clearly Lemma 2.4 holds for $x = y$. For $x, y \in B$ with $x \ne y$, by Lemma 2.2 and Lemma 2.3,

$$G_B(x, y) \le \frac{C \, \delta_B(x)^{\alpha/2}}{|x - y|^{n-\alpha/2}} \left(\frac{|x - y|^{\alpha/2}}{\delta_B(y)^{\alpha/2}} \wedge \frac{\delta_B(y)^{\alpha/2}}{|x - y|^{\alpha/2}} \right) \le \frac{C \, \delta_B(x)^{\alpha/2}}{|x - y|^{n-\alpha/2}}.$$

$\square$

Lemma 2.5. *There exists a constant $C = C(n, \alpha) > 0$ such that for any $B = B(a, r) \subset \mathbf{R}^n$ we have*

$$G_{B^c}(x, y) \le C \, |y - a|^{\alpha/2} \frac{\delta_B(x)^{\alpha/2}}{|x - y|^{n-\alpha/2}}, \quad x, y \in B^c,$$

where G_{B^c} is the Green function of B^c.

Proof. In view of Proposition 2.1, we may assume that $B = B(0, 1)$. Then from the discussions on pages 263-265 of [16] one can easily show that

$$G_{B^c}(x, y) = |x^*|^{n-\alpha}|y^*|^{n-\alpha}G_B(x^*, y^*)$$

where $x^* = x/|x|^2$ and $y^* = y/|y|^2$. Since

$$|x^* - y^*| = \frac{|x - y|}{|x| \, |y|} \quad \text{and} \quad \delta_B(x^*) = \frac{\delta_B(x)}{|x|},$$

we have by Lemma 2.4 that

$$\begin{aligned}
G_{B^c}(x, y) &\leq C|x^*|^{n-\alpha}|y^*|^{n-\alpha}\frac{\delta_B(x^*)^{\alpha/2}}{|x^* - y^*|^{n-\alpha/2}} \\
&= C|y|^{\alpha/2}\frac{\delta_B(x)^{\alpha/2}}{|x - y|^{n-\alpha/2}}.
\end{aligned}$$

$\square$

Now let D be a bounded $C^{1,1}$ domain and let G_D be the Green function of D. It is well known that there exists $r_0 > 0$ depending only on D such that for any $z \in \partial D$, $0 < r \leq r_0$, there exist two balls $B_1^z(r)$ and $B_2^z(r)$ of radius r such that $B_1^z(r) \subset D$, $B_2^z(r) \subset \mathbf{R}^n \setminus \overline{D}$ and $\{z\} = \partial B_1^z(r) \cap \partial B_2^z(r)$. Let d_D be the diameter of D. In the following, we are going to assume that $r_0 < \frac{1}{2}d_D$.

Proof of (1.4). Let $x_0 \in \partial D$ be such that $|x - x_0| = \delta(x)$. Consider the ball $B = B_2^{x_0}(r_0) = B(a, r_0)$. From Lemma 2.5, there is a constant $C = C(n, \alpha) > 0$ such that

$$\begin{aligned}
G_D(x, y) &\leq G_{B^c}(x, y) \leq C|y - a|^{\alpha/2}\frac{\delta_B(x)^{\alpha/2}}{|x - y|^{n-\alpha/2}} \\
&= C|y - a|^{\alpha/2}\frac{\delta(x)^{\alpha/2}}{|x - y|^{n-\alpha/2}}.
\end{aligned}$$

$\square$

Proof of (1.6). From (1.3) and (1.4) we know that

$$G_D(x, y) \leq C_1\frac{1}{|x - y|^{n-\alpha}},$$

$$G_D(x, y) \leq C_2\frac{\delta(x)^{\alpha/2}}{|x - y|^{n-\alpha/2}}$$

Since $\delta(y) \leq \delta(x) + |x - y|$, we have $\delta(y)^{\alpha/2} \leq \delta^{\alpha/2}(x) + |x - y|^{\alpha/2}$. Therefore,

$$\begin{aligned}
\delta(y)^{\alpha/2}G_D(x, y) &\leq \delta(x)^{\alpha/2}G_D(x, y) + |x - y|^{\alpha/2}G_D(x, y) \\
&\leq C_1\frac{\delta(x)^{\alpha/2}}{|x - y|^{n-\alpha}} + C_2\frac{\delta(x)^{\alpha/2}}{|x - y|^{n-\alpha}}.
\end{aligned}$$

Hence

$$G_D(x,y) \le (C_1 + C_2)\frac{\delta(x)^{\alpha/2}}{\delta(y)^{\alpha/2}}\frac{1}{|x-y|^{n-\alpha}}.$$

$\square$

Proof of (1.5). If $\delta(y) \ge r_0$ or $\delta(y) < r_0$ and $|x - y| \le 8\delta(y)$, then (1.5) follows from (1.4). So we assume that $\delta(y) < r_0$ and $|x - y| > 8\delta(y)$. Set $r = \min(|x - y|/8, r_0)$. Let $y_0 \in \partial D$ be such that $|y - y_0| = \delta(y)$ and let $B(a,r) = B_1^{y_0}(r)$. Without loss of generality, we can assume that a is at the origin and that $y_0 = (0, \cdots, 0, -r)$. Then $B_2^{y_0}(r) = B(b,r)$ where $b = (0, \cdots, 0, -2r)$. By the explicit formula for the Poisson kernel on the ball $B(0,r)$ given in [3] and the strong Markov property, we have

$$
\begin{aligned}
G_D(x,y) &= E_y[G_D(x, X^D_{\tau_{B(0,r)}})]\\
&= C\int_{D\setminus B(0,r)} \frac{(r^2 - |y|^2)^{\alpha/2}}{(|u|^2 - r^2)^{\alpha/2}} \frac{1}{|y - u|^n} G_D(x,u)du\\
&\le C\delta(x)^{\alpha/2}\delta(y)^{\alpha/2} \int_{D\setminus B(0,r)} \frac{du}{(|u| - r)^{\alpha/2}|y - u|^n|x - u|^{n - \frac{\alpha}{2}}}\\
&\le C\delta(x)^{\alpha/2}\delta(y)^{\alpha/2} \int_{D_0} \frac{du}{(|u| - r)^{\alpha/2}|u - y|^n|u - x|^{n - \frac{\alpha}{2}}},
\end{aligned}
$$

where in the first inequality above we used (1.4) and in the last expression above D_0 is the set $B(b,r)^c \setminus B(0,r)$. Let $u = rv$, $\tilde{x} = \frac{x}{r}$, $\tilde{y} = \frac{y}{r}$. Then

$$\int_{D_0} \frac{du}{(|u| - r)^{\alpha/2}|u - y|^n|u - x|^{n - \frac{\alpha}{2}}} = \frac{1}{r^n}\int_{D_1} \frac{dv}{(|v| - 1)^{\alpha/2}|v - \tilde{y}|^n|v - \tilde{x}|^{n - \frac{\alpha}{2}}},$$

where $D_1 = \frac{1}{r}D_0 = B((0, \cdots, 0, -2), 1)^c \setminus B(0,1))$. Note that $|\tilde{y}| < 1$ and $|\tilde{x}| > 7$. The proof will be finished if we can show that the function

$$F(x,y) = \int_{D_1} \frac{dv}{(|v| - 1)^{\alpha/2}|v - y|^n|v - x|^{n - \frac{\alpha}{2}}}$$

is bounded on the set

$$\overline{B((0, \cdots, 0, -2), 1)^c \cap \{|x| > 7\}} \times \overline{B(0,1) \cap \{(0, \cdots, 0, y_n) : y_n < 0\}}$$

by a constant depending only on D and α. In order to accomplish this, one only has to show that the function $F(x, (0, \cdots, 0, -1))$ is bounded on the set $\overline{B((0, \cdots, 0, -2), 1)^c \cap \{|x| > 7\}}$, which can be accomplished by elementary analysis. We omit the details here. $\square$

3. Poisson kernel estimates

Let D be a bounded domain in $\mathbf{R}^n$ and let X be the symmetric α–stable process on $\mathbf{R}^n$. In this Sect. we are going to prove Theorem 1.4 first, and then we will use Theorem 1.4 to prove Theorem 1.5. To prove Theorem 1.4, we need the following result:

Theorem 3.1. *For any Borel measurable $\phi \geq 0$ on D^c*

$$(3.1) \qquad E_x[\phi(X_{\tau_D}); X_{\tau_D} \neq X_{\tau_D-}] = \int_{D^c} \phi(z)dz \int_D \frac{G_D(x,y)}{|y-z|^{n+\alpha}}dy.$$

Proof. It is known (see, e.g., page 19 of [2]) that the Lévy system (N, H) of the α-symmetric stable process X is given by

$$N(x,dy) = A(n,\alpha)\frac{1}{|x-y|^{n+\alpha}}dy, \quad H_t = t,$$

where $A(n,\alpha)$ is given by (1.10). Extend ϕ to $\mathbf{R}^n$ by defining it to be zero in D. For any non-negative Borel measurable function f on $\mathbf{R}^n$, set

$$F(x) = A(n,\alpha)\int_{\mathbf{R}^n} \frac{f(x)\phi(y)}{|x-y|^{n+\alpha}}dy = A(n,\alpha)\int_{D^c} \frac{f(x)\phi(y)}{|x-y|^{n+\alpha}}dy.$$

Define

$$\kappa(\omega, dt) = \sum_{s>0} f(X_{s-}(\omega))\phi(X_s(\omega))\delta_s(dt)1_{\{X_{s-}(\omega)\neq X_s(\omega)\}}.$$

By Theorems (73.1) and (73.4) of [19] we know that the dual predictable projection of κ is given by

$$\kappa^P(dt) = F(X_t)dt.$$

Since the random process $1_{\{(t,\omega):0<t\leq\tau_D(\omega)\}}$ is left-continuous and is therefore predictable,

$$
\begin{aligned}
E_x \sum_{0<s\leq\tau_D} f(X_{s-})\phi(X_s)1_{\{X_{s-}\neq X_s\}} &= E_x \int_0^{\tau_D} F(X_s)ds \\
&= \int_D G_D(x,y)F(y)dy \\
&= A(n,\alpha)\int_D G_D(x,y)\int_{D^c} \frac{f(y)\phi(z)}{|y-z|^{n+\alpha}}dzdy \\
&= A(n,\alpha)\int_{D^c} \phi(z)dz \int_D \frac{G_D(x,y)f(y)}{|y-z|^{n+\alpha}}dy.
\end{aligned}
$$

Therefore

$$E_x\left(f(X_{\tau_D-})\phi(X_{\tau_D})1_{\{X_{\tau_D-}\neq X_{\tau_D}\}}\right) = A(n,\alpha)\int_{D^c} \phi(z)dz \int_D \frac{G_D(x,y)f(y)}{|y-z|^{n+\alpha}}dy.$$

In particular,

$$E_x[\phi(X_{\tau_D}); X_{\tau_D} \neq X_{\tau_D-}] = A(n,\alpha) \int_{D^c} \phi(z)dz \int_D \frac{G_D(x,y)}{|y-z|^{n+\alpha}} dy.$$

$\square$

Proof of Theorem 1.4. It follows from Lemma 6 of Bogdan [4] that for any bounded domain D satisfying uniform exterior cone condition,

(3.2) $$P_x(X_{\tau_D} \neq X_{\tau_D-}) = 1, \qquad x \in D.$$

The theorem now follows immediately from (3.1) and (3.2). $\square$

Theorem 3.2. *Suppose D is a bounded $C^{1,1}$ domain in $\mathbf{R}^n$. Then there is a constant $C = C(D,\alpha) > 0$ such that*

$$K_D(x,z) \leq C \frac{\delta(x)^{\alpha/2}}{(1+\delta(z))^{\alpha/2}\delta(z)^{\alpha/2}} \frac{1}{|x-z|^n}, \qquad x \in D, \ z \in \overline{D}^c.$$

Proof. By Theorem 1.4

$$
\begin{aligned}
K_D(x,z) &= A(n,\alpha) \int_D \frac{G_D(x,y)}{|y-z|^{n+\alpha}} \, dy \\
&\leq A(n,\alpha) \frac{2^n}{|x-z|^n} \int_D \frac{G_D(x,y)}{|y-z|^{n+\alpha}}(|x-y|^n + |y-z|^n) \, dy \\
&\leq A(n,\alpha) \frac{2^n}{|x-z|^n} \left(\int_D \frac{G_D(x,y)|x-y|^n}{|y-z|^{n+\alpha}} \, dy + \int_D \frac{G_D(x,y)}{|y-z|^\alpha} \, dy \right) \\
&= A(n,\alpha) \frac{2^n}{|x-z|^n} (I + II).
\end{aligned}
$$

(3.3)

By Theorem 1.1, there is a constant $C = C(D,\alpha) > 0$ such that

$$G_D(x,y) \leq C \frac{\delta(x)^{\alpha/2}\delta(y)^{\alpha/2}}{|x-y|^n}.$$

We have

$$
\begin{aligned}
I &\leq C\delta(x)^{\alpha/2} \int_D \frac{\delta(y)^{\alpha/2}}{|y-z|^{n+\alpha}} \, dy \\
&\leq C\delta(x)^{\alpha/2} \int_D \frac{1}{|y-z|^{n+\alpha/2}} \, dy \\
&\leq C\delta(x)^{\alpha/2}\omega_n \int_{\delta(z)}^{\delta(z)+d_D} \frac{1}{r^{n+\alpha/2}} r^{n-1} dr \\
&\leq \frac{2C\omega_n}{\alpha} \delta(x)^{\alpha/2}(-r^{-\alpha/2})|_{\delta(z)}^{\delta(z)+d_D} \\
&\leq \frac{2C\omega_n}{\alpha} \frac{\delta(x)^{\alpha/2}}{\delta(z)^{\alpha/2}}.
\end{aligned}
$$

(3.4)

Using

$$G_D(x,y) \le C \frac{\delta(x)^{\alpha/2}}{|x-y|^{n-\alpha/2}},$$

we get

$$II \le C\delta(x)^{\alpha/2} \int_D \frac{1}{|x-y|^{n-\alpha/2}|y-z|^\alpha}\, dy$$

$$\le C\delta(x)^{\alpha/2} \left\{ \int_{\{y\in D: |x-y| \ge |y-z|\}} \frac{1}{|y-z|^{n+\alpha/2}} dy + \right.$$

$$+ \int_{\{y\in D: |x-y| < |y-z|\}} \frac{1}{|x-y|^{n-\alpha/2}} \frac{1}{|y-z|^\alpha} dy \left. \right\}$$

$$\le C\delta(x)^{\alpha/2} \left\{ \frac{1}{\delta(z)^{\alpha/2}} \right.$$

$$+ \int_{\{y\in D: |x-y| < |y-z|\}} \frac{1}{|x-y|^{n-\alpha/2}\left(\big||x-z|-|x-y|\big| \vee |x-y|\right)^\alpha}\, dy \left. \right\}.$$

In the last inequality, we used (3.4) and the triangle inequality. Using spherical coordinates centered at x,

$$II \le C\delta(x)^{\alpha/2} \left\{ \frac{1}{\delta(z)^{\alpha/2}} + \int_0^{d_D} \frac{r^{n-1}}{r^{n-\alpha/2}(|a-r| \vee r)^\alpha}\, dr \right\}$$

$$\text{where } a = |x-z|$$

$$\overset{r=as}{\le} c\delta(x)^{\alpha/2} \left\{ \frac{1}{\delta(z)^{\alpha/2}} + \frac{1}{a^{\alpha/2}} \int_0^{d_D/a} \frac{s^{\alpha/2-1}}{(|1-s| \vee s)^\alpha}\, ds \right\}.$$

Since $0 < \alpha < 2$,

$$\int_0^{d_D/a} \frac{s^{\alpha/2-1}}{(|1-s| \vee s)^\alpha}\, ds \le \int_0^\infty \frac{s^{\alpha/2-1}}{(|1-s| \vee s)^\alpha} ds < \infty,$$

thus

$$II \le C \frac{\delta(x)^{\alpha/2}}{\delta(z)^{\alpha/2}}$$

for some constant $C > 0$. This together with (3.3) and (3.4), implies

$$K_D(x,z) \le A(n,\alpha)C \frac{\delta(x)^{\alpha/2}}{\delta(z)^{\alpha/2}} \frac{1}{|x-z|^n} \quad \text{for} \quad x \in D, \ z \in D^c.$$

In particular, if $z \in D^c$ with $\delta(z) \le d_D$, then

$$K_D(x,z) \le A(n,\alpha)C(1+d_D)^{\alpha/2} \frac{\delta(x)^{\alpha/2}}{\delta(z)^{\alpha/2}(1+\delta(z))^{\alpha/2}} \frac{1}{|x-z|^n}.$$

If $\delta(z) > d_D$, then for $y \in D$, $\delta(z) \le |y-z| \le 2\delta(z)$. Therefore

$$
\begin{aligned}
K(x, z) &= A(n, \alpha) \int_D \frac{G_D(x, y)}{|y - z|^{n+\alpha}} \, dy \\
&\le A(n, \alpha) \frac{1}{\delta(z)^{n+\alpha}} \int_D G_D(x, y) dy \\
&\le A(n, \alpha) \frac{1}{\delta(z)^{n+\alpha}} \int_D \frac{C_1 \, \delta(x)^{\alpha/2}}{|x - y|^{n-\alpha/2}} \, dy \\
&\le A(n, \alpha) C \frac{\delta(x)^{\alpha/2}}{\delta(z)^{\alpha}} \frac{1}{\delta(z)^n} \\
&\le A(n, \alpha) C \frac{\delta(x)^{\alpha/2}}{\delta(z)^{\alpha}} \frac{1}{|x - z|^n} \\
&\le A(n, \alpha) C \frac{\delta(x)^{\alpha/2}}{\delta(z)^{\alpha/2}(1 + \delta(z))^{\alpha/2}} \frac{1}{|x - z|^n}.
\end{aligned}
$$

This completes the proof of the theorem.   □

Before going to estimate the lower bound of K_D, we first record three simple facts about bounded Lipschitz domains in the following lemma. The proof of this lemma is straightforward and thus omitted. We say a Lipschitz domain D has Lipschitz characteristic constants (r_0, A_0) if for every $z \in \partial D$, there is a local coordinate system $(\xi_1, \xi^{(1)}) \in \mathbf{R} \times \mathbf{R}^{n-1}$ with origin sitting at z and there is a Lipschitz function f defined on $\mathbf{R}^{n-1}$ with Lipschitz constant A_0 such that $D \cap B(z, r_0) = B(z, r_0) \cap \{\xi = (\xi_1, \xi^{(1)}) : \xi_1 > f(\xi^{(1)})\}$.

Lemma 3.3. *(1) If D is a Lipschitz domain with characteristic constants (r_0, A_0) and $a > 0$ is a constant, then the dilation aD has Lipschitz characteristic constant (ar_0, A_0).*

(2) Suppose that $0 \in \overline{D}$ and $r > 0$, $0 < \alpha < 2$. Then there is a constant $C_0 > 0$ depending only on r, α, and the Lipschitz characteristic constants (r_0, A_0) of D such that

$$
\int_{D \cap B(0,r)} \frac{1}{|y|^{n-\alpha}} \left(1 \wedge \frac{\delta(y)^{\alpha/2}}{|y|^{\alpha}} \right) dy \ge C_0 > 0.
$$

(3) There is a constant $C_0 = C_0(r_0, A_0, r) > 0$ such that

$$
\int_{D \cap B(x,r)} \delta(y)^{\alpha/2} dy \ge C_0 > 0 \quad \text{for all} \quad x \in \partial D.
$$

Theorem 3.4. *Suppose D is a bounded $C^{1,1}$ domain in $\mathbf{R}^n$. Then there is a constant $C = C(D, \alpha) > 0$ such that*

$$
K_D(x, z) \ge C \frac{\delta(x)^{\alpha/2}}{\delta(z)^{\alpha/2}(1 + \delta(z))^{\alpha/2}} \frac{1}{|x - z|^{n+\alpha}}, \quad x \in D, \, z \in \overline{D}^c.
$$

Proof. Let (r_0, A_0) denote the Lipschitz characteristic constants for D. By Theorem 1.2, we have

$$G_D(x,y) \geq C \min \left\{ \frac{1}{|x-y|^{n-\alpha}}, \frac{\delta(x)^{\alpha/2}\delta(y)^{\alpha/2}}{|x-y|^n} \right\}, \quad x,y \in D.$$

Recall that d_D is the diameter of D. We derive the estimate by considering two cases.

Case 1. $\delta(z) \leq d_D$.

Using change of variable, we have

$$K_D(x,z)$$

$$= A(n,\alpha) \int_D \frac{G_D(x,y)}{|y-z|^{n+\alpha}} \, dy$$

$$\geq A(n,\alpha) C\delta(x)^{\alpha/2} \int_D \frac{1}{|y-z|^{n+\alpha}} \frac{1}{|x-y|^{n-\alpha}} \left(\frac{1}{\delta(x)^{\alpha/2}} \wedge \frac{\delta(y)^{\alpha/2}}{|x-y|^\alpha} \right) dy$$

$$\overset{y-x=|x-z|\widetilde{y}}{\geq} A(n,\alpha) \frac{C\delta(x)^{\alpha/2}}{|x-z|^{n+\alpha/2}} \int_{\widetilde{D}} \frac{1}{(1+|\widetilde{y}|)^{n+\alpha}} \frac{1}{|\widetilde{y}|^{n-\alpha}} \left(\frac{1}{\widetilde{\delta}(\widetilde{x})^{\alpha/2}} \wedge \frac{\widetilde{\delta}(\widetilde{y})^{\alpha/2}}{|\widetilde{y}|^\alpha} \right) d\widetilde{y}$$

where $\widetilde{D} = \frac{1}{|x-z|}(D-x)$ and $\widetilde{\delta}(\widetilde{y}) = d(\widetilde{y},\partial\widetilde{D})$. Since $\widetilde{\delta}(\widetilde{x}) = \frac{\delta(x)}{|x-z|} \leq 1$,

$$K_D(x,z) \geq A(n,\alpha) \frac{C\delta(x)^{\alpha/2}}{|x-z|^{n+\alpha/2}} \int_{\widetilde{D}} \frac{1}{(1+|\widetilde{y}|)^{n+\alpha}} \frac{1}{|\widetilde{y}|^{n-\alpha}} \left(1 \wedge \frac{\widetilde{\delta}(\widetilde{y})^{\alpha/2}}{|\widetilde{y}|^\alpha} \right) d\widetilde{y}.$$

Note that $|x-z| \leq 2d_D$, so $\widetilde{D}$ has Lipschitz characteristic constants $\left(\frac{r_0}{2d_D}, A_0 \right)$. Let $0 < r < \frac{r_0}{2d_D}$ be fixed. Then

$$K_D(x,z)$$

$$\geq A(n,\alpha) \frac{C\delta(x)^{\alpha/2}}{|x-z|^{n+\alpha/2}} \int_{\widetilde{D}\cap B(0,r)} \frac{1}{(1+r)^{n+\alpha}} \frac{1}{|\widetilde{y}|^{n-\alpha}} \left(1 \wedge \frac{\widetilde{\delta}(\widetilde{y})^{\alpha/2}}{|\widetilde{y}|^\alpha} \right) d\widetilde{y}$$

$$\geq \frac{C\delta(x)^{\alpha/2}}{|x-z|^{n+\alpha/2}} \quad \text{by Lemma 3.3(2)}. \tag{3.5}$$

1°. If $\delta(z) \geq r_0$, then by (3.5)

$$K_D(x,z) \geq A(n,\alpha) \frac{Cr_0^{\alpha/2}}{\delta(z)^{\alpha/2}} \frac{\delta(x)^{\alpha/2}}{|x-z|^n (2d_D)^{\alpha/2}} \geq A(n,\alpha)C \frac{\delta(x)^{\alpha/2}}{\delta(z)^{\alpha/2}|x-z|^n}.$$

2°. If $|x-z| \leq 4\delta(z)$, then by (3.5)

$$K_D(x,z) \geq A(n,\alpha) \frac{C\delta(x)^{\alpha/2}}{|x-z|^n (4\delta(z))^{\alpha/2}} \geq A(n,\alpha) \frac{C\delta(x)^{\alpha/2}}{\delta(z)^{\alpha/2}|x-z|^n}.$$

3°. Lastly, if $|x-z| > 4\delta(z)$ and $\delta(z) < r_0$, then by a change of variable

$$K_D(x,z) = A(n,\alpha)\int_D \frac{G_D(x,y)}{|y-z|^{n+\alpha}}\,dy$$

$$\geq A(n,\alpha)C\delta(x)^{\alpha/2}\int_{\{y\in D:|y-z|<\frac{1}{2}|x-z|\}}\frac{\delta(y)^{\alpha/2}}{|x-y|^n|y-z|^{n+\alpha}}\,dy$$

$$\overset{y-z=\delta(z)\widetilde{y}}{=} A(n,\alpha)\frac{C\delta(x)^{\alpha/2}}{\delta(z)^{\alpha/2}|x-z|^n}\int_{\{\widetilde{y}\in\widetilde{D}:|\widetilde{y}|<\frac{1}{2}|\widetilde{x}|\}}\frac{|\widetilde{x}|^n\widetilde{\delta}(\widetilde{y})^{\alpha/2}}{|\widetilde{x}-\widetilde{y}|^n|\widetilde{y}|^{n+\alpha}}\,d\widetilde{y}$$

where

$$\widetilde{D} = \frac{1}{\delta(z)}(D-z) \quad \text{and} \quad \widetilde{x} = \frac{x-z}{\delta(z)}.$$

Thus

$$K_D(x,z)$$

$$\geq A(n,\alpha)\frac{C\delta(x)^{\alpha/2}}{\delta(z)^{\alpha/2}|x-z|^n}\int_{\{\widetilde{y}\in\widetilde{D}:|\widetilde{y}|<\frac{1}{2}|\widetilde{x}|\}}\frac{|\widetilde{x}|^n\widetilde{\delta}(\widetilde{y})^{\alpha/2}}{(|\widetilde{x}|+|\widetilde{y}|)^n|\widetilde{y}|^{n+\alpha}}\,d\widetilde{y}$$

$$\overset{\text{since }|\widetilde{x}|>4}{\geq} A(n,\alpha)\frac{C\delta(x)^{\alpha/2}}{\delta(z)^{\alpha/2}|x-z|^n}\int_{\{\widetilde{y}\in\widetilde{D}:|\widetilde{y}|\leq2\}}\widetilde{\delta}(\widetilde{y})^{\alpha/2}d\widetilde{y}. \tag{3.6}$$

Note that $\widetilde{D}$ has Lipschitz characteristic constants $\left(\frac{r_0}{\delta(z)},A_0\right)$ with $\frac{r_0}{\delta(z)}>1$. Let $y^*\in\partial\widetilde{D}$ such that $|y^*|=d(0,\partial\widetilde{D})=1$. Then by (3.6), and Lemma 3.3(3),

$$K_D(x,z) \geq A(n,\alpha)\frac{C\delta(x)^{\alpha/2}}{\delta(z)^{\alpha/2}|x-z|^n}\int_{\widetilde{D}\cap B(y^*,1)}\widetilde{\delta}(\widetilde{y})^{\alpha/2}d\widetilde{y}$$

$$\geq A(n,\alpha)\frac{C\delta(x)^{\alpha/2}}{\delta(z)^{\alpha/2}|x-y|^n}.$$

In summary, we have for $\delta(z)\leq d_D$,

$$K_D(x,z)\geq A(n,\alpha)\frac{C\delta(x)^{\alpha/2}}{\delta(z)^{\alpha/2}|x-y|^n}\geq A(n,\alpha)\frac{C\delta(x)^{\alpha/2}}{\delta(z)^{\alpha/2}(1+\delta(z))^{\alpha/2}}\,\frac{1}{|x-z|^n}.$$

Case 2. $\delta(z)>d_D$

In this remaining case, for any $y\in D$, we have $\delta(z)\leq|y-z|\leq2\delta(z)$. Thus

$$K_D(x,z) = A(n,\alpha)\int_D\frac{G_D(x,y)}{|y-z|^{n+\alpha}}\,dy$$

$$\geq A(n,\alpha)\frac{1}{2^{n+\alpha}\delta(z)^{n+\alpha}}\int_D G_D(x,y)dy$$

$$\geq A(n,\alpha)\frac{C\delta(x)^{\alpha/2}}{\delta(z)^{n+\alpha}}\int_D\min\left\{\frac{1}{|x-y|^{n-\alpha}\delta(x)^{\alpha/2}},\frac{\delta(y)^{\alpha/2}}{|x-y|^n}\right\}dy$$

$$\geq A(n,\alpha)\frac{C\delta(x)^{\alpha/2}}{\delta(z)^{\alpha}|x-z|^n}\int_D\left(\frac{1}{d_D^{\alpha/2}|x-y|^{n-\alpha}}\wedge\frac{\delta(y)^{\alpha/2}}{|x-y|^n}\right)dy. \tag{3.7}$$

Note that

$$\varphi(x) = \int_D \left(\frac{1}{d_D^{\alpha/2}|x-y|^{n-\alpha}} \wedge \frac{\delta(y)^{\alpha/2}}{|x-y|^n} \right) dy$$

is strictly positive and continuous on $\overline{D}$, thus $\inf_{x \in D} \varphi(x) \geq C_0 > 0$. So by (3.7)

$$K_D(x,z) \geq A(n,\alpha) \frac{C\delta(x)^{\alpha/2}}{\delta(z)^{\alpha/2}(1+\delta(z))^{\alpha/2}} \frac{1}{|x-z|^n}.$$

$\square$

4. Boundary Harnack inequality

By using the strong Markov property and the quasi-left-continuity of X, one can show that if a function u defined on $\mathbf{R}^n$ is $(-\Delta)^{\alpha/2}$–harmonic in D, then for any open set D_1 with $\overline{D}_1 \subset D$,

$$u(x) = E_x u(X(\tau_{D_1})), \quad \text{for } x \in D_1.$$

Proof of Theorem 1.7. Without loss of generality, we can assume that for any $x \in D$,

$$u(x) = E_x u(X(\tau_D)), \quad v(x) = E_x v(X(\tau_D)).$$

Otherwise, we can take a $C^{1,1}$ domain $D_1 \subset D$ and a set V_1 with $K \subset V_1 \subset V$ such that $(\partial D) \cap V_1 \subset (\partial D) \cap (\partial D_1) \subset V$. Then u and v are bounded and continuous in D_1 and vanishes in $D_1^c \cap V_1$. From Theorem 1.4 and the quasi-left-continuity of X one can show that

$$u(x) = E_x u(X(\tau_{D_1})), \quad v(x) = E_x v(X(\tau_{D_1})).$$

Then we can work with D_1, V_1, K instead of K, V, D.

Let $r_0 = d(K \cap D, D^c \setminus V) > 0$ and d_0 be the diameter of $D \cap K$. Then by Theorem 1.5 for $x \in D$

$$
\begin{aligned}
u(x) &= \int_{D^c} K_D(x,z)u(z)dz \\
&\geq C^{-1}\delta(x)^{\alpha/2} \int_{D^c} \frac{u(z)}{\delta(z)^{\alpha/2}(1+\delta(z))^{\alpha/2}|x-z|^n} dz \\
&= C^{-1}\delta(x)^{\alpha/2} \int_{D^c \setminus V} \frac{u(z)}{\delta(z)^{\alpha/2}(1+\delta(z))^{\alpha/2}|x-z|^n} dz. \quad (4.1)
\end{aligned}
$$

Similarly

$$(4.2) \qquad u(x) \leq C\delta(x)^{\alpha/2} \int_{D^c \setminus V} \frac{u(z)}{\delta(z)^{\alpha/2}(1+\delta(z))^{\alpha/2}|x-z|^n} dz.$$

For $x, y \in D \cap K$ and $z \in D^c \setminus V$

$$|x - z| \leq |y - z| + |x - y| \leq |y - z| + d_0 \frac{r_0}{r_0}$$

$$\leq |y - z| + \frac{d_0}{r_0}|y - z| \leq \left(1 + \frac{d_0}{r_0}\right)|y - z|.$$

Hence

$$\frac{|x - z|}{|y - z|} \leq \left(1 + \frac{d_0}{r_0}\right).$$

Interchange the role of x and y, we have

$$\left(1 + \frac{d_0}{r_0}\right)^{-1} \leq \frac{|x - z|}{|y - z|} \leq 1 + \frac{d_0}{r_0}.$$

Thus by (4.1) and (4.2)

$$C^{-1}\left(1 + \frac{d_0}{r_0}\right)^{-n} \leq \frac{u(x)}{u(y)} \leq C\left(1 + \frac{d_0}{r_0}\right)^n.$$

Similarly

$$C^{-1}\left(1 + \frac{d_0}{r_0}\right)^{-n} \leq \frac{v(x)}{v(y)} \leq C\left(1 + \frac{d_0}{r_0}\right)^n.$$

Thus there is a constant $C = C(D, V, K, \alpha) > 0$ such that

$$\frac{u(x)}{v(x)} \leq C\frac{u(y)}{v(y)} \quad \text{for } x, y \in D \cap K.$$

$\square$

5. 3G Theorem and conditional lifetimes

Proof of Theorem 1.6. For $x, y, w \in D$, if $|x - w| < \frac{1}{2}\max\{\delta(x), \delta(w)\}$, then by (1.3) and (1.8)

$$\frac{G_D(x, y)G_D(y, w)}{G_D(x, w)} \leq C\frac{|x - w|^{n-\alpha}}{|x - y|^{n-\alpha}|y - w|^{n-\alpha}}.$$

If $|x - w| > \frac{1}{2}\max\{\delta(x), \delta(w)\}$, then by (1.5), (1.7) and (1.9) we have

$$\frac{G_D(x, y)G_D(y, w)}{G_D(x, w)} \leq C\frac{|x - w|^n}{\delta(x)^{\alpha/2}\delta(w)^{\alpha/2}}\frac{\delta(x)^{\alpha/2}\delta(y)^{\alpha/2}}{|x - y|^n}\frac{\delta(w)^{\alpha/2}}{\delta(y)^{\alpha/2}|y - w|^{n-\alpha}}$$

$$= C\frac{|x - w|^n}{|x - y|^n|y - w|^{n-\alpha}}. \tag{5.1}$$

Interchange the role of x and w,

$$\tag{5.2} \frac{G_D(x, y)G_D(y, w)}{G_D(x, w)} \leq C\frac{|x - w|^n}{|x - y|^{n-\alpha}|y - w|^n}.$$

If $|x - y| \geq \frac{1}{2}|x - w|$, we see from (5.1) that (1.11) holds. If $|x - y| < \frac{1}{2}|x - w|$, since $|y - w| \geq |x - w| - |x - y| > \frac{1}{2}|x - w|$, we get (1.1) from (5.2). Thus (1.11) is proved.

For $x, y \in D$ and $z \in \overline{D}^c$, by Theorem 1.5

$$(5.3) \qquad \frac{G_D(x,y)K_D(y,z)}{K_0(x,z)} \leq C G_D(x,y) \frac{\delta(y)^{\alpha/2}}{\delta(x)^{\alpha/2}} \frac{|x-z|^n}{|y-z|^n}.$$

If $|x - z| \leq 2|y - z|$, by (5.3) and (1.6) we have

$$\frac{G_D(x,y)K_D(y,z)}{K_D(x,z)} \leq C \frac{|x-z|^{n-\alpha}}{|x-y|^{n-\alpha}|y-z|^{n-\alpha}}.$$

If $|x - z| > 2|y - z|$, then $|x - z| \leq |x - y| + |y - z| \leq |x - y| + \frac{1}{2}|x - z|$. So $|x - z| \leq 2|x - y|$. By (4.3) and (1.5)

$$\frac{G_D(x,y)K_D(x,y)}{K_D(x,y)} \leq C \frac{\delta(y)^\alpha}{|x-y|^n} \frac{|x-z|^n}{|y-z|^n} \leq C \frac{|x-z|^{n-\alpha}}{|x-y|^{n-\alpha}|y-z|^{n-\alpha}}.$$

$\square$

Proof of Theorem 1.8. For $y \in D$ and $x \in D \setminus \{y\}$, by (1.11)

$$
\begin{aligned}
E_x^y[\tau_{D\setminus\{y\}}] &= \int_0^\infty P_x^y\left(X_t^D \in D \setminus \{y\}\right) dt \\
&= \frac{1}{G_D(x,y)} \int_0^\infty E_x\left[G_D(X_t,y); X_t^D \in D \setminus \{y\}\right] dt \\
&= \int_D \frac{G_D(x,w)G_D(w,y)}{G_D(x,y)} dw \\
&\leq C \int_D \left(\frac{1}{|x-w|^{n-\alpha}} + \frac{1}{|y-w|^{n-\alpha}}\right) dw \\
&\leq 2C \sup_{x \in \mathbf{R}^n} \int_D \frac{1}{|x-w|^{n-\alpha}} dw \\
&\equiv C_0 < \infty.
\end{aligned}
$$

For $x \in D$, $z \in \overline{D}^c$, by (1.12)

$$
\begin{aligned}
E_x^z[\tau_D] &= \int_D \frac{G_D(x,w)K_D(w,z)}{K_D(x,z)} dw \\
&\leq C \int_D \left(\frac{1}{|x-w|^{n-\alpha}} + \frac{1}{|w-z|^{n-\alpha}}\right) dw \\
&\leq C_0 < \infty.
\end{aligned}
$$

$\square$

6. Lower bound estimates for Green functions

Since D is a bounded $C^{1,1}$ domain, we know that there exist positive constants C_0 and r_0 depending only on D such that for any $z, w \in \partial D$, $|n_z - n_w| \leq C_0 |z - w|$ (where n_z and n_w are the inward unit normal vector to ∂D at z and w respectively) and for any $z \in \partial D$, $0 < r \leq r_0$, there exist two balls $B_1^z(r)$ and $B_2^z(r)$ of radius r such that $B_1^z(r) \subset D$, $B_2^z(r) \subset \mathbf{R}^n \setminus \overline{D}$ and $\partial B_1^z(r) \cap \partial B_2^z(r) = \{z\}$.

Without loss of generality, we can assume that

$$(6.1) \qquad\qquad\qquad r_0 \leq \frac{1}{2C_0}.$$

Lemma 6.1. *For any $x \in [-1, 1]$,*

$$(2x - 1) + \sqrt{(2x - 1)^2 + 4(1 - x)^2} \geq 1.$$

Proof. Using first year calculus it can be shown that the function

$$f(x) := (2x - 1) + \sqrt{(2x - 1)^2 + 4(1 - x)^2}$$

achieves its minimum on $[-1, 1]$ at $x = \frac{1}{2}$. Thus $f(x) \geq f\left(\frac{1}{2}\right) = 1$.   □

Corollary 6.2. *For any $0 \leq s \leq r$, we have*

$$(6.2) \qquad\qquad s^2 + 2rs - 4rs \cos \varphi_1 \leq 4r^2 (1 - \cos \varphi_1)^2.$$

Proof. It is easy to see that (6.2) is true when

$$0 \leq s \leq r \left[(2 \cos \varphi_1 - 1) + \sqrt{(2 \cos \varphi_1 - 1)^2 + 4(1 - \cos \varphi_1)^2} \right].$$

From Lemma 6.1 we know that

$$(2 \cos \varphi_1 - 1) + \sqrt{(2 \cos \varphi_1 - 1)^2 + 4(1 - \cos \varphi_1)^2} \geq 1.$$

Therefore (6.2) is true for any $0 \leq s \leq r$.   □

Lemma 6.3. *For any $z, w \in \partial D$ with $z \neq w$,*

$$\liminf_{\substack{D \ni x \to z \\ D \ni y \to w}} \frac{G_D(x, y)}{\delta(x)^{\alpha/2} \delta(y)^{\alpha/2}} > 0.$$

Proof. For any fixed $0 < \beta < d_D$, the diameter of D, let $r = \min\left\{r_0, \frac{\beta}{8}\right\}$. For any $z, w \in \partial D$ with $|z - w| > \beta$, when $|x - z| < \frac{r}{2}$ and $|y - w| < \frac{r}{2}$, let x^* and y^* be the points on ∂D such that $|x - x^*| = \delta(x)$ and $|y - y^*| = \delta(y)$. Write

$$B_1^{x^*}(r) = B(O_x, r), \qquad B_1^{y^*}(r) = B(O_y, r)$$
$$B_1^z(r) = B(O_z, r), \qquad\quad B_1^w(r) = B(O_w, r).$$

Note that $x \in B(O_x, r)$. By the explicit formula for the Poisson kernel on the ball $B(O_x, r)$ given in [3] and the strong Markov property , we have

$$G_D(x,y)$$
$$= E_x\left[G_D(X_{\tau_{B(O_x,r)}}, y)\right]$$
$$= C_1 \int_{B(O_x,r)^c \cap D} \frac{(r^2 - |x - O_x|^2)^{\alpha/2}}{(|u - O_x|^2 - r^2)^{\alpha/2}} \frac{1}{|x-u|^n} G_D(u,y)du$$
$$\geq C_1^2 \int_{B(O_x,r)^c \cap D} \int_{B(O_y,r)^c \cap D} \frac{(r^2 - |x - O_x|^2)^{\alpha/2}(r^2 - |y - O_y|^2)^{\alpha/2}}{(|u - O_x|^2 - r^2)^{\alpha/2}(|v - O_y|^2 - r^2)^{\alpha/2}} \cdot$$
$$\cdot \frac{G_D(u,v)}{|x-u|^n|y-v|^n} \, du dv$$
$$= C_1^2 \int_D \int_D \frac{(r^2 - |x - O_x|^2)^{\alpha/2}(r^2 - |y - O_y|^2)^{\alpha/2}}{(|u - O_x|^2 - r^2)^{\alpha/2}(|v - O_y|^2 - r^2)^{\alpha/2}} \cdot$$
$$\cdot \frac{G_D(u,v)1_{B(O_x,r)^c}(u)1_{B(O_y,r)^c}(v)}{|x-u|^n|y-v|^n} \, du dv$$

where

$$C_1 = \pi^{-(\frac{n}{2}+1)}\Gamma\left(\frac{n}{2}\right)\sin\frac{\alpha\pi}{2}.$$

Thus

$$\frac{G_D(x,y)}{\delta(x)^{\alpha/2}\delta(y)^{\alpha/2}} \qquad \geq \qquad C_1^2 \int_D \int_D \frac{(r + |x - O_x|)^{\alpha/2}(r + |y - O_y|)^{\alpha/2}}{(|u - O_x|^2 - r^2)^{\alpha/2}(|v - O_y|^2 - r^2)^{\alpha/2}}$$
$$\cdot \frac{G_D(u,v)1_{B(O_x,r)^c}(u)1_{B(O_y,r)^c}(v)}{|x-u|^n|y-v|^n} \, du \, dv$$
$$:= h(x,y) \tag{6.3}$$

We show next that $\lim_{\substack{D \ni x \to z \\ D \ni y \to w}} h(x,y)$ exists and forms a positive and continuous function on $\{(z,w) \in \partial D \times \partial D : |z - w| > \beta\}$. For this, we set up a spherical coordinate system $(\rho, \varphi_1, \ldots, \varphi_{n-1})$ with origin O_x and principal axis $\overrightarrow{O_x x^*}$. Then for any $u = u(\rho, \varphi_1, \ldots, \varphi_{n-1}) \in B(O_x, r)^c$, we have

$$|u - x|^2 \quad = \quad \rho^2 + (r - \delta(x))^2 - 2\rho(r - \delta(x))\cos\varphi_1$$
$$\overset{\rho=r+s}{=} \quad 2r(r + s - \delta(x))(1 - \cos\varphi_1) + s^2 + \delta^2(x) + 2s\delta(x)\cos\varphi_1$$
$$\geq \quad 4r(r + s - \delta(x))\sin^2\frac{\varphi_1}{2} + s^2$$
$$\overset{\delta(x)<\frac{r}{2}}{\geq} \quad 2r^2 \sin^2\frac{\varphi_1}{2} + s^2$$
$$\geq \quad \left(2r^2 \sin^2\frac{\varphi_1}{2}\right) \vee (s^2). \tag{6.4}$$

Similarly, if we set up a spherical coordinate system $(\gamma; \theta_1, \ldots, \theta_{n-1})$ with origin O_y and principal axis $\overrightarrow{O_y y^*}$, we have for any $v = v(\gamma; \theta_1, \ldots, \theta_{n-1}) \in B(O_y, r)^c$,

$$|v - y|^2 \geq \left(2r^2 \sin^2\frac{\theta_1}{2}\right) \vee (t^2) \tag{6.5}$$

where $t = \gamma - r$. Let

$$D_1 = \big(B(O_x, 2r) \setminus B(O_x, r)\big) \cap D, \quad D_2 = D \setminus B(O_x, 2r)$$
$$D_3 = \big(B(O_y, 2r) \setminus B(O_y, r)\big) \cap D, \quad D_4 = D \setminus B(O_y, 2r).$$

Then, for any $u \in D_1$,

$$
\begin{aligned}
d(u, \partial B_2^{x^*}(r)) \quad &= \quad [4r^2 + \rho^2 - 4r\rho \cos \varphi_1]^{1/2} - r \\
&\overset{\rho = r+s}{=} \quad [r^2(5 - 4\cos \varphi_1) + (s^2 + 2rs - 4rs \cos \varphi_1)]^{1/2} - r \\
&\overset{\text{Corollary 6.2}}{\leq} \quad [r^2(5 - 4\cos \varphi_1) + 4r^2(1 - \cos \varphi_1)^2]^{1/2} - r \\
&\leq \quad r(\sqrt{5 - 4\cos \varphi_1} - 1) + 2r(1 - \cos \varphi_1) \\
&\leq \quad 4r(1 - \cos \varphi_1) + 2r(1 - \cos \varphi_1) \\
&= \quad 12r \sin^2 \frac{\varphi_1}{2}.
\end{aligned}
\tag{6.6}
$$

Similarly, for any $v \in D_3$

$$d(v, \partial B_2^{y^*}(r)) \leq 12r \sin^2 \frac{\theta_1}{2}.$$

For $u \in D_1$, $v \in D_3$

$$
\begin{aligned}
\beta < |z - w| \quad &\leq \quad |z - x| + |x - u| + |u - v| + |v - y| + |y - w| \\
&\leq \quad \frac{r}{2} + 3r + |u - v| + 3r + \frac{r}{2} \\
&\leq \quad 7r + |u - v| \\
&\leq \quad \frac{7\beta}{8} + |u - v|,
\end{aligned}
$$

and hence

$$|u - v| \geq \frac{\beta}{8}, \quad \forall\, u \in D_1, \ \forall\, v \in D_3.$$

By Theorem 1.1, we have

$$
\begin{aligned}
G_D(u, v) \quad &\leq \quad C \frac{\delta(u)^{\alpha/2}\delta(v)^{\alpha/2}}{|u - v|^n} \\
&\leq \quad C \left(\frac{8}{\beta}\right)^n d(u, \partial B_2^{x^*}(r))^{\alpha/2} d(v, \partial B_2^{y^*}(r))^{\alpha/2} \\
&\leq \quad C(12r)^\alpha \left(\frac{8}{\beta}\right)^n \sin^\alpha \frac{\varphi_1}{2} \sin^\alpha \frac{\theta_1}{2} \qquad \text{for } (u, v) \in D_1 \times D_3,
\end{aligned}
$$

$$G_D(u, v) \quad \leq \quad C \frac{\delta(u)^{\alpha/2}}{|u - v|^{n-\alpha/2}} \leq \frac{C(12r)^{\alpha/2} \sin^\alpha \frac{\varphi_1}{2}}{|u - v|^{n-\alpha/2}} \qquad \text{for } (u, v) \in D_1 \times D_4,$$

$$G_D(u, v) \quad \leq \quad \frac{C(12r)^{\alpha/2} \sin^\alpha \frac{\theta_1}{2}}{|u - v|^{n-\alpha/2}} \qquad \text{for } (u, v) \in D_2 \times D_3,$$

and

$$G_D(u, v) \leq C \frac{1}{|u - v|^{n-\alpha}} \qquad \text{for } (u, v) \in D_2 \times D_4.$$

Thus for $(u, v) \in B(O_x, r)^c \times B(O_y, r)^c$

$$\frac{(r + |x - O_x|)^{\alpha/2}(r + |y - O_y|)^{\alpha/2}}{(|u - O_x|^2 - r^2)^{\alpha/2}(|v - O_y|^2 - r^2)^{\alpha/2}} \frac{G_D(u, v)}{|u - x|^n|v - y|^n}$$

$$\leq \frac{1}{(|u - O_x| - r)^{\alpha/2}(|v - O_y| - r)^{\alpha/2}} \cdot$$

$$\left\{ \frac{C(12r)^{\alpha}(\frac{8}{\beta})^n \sin^{\alpha}\frac{\varphi_1}{2}\sin^{\alpha}\frac{\theta_1}{2} 1_{D_1}(u) 1_{D_3}(v)}{[(2r^2 \sin^2 \frac{\varphi_1}{2}) \vee (|u - O_x| - r)^2]^{n/2} \cdot [(2r^2 \sin^2 \frac{\theta_1}{2}) \vee (|v - O_y| - r)^2]^{n/2}} \right.$$

$$+ \frac{C(12r)^{\alpha/2} \sin^{\alpha}\frac{\varphi_1}{2} 1_{D_1}(u) 1_{D_4}(v)}{r^n[(2r^2 \sin^2 \frac{\varphi_1}{2}) \vee (|u - O_x| - r)^2]^{n/2}} \frac{1}{|u - v|^{n - \alpha/2}}$$

$$+ \frac{C(12r)^{\alpha/2} \sin^{\alpha}\frac{\theta_1}{2} 1_{D_2}(u) 1_{D_3}(v)}{r^n[(2r^2 \sin^2 \frac{\theta_1}{2}) \vee (|v - O_y| - r)^2]^{n/2}} \frac{1}{|u - v|^{n - \alpha/2}}$$

$$\left. + \frac{1}{r^{2n}} \frac{1_{D_2}(u) 1_{D_4}(v)}{|u - v|^{n - \alpha}} \right\}$$

$$\leq C(12r)^{\alpha}\left(\frac{8}{\beta}\right)^n \frac{\sin^{\alpha}\frac{\varphi_1}{2} \cdot 1_{D_1}(u)}{[(2r^2 \sin^2 \frac{\varphi_1}{2}) \vee (|u - O_x| - r)^2]^{n/2}} \frac{1}{(|u - O_x| - r)^{\alpha/2}} \cdot$$

$$\cdot \frac{\sin^{\alpha}\frac{\theta_1}{2} \cdot 1_{D_3}(v)}{[(2r^2 \sin^2 \frac{\theta_1}{2}) \vee (|v - O_y| - r)^2]^{n/2}} \frac{1}{(|v - O_y| - r)^{\alpha/2}}$$

$$+ C(12)^{\alpha/2} r^{-n} \frac{\sin^{\alpha}\frac{\varphi_1}{2} \cdot 1_{D_1}(u)}{[(2r^2 \sin^2 \frac{\varphi_1}{2}) \vee (|u - O_x| - r)^2]^{n/2}} \frac{1}{(|u - O_x| - r)^{\alpha/2}}$$

$$\cdot \frac{1_{D_4}(v)}{|u - v|^{n - \alpha/2}}$$

$$+ C\, 12^{\alpha/2} r^{-n} \frac{\sin^{\alpha}\frac{\theta_1}{2} \cdot 1_{D_3}(v)}{[(2r^2 \sin^2 \frac{\theta_1}{2}) \vee (|v - O_y| - r)^2]^{n/2}} \frac{1}{(|v - O_y| - r)^{\alpha/2}}$$

$$\cdot \frac{1_{D_2}(u)}{|u - v|^{n - \alpha/2}}$$

$$+ C \frac{1}{r^{2n+\alpha}} \frac{1_{D_2}(u) 1_{D_4}(v)}{|u - v|^{n - \alpha}}$$

$$:= f(u, v; x, y, z, w)$$

$$:= C(12r)^{\alpha}\left(\frac{8}{\beta}\right)^n f_1(u, v; x, y, z, w) + C \cdot 12^{\alpha/2} r^{-n} f_2(u, v; x, y, z, u)$$

$$+ C\, 12^{\alpha} r^{-n} f_3(u, v; x, y, z, w) + C \frac{1}{r^{2n+\alpha}} f_4(u, v; x, y, z, w).$$

From Lemma 6.4 below we know that the family of functions of (u, v)

$$\mathcal{A} = \left\{ f(u, v, x, y, z, w) \quad : \quad x, y \in D, z, w \in \partial D, |z - w| > \beta, \right.$$

$$\left. |x - z| < \frac{r}{2}, |y - w| < \frac{r}{2} \right\} \tag{6.7}$$

is uniformly integrable on $D \times D$.

When $x \to z$, $y \to w$,

$$
\begin{aligned}
|O_x - O_z| &= |(x^* + rn_{x^*}) - (z - rn_z)| \\
&\le |x^* - z| + rC_0|x^* - z| \\
&\le (1 + rC_0)(|x^* - x| + |x - z|) \\
&\le 2(1 + rC_0)|x - z| \to 0.
\end{aligned}
$$

Similarly

$$
|O_y - O_w| \to 0.
$$

Thus by the uniform integrability of $\mathscr{A}$ in (6.7), we have

$$
\lim_{\substack{D \ni x \to z \\ D \ni y \to w}} h(x, y) =
$$

$$
C_1^2 (2r)^\alpha \int_D \int_D \frac{G_D(u, v)1_{B(O_z, r)^c}(u)1_{B(O_z, r)^c}(v)}{(|u - O_z|^2 - r^2)^{\alpha/2}(|v - O_w|^2 - r^2)^{\alpha/2}|u - z|^n|v - w|^n} \, du \, dv
$$
$$
> 0
$$

and is a continuous function in $\{(z, w) \in \partial D \times \partial D : |z - w| > \beta\}$. Since $\beta > 0$ is arbitrary, we have by (6.3) that for any $(z, w) \in \partial D \times \partial D$ with $z \ne w$,

$$
\liminf_{\substack{D \ni x \to z \\ D \ni y \to w}} \frac{G_D(x, y)}{\delta(x)^{\alpha/2}\delta(y)^{\alpha/2}} \ge \lim_{\substack{D \ni x \to z \\ D \ni y \to w}} h(x, y) > 0.
$$

$\square$

Lemma 6.4. *There exists $\eta = \eta(n, \alpha) > 1$ such that*

$$
\sup_{f \in \mathscr{A}} \int_D \int_D (f(u, v; x, y, z, w))^\eta \, du \, dv < \infty,
$$

where $\mathscr{A}$ is the set of functions defined above in (6.7).

Proof. It is easy to see that for any $1 < \eta < \frac{n}{n - \alpha/2}$, and any $u \in D$

$$
\text{(6.8)} \qquad \int_D \frac{1}{|u - v|^{(n - \alpha/2)\eta}} dv \le \omega_n \int_0^{d_D} \frac{\tau^{n-1}}{\tau^{\eta(n - \alpha/2)}} \, d\tau < \infty
$$

and

$$
\text{(6.9)} \qquad \int_D \int_D \frac{1}{|u - v|^{(n - \alpha)\eta}} \, du \, dv \le |D|\omega_n \int_0^{d_D} \frac{\tau^{n-1}}{\tau^{\eta(n - \alpha)}} \, d\tau < \infty.
$$

Case 1. $1 < \alpha < 2$.

In this case, we take $1 < \eta < \min\left\{\frac{2}{\alpha}, \frac{n-1}{n-\alpha}, \frac{n}{n-\alpha/2}\right\}$. Then

$$\int_D \left[\frac{\sin^\alpha \frac{\varphi_1}{2} 1_{D_1}(u)}{[(2r^2 \sin^2 \frac{\varphi_1}{2}) \vee (|u - O_x| - r)^2]^{n/2}} \frac{1}{(|u - O_x| - r)^{\alpha/2}} \right]^\eta du$$

$$\leq \int_D \left(\frac{1_{D_1}(u)}{2^{n/2} r^n \sin^{n-\alpha} \frac{\varphi_1}{2}} \frac{1}{(|u - O_x| - r)^{\alpha/2}} \right)^\eta du$$

$$\leq \omega_{n-1} \int_r^{2r} \frac{1}{(\rho - r)^{\alpha\eta/2}} \int_0^\pi \frac{1}{(2^{n/2} r^n \sin^{n-\alpha} \frac{\varphi_1}{2})^\eta} \rho^{n-1} \sin^{n-2} \varphi_1 d\varphi_1 d\rho$$

$$\leq \frac{\omega_{n-1}(2r)^{n-1}}{(2^{n/2} r^n)^\eta} \int_0^r \frac{1}{s^{\alpha\eta/2}} ds \int_0^\pi \frac{\sin^{n-2} \varphi_1}{\sin^{\eta(n-\alpha)} \frac{\varphi_1}{2}} d\varphi_1 < \infty. \tag{6.10}$$

Similarly

$$\int_D \left[\frac{\sin^\alpha \frac{\theta_1}{2} 1_{D_3}(v)}{[(2r^2 \sin^2 \frac{\theta_1}{2}) \vee (|v - O_y| - r)^2]^{n/2}} \frac{1}{(|v - O_y| - r)^{\alpha/2}} \right]^\eta dv$$

$$\leq \frac{\omega_{n-1}(2r)^{n-1}}{(2^{n/2} r^n)\eta} \int_0^r \frac{1}{s^{\alpha\eta/2}} ds \int_0^\pi \frac{\sin^{n-2} \theta_1}{\sin^{\eta(n-\alpha)} \frac{\theta_1}{2}} d\theta_1 < \infty. \tag{6.11}$$

Thus, from (6.8)–(6.11), we have

$$\int_D \int_D f_1(u, v; x, y, z, w)^\eta du dv$$

$$\leq \frac{\omega_{n-1}^2 (2r)^{2n-2}}{(2^n r^{2n})^\eta} \left(\int_0^r s^{-\frac{\alpha\eta}{2}} ds \right)^2 \left(\int_0^\pi \frac{\sin^{n-2} \varphi}{\sin^{\eta(n-\alpha)} \frac{\varphi}{2}} d\varphi \right)^2 < \infty,$$

$$\int_D \int_D f_2(u, v; x, y, z, w)^\eta du dv$$

$$\leq \frac{\omega_n \omega_{n-1} (2r)^{n-1}}{(2^{n/2} r^n)^\eta} \left(\int_0^r s^{-\frac{\alpha\eta}{2}} ds \right) \left(\int_0^{d_D} \frac{\tau^{n-1}}{\tau^{\eta(n-\alpha/2)}} d\tau \right) \left(\int_0^\pi \frac{\sin^{n-\alpha} \varphi}{\sin^{\eta(n-2)} \frac{\varphi}{2}} d\varphi \right)$$

$$< \infty,$$

$$\int_D \int_D f_3(u, v; x, y, z, w)^\eta du dv$$

$$\leq \frac{\omega_n \omega_{n-1} (2r)^{n-1}}{(2^{n/2} r^n)^\eta} \left(\int_0^r s^{-\frac{\alpha\eta}{2}} ds \right) \left(\int_0^{d_D} \frac{\tau^{n-1}}{\tau^{\eta(n-\alpha/2)}} d\tau \right) \left(\int_0^\pi \frac{\sin^{n-2} \varphi}{\sin^{\eta(n-\alpha)} \frac{\varphi}{2}} d\varphi \right)$$

$$< \infty,$$

and

$$\int_D \int_D f_4(u, v; x, y, z, w)^\eta du dv \leq |D| \omega_n \left(\int_0^{d_D} \frac{\tau^{n-1}}{\tau^{\eta(n-\alpha)}} d\tau \right) < \infty.$$

Therefore the assertion of Lemma 6.4 is valid when $1 < \alpha < 2$.

Case 2. $0 < \alpha \leq 1$.

In this case we take $1 < \eta < \min\left\{\frac{n-1}{n-\left(\frac{\alpha}{4}+1\right)}, \frac{4}{3\alpha}, \frac{n}{n-\frac{\alpha}{2}}\right\}$. Then

$$
\int_D \left[\frac{\sin^2 \frac{\varphi_1}{2}}{[(2r^2 \sin^2 \frac{\varphi_1}{2}) \vee (|u - O_x| - r)^2]^{n/2}} \frac{1_{D_1}(u)}{(|u - O_x| - r)^{\alpha/2}}\right]^{\eta} du
$$

$$
\leq \int_D \left[\frac{\sin^\alpha \frac{\varphi_1}{2}}{(2r^2)^{\frac{n-(1-\frac{3\alpha}{4})}{2}} \sin^{n-(1-\frac{3\alpha}{4})} \frac{\varphi_1}{2}} \frac{1_{D_1}(u)}{(|u - O_x| - r)^{1-\frac{\alpha}{4}}}\right]^{\eta} du
$$

$$
\leq (2r^2)^{-\frac{n(1-\frac{3\alpha}{4})}{2}\eta} \omega_{n-1} \int_r^{2r} \int_0^\pi \frac{\sin^{n-2} \varphi_1}{\sin^{\eta(n-(1+\frac{\alpha}{4}))} \frac{\varphi_1}{2}} \frac{1}{(\rho - r)^{(1-\frac{\alpha}{4})\eta}} \rho^{n-1} d\varphi_1 d\rho
$$

$$
\leq (2r^2)^{-\frac{n-(1-\frac{3\alpha}{4})}{2}\eta} (2r)^{n-1} \omega_{n-1} \left(\int_0^r s^{-(1-\frac{\alpha}{4})\eta} ds\right) \int_0^\pi \frac{\sin^{n-2} \varphi_1}{\sin^{\eta(n-(1+\frac{\alpha}{4}))} \frac{\varphi_1}{2}} d\varphi
$$

$$
< \infty. \tag{6.12}
$$

Similarly

$$
\int_D \left[\frac{\sin^2 \frac{\theta_1}{2} 1_{D_3}(v)}{[(2r^2 \sin^2 \frac{\theta_1}{2}) \vee (|v - O_y| - r)^2]^{n/2}} \frac{1}{(|v - O_y| - r)^{\alpha/2}}\right]^{\eta} dv
$$

$$
\leq (2r^2)^{-\frac{n-(1-\frac{3\alpha}{4})}{2}\eta} (2r)^{n-1} \omega_{n-1} \left(\int_0^r s^{-(1-\frac{\alpha}{4})\eta} ds\right) \left(\int_0^\pi \frac{\sin^{n-2} \varphi}{\sin^{\eta(n-(1+\frac{\alpha}{4}))} \frac{\varphi}{2}} d\varphi\right)
$$

$$
< \infty. \tag{6.13}
$$

Therefore, from (6.8), (6.9), (6.12) and (6.13), we have

$$
\int_D \int_D f_1(u, v; x, y, z, w)^\eta du dv
$$

$$
\leq (2r^2)^{-(n-(1-\frac{3\alpha}{4}))\eta} (2r)^{2n-2} \omega_{n-1}^2 \left(\int_0^r s^{-(1-\frac{\alpha}{4})\eta} ds\right)^2
$$

$$
\times \left(\int_0^\pi \frac{\sin^{n-2} \varphi}{\sin^{\eta(n-(1+\frac{\alpha}{4}))} \frac{\varphi}{2}} d\varphi\right)^2
$$

$$
< \infty,
$$

$$
\int_D \int_D f_2(u, v; x, y, z, w)^\eta du dv
$$

$$
\leq (2r^2)^{-(n-(1-\frac{3\alpha}{4}))\eta} (2r)^{n-1} \omega_n \omega_{n-1} \left(\int_0^r s^{-(1-\frac{\alpha}{4})\eta} ds\right) \left(\int_0^{d_D} \frac{\tau^{n-1}}{\tau^{\eta(n-\frac{\alpha}{2})}} d\tau\right)
$$

$$
\cdot \left(\int_0^\pi \frac{\sin^{n-2} \varphi}{\sin^{\eta(n-(1+\frac{\alpha}{4}))} \frac{\varphi}{2}} d\varphi\right)
$$

$$
< \infty,
$$

$$\int_D \int_D f_3(u,v;x,y,z,w)^\eta \, du \, dv$$

$$\leq \quad (2r^2)^{-\frac{n-(1-\frac{3\alpha}{4})}{2}\eta}(2r)^{n-1}\omega_n\omega_{n-1}\left(\int_0^r s^{-(1-\frac{\alpha}{4})\eta} ds\right)$$

$$\cdot \left(\int_0^{d_D} \frac{\tau^{n-1}}{\tau^{\eta(n-\frac{\alpha}{2})}} d\tau\right)\left(\int_0^\pi \frac{\sin^{n-2}\varphi}{\sin^{\eta(n-(1+\frac{\alpha}{4}))}\frac{\varphi}{2}} d\varphi\right)$$

$$< \quad \infty,$$

and

$$\int_D \int_D f_4(u,v;x,y,z,w)^\eta \, du \, dv \leq |D|\omega_n(1)\left(\int_0^{d_D} \frac{\tau^{n-1}}{\tau^{\eta(n-\alpha)}} d\tau\right) < \infty.$$

Hence the assertion of Lemma 6.4 is also valid when $0 < \alpha \leq 1$. □

Lemma 6.5. *For any $x_0 \in D$ and $z \in \partial D$, the limit*

$$\lim_{\substack{x \to x_0 \\ D \ni y \to z}} \frac{G_D(x,y)}{\delta(y)^{\alpha/2}}$$

exists and forms a positive and continuous function on $D \times \partial D$.

Proof. Using the same notations as in the proof of Lemma 6.3, we have

$$G_D(x,y) = E_x\left[G_D(X_{\tau_{B(O_x,r)}}, y)\right]$$

$$= C_1 \int_{B(O_x,r)^c \cap D} \frac{(r^2 - |x - O_x|^2)^{\alpha/2}}{(|u - O_x|^2 - r^2)^{\alpha/2}} \frac{1}{|x-u|^n} G_D(u,y) du.$$

From here on we can use a similar but simpler argument as that of Lemma 6.3 to finish the proof of this lemma. □

Lemma 6.6. *For any ball $B \subset \mathbf{R}^n$ of radius r, there is a constant $C_2 = 2^{-\alpha}\pi^{-\frac{n}{2}}\Gamma\left(\frac{\alpha}{2}\right)^{-2}\Gamma\left(\frac{n}{2}\right)$ such that*

$$G_B(x,y) \geq C_2 \left(\frac{|x-y|^n}{\delta_B(x)^{\alpha/2}\delta_B(y)^{\alpha/2}} + |x-y|^{n-\alpha}\right)^{-1},$$

where $\delta_B(x) = d(x, \partial B)$.

Proof. Without loss of generality, we can assume that $B = B(0,1)$. We know that

$$G_B(x,y) = C_2 \int_0^z (u+1)^{-n/2}u^{\alpha/2-1} du |x-y|^{\alpha-n}$$

where $z = (1 - |x|^2)(1 - |y|^2)|x-y|^{-2}$. If $z \geq 1$, then

(6.14) $$G_B(x,y) \geq C_2 \int_0^1 (u+1)^{-n/2}u^{\alpha/2-1} du |x-y|^{\alpha-n}.$$

If $z < 1$, since

$$\int_0^z (u + 1)^{-n/2} u^{\alpha/2-1} du = z^{\alpha/2} \int_0^1 (1 + vz)^{-n/2} v^{\alpha/2-1} dv$$

$$\geq \quad z^{\alpha/2} \int_0^1 (1 + v)^{-n/2} v^{\alpha/2-1} dv$$

$$\geq \quad (1 - |x|)^{\alpha/2} (1 - |y|)^{\alpha/2} |x - y|^{-\alpha} \int_0^1 (1 + v)^{-n/2} v^{\alpha/2-1} dv$$

we have

(6.15) $\qquad G_B(x, y) \geq C_2 \int_0^1 (1 + v)^{-n/2} v^{\alpha/2-1} dv \dfrac{\delta(x)^{\alpha/2} \delta(y)^{\alpha/2}}{|x - y|^n}$

Combining (6.14) and (6.15) we get the assertion of Lemma 6.6. $\qquad\qquad\square$

Proof of Theorem 1.2. If $|x - y| \leq \frac{\delta(x)}{2}$, take $B = B(x, \delta(x))$. Then $y \in B \subset D$. Let G_B be the Green function of B. Then it follows from [3] that we have

$$\begin{aligned}
G_D(x, y) &\geq G_B(x, y) \\
&= c_1 \left(|x - y|^{\alpha-n} - \int_{|u-x| \geq \delta(x)} |u - x|^{\alpha-n} \mu_y(du) \right) \\
&\geq c_1 (|x - y|^{\alpha-n} - \delta(x)^{\alpha-n}) \\
&\geq c_1 (1 - 2^{\alpha-n}) \frac{1}{|x - y|^{n-\alpha}},
\end{aligned}$$

where $c_1 = \Gamma\left(\frac{n-\alpha}{2}\right) \left[2^{\alpha} \pi^{n/2} \Gamma\left(\frac{\alpha}{2}\right) \right]^{-1}$ and μ_y is the distribution of first hitting position of $B(x, \delta(x))^c$ when the process starts from y.

If $|x - y| \leq \frac{\delta(y)}{2}$, the proof is the same.

In the case that

$$|x - y| > \max \left\{ \frac{\delta(x)}{2}, \frac{\delta(y)}{2} \right\},$$

if the conclusion of the theorem were false, there must be a sequence $\{(x_k, y_k)\} \subset D \times D$ such that

$$|x_k - y_k| > \max \left\{ \frac{\delta(x_k)}{2}, \frac{\delta(y_k)}{2} \right\}, \quad k \geq 1$$

and

$$\lim_{k \to \infty} \frac{G_D(x_k, y_k)}{\delta(x_k)^{\alpha/2} \delta(y_k)^{\alpha/2}} |x_k - y_k|^n = 0.$$

By taking subsequence, we may assume $(x_k, y_k) \to (z, w) \in \overline{D} \times \overline{D}$ as $n \to \infty$.

If $z \neq w$, $z, w \in D$, then

$$\lim_{k \to \infty} \frac{G_D(x_k, y_k)}{\delta(x_k)^{\alpha/2} \delta(y_k)^{\alpha/2}} |x_k - y_k|^n = \frac{G_D(z, w)}{\delta(z)^{\alpha/2} \delta(w)^{\alpha/2}} |z - w|^n > 0.$$

If $z \neq w$, $z, w \in \partial D$, then by Lemma 6.3

$$\liminf_{k \to \infty} \frac{G_D(x_k, y_k)}{\delta(x_k)^{\alpha/2} \delta(y_k)^{\alpha/2}} |x_k - y_k|^n = |z - w|^n \liminf_{k \to \infty} \frac{G_D(x_k, y_k)}{\delta(x)^{\alpha/2} \delta(y_k)^{\alpha/2}} > 0.$$

If $z \neq w$, $z \in D$ and $w \in \partial D$, then by Lemma 6.5

$$\lim_{k \to \infty} \frac{G_D(x_k, y_k)}{\delta(x_k)^{\alpha/2} \delta(y_k)^{\alpha/2}} |x_k - y_k|^n = \frac{|z - w|^n}{\delta(z)^{\alpha/2}} \lim_{k \to \infty} \frac{G_D(x_k, y_k)}{\delta(y_k)^{\alpha/2}} > 0.$$

Similarly, if $z \neq w$, $z \in \partial D$ and $w \in D$, then

$$\lim_{k \to \infty} \frac{G_D(x_k, y_k)}{\delta(x_k)^{\alpha/2} \delta(y_k)^{\alpha/2}} |x_k - y_k|^n > 0.$$

To deal with the case $z = w$ ($x_k - y_k \to 0$), we want to prove that there exists a constant $C > 0$ such that for any x and y in D satisfying

$$(6.16) \qquad \max\left\{\frac{\delta(x)}{2}, \frac{\delta(y)}{2}\right\} \leq |x - y| < \frac{r_0}{10(1 + r_0 C_0)},$$

the inequality

$$(6.17) \qquad G_D(x, y) \geq C \frac{\delta(x)^{\alpha/2} \delta(y)^{\alpha/2}}{|x - y|^n}$$

holds. This fact leads to the final contradiction.

For any fixed points x and y satisfying (6.16), let x^* and y^* be the points on ∂D such that $|x - x^*| = \delta(x)$ and $|y - y^*| = \delta(y)$. Set $r = r_0$, $B_x = B_1^{x^*}(r) = B(O_x, r)$, $B_y = B_1^{y^*}(r) = B(O_y, r)$. Then

$$
\begin{aligned}
|x^* - y^*| &\leq |x - y| + \delta(x) + \delta(y) \leq 5|x - y| \\
|O_x - O_y| &\leq |x^* - y^*| + r|n_{x^*} - n_{y^*}| \\
&\leq (1 + r_0 C_0)|x^* - y^*| \\
&\leq 5(1 + r_0 C_0)|x - y| \leq \frac{r}{2}.
\end{aligned}
$$

Set up a spherical coordinate system $(\rho; \varphi_1, \ldots, \varphi_{n-1})$ with origin O_x and principal axis $\overrightarrow{O_x y}$ (see Fig. 1 below) and set

$$
\begin{aligned}
\theta &= \varphi_1 \text{ coordinate of } O_y \\
\overline{\alpha} &= \max(\varphi_1 : (r; \varphi_1, \ldots, \varphi_{n-1}) \in \overline{B}_y) \\
S &= \left\{(r; \varphi_1, \ldots, \varphi_{n-1}) \in \partial B_x : 0 \leq \varphi_1 \leq \frac{\overline{\alpha}}{2}\right\} \\
I &= |y - O_x| \\
J &= |y - O_y| \\
\beta &= \cos(\theta + \overline{\alpha}) \\
\eta &= \cos \theta.
\end{aligned}
$$

By some elementary geometric arguments, we can get

$$
\begin{aligned}
|O_x - O_y| &= 2r\cos(\theta + \overline{\alpha}) \\
|y - O_y|^2 &= |y - O_x|^2 + |O_x - O_y|^2 - 2|y - O_x|\,|O_x - O_y|\cos\theta. \quad (6.18)
\end{aligned}
$$

Thus

$$
\begin{aligned}
I^2 - 4r\beta\eta I + (4r^2\beta^2 - J^2) &= 0 \\
I &= 2r\beta\eta + (4r^2\beta^2\eta^2 - 4r^2\beta^2 + J^2)^{1/2} \\
I - r &= \frac{(J^2 - r^2) + 4r^2\beta(\eta - \beta)}{(r - 2r\beta\eta) + (4r^2\beta^2\eta^2 - 4r^2\beta^2 + J^2)^{1/2}}. \quad (6.19)
\end{aligned}
$$

Set

$$
\begin{aligned}
h &= 16r\cos(\theta + \overline{\alpha})\sin\frac{\overline{\alpha}}{2}\sin\left(\theta + \frac{\overline{\alpha}}{2}\right) \\
&= 8|O_x - O_y|\sin\frac{\overline{\alpha}}{2}\sin\left(\theta + \frac{\overline{\alpha}}{2}\right).
\end{aligned}
$$

There are three possible cases:

Case 1. $y \notin B_x$.

Then $I \geq r$. By (6.19) we get

$$
r^2 - J^2 \leq 4r^2\beta(\eta - \beta) = 8r^2\cos(\theta + \overline{\alpha})\sin\frac{\overline{\alpha}}{2}\sin\left(\theta + \frac{\overline{\alpha}}{2}\right) = \frac{rh}{2},
$$

$$
\delta(y) = r - J \leq (r^2 - J^2)/r \leq h.
$$

Noting $\beta = |O_x - O_y|/2r \leq \frac{1}{4}$ and using (6.19), we have

$$
|I - r| \leq \frac{4r\beta(\eta - \beta)}{1 - 2\eta\beta} \leq 8r\beta(\eta - \beta) = h.
$$

Case 2. $y \in B_x$ and $\delta(y) \leq h$.

Then

$$
|I - r| = \delta_{B_x}(y) \leq \delta(y) \leq h.
$$

So in both Case 1 and Case 2, the following inequalities

(6.20) $$\delta(y) \leq h$$

(6.21) $$|I - r| \leq h$$

hold.

Since $G_D(\cdot, y)$ is superharmonic on B_x, we have

$$G_D(x,y) \geq C_1 \int_{B_x^c} \frac{(r^2 - |x - O_x|^2)^{\alpha/2}}{(|u - O_x|^2 - r^2)^{\alpha/2}} \frac{G_D(u,y)}{|u - x|^n} \, du$$

$$\geq C_1 \int_{B_x^c} \frac{(r^2 - |x - O_x|^2)^{\alpha/2}}{(|u - O_x|^2 - r^2)^{\alpha/2}} \frac{G_{B_y}(u,y)}{|u - x|^n} \, du$$

$$\geq C_1 \int_{B_x^c} \frac{(r^2 - |x - O_x|^2)^{\alpha/2}}{(|u - O_x|^2 - r^2)^{\alpha/2}} \frac{1_{B_y}(u)}{|u - x|^n} \cdot$$

$$\cdot \left(\frac{|u - y|^n}{\delta_{B_y}(u)^{\alpha/2} \delta_{B_y}(y)^{\alpha/2}} + |u - y|^{n - \alpha} \right)^{-1} du$$

where in the last inequality we used Lemma 6.6.

Put

$$D_0 = \left\{ u(\rho; \varphi_1, \ldots, \varphi_{n-1}) : 0 \leq \varphi_1 \leq \frac{\overline{\alpha}}{2}, \ r \leq \rho \leq r + \frac{h}{24\pi} \right\},$$

and let v be the point on ∂B_y such that the angle between $\overrightarrow{O_x v}$ and $\overrightarrow{O_x y}$ is $\frac{\overline{\alpha}}{2}$. (See Fig. 1.)
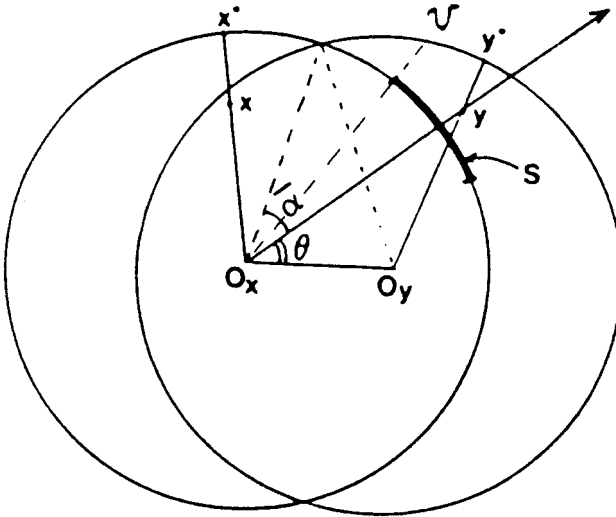

Fig. 1

By the law of cosine we have

$$|v - O_x|^2 + |O_x - O_y|^2 - 2|v - O_x| \, |O_x - O_y| \cos \left(\theta + \frac{\overline{\alpha}}{2} \right) = r^2,$$

i.e.,

$$|v - O_x|^2 + 4r^2 \cos^2(\theta + \overline{\alpha})] - 4r|v - O_x| \cos(\theta + \overline{\alpha}) \cos \left(\theta + \frac{\overline{\alpha}}{2} \right) = r^2$$

which can be rewritten as

$$(|v - O_x| - r)^2 + 2(|v - O_x| - r)r \left(1 - 2\cos(\theta + \overline{\alpha})\cos\left(\theta + \frac{\overline{\alpha}}{2}\right)\right)$$

$$+4r^2\cos(\theta + \overline{\alpha})\left(\cos(\theta + \overline{\alpha}) - \cos\left(\theta + \frac{\overline{\alpha}}{2}\right)\right) = 0.$$

Solving the above equation, we get

$$|v - O_x| - r$$

$$= -r\left(1 - 2\cos(\theta + \overline{\alpha})\cos\left(\theta + \frac{\overline{\alpha}}{2}\right)\right) +$$

$$+r\sqrt{\left(1 - 2\cos(\theta + \overline{\alpha})\cos\left(\theta + \frac{\overline{\alpha}}{2}\right)\right)^2 + 4\cos(\theta + \overline{\alpha})\left(\cos\left(\theta + \frac{\overline{\alpha}}{2}\right) - \cos(\theta + \overline{\alpha})\right)}$$

$$= \frac{4r\cos(\theta + \overline{\alpha})\left(\cos\left(\theta + \frac{\overline{\alpha}}{2}\right) - \cos(\theta + \overline{\alpha})\right)}{\left(1 - 2\cos(\theta + \overline{\alpha})\cos\left(\theta + \frac{\overline{\alpha}}{2}\right)\right) + \sqrt{1 - 4\cos^2(\theta + \overline{\alpha})\sin^2\left(\theta + \frac{\overline{\alpha}}{2}\right)}}$$

$$\geq 2r\cos(\theta + \overline{\alpha})\left(\cos\left(\theta + \frac{\overline{\alpha}}{2}\right) - \cos(\theta + \overline{\alpha})\right)$$

$$= 4r\cos(\theta + \overline{\alpha})\sin\left(\theta + \frac{3\overline{\alpha}}{4}\right)\sin\frac{\overline{\alpha}}{4}$$

$$\geq 4r\cos(\theta + \overline{\alpha})\sin\left(\theta + \frac{\overline{\alpha}}{2}\right)\sin\frac{\overline{\alpha}}{2}\frac{\sin\frac{\overline{\alpha}}{4}}{\sin\frac{\overline{\alpha}}{2}}$$

$$\geq \frac{1}{4\pi}h, \qquad \text{since} \quad \frac{\sin\frac{\overline{\alpha}}{4}}{\sin\frac{\overline{\alpha}}{2}} \geq \frac{1}{\pi}.$$

Therefore $D_0 \subset B_y$, which implies that

$$G_D(x, y) \geq C_1 \int_{D_0} \frac{(r^2 - |x - O_x|^2)^{\alpha/2}}{(|u - O_x|^2 - r^2)^{\alpha/2}} \frac{1}{|x - u|^n} \cdot$$

$$\cdot \left(\frac{|u - y|^n}{\delta_{B_y}(u)^{\alpha/2}\delta_{B_y}(y)^{\alpha/2}} + |u - y|^{n-\alpha}\right)^{-1} du. \quad (6.22)$$

From Zhao [23] we know that for any $\widetilde{u} \in S$, $|x - \widetilde{u}| \leq 4|x - y|$. For any $u \in D_0$, $\widetilde{u} = \frac{r}{|u - O_x|}(u - O_x) + O_x \in S$, thus

$$|x - u| \leq |x - \widetilde{u}| + |\widetilde{u} - u| \leq 4|x - y| + \frac{1}{24\pi}h$$

$$\leq 4|x - y| + \frac{1}{24\pi}8|O_x - O_y|$$

$$\leq 4|x - y| + \frac{40}{24\pi}(1 + r_0 C_0)|x - y|$$

$$< 5|x - y|, \qquad (6.23)$$

due to the fact that $r_0 \leq \frac{1}{2C_0}$. Let θ_u be the angle between $\overrightarrow{O_x u}$ and $\overrightarrow{O_x O_y}$. Then $\theta_u \leq \theta + \varphi_1(u) \leq \theta + \frac{\overline{\alpha}}{2}$. Since

$$\frac{|u - O_y|^2}{}$$

$$= \quad |u - O_x|^2 + |O_x - O_y|^2 - 2|u - O_x| \, |O_x - O_y| \cos \theta_u$$

$$\leq \quad |u - O_x|^2 + |O_x - O_y|^2 - 2|u - O_x| \, |O_x - O_y| \cos \left(\theta + \frac{\overline{\alpha}}{2} \right)$$

$$\overset{\text{put } |u - O_x| = r + s}{=} \quad (r + s)^2 + 4r^2 \cos^2(\theta + \overline{\alpha}) - 4r(r + s) \cos(\theta + \overline{\alpha}) \cos \left(\theta + \frac{\overline{\alpha}}{2} \right),$$

we have

$$r^2 - |u - O_y|^2$$

$$\geq \quad 4r^2 \cos(\theta + \overline{\alpha}) \left(\cos \left(\theta + \frac{\overline{\alpha}}{2} \right) - \cos(\theta + \overline{\alpha}) \right)$$

$$\quad + 4rs \cos(\theta + \overline{\alpha}) \cos \left(\theta + \frac{\overline{\alpha}}{2} \right) - (2rs + s^2)$$

$$\geq \quad 3r^2 \cos(\theta + \overline{\alpha}) \left(\cos \left(\theta + \frac{\overline{\alpha}}{2} \right) - \cos(\theta + \overline{\alpha}) \right)$$

$$= \quad 6r^2 \cos(\theta + \overline{\alpha}) \sin \left(\theta + \frac{3\overline{\alpha}}{4} \right) \sin \frac{\overline{\alpha}}{4}$$

$$\geq \quad \frac{6r^2}{\pi} \cos(\theta + \overline{\alpha}) \sin \left(\theta + \frac{\overline{\alpha}}{2} \right) \sin \frac{\overline{\alpha}}{2}$$

$$= \quad \frac{3}{8\pi} \, rh$$

where in the second inequality from above we used the fact that for $0 \leq s \leq \frac{1}{24\pi} h$,

$$s^2 + 2rs - 4rs \cos(\theta + \overline{\alpha}) \cos \left(\theta + \frac{\overline{\alpha}}{2} \right) \leq r^2 \cos(\theta + \overline{\alpha}) \left(\cos \left(\theta + \frac{\overline{\alpha}}{2} \right) - \cos(\theta + \overline{\alpha}) \right)$$

which can be proven by using elementary algebra. Note that

(6.24) $$\delta_{B_y}(y) = \delta(y)$$

and

(6.25) $$\delta_{B_y}(u) = r - |u - O_y| \geq (r^2 - |u - O_y|^2)/2r \geq \frac{3}{16\pi} \, h.$$

$$|u - y|^2$$

$$= |y - O_x|^2 + |u - O_x|^2 - 2|y - O_x| \, |u - O_x| \cos \varphi_1(u)$$

$$\overset{|u - O_x| = r + s}{\longrightarrow} = I^2 + (r + s)^2 - 2I(r + s) \cos \varphi_1$$

$$= (I - r)^2 + 2Ir(1 - \cos \varphi_1) + 2rs(1 - \cos \varphi_1) + 2(r - I)s \cos \varphi_1 + s^2$$

$$\leq h^2 + 4r^2(1 - \cos \varphi_1) + \frac{rh}{12\pi}(1 - \cos \varphi_1) + \frac{h^2}{12\pi} + \frac{h^2}{(24\pi)^2},$$

$$I \leq 2r, \ |r - I| \leq h$$

$$\leq \left(\frac{3h}{2}\right)^2 + 5r^2(1 - \cos \varphi_1)$$

$$\leq \left(\frac{3}{2}h\right)^2 + 5r^2 \sin^2 \varphi_1. \tag{6.26}$$

From (6.20)–(6.26), we have

$$G_D(x, y)$$

$$\geq C_1 \frac{(r^2 - |x - O_x|^2)^{\alpha/2}}{5^n |x - y|^n} \int_{D_0} \frac{1}{(|u - O_x|^2 - r^2)^{\alpha/2}} \cdot$$

$$\cdot \left(\left(\frac{16\pi}{3}\right)^{\alpha/2} \frac{1}{h^{\alpha/2}\delta(y)^{\alpha/2}} \left[\left(\frac{3}{2}h\right)^2 + 5r^2 \sin^2 \varphi_1\right]^{n/2}\right.$$

$$\left. + \left[\left(\frac{3}{2}h\right)^2 + 5r^2 \sin^2 \varphi_1\right]^{n/2 - \alpha/2}\right)^{-1} du$$

$$\geq \frac{C_1 \, \delta(x)^{\alpha/2}}{3^{\alpha/2} 5^n |x - y|^n} \int_{D_0} \frac{1}{(|u - O_x| - r)^{\alpha/2}} \cdot \left(\left(\frac{16\pi}{3}\right)^{\alpha/2} \frac{1}{h^{\alpha/2}\delta(y)^{\alpha/2}} \cdot\right.$$

$$\left. \cdot \left[\left(\frac{3h}{2}\right)^2 + 5r^2 \sin^2 \varphi_1\right]^{n/2} + \left[\left(\frac{3h}{2}\right)^2 + 5r^2 \sin^2 \varphi_1\right]^{n/2 - \alpha/2}\right)^{-1} du$$

$$= \frac{C_1 \, \omega_{n-1}}{3^{\alpha/2} 5^n} \frac{\delta(x)^{\alpha/2}}{|x - y|^n} \int_r^{r + \frac{h}{24\pi}} \int_0^{\frac{\overline{\alpha}}{2}} \frac{1}{(\rho - r)^{\alpha/2}} \rho^{n-1} \sin^{n-2} \varphi_1$$

$$\cdot \left(\left(\frac{16\pi}{3}\right)^{\alpha/2} \frac{1}{h^{\alpha/2}\delta(y)^{\alpha/2}} \left[\left(\frac{3}{2}h\right)^2 + 5r^2 \sin^2 \varphi_1\right]^{n/2}\right.$$

$$\left. + \left[\left(\frac{3}{2}h\right)^2 + 5r^2 \sin^2 \varphi_1\right]^{n/2 - \alpha/2}\right)^{-1} d\varphi_1 d\varrho$$

$$\geq \frac{C_1 \, \omega_{n-1}}{3^{\alpha/2} 5^n} \frac{\delta(x)^{\alpha/2}}{|x - y|^n} \int_0^{\frac{h}{24\pi}} s^{-\alpha/2} ds \int_0^{\frac{\overline{\alpha}}{2}} r^{n-1} \sin^{n-2} \varphi_1$$

$$\cdot \left(\left(\frac{16\pi}{3}\right)^{\alpha/2} \frac{1}{h^{\alpha/2}\delta(y)^{\alpha/2}} \left[\left(\frac{3}{2}h\right)^2 + 5r^2 \sin^2 \varphi_1\right]^{n/2}\right.$$

$$\left. + \left[\left(\frac{3}{2}h\right)^2 + 5r^2 \sin^2 \varphi_1\right]^{n/2 - \alpha/2}\right)^{-1} d\varphi_1.$$

Note that

$$
\int_0^{\frac{\overline{\alpha}}{2}} \left(\left(\frac{16\pi}{3} \right)^{\alpha/2} \frac{1}{h^{\alpha/2}\delta(y)^{\alpha/2}} \left[\left(\frac{3}{2}h \right)^2 + 5r^2 \sin^2 \varphi_1 \right]^{n/2} \right.
$$

$$
\left. + \left[\left(\frac{3}{2}h \right)^2 + 5r^2 \sin^2 \varphi_1 \right]^{n/2-\alpha/2} \right)^{-1} r^{n-1} \sin^{n-2} \varphi_1 d\varphi_1
$$

$$
\geq \int_0^{r \sin \frac{\overline{\alpha}}{2}} \left(\left(\frac{16\pi}{3} \right)^{\alpha/2} \frac{1}{h^{\alpha/2}\delta(y)^{\alpha/2}} \left[\left(\frac{3}{2}h \right)^2 + 5v^2 \right]^{n/2} + \right.
$$

$$
\left. + \left[\left(\frac{3}{2}h \right) + 5v^2 \right]^{n/2-\alpha/2} \right)^{-1} v^{n-2} dv
$$

$$
= \int_0^{\frac{2r \sin \frac{\overline{\alpha}}{2}}{3h}} \left(\frac{3}{2}h \right)^{\alpha-1} \left(\left(\frac{16\pi}{3} \right)^{\alpha/2} \left(\frac{3}{2} \right)^\alpha \frac{h^{\alpha/2}}{\delta(y)^{\alpha/2}} (1+5w^2)^{n/2} + \right.
$$

$$
\left. + (1+5w^2)^{n/2-\alpha/2} \right)^{-1} w^{n-2} dw
$$

$$
= \delta(y)^{\alpha/2} \left(\frac{3}{2}h \right)^{\alpha/2-1} \left(\frac{3}{2} \right)^{\alpha/2} \int_0^{\frac{2r \sin \frac{\overline{\alpha}}{2}}{3h}} \frac{h^{\alpha/2}}{\delta(y)^{\alpha/2}} \cdot
$$

$$
\cdot \left(\left(\frac{16\pi}{3} \right)^{\alpha/2} \left(\frac{3}{2} \right)^\alpha \frac{h^{\alpha/2}}{\delta(y)^{\alpha/2}} (1+5w^2)^{n/2} + (1+5w^2)^{n/2-\alpha/2} \right)^{-1} w^{n-2} dw
$$

$$
\geq \delta(y)^{\alpha/2} \left(\frac{3}{2}h \right)^{\alpha/2-1} \left(\frac{3}{2} \right)^{\alpha/2} \int_0^{1/6} \left(\left(\frac{16\pi}{3} \right)^{\alpha/2} \left(\frac{3}{2} \right)^\alpha (1+5w^2)^{n/2} + \right.
$$

$$
\left. + (1+5w^2)^{n/2-\alpha/2} \right)^{-1} w^{n-2} dw
$$

$$
= C\, \delta(y)^{\alpha/2} h^{\alpha/2-1}.
$$

Therefore

$$
G_D(x,y) \;\geq\; C\, \frac{\delta(x)^{\alpha/2}\delta(y)^{\alpha/2}}{|x-y|^n} h^{\alpha/2-1} \int_0^{\frac{h}{24\pi}} s^{-\alpha/2} ds
$$

$$
= C\, \frac{\delta(x)^{\alpha/2}\delta(y)^{\alpha/2}}{|x-y|^n}. \tag{6.27}
$$

Case 3. $y \in B_x$, $\delta(y) > h$.

Since $I < r$ and $(r - 2r\beta\eta) + (4r^2\beta^2\eta^2 - 4r^2\beta^2 + J^2)^{1/2} \leq 2r$. We have from (6.19) that

$$
\delta_{B_x}(y) = r - I \;\geq\; \frac{(r^2 - J^2) - 4r^2\beta(\eta - \beta)}{2r}
$$

$$
\geq\; \frac{1}{2}[(r - J) - 4r\beta(\eta - \beta)]
$$

$$
=\; \frac{1}{2}\left[\delta(y) - \frac{h}{2} \right].
$$

Thus

$$\begin{aligned}
\delta(y) &\leq 2\delta_{B_x}(y) + \frac{h}{2} \leq 2\delta_{B_x}(y) + \frac{\delta(y)}{2} \\
\delta(y) &\leq 4\delta_{B_x}(y) \\
\delta(x) &= \delta_{B_x}(x).
\end{aligned}$$

Therefore by Lemma 6.6 we have

$$\begin{aligned}
G_D(x,y) &\geq G_{B_x}(x,y) \\
&\geq C_2\Big(\frac{|x-y|^n}{\delta_{B_x}(x)^{\alpha/2}\delta_{B_x}(y)^{\alpha/2}} + |x-y|^{n-\alpha}\Big)^{-1} \\
&\geq C_2\Big(\frac{4^{\alpha/2}|x-y|^n}{\delta(x)^{\alpha/2}\delta(y)^{\alpha/2}} + |x-y|^{n-\alpha}\Big)^{-1} \\
&\geq C_2\Big(\frac{4^{\alpha/2}|x-y|^n}{\delta(x)^{\alpha/2}\delta(y)^{\alpha/2}} + \frac{4^{\alpha/2}|x-y|^\alpha}{\delta(x)^{\alpha/2}\delta(y)^{\alpha/2}}|x-y|^{n-\alpha}\Big)^{-1} \\
&= \frac{C_2}{2^{\alpha+1}}\frac{\delta(x)^{\alpha/2}\delta(y)^{\alpha/2}}{|x-y|^n}. \qquad\qquad (6.28)
\end{aligned}$$

Inequality (6.17) follows from (6.27) and (6.28). The proof is now completed. □

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Note added in proof. The first version of this paper was finished in September of 1996. The results of this paper were presented by the authors at the probability seminars of Cornell University, Duke University, New York University in October of 1996. After this paper was accepted, we found out that part of the results of this paper were independently obtained by T. Kulczcki in the paper "Properties of Green functions of symmetric stable processes", Probability and Mathematical Statistics, Vol. 17, 1997, pp. 339–364. We thank K. Bogdan for bringing this paper to our attention.