Estimating Quadratic Variation Consistently in the Presence of Correlated Measurement Error*

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Abstract

We propose an econometric model that captures the e¤ects of market microstructure on a latent price process. In particular, we allow for correlation between the measurement error and the return process and we allow the measurement error process to have a diurnal heteroskedasticity. We propose a modification of the TSRV estimator of quadratic variation. We show that this estimator is consistent, with a rate of convergence that depends on the size of the measurement error, but is no worse than n1=6. We investigate in simulation experiments the finite sample performance of various proposed implementations.

Keywords: Endogenous noise; Market Microstructure; Realised Volatility; Semimartingale

JEL Classification: C12

1 Introduction

It has been widely recognized that using very high frequency data requires taking into account the effect of market microstructure (MS) noise. We are interested in estimation of the quadratic variation of the latent price in the case where the observed log-price is a sum of the latent log-price that evolves in continuous time and an error that captures the effect of MS noise.

There is by now a large literature that uses realized variance as a nonparametric measure of volatility. The justification is that in the absence of market microstructure noise it is a consistent estimator of the quadratic variation as the time between observations goes to zero. For a literature review, see Barndorff-Nielsen and Shephard (2007). In practice, ignoring microstructure noise seems to work well for frequencies below 10 minutes. For higher frequencies realized variance is not robust, as has been evidenced in the so-called 'volatility signature plots'.

The first consistent estimator of quadratic variation of the latent price in the presence of MS noise was proposed by Zhang, Mykland, and Aït-Sahalia (2005) who introduced the Two Scales Realized Volatility (TSRV) estimator, and derived the appropriate central limit theory. TSRV estimates the quadratic variation using a combination of realized variances computed on two different time scales, performing an additive bias correction. They assumed that the MS noise was i.i.d. and independent of the latent price. The rate of convergence of the TSRV estimator is $n^{1/6}$. Zhang (2004) introduced the more complicated Multiple Scales Realized Volatility (MSRV) estimator that combines multiple ($\sim n^{1/2}$) time scales, which has a convergence rate $n^{1/4}$. This has been shown to be the optimal rate. Aït-Sahalia, Mykland and Zhang (2006a) modify TSRV and MSRV estimators and achieve consistency in the presence of serially correlated microstructure noise. Another class of consistent estimators of the quadratic variation was proposed by Barndorff-Nielsen, Hansen, Lunde, and Shephard (2006). They introduce realized kernels, a general class of estimators that extends the unbiased but inconsistent estimator of Zhou (1996), and is based on a general weighting of realized autocovariances as well as realized variances. They show that they can be designed to be consistent and derive the central limit theory. They show that for particular choices of weight functions they can be asymptotically equivalent to TSRV and MSRV estimators, or even more efficient. Apart from the benchmark setup where the noise is i.i.d. and independent from the latent price Barndorff-Nielsen et al. have two additional sections, one allowing for AR(1) structure in the noise, another with an additional endogenous term albeit one that is asymptotically degenerate.

We generalize the initial case of the noise being i.i.d. and independent from the latent price in three directions. The first generalization is allowing for (asymptotically non-degenerate) correlation between MS noise and the latent returns. This is one of the stylized facts found in Hansen and Lunde (2006) where they show that this correlation is present and is negative.

Another generalization concerns the magnitude of the MS noise, which we model explicitly. All of the papers above, like most of the literature that takes account of MS noise, assume that the variance of the MS noise is constant across sampling frequencies. An exception is Zhang et al. (2005) in the part where they derive the optimal sparse sampling frequency. We explicitly model the magnitude of the MS noise via a parameter α, where $\alpha = 0$ case corresponds to the benchmark case of the variance of MS noise being constant as n goes to infinity. The rate of convergence of our estimator depends on the magnitude of the noise, and can be from $n^{1/6}$ to $n^{1/3}$, where $n^{1/6}$ is the rate of convergence corresponding to "big" noise when $\alpha = 0$.

The third feature of our model is that we allow the MS noise to exhibit diurnal heteroscedasticity. This is motivated by the stylised fact in market microstructure literature that intradaily spreads (one of the most important components of the market microstructure noise) and intradaily financial asset price volatility are described typically by a U-shape. See Andersen, Bollerslev, and Cai (2000), McInish and Wood (1992). For example, Engle and Russell (1998) use a diurnal factor with splines to adjust the stock price volatility. Allowing for diurnal heteroscedasticity in our model has the effect that the original TSRV estimator may not be consistent because of end effects. In some cases, instead of estimating the quadratic variation, it would be estimating some function of the noise. We propose a modification of the TSRV estimator that is consistent, without introducing new parameters to be chosen. Our model is not meant to be definitive and can be generalized in a number of ways.

The structure of the paper is as follows. Section 2 introduces the model. Section 3 describes the estimator. Section 4 gives the main result and the intuition behind it. Section 5 gives the modification of the main result that arises when the noise is particularly small ($\alpha \geq 1/2$). Section 6 investigates the numerical properties of the estimator in a set of simulation experiments. Section 7 concludes.

2 The Model

Suppose that the latent (log) price process X_t is a Brownian semimartingale solving the stochastic differential equation

$$dX_t = \mu_t dt + \sigma_t dW_t, \tag{1}$$

where W_t is standard Brownian motion, and σ_t and μ_t are predictable and locally bounded processes, independent of the process W. The (no leverage) assumption of σ_t and μ_t being independent of W is unrealistic, but is frequently used and makes the analysis more tractable. The simulation results suggest that it does not change the result.

We observe the noisy price Y at fixed equidistant times $t_1, \ldots, t_n$ on $[0, 1]$, that is,

$$Y_{t_i} = X_{t_i} + u_{t_i}, \tag{2}$$

where the measurement error process is

$$\begin{aligned} u_{t_i} &= v_{t_i} + \varepsilon_{t_i} \\ v_{t_i} &= \delta\gamma_n \left(W_{t_i} - W_{t_{i-1}}\right) \\ \varepsilon_{t_i} &= m(t_i) + n^{-\alpha/2}\omega(t_i)\,\epsilon_{t_i},\ \alpha \in [0, 1/2) \end{aligned} \tag{3}$$

with ϵ_{t_i} i.i.d. mean zero and variance one and independent of the Gaussian process W with $E\epsilon_{t_i}^4 < \infty$. The functions m and ω are differentiable, nonstochastic functions of time. They are unknown as is the constant δ and the rate α. The usual benchmark measurement error model with noise being i.i.d. and independent from the latent price has $\alpha = 0$, $\gamma_n = 0$ and $\omega(.)$ and $m(.)$ constant.

There are two key parts to our model: the correlation between u and X and the relative magnitudes of u and X. The term v in u induces a correlation between latent returns and the change in the measurement error, which can be of either sign depending on δ. Correlation between u and X is plausible due to rounding effects, asymmetric information, or other reasons [Hansen and Lunde (2006), Diebold (2006)].[1] We have $E[u_{t_i}] = m(t_i)$ and $\text{var}[u_{t_i}] = \delta^2\gamma_n^2(t_i - t_{i-1}) + 2n^{-\alpha}\sigma_\epsilon^2(i/n)$. To have the variance of both terms in u equal, set $\gamma_n^2 = n^{1-\alpha}$. This seems like a reasonable restriction if both components are generated by the same mechanism. In this case, the size of the variance of the measurement error can be measured by α alone. In the special case that $\sigma_t = \sigma$ and $\omega(t_i) = \omega$, we find

$$\text{corr}(\Delta X_{t_i}, \Delta u_{t_i}) \simeq \frac{\delta}{\sqrt{[2\delta^2 + 2\omega^2]}}.$$

In this case, the range of correlation is limited, although it is quite wide - one can obtain up to a correlation of $\pm 1/\sqrt{2}$ depending on the relative magnitudes of δ, ω.

[1] In a recent survey of measurement error in microeconometrics models, Bound, Brown, and Mathiowetz (2001) emphasize 'mean-reverting' measurement error that is correlated with the signal.

An alternative model for endogenous noise has been developed by Barndorff-Nielsen, Hansen, Lunde, and Shephard (2006). In our notation, they have the endogenous noise part such that $\text{var}(v_{t_i}) = O(1/n)$, and an i.i.d., independent from X part with $\text{var}(\varepsilon_{t_i}) = O(1)$. They conclude robustness of their estimator to this type of endogeneity, with no change to the first order asymptotic properties compared to the case where $v_{t_i} = 0$.

The process ε_{t_i} is a special case of the more general class of locally stationary processes of Dahlhaus (1997). The generalization to allowing time varying mean and variance in the measurement error allows one to capture diurnal variation in the measurement error process, which is likely to exist in calendar time. Nevertheless, the measurement error in prices is approximately stationary under our conditions, which seems reasonable. Allowing a non-constant scaling factor ($\alpha > 0$) seems natural from a statistical point of view since the u_{t_i} represent outcomes that have happened in the small interval $[(i-1)/n, i/n]$; the scale of this distribution ought to reduce as the interval shrinks, i.e., as $n \to \infty$, at least for some of the components of the market microstructure noise. Many authors argue that measurement error is small; small is what the sampling interval is also argued to be and asymptotics are built off this assumption so why not also apply this to the scale of the measurement error.

The focus of this paper is estimating increments in quadratic variation of the latent price process,[2] but estimation of parameters of the MS noise in our model is also of interest. We acknowledge that not all the parameters of our model are identifiable, but some are. We have provided elsewhere consistent estimators of α, Linton and Kalnina (2005), and do not pursue this here. Estimating the function $\omega(\tau)$ would allow us to measure the diurnal variation of the MS noise. In the benchmark measurement error model this is a constant $\omega(\tau) \equiv \omega$ that can be estimated consistently by $\sum_{i=1}^{n-1}\left(Y_{t_{i+1}} - Y_{t_i}\right)^2/2n$ (Bandi and Russell (2006a), Barndorff-Nielsen et al. (2006), Zhang et al. (2005)). In our model, instead of n^{-1}, the appropriate scaling is $n^{\alpha-1}$. Such an estimator would converge to $(\delta^2/2) + \int \omega^2(u)\,du$. Hence, in the special case $\delta = 0$ this estimator would converge asymptotically to the integrated variance of the MS noise. Following Kristensen (2006) we could also estimate $\omega(\cdot)$ at some fixed point τ using kernel smoothing,

$$\widehat{\omega}^2(\tau) = \frac{1}{2n^{1-\alpha}} \frac{\sum_{i=1}^{n} K_h(t_{t-1} - \tau)\left(\Delta Y_{t_{i-1}}\right)^2}{\sum_{i=1}^{n} K_h(t_{t-1} - \tau)(t_t - t_{t-1})}.$$

[2] There is a question about whether one should care about the latent price or the actual price. This has been raised elsewhere, see Zhang, Mykland, and Aït-Sahalia (2005). We stick with the usual practice here, acknowledging that the presence of correlation between the noise and efficient price makes this even more debatable, Aït-Sahalia, Mykland, and Zhang (2006b).

Under equidistant observations, this simplifies to $\widehat{\omega}^2(\tau) = \sum_{i=1}^{n} K_h(t_{t-1} - \tau)\left(\Delta Y_{t_{i-1}}\right)^2 / 2n^{1-\alpha}$. In the above, h is a bandwidth that tends to zero asymptotically and $K_h(.) = K(./h)/h$, where $K(.)$ is a kernel function satisfying some regularity conditions. If we also allow for endogeneity ($\delta \neq 0$), $\widehat{\omega}^2(\tau)$ estimates $\omega^2(\tau)$ plus a constant, and so still allows to investigate the pattern of diurnal variation.

3 Estimation

We suppose that the parameter of interest is the quadratic variation of X on $[0,1]$, denoted $QV_X = \int_0^1 \sigma_t^2 dt$. Let

$$[Y,Y]^n = \sum_{i=1}^{n-1} \left(Y_{t_{i+1}} - Y_{t_i}\right)^2$$

be the realized variation of Y, and introduce a modified version of it (*jittered* RV) as follows,

$$[Y,Y]^{\{n\}} = \frac{1}{2}\left(\sum_{i=1}^{n-K} \left(Y_{t_{i+1}} - Y_{t_i}\right)^2 + \sum_{i=K}^{n-1} \left(Y_{t_{i+1}} - Y_{t_i}\right)^2\right). \qquad (4)$$

This modification is useful for controlling the end effects that arise under our sampling scheme.

Our estimator of QV_X makes use of the same principles as the TSRV estimator in Zhang et al. (2005). We split the original sample of size n into K subsamples, each of size $\overline{n} = n/K$. Introduce a constant β and c such that $K = cn^{\beta}$. For consistency we will need $\beta > 1/2 - \alpha$. The optimal choice of β is discussed in the next section. By setting $\alpha = 0$, we get the condition for consistency in Zhang et al. (2005), that $\beta > 1/2$.[3]

Let $[Y,Y]^{n_j}$ denote the j^{th} subsample estimator based on a K-spaced subsample of size n_j,

$$[Y,Y]^{n_j} = \sum_{i=1}^{n_j-1} \left(Y_{t_{iK+j}} - Y_{t_{(i-1)K+j}}\right)^2, \; j = 1, \ldots, K,$$

and let

$$[Y,Y]^{avg} = \frac{1}{K}\sum_{j=1}^{K} [Y,Y]^{n_j}$$

be the averaged subsample estimator. To simplify the notation, we assume that n is divisible by K and hence the number of data points is the same across subsamples, $n_1 = n_2 = ... = n_K = n/K$. Let $\overline{n} = n/K$.

[3] This condition is implicit in Zhang et al. (2005) in Theorem 1 (page 16) where the rate of convergence is $\sqrt{K/\overline{n}} = c\sqrt{n^{2\beta-1}}$.

Define the adjusted TSRV estimator (*jittered* TSRV) as

$$\widehat{QV}_X = [Y,Y]^{avg} - \left(\frac{\overline{n}}{n}\right)[Y,Y]^{\{n\}}. \tag{5}$$

Compared to the TSRV estimator, this estimator does not involve any new parameters that would have to be chosen by the econometrician, so it is as easy to implement. The need to adjust the TSRV estimator arises from the fact that under our assumptions TSRV is not always consistent. The problem arises due to end-of-sample effects. For a very simple example where the TSRV estimator is inconsistent, let us simplify the model to the framework of Zhang et al. (2005), and introduce only heteroscedasticity, the exact form of which is to be chosen below. Let us evaluate the asymptotic bias of TSRV estimator,[4]

$$\begin{aligned}
& n^{1/6}E\left\{\widehat{QV}_X^{TSRV} - QV_X\right\} \\
= \; & n^{1/6}\left\{E[u,u]^{avg} - c^{-1}n^{-2/3}E\left[u,u\right]^n\right\} + n^{1/6}\left\{E[X,X]^{\overline{n}}_{avg} - c^{-1}n^{-2/3}E\left[X,X\right]^n\right\} - n^{1/6}QV_X \\
= \; & c^{-1}n^{-1/2}\sum_{i=1}^{n-K}\left(\omega^2_{t_{i+K}} + \omega^2_{t_i}\right) - c^{-1}n^{-1/2}\sum_{i=1}^{n-1}\left(\omega^2_{t_{i+1}} + \omega^2_{t_i}\right) + o\left(1\right).
\end{aligned}$$

A simple example of heteroscedasticity would be $\omega^2\left(i/n\right) = a + i/n$, where a is any constant. In this case simple calculations give that $n^{1/6}E(\widehat{QV}_X^{TSRV} - QV_X)$ diverges to infinity. The rate $n^{1/6}$ is achieved by choosing β (that enters into $\overline{n}$) optimally (so $\beta = 2/3$). Even if any other β is used, we get in this example that TSRV estimator is asymptotically biased.

We remark that (5) is an additive bias correction and there is a nonzero probability that $\widehat{QV}_X < 0$. One can ensure positivity by replacing $\widehat{QV}_X$ by $\max\{\widehat{QV}_X, 0\}$, but this is not very satisfactory.

4 Asymptotic Properties

The expansion for $[Y,Y]^{avg}$ and $[Y,Y]^n$ both contain additional terms due to the correlation between the measurement error and the latent returns. The main issues can be illustrated using the expansion of $[Y,Y]^{avg}$, conditional on the path of σ_t:

$$[Y,Y]^{avg} = \underbrace{QV_X}_{(a)} + \underbrace{2\frac{\delta\gamma_n}{K}\int_0^1 \sigma_t dt}_{(b)} + \underbrace{E\left[u,u\right]^{avg}}_{(c)} + O\left(\underbrace{\overline{n}^{-1/2}}_{(d)} + \underbrace{\sqrt{\frac{\overline{n}}{Kn^{2\alpha}}}}_{(e)}\right)Z, \tag{6}$$

[4] For intermediate steps, see proof of Lemma A7, which uses very similar calculations for the full model.

where $Z \sim N(0,1)$, while the terms in curly braces are as follows: (a) the probability limit of $[X,X]^{avg}$, which we aim to estimate; (b) the bias due to correlation between the latent returns and the measurement error; (c) the bias due to measurement error; (d) the variance due to discretization; (e) the variance due to measurement error.

Should we observe the latent price without measurement error, (a) and (d) would be the only terms. In this case, of course, it is better to use $[X,X]^n$, since that has an error of smaller order $n^{-1/2}$. In the presence of the measurement error, however, both $[Y,Y]^{avg}$ and $[Y,Y]^n$ are badly biased, the bias arising both from correlation between the latent returns and the measurement error, and from the variance of the measurement error. The largest term is (c), which satisfies

$$E[u,u]^{avg} = 2\overline{n}n^{-\alpha}\left(\int_0^1 \omega^2(u)\,du + \delta^2\right) + O\left(n^{-\alpha} + \overline{n}^{-1}\right) = O\left(\overline{n}n^{-\alpha}\right),$$

i.e., it is of order $\overline{n}n^{-\alpha}$. So without further modifications, this is what $[Y,Y]^{avg}$ would be estimating. Should we be able to correct that, the next term would be $2(\delta\gamma_n/K)\int \sigma_t dt$ arising from $E[X,u]^{avg}$. This second term is zero, however, if there is no correlation between the latent price and the MS noise, i.e., if $\delta = 0$. Interestingly when we use the TSRV estimator for bias correction of $E[u,u]^{avg}$, we also cancel this second term.

The asymptotic distribution of our estimator arises as a combination of two effects, measurement error and discretization effect. First, quadratic variation of the observed price is only a proxy for the quadratic variation of latent price. After correcting for the bias, we still have the variation due to the measurement error. The convergence of the estimator to the realized variance of the latent price X is the following,

$$\sqrt{\frac{Kn^{2\alpha}}{\overline{n}}}\left(\widehat{QV}_X - [X,X]^{avg}\right) \Longrightarrow N\left(0, 8\delta^4 + 48\delta^2\int \omega^2(u)\,du + 8\int \omega^4(u)\,du\right). \tag{7}$$

The rate of convergence arises from $\mathrm{var}[u,u]^{avg} = O\left(\overline{n}/Kn^{2\alpha}\right)$. Both parts of the noise u, which are v and ε, contribute to the asymptotic variance. The first part of the asymptotic variance roughly arises from $\mathrm{var}[v,v]$, the second part from $\mathrm{var}[v,\varepsilon]$ (which is nonzero even though correlation between both terms is zero), and the third part from $\mathrm{var}[\varepsilon,\varepsilon]$. If the measurement error is uncorrelated with the latent price, the first two terms disappear.

Should we observe the latent price without any error, we would still not know its quadratic variation due to observing the latent price only at discrete time intervals. This is another source of estimation error. From Theorem 3 in Zhang et al. (2005) we have

$$\overline{n}^{1/2}\left([X,X]^{avg}-QV_X\right)\Longrightarrow N\left(0,\frac{4}{3}\int_0^1\sigma_t^4dt\right). \tag{8}$$

The final result is a combination of the two results (7) and (8), and the fact that they are asymptotically independent. The fastest rate of convergence is achieved by choosing K so that the variance from the discretization is of the same order as the variance arising from the MS noise, so set $\overline{n}^{-1/2}=\sqrt{\overline{n}/Kn^{2\alpha}}$. The resulting optimal magnitude of K is such that $\beta=2(1-\alpha)/3$. The rate of convergence with this rule is $\overline{n}^{-1/2}=n^{-1/6-\alpha/3}$. The slowest rate of convergence is $n^{-1/6}$, and it corresponds to large MS noise case, $\alpha=0$. The fastest rate of convergence is $n^{-1/3}$, which corresponds to $\alpha=1/2$ case. If we pick a larger β (and hence more subsamples K) than optimal, the rate of convergence in (7) increases, and the rate in (8) decreases so dominates the final convergence result. In this case the final convergence is slower and only the first term due to discretization appears in the asymptotic variance (see (9)). Conversely, if we pick a smaller β (and hence K) than optimal, we get a slower rate of convergence and only the second term in the asymptotic variance ("measurement error" in (9)), which is due to MS noise.

We obtain the asymptotic distribution of $\widehat{QV}_X$ in the following theorem

Theorem 1. *Suppose $\{X_t\}$ is a Brownian semimartingale satisfying (1). Suppose $\{\sigma_t\}$ and $\{\sigma_t\}$ are measurable and càdlàg processes, independent of the process $\{W_t\}$. Suppose further that the observed price arises as in (2). Let measurement error u_{t_i} be generated by (3), with ϵ_{t_i} i.i.d. mean zero and variance one and independent of the Gaussian process $\{W_t\}$ with $E\epsilon_{t_i}^4<\infty$. Then, conditional on the sample path $\{\sigma_t^2\}$ with probability one*

$$\overline{n}^{1/2}\left(\widehat{QV}_X-QV_X\right)\Longrightarrow N(0,V),$$

$$V=\underbrace{\frac{4}{3}\int_0^1\sigma_t^4dt}_{discretization}+\underbrace{c^{-3}\left(8\delta^4+48\delta^2\int\omega^2(u)\,du+8\int\omega^4(u)\,du\right)}_{measurement\ error}. \tag{9}$$

Remarks.

1. The main statement of the theorem can also be written as

$$n^{1/6+\alpha/3}\left(\widehat{QV}_X-QV_X\right)\Longrightarrow N(0,cV),$$

where $V = V_1 + c^{-3}V_2$, with V_1 being the discretization error. We can use this to find the value of c that would minimize the asymptotic variance, $c_{opt} = (2V_2/V_1)^{1/3}$, resulting in asymptotic variance $(3/2^{2/3})V_2^{1/3}V_1^{2/3}$.

2. Suppose the parameter of interest is $\int_{T_1}^{T_2} \sigma_t^2 dt$, the quadratic variation of X on $[T_1, T_2]$. Then the asymptotic variance of Theorem 1 becomes

$$V = \frac{4}{3}(T_2 - T_1) \int_{T_1}^{T_2} \sigma_t^4 dt + c^{-3} \left(8 (T_2 - T_1)^2 \delta^4 + 48\delta^2 \int_{T_1}^{T_2} \omega^2 (u) \, du + 8 (T_2 - T_1)^{-1} \int_{T_1}^{T_2} \omega^4 (u) \, du \right).$$

This follows by simple adjustments in the proofs.

5 Asymptotic properties in the case $\alpha \geq 1/2$

So far we have worked with the model, in which $\alpha \in [0, 1/2)$. In this section we consider cases when the measurement error is even smaller, i.e., the case $\alpha \in [1/2, 1)$ and the case $\alpha \geq 1$.

For high α the convergence rate $\sqrt{Kn^{2\alpha}/\overline{n}}$ of the measurement error in (7) is faster, so optimally we would like to pick larger $\overline{n}$ (i.e., smaller β) to increase also the rate of convergence $\overline{n}^{1/2}$ of the discretisation error in (8). However, there is a bias term that prevents us choosing very small β. It is $E\left(\frac{\overline{n}}{n}[X, X]^n\right) = \frac{\overline{n}}{n} QV_X$ and with rate of convergence $\overline{n}^{1/2}$ it is only negligible if $\beta > 1/3$. This latter condition is incompatible (for $\alpha \geq 1/2$) with setting $\beta = 2(1 - \alpha)/3$ to balance both error terms. This results in the measurement error converging faster than the discretisation error. Hence, the convergence rate $\overline{n}^{1/2} = n^{(1-\beta)/2}$ and asymptotic variance of the estimator now come only from the measurement error. Since the rate of convergence is decreasing in β, but consistency requires $\beta > 1/3$, we have that to the first order asymptotically optimal β is slightly above 1/3. For $\beta = 1/3 + \Delta$ (where Δ small and positive) the rate of convergence is $\overline{n}^{1/2} = n^{(1-\beta)/2} = n^{1/3 - \Delta/2}$. Note that this is exactly the rate that occurs when there is no measurement error at all.

To summarize, we have the following theorem.

THEOREM 2. *Suppose that the conditions for Theorem 1 hold, except* $\alpha \in [1/2, 1)$. *Choose* $\beta \in (1/3, 1)$. *Then, conditional on the sample path* $\{\sigma_t^2\}$ *with probability one*

$$n^{(1-\beta)/2} \left(\widehat{QV}_X - QV_X \right) \Longrightarrow N(0, V),$$

$$V = \underbrace{\frac{4}{3}\int_0^1 \sigma_t^4 dt}_{discretization} . \tag{10}$$

Finally note that $\alpha \geq 1$ means $[u,u]$ is of the same or smaller magnitude than $[X,X]$. In the case $\alpha = 1$ they are of the same order and identification breaks down. When $\alpha > 1$, realized volatility of observed prices is a consistent estimator of quadratic variation of latent prices, as measurement error is of smaller order. This is an artificial case and does not seem to appear in the real data.

How can we put this analysis in context? A useful benchmark for evaluation of the asymptotic properties of nonparametric estimators is the performance of parametric estimators. Gloter and Jacod (2001) allow for the dependence of the variance of i.i.d. Gaussian measurement error ρ_n on n and establish the Local Asymptotic Normality (LAN) property of the likelihood, which is a precondition to asymptotic optimality of the MLE. For the special case $\rho_n = \rho$ they obtain a convergence rate $n^{-1/4}$, thus allowing us to conclude that the MSRV and realized kernels can achieve the fastest possible rate. They also show that the rate of convergence is $n^{-1/2}$ if ρ_n goes to zero sufficiently fast, which is the rate when there is no measurement error at all. Our estimator has a rate $n^{-1/3+\Delta}$ when there is no measurement error, which is also the rate of convergence when the noise is sufficiently small. Also, Gloter and Jacod have that for "large" noise, the rate of convergence depends on the magnitude of the noise, which is in line with what we showed in the previous section. The rate of convergence and the threshold for the magnitude of the variance of the noise is different, though.

6 Simulation study

In this section we explore the behaviour of the estimator (5) in finite samples. We simulate the Heston (1993) model:

$$\begin{aligned} dX_t &= (\mu_t - v_t/2)\, dt + \sigma_t dB_t \\ dv_t &= \kappa\,(\theta - v_t)\, dt + \gamma v_t^{1/2} dW_t, \end{aligned}$$

where $v_t = \sigma_t^2$, and B_t, W_t are independent standard Brownian motions.[5]

For the benchmark model, we take the parameters of Zhang et al. (2005): $\mu = 0.05$, $\kappa = 5$, $\theta = 0.04$, $\gamma = 0.5$. We set the length of the sample path to 23400 corresponding to the number of

[5] Simulations with nonzero correlation yield the same conclusions, but we have assumed it away in our framework.

seconds in a business day, the time between observations corresponding to one second when a year is one unit, and the number of replications to be 100,000.[6] We set $\alpha = 0$. We choose the values of ω and δ so as to have a homoscedastic measurement error with variance equal to 0.0005^2 (again from the Zhang et al. (2005)), and correlation between the latent returns and the measurement error equal to -0.1. For this we use the identity

$$\operatorname{corr}(\Delta X_{t_i}, \Delta u_{t_i}) = \frac{E(\sigma)}{\sqrt{2E(\sigma^2)}} \frac{\delta}{\sqrt{\delta^2 + \omega^2}}$$

and the fact that for our volatility we have $E(\sigma) = \theta$, $\operatorname{var}(\sigma) = \theta\gamma^2/2\kappa$. We set $\beta = 2(1-\alpha)/3$.

First, we construct different models to see the effect of varying α and the number of observations within a day. We take the values of δ and ω that arise from the benchmark model, and then do simulations for the following combinations of α and n. When interpreting the results, we should also take into account that both of these parameters change the size of the variance of the measurement error. The measure to assess the proximity of the finite sample distribution to the asymptotic distribution will be percentage errors of the interquartile range of $\overline{n}^{1/2}(\widehat{QV}_X - QV_X)$ compared to $1.3\sqrt{V}$. This measure is easiest to interpret if we work with a fixed variance, i.e., when we condition on the volatility path. Hence, we simulate the volatility path for the largest number of observations, 23400, and perform all simulations using this one sample path of volatility. The last parameter to choose is K, the number of subsamples. This is the only parameter that an econometrician has to choose in real life. Therefore, we use four different values as follows (the first three expressions are also rounded to the closest integer):

Table 1. Choices of K

$(2V_2/V_1)^{1/3} n^{\frac{2}{3}(1-\alpha)}$	asymptotically optimal rate and c	*Tables 2 and 3*
$n^{\frac{2}{3}(1-\alpha)}$	variation of above	*Tables 4 and 5*
$n^{\frac{2}{3}}$	variation of above	*Table 6*
$\lfloor \phi n \rfloor, \phi = \left(\frac{3\frac{RV^2}{n^2}}{2RQ} \right)^{1/3}$	Bandi and Russell (2006, eq. 24)	*Table 7*

Table 2 contains the interquartile range errors (IQRs), in per cent, with asymptotically optimal rate and constant for K. That is, we use $K = (2V_2/V_1)^{1/3} n^{2(1-\alpha)/3}$, rounded to the nearest integer, where V_1 and V_2 are discretisation and measurement errors from (9). Table 3 contains the values of K.

[6]Note that in the theoretical part of the paper we had for brevity taken interval [0,1]. For the simulations we need the interval [0,1/250]. See remark 2 for the relevant expression of Theorem 1, with $T_1 = 0$ and $T_2 = 1/250$.

Tables 2 and 3 here

We see large errors, and from the values of K in Table 3 we can guess this is due to the asymptotically optimal rule selecting very low c_{opt}. In fact, for the volatility path used here, $c_{opt} = (2V_2/V_1)^{1/3} = 0.0242$. Hence, another experiment we consider is an arbitrary choice $c = 1$. The next two tables (Table 4 and 5) contain the percentage errors and values of K that result from using $K = n^{2(1-\alpha)/3}$.

Table 4 and 5 here

The performance of this choice is much better. We can see from Table 4 that for small values of α, the asymptotic approximation improves with sample size. The sign of the error changes as α increases for given n, meaning that the actual IQR is below that predicted by the asymptotic distribution for small α and small n but this changes into the actual IQR being above the asymptotic prediction.

Another variant that does not include the unobservable α would be to use $K = n^{2/3}$. Table 6 shows the corresponding results, which are slightly worse for small n. The choice of K of course does not depend on α, see the last column of Table 6 for the resulting values.

Table 6

Finally, we consider the choice of K proposed by Bandi and Russell (2006, eqn 24). Table 7 contains the IQR percentage errors and values of K that result from using $K^{BR} = \lfloor \phi n \rfloor, \phi = \left(\frac{3\frac{RV^2}{n^2}}{2RQ} \right)^{1/3}$, where RV is the realised variance, $RV = \sum (\Delta Y_{low})^2$ and RQ is the realised quarticity, $RQ = \frac{S}{3} \sum (\Delta Y_{low})^4$. Here, Y_{low} is low frequency (15 minute) returns, which gives $S = 24$ to be the number of low frequency observations during one day.

Table 7 here

We see that the IQR errors of this choice are generally smaller than with asymptotically optimal K, except for cases that have both large n and small α, including the case $\alpha = 0$ usually considered in the literature. We notice that K^{BR} rule gives better results than asymptotically optimal when it chooses a larger K, which is in most cases, but not all. In comparison to rules $K = n^{2(1-\alpha)/3}$ and $K = n^{2/3}$ (Tables 4 and 6, respectively), the performance of this choice is still disappointing, especially for small α. The reason seems to be that values of K^{BR} are in general too small, and appropriate only for very small MS noise cases, i.e., when α is very large. For these cases (as in columns to the right) performance of K^{BR} (Table 7) approaches to one for $K = n^{2(1-\alpha)/3}$ (Table 4). One possible reason of these disappointing results could be that the assumption of constant volatility in Bandi and Russell (2006) is not representative of the volatility path in above simulations, see Figure 1. One might also note that this is not exactly the TSRV estimator that Bandi and Russell derived the optimal value of K for, due to our end-of-sample adjustments (*jittering*). However, under homoscedastic MS noise as in our simulation setup, the differences between $\widehat{QV}$ and $\widehat{QV}^{TSRV}$ are negligible in practice. Hence, this cannot be a cause for substantial worsening of the results.

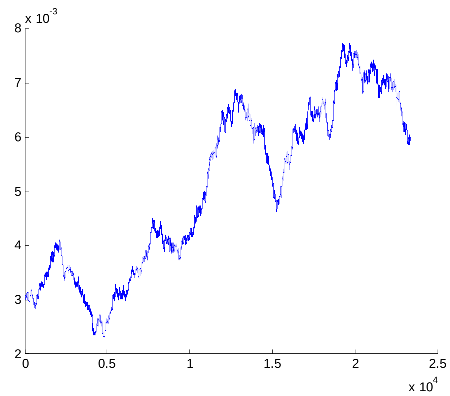
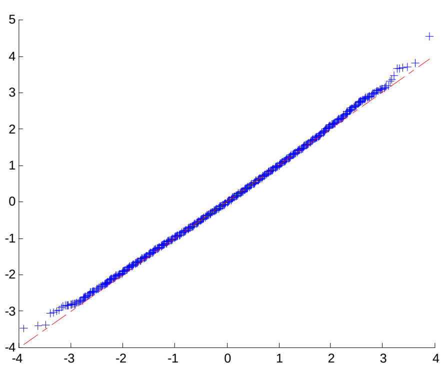


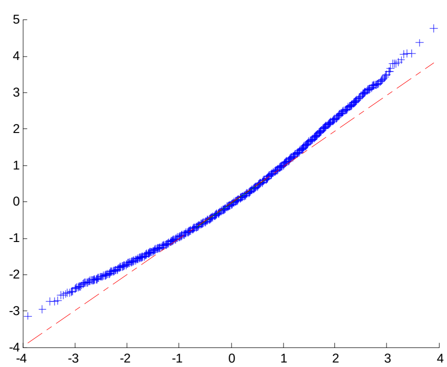
Figure 1. The common volatility path for all simulations.

It has been noted elsewhere that the asymptotic approximation can perform poorly, see Gonçalves and Meddahi (2005) and Aït-Sahalia, Zhang and Mykland (2005).

In the following, we use the rule $K = n^{2(1-\alpha)/3}$.We show below that the size of α can have an effect of how close is the finite sample distribution to normal. Figure 2 shows two Normal Q-Q plots for $n = 23400$ and two different values of α.



(a) $\alpha = 0$



(b) $\alpha = 0.4$

Figure 2. Normal QQ plot of studentized $\widehat{QV}_X$ and $n = 23400$.

If we use errors unscaled by n, we can see that they are becoming smaller as we increase α. Figure 4 contains box plots of $(\widehat{QV}_X - QV_X)/QV_X$ in models corresponding to the last row of Table 1, i.e., we take a range of values for α and set $n = 23400$.

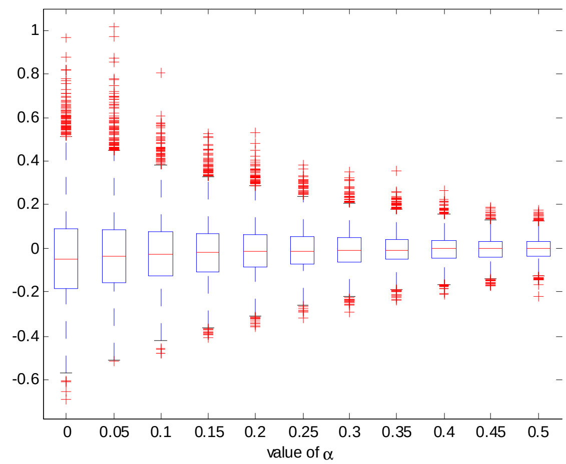


Figure 4. Boxplots of $(\widehat{QV}_X - QV_X)/QV_X$ corresponding to different values of α and for $n = 23400$.

In a second set of experiments we investigate the effect of varying ω, which controls the variance of the second part of the measurement error, for the largest sample size. Denoting by ω_b^2 the value of ω^2 in the benchmark model, we construct models with $\omega^2 = \omega_b^2, 4\omega_b^2, 8\omega_b^2, 10\omega_b^2$, and $20\omega_b^2$. The corresponding interquartile errors are 0.96%, 1.26%, 1.93%, 2.29%, and 4.64%.

In a third set of experiments we investigate the effect of varying δ, which controls the size of the correlation of the latent returns and measurement error. Denoting by δ_b^2 the value of δ^2 in the benchmark model, we construct models with δ^2 being from $0.01 \times \delta^2$ to $20 \times \delta^2$. The exact values of δ^2, as well as corresponding correlation between returns and increments of the noise, and the resulting interquartile errors are reported in Table 7.

Table 8 here

We can see that when the number of observations is 23400, there is no strong effect from the correlation of the latent returns and measurement error on the approximation of the asymptotic interquartile range of the estimator.

7 Conclusions and Extensions

In this paper we showed that the TSRV estimator is consistent for quadratic variation of the latent (log) price process when measurement error is correlated with the latent price, although some adjustment is necessary when measurement error is heteroscedastic. We also showed how the rate of convergence of the estimator depends on the magnitude of the measurement error. Given that robustness of TSRV estimator under the case of autocorrelated noise has been shown before, we see that TSRV paradigm of estimators seems to be very stable to different assumptions about the additive measurement error. What is less easy to answer is the question of valid inference. So far, this has been only solved for the benchmark case of i.i.d. and independent from the latent price measurement error, or asymptotically equivalent specifications. The case of $\alpha \geq 1/2$ analysed in Section 6 falls under this scenario, and, incidentally, valid inference can be achieved there by, e.g., the same estimator of integrated quarticity as in Zhang et al. (2005). However, this question has not been solved when additional terms arise in asymptotic variance due to endogeneity (this paper) or autocorrelation in measurement error (as in Aït-Sahalia, Mykland, and Zhang (2006a)). We plan to investigate this question further. Gonçalves and Meddahi (2005) have recently proposed a bootstrap methodology for conducting inference under the assumption of no noise (and no leverage) and shown that it has good small sample performance in their model. Zhang, Mykland, and Aït-Sahalia (2005) have developed Edgeworth expansions for the TSRV estimator, and it would be very interesting to use this for analysis of inference using bootstrap. It is also of interest to estimate the parameters of (2), namely the size of the measurement error, the correlation of the measurement error with the latent price, and the intraday heteroskedasticity in the measurement error. We have discussed how some of these quantities may be estimated. The results we have presented may be generalized to cover MSRV estimators and to allow for serial correlation in the error terms, although in both cases the notation becomes very complicated.

A Appendix

For deterministic sequences A_n, B_n we use the notation $A_n \sim B_n$ to mean that A_n is equal to B_n plus something of smaller order than B_n, i.e., $A_n/B_n \to 1$.

To follow easier the notation regarding all subscripts, it is convenient to think in terms of grids. The time indices of the full dataset with n data points are on a grid $\mathcal{G} = \{1, 2, 3, \ldots, n\}$. For the first few lemmas we take the first subsample only, which has time indices on the first subgrid $\mathcal{G}_1 = \{1, K+1, 2K+1, \ldots, (\overline{n}-1)K+1\}$, where $\overline{n} = n/K$. This translates into $(i-1)K+1$, $i = 1, \ldots, \overline{n}$. Hence, for summations like the one defining $[Y,Y]^{n_1}$ we will need to take

$$\left\{\begin{matrix} K+1, 2K+1, \ldots, (\overline{n}-1)K+1 \\ 1, K+1, \ldots, (\overline{n}-2)K+1 \end{matrix}\right\} = \left\{\begin{matrix} iK+1, i=1,\ldots,\overline{n}-1 \\ (i-1)K+1, i=1,\ldots,\overline{n}-1 \end{matrix}\right\}.$$

Similarly, the j^{th} subgrid is $\mathcal{G}_j = \{j, K+j, 2K+j, \ldots, (\overline{n}-1)K+j\}, j = 1, 2, \ldots, K$. This translates into $(i-1)K+j$, $i = 1, \ldots, \overline{n}$. Hence,

$$\left\{\begin{matrix} K+j, 2K+j, \ldots, (\overline{n}-1)K+j \\ j, K+j, \ldots, (\overline{n}-2)K+j \end{matrix}\right\} = \left\{\begin{matrix} iK+j, i=1,\ldots,\overline{n}-1 \\ (i-1)K+j, i=1,\ldots,\overline{n}-1 \end{matrix}\right\}.$$

We assume for simplicity that $\mu \equiv 0$ in the sequel. Drift is not important in high frequencies as it is of order dt, while the diffusion term is of order $\sqrt{dt}$ (see, for example Aït-Sahalia et al.(2006)).With the assumptions of Theorem 1, the same method as in the proof can be applied to the drift, yielding the conclusion that it is not important statistically.

Proof of Theorem. Expectations are taken conditional on the whole path of σ_t. We have

$$\begin{aligned} \overline{n}^{1/2}\left(\widehat{QV}_X - QV_X\right) &= \overline{n}^{1/2}[Y,Y]^{avg} - \frac{\overline{n}^{3/2}}{n}[Y,Y]^n - \overline{n}^{1/2}QV_X \\ &= \overline{n}^{1/2}\{[X,X]^{avg} + 2[X,u]^{avg} + [u,u]^{avg}\} \\ &\quad -\frac{\overline{n}^{3/2}}{n}\left\{[X,X]^{\{n\}} + 2[X,u]^{\{n\}} + [u,u]^{\{n\}}\right\} - \overline{n}^{1/2}QV_X \\ &\equiv C1 + C2 + C3 - C4 - C5 - C6. \end{aligned}$$

We calculate the order in probability of these terms by computing their means and variances and using Chebychev's inequality. We show below that:

		mean	variance
C1	$\overline{n}^{1/2}\,[X,X]^{avg} - \overline{n}^{1/2}QV_X$	0	$O(1)$
C2	$2\overline{n}^{1/2}\,[X,u]^{avg}$	$O(n^{\alpha/2})$	$o(1)$
C3	$\overline{n}^{1/2}\,[u,u]^{avg}$	$O(n^{1/2})$	$O(1)$
C4	$\frac{\overline{n}^{3/2}}{n}\,[X,X]^{\{n\}}$	$o(1)$	$o(1)$
C5	$\frac{\overline{n}^{3/2}}{n}2\,[X,u]^{\{n\}}$	$O(n^{\alpha/2})$	$o(1)$
C6	$\frac{\overline{n}^{3/2}}{n}\,[u,u]^{\{n\}}$	$O(n^{1/2})$	$O(1)$.

(C1) The term $\overline{n}^{1/2}\,[X,X]^{\overline{n}}_{avg} - \overline{n}^{1/2}QV_X$ has zero mean and variance $O(1)$ from the result in Zhang et al. (2005) (eqn. 49, pp. 1401), and:

$$\overline{n}^{1/2}\left([X,X]^{\overline{n}}_{avg} - QV_X\right) \Longrightarrow N\left(0, \frac{4}{3}\int_0^1 \sigma_t^4 dt\right).$$

(C2,C5) In Lemma A5 we show that $E\,[C2 - C5] = o_p(1)$. Lemma A2 shows that variance of $C5$ is small. From Lemma A6, $4\overline{n}\text{var}[X,u]^{avg} = O\left(\overline{n}n^{\beta-2}\right) = O\left(n^{-1}\right).$

(C3,C6) In Lemma A7 we show that $E\,[C3 - C6] = o_p(1)$. From Lemma A4, var $\left(\frac{\overline{n}^{3/2}}{n}\,[u,u]^{\{n\}}\right) = O(1)$ and from Lemma A8, var $\left(\overline{n}^{1/2}\,[u,u]^{\overline{n}}_{avg}\right) = O(1)$.

(C4) By Jacod and Protter (1998), $[X,X]^n - QV_X = O_p\left(n^{-1/2}\right)$ and so $\frac{\overline{n}^{3/2}}{n}\,[X,X]^n = O_p\left(\frac{\overline{n}^{3/2}}{n}\right) = O_p\left(n^{\alpha-1/2}\right) = o_p\,(1)$. Since $[X,X]^n - [X,X]^{\{n\}} = O_p\,(K/n)$, we have also $\frac{\overline{n}^{3/2}}{n}\,[X,X]^{\{n\}} = o_p\,(1)$.

It follows that the limiting distribution of $\overline{n}^{1/2}(\widehat{QV}_X - QV_X)$ is that of $C1 + C3 - C6$. The covariances between these terms are calculated in Lemmas B1, B2, and B3. Therefore, the asymptotic variance is

$$\begin{aligned} V &= \lim_{n\to\infty}\left\{\overline{n}\text{var}\,[X,X]^{avg} + \overline{n}\text{var}\,[u,u]^{avg} + \left(\frac{\overline{n}^{3/2}}{n}\right)^2 \text{var}\,[u,u]^{\{n\}} - 2\frac{\overline{n}^2}{n}\text{cov}\left([u,u]^{\{n\}},[u,u]^{avg}\right)\right\} \\ &= \frac{4}{3}\int_0^1 \sigma_t^4 dt + 2c^{-3}\left(12\delta^4 + 4E\epsilon^4 \int \omega^4(u)\,du + 24\delta^2 \int \omega^2(u)\,du\right) - 2c^{-3}\left(8\delta^4 + 4\left(E\epsilon^4 - 1\right)\int \omega^4(u)\,du\right) \\ &= \frac{4}{3}\int_0^1 \sigma_t^4 dt + c^{-3}\left(8\delta^4 + 48\delta^2 \int \omega^2(u)\,du + 8\int \omega^4(u)\,du\right). \end{aligned}$$

■

A.1 Lemmas

Here we give the lemmas needed in the proof of the Theorem.

LEMMA A1. *For all n*

$$E[X,u]^{n_1} = \delta\gamma_n \sum_{i=1}^{n_1-1} \int_{t_{iK}}^{t_{iK+1}} \sigma_t dt.$$

LEMMA A2. *As* $n \to \infty$,

$$\begin{aligned} \text{var}[X,u]^{n_1} &= O\left(n^{-\alpha}\right) + O\left(\frac{1}{n_1^2}\right) \\ 4\left(\frac{\overline{n}^{3/2}}{n}\right)^2 \text{var}\left[X,u\right]^{\{n\}} &= o\left(1\right). \end{aligned}$$

LEMMA A3. *As* $n \to \infty$,

$$E[u,u]^{n_1} = [m,m]^{n_1} + 2n_1 n^{-\alpha} \int_0^1 \omega^2\left(u\right) du + 2n_1\delta^2 n^{-\alpha} + O\left(n^{-\alpha}\right).$$

LEMMA A4. *As* $n \to \infty$,

$$\begin{aligned} \text{var}[u,u]^{n_1} &= \frac{n_1}{n^{2\alpha}}\left\{12\delta^4 + 4E\epsilon^4 \int \omega^4\left(u\right) du + 24\delta^2 \int \omega^2\left(u\right) du + O\left(\frac{n^{\alpha/2}}{n_1}\right) + O\left(\frac{n^{\alpha}}{n_1^2}\right)\right\} \\ \left(\frac{\overline{n}^{3/2}}{n}\right)^2 \text{var}[u,u]^{\{n\}} &= c^{-3}\left\{12\delta^4 + 4E\epsilon^4 \int \omega^4\left(u\right) du + 24\delta^2 \int \omega^2\left(u\right) du\right\} + o\left(1\right) \end{aligned}$$

LEMMA A5. *As* $n \to \infty$,

$$\begin{aligned} E[X,u]^{avg} &= \frac{\delta\gamma_n}{K} \int_0^1 \sigma_t dt + O\left(n^{-\frac{1+\alpha}{2}}\right) \\ \overline{n}^{1/2} E[X,u]^{avg} &= \frac{\overline{n}^{3/2}}{n} E[X,u]^{\{n\}} + o\left(1\right) \end{aligned}$$

LEMMA A6. *As* $n \to \infty$,

$$\text{var}[X,u]^{avg} \sim \frac{1}{K}\text{var}[X,u]^{n_1} = O\left(n^{\beta-2}\right).$$

LEMMA A7. *As* $n \to \infty$,

$$\overline{n}^{1/2} E\left[u,u\right]^{avg} = \frac{\overline{n}^{3/2}}{n} E\left[u,u\right]^{\{n\}} + o(1)$$

Lemma A8. *As $n \to \infty$,*

$$\begin{aligned}\mathrm{var}[u,u]^{avg} &= \frac{n_1}{Kn^{2\alpha}}\left\{12\delta^4 + 4E\epsilon^4\int\omega^4(u)\,du + 24\delta^2\int\omega^2(u)\,du + O\left(\frac{n^{\alpha/2}}{n_1}\right) + O\left(\frac{n^{\alpha}}{n_1^2}\right)\right\} \\ &= c^{-3}\overline{n}^{-1}\left\{12\delta^4 + 4E\epsilon^4\int\omega^4(u)\,du + 24\delta^2\int\omega^2(u)\,du + o(1)\right\} \text{ if } \beta = \frac{2}{3}(1-\alpha).\end{aligned}$$

Lemma B1. *As $n \to \infty$,*

$$\mathrm{cov}\left([X,X]^{avg},[u,u]^{avg}\right) = O\left(n^{-\alpha}K^{-1}\right) = o\left(\overline{n}^{-1/2}\right).$$

Lemma B2. *As $n \to \infty$,*

$$\mathrm{cov}\left([X,X]^{avg},[u,u]^{\{n\}}\right) = O\left(n^{-\alpha}K^{-1}\right) = o\left(\overline{n}^{-1/2}\right).$$

Lemma B3. *As $n \to \infty$,*

$$\mathrm{cov}\left([u,u]^{\{n\}},[u,u]^{avg}\right) = \frac{n}{\overline{n}^2}c^{-3}\left\{8\delta^4 + 4\left(E\epsilon^4 - 1\right)\int\omega^4(u)\,du + o(1)\right\}.$$

A.2 Proofs of Lemmas

Write symbolically $[X,u] = [X,v] + [X,\varepsilon]$ and $[u,u] = [v,v] + [\varepsilon,\varepsilon] + 2[v,\varepsilon]$, where the process X is independent of the process ε and the process v is also independent of the process ε. Also use for a function g and lag $J = 1,\ldots,K$, $\Delta_J g(t_i) = g(t_i) - g(t_{i-J})$ with $\Delta = \Delta_1$ for simplicity.

Proof of Lemma A1. We have

$$\begin{aligned}&E[X,u]^{n_1} \\ =& \sum_{i=1}^{n_1-1} E\left[\left(u_{t_{iK+1}} - u_{t_{(i-1)K+1}}\right)\left(X_{t_{iK+1}} - X_{t_{(i-1)K+1}}\right)\right] \\ =& \sum_{i=1}^{n_1-1} E\left[\int_{t_{(i-1)K+1}}^{t_{iK+1}} \sigma_t dW_t\left[\delta\gamma_n\left(W_{t_{iK+1}} - W_{t_{iK}}\right) - \delta\gamma_n\left(W_{t_{(i-1)K+1}} - W_{t_{(i-1)K}}\right) + \left(\varepsilon_{t_{iK+1}} - \varepsilon_{t_{(i-1)K+1}}\right)\right]\right] \\ =& \delta\gamma_n \sum_{i=1}^{n_1-1}\int_{t_{iK}}^{t_{iK+1}} \sigma_t dt.\end{aligned}$$

■

Proof of Lemma A2. We have

$$\begin{aligned}
\operatorname{var}[X,u]^{n_1} &= \operatorname{var}[X,v]^{n_1} + \operatorname{var}[X,\varepsilon]^{n_1} + 2\operatorname{cov}\left([X,v]^{n_1},[X,\varepsilon]^{n_1}\right) \\
&= \operatorname{var}[X,v]^{n_1} + \operatorname{var}[X,\varepsilon]^{n_1} \text{ since } \mathrm{E}\left((\Delta X)^2 \Delta W\right) = 0 \text{ by normality} \\
&= O\left(n^{-\alpha}\right) + O\left(\frac{1}{\overline{n}^2}\right). \qquad (11)
\end{aligned}$$

We prove (11) below. First part is $\operatorname{var}[X,v]^{n_1} \sim n^{1-\alpha}\left(\frac{1}{n}+\frac{n_1}{n^2}\right) \sim n^{-\alpha}$ by

$$\begin{aligned}
&\frac{1}{\delta^2\gamma_n^2}\operatorname{var}[X,v]^{n_1} \\
=\ &\frac{1}{\delta^2\gamma_n^2}\operatorname{var}\left[\sum_{i=1}^{n_1-1}\left(X_{t_{iK+1}} - X_{t_{(i-1)K+1}}\right)\left(v_{t_{iK+1}} - v_{t_{(i-1)K+1}}\right)\right] \\
=\ &\operatorname{var}\left[\sum_{i=1}^{n_1-1}\left(X_{t_{iK+1}} - X_{t_{(i-1)K+1}}\right)\left(\left\{W_{t_{iK+1}} - W_{t_{iK}}\right\} - \left\{W_{t_{(i-1)K+1}} - W_{t_{(i-1)K}}\right\}\right)\right] \\
=\ &\sum_{i=1}^{n_1-1}\operatorname{var}\left[\left(X_{t_{iK+1}} - X_{t_{(i-1)K+1}}\right)\left(W_{t_{iK+1}} - W_{t_{iK}}\right)\right] \\
&+\sum_{i=1}^{n_1-1}\operatorname{var}\left[\left(X_{t_{iK+1}} - X_{t_{(i-1)K+1}}\right)\left(W_{t_{(i-1)K+1}} - W_{t_{(i-1)K}}\right)\right] \\
&+2\sum_{i=1}^{n_1-2}\mathrm{E}\left[\Delta_K X_{t_{iK+1}}\left(\Delta W_{t_{iK+1}} - \Delta W_{t_{(i-1)K+1}}\right)\Delta_K X_{t_{(i+1)K+1}}\left(\Delta W_{t_{(i+1)K+1}} - \Delta W_{t_{iK+1}}\right)\right] \\
=\ &\frac{2}{n}\int_{t_1}^{t_{n-K+1}}\sigma_t^2 dt + O\left(\frac{n_1}{n^2}\right),
\end{aligned}$$

where for the final equality we use:

$$
\begin{aligned}
& \sum_{i=1}^{n_1-1} \operatorname{var}\left[\left(X_{t_{iK+1}} - X_{t_{(i-1)K+1}}\right)\left(W_{t_{iK+1}} - W_{t_{iK}}\right)\right] \\
= & \sum_{i=1}^{n_1-1} E\left(X_{t_{iK+1}} - X_{t_{(i-1)K+1}}\right)^2 E\left(W_{t_{iK+1}} - W_{t_{iK}}\right)^2 \\
& + \sum_{i=1}^{n_1-1} E^2\left(X_{t_{iK+1}} - X_{t_{(i-1)K+1}}\right)\left(W_{t_{iK+1}} - W_{t_{iK}}\right) \text{ by normality} \\
= & \sum_{i=1}^{n_1-1}\left[\int\limits_{t_{(i-1)K+1}}^{t_{iK+1}} \sigma_t^2 dt \frac{1}{n} + \left(\int\limits_{t_{iK}}^{t_{iK+1}} \sigma_t dt\right)^2\right] = \frac{1}{n}\int\limits_{t_1}^{t_{n-K+1}} \sigma_t^2 dt + O\left(\frac{n_1}{n^2}\right),
\end{aligned}
$$

$$
\begin{aligned}
& \sum_{i=1}^{n_1-1} \operatorname{var}\left[\left(X_{t_{iK+1}} - X_{t_{(i-1)K+1}}\right)\left(W_{t_{(i-1)K+1}} - W_{t_{(i-1)K}}\right)\right] \\
= & \sum_{i=1}^{n_1-1} \int\limits_{t_{(i-1)K+1}}^{t_{iK+1}} \sigma_t^2 dt \frac{1}{n} = \frac{1}{n}\int\limits_{t_1}^{t_{n-K+1}} \sigma_t^2 dt,
\end{aligned}
$$

and

$$
\begin{aligned}
& 2\sum_{i=1}^{n_1-2} \mathrm{E}\left[\Delta_K X_{t_{iK+1}}\left(\Delta W_{t_{iK+1}} - \Delta W_{t_{(i-1)K+1}}\right)\Delta_K X_{t_{(i+1)K+1}}\left(\Delta W_{t_{(i+1)K+1}} - \Delta W_{t_{iK+1}}\right)\right] \\
= & 2\sum_{i=1}^{n_1-2} \mathrm{E}\left(\Delta_K X_{t_{iK+1}} \Delta W_{t_{iK+1}}\right) \mathrm{E}\left(\Delta_K X_{t_{(i+1)K+1}} \Delta W_{t_{(i+1)K+1}}\right) \\
= & 2\sum_{i=1}^{n_1-2}\left[\int\limits_{t_{iK}}^{t_{iK+1}} \sigma_t dt \int\limits_{t_{(i+1)K}}^{t_{(i+1)K+1}} \sigma_t dt\right] = O\left(\frac{n_1}{n^2}\right).
\end{aligned}
$$

For the second part of (11), we have $\mathrm{var}[X,\varepsilon]^{n_1} = O\left(\frac{1}{\overline{n}^2}\right) + O\left(n^{-\alpha}\right)$ by

$$\begin{aligned}
& \mathrm{var}[X,\varepsilon]^{n_1} \\
= & \mathrm{var}\left[\sum_{i=1}^{n_1-1}\left(X_{t_{iK+1}} - X_{t_{(i-1)K+1}}\right)\left(\varepsilon_{t_{iK+1}} - \varepsilon_{t_{(i-1)K+1}}\right)\right] \\
= & \sum_{i=1}^{n_1-1} \mathrm{var}\left[\Delta_K X_{t_{iK+1}} \Delta_K \varepsilon_{t_{iK+1}}\right] + 2\sum_{i=2}^{n_1-1} \mathrm{cov}\left[\Delta_K X_{t_{iK+1}} \Delta_K \varepsilon_{t_{iK+1}}, \Delta_K X_{t_{(i-1)K+1}} \Delta_K \varepsilon_{t_{(i-1)K+1}}\right] \\
= & \sum_{i=1}^{n_1-1} \mathrm{E}\left(\Delta_K X_{t_{iK+1}}\right)^2 \mathrm{E}\left(\Delta_K \varepsilon_{t_{iK+1}}\right)^2 + 2\sum_{i=2}^{n_1-1} \mathrm{E}\left(\Delta_K X_{t_{iK+1}}\right) \mathrm{E}\left(\Delta_K X_{t_{(i-1)K+1}}\right) \mathrm{E}\left(\Delta_K \varepsilon_{t_{iK+1}} \Delta_K \varepsilon_{t_{(i-1)K+1}}\right) \\
= & \sum_{i=1}^{n_1-1} \int_{t_{(i-1)K+1}}^{t_{iK+1}} \sigma_t^2 dt \left[\left(\Delta_K m_{t_{iK+1}}\right)^2 + n^{-\alpha}\left(\omega_{t_{iK+1}}^2 + \omega_{t_{(i-1)K+1}}^2\right)\right] \\
& +2\sum_{i=2}^{n_1-1} \int_{t_{(i-1)K+1}}^{t_{iK+1}} \sigma_t dt \int_{t_{(i-1)K+1}}^{t_{iK+1}} \sigma_t dt \left[\Delta_K m_{t_{iK+1}} \Delta_K m_{t_{(i-1)K+1}} + n^{-\alpha} \omega_{t_{(i-1)K+1}}^2\right] \\
= & O\left(\frac{1}{\overline{n}^2}\right) + O\left(n^{-\alpha}\right).
\end{aligned}$$

Now we prove the second part of Lemma A2. Note that by substituting n for n_1 we get $\mathrm{var}[X,u]^n = O\left(n^{-\alpha}\right) + O\left(\frac{1}{n^2}\right)$, and so $\left(\frac{\overline{n}^{3/2}}{n}\right)^2 \mathrm{var}[X,u]^n \sim (\overline{n}^3/n^2 n^\alpha) + (\overline{n}^3/n^5) \sim n^{3(1-\beta)-2-\alpha} = n^{1-\alpha-3\beta} = n^{1-\alpha-2(1-\alpha)} = n^{\alpha-1} = o(1)$. Since $[X,u]^n - [X,u]^{\{n\}}$ is of smaller order than $[X,u]^{\{n\}}$, the same holds for $\left(\frac{\overline{n}^{3/2}}{n}\right)^2 \mathrm{var}[X,u]^{\{n\}}$.

■

Proof of Lemma A3. We have

$$\begin{aligned}
E[u,u]^{n_1} &= \sum_{i=1}^{n_1-1} E\left[\left(u_{t_{iK+1}} - u_{t_{(i-1)K+1}}\right)^2\right] \\
&= \sum_{i=1}^{n_1-1} E\left[\delta^2\gamma_n^2\left(W_{t_{iK+1}} - W_{t_{iK}}\right)^2 + \delta^2\gamma_n^2\left(W_{t_{(i-1)K+1}} - W_{t_{(i-1)K}}\right)^2 + \left(\varepsilon_{t_{iK+1}} - \varepsilon_{t_{(i-1)K+1}}\right)^2\right] \\
&= \sum_{i=1}^{n_1-1} E\left[\left(\varepsilon_{t_{iK+1}} - \varepsilon_{t_{(i-1)K+1}}\right)^2\right] + \frac{2}{n}\left(n_1-1\right)\delta^2\gamma_n^2 \\
&= [m,m]^{n_1} + 0 + n^{-\alpha}\sum_{i=1}^{n_1-1}\left[\omega^2\left(\frac{iK+1}{n}\right) + \omega^2\left(\frac{(i-1)K+1}{n}\right)\right] + \frac{2}{n}\left(n_1-1\right)\delta^2\gamma_n^2 \qquad (12) \\
&= [m,m]^{n_1} + 2n_1 n^{-\alpha}\int_0^1 \omega^2(u)\,du + O\left(n^{-\alpha}\right) + 2n_1\delta^2 n^{-\alpha}.
\end{aligned}$$

We prove (12) below.

$$
\begin{aligned}
& \sum_{i=1}^{n_1-1} E\left[\left(\varepsilon_{t_{iK+1}}-\varepsilon_{t_{(i-1)K+1}}\right)^2\right] \\
= & \sum_{i=1}^{n_1-1} E\left[\left(m_{t_{iK+1}}-m_{t_{(i-1)K+1}}+n^{-\alpha/2}\omega\left(\frac{iK+1}{n}\right)\epsilon_{t_{iK+1}}-n^{-\alpha/2}\omega\left(\frac{(i-1)K+1}{n}\right)\epsilon_{t_{(i-1)K+1}}\right)^2\right] \\
= & \left(m_{t_{iK+1}}-m_{t_{(i-1)K+1}}\right)^2+n^{-\alpha}\left[\omega^2\left(\frac{iK+1}{n}\right)+\omega^2\left(\frac{(i-1)K+1}{n}\right)\right].
\end{aligned}
$$

■

Proof of Lemma A4. We have

$$
\begin{aligned}
& \operatorname{var}[u,u]^{n_1} \\
= & \operatorname{var}[v,v]^{n_1}+\operatorname{var}[e,e]^{n_1}+4\operatorname{var}[v,e]^{n_1} \\
= & 12\delta^4 n^{-2\alpha}n_1+O\left(n^{-2\alpha}\right) \qquad (13)\\
& +4n_1 n^{-2\alpha}E\epsilon^4\int\omega^4(u)\,du+O\left(n^{-3\alpha/2}\right)+O\left(n^{-\alpha}\overline{n}^{-1}\right) \\
& +24\delta^2 n^{-2\alpha}n_1\int\omega^2(u)\,du+O\left(n^{-2\alpha}\right)+O\left(n^{-\alpha}\overline{n}^{-1}\right) \\
= & 12\delta^4 n^{-2\alpha}n_1+24\delta^2 n^{-2\alpha}n_1\int\omega^2(u)\,du+4n_1 n^{-2\alpha}E\epsilon^4\int\omega^4(u)\,du+O\left(n^{-3\alpha/2}\right)+O\left(n^{-\alpha}\overline{n}^{-1}\right) \\
= & \frac{n_1}{n^{2\alpha}}\left\{12\delta^4+4E\epsilon^4\int\omega^4(u)\,du+24\delta^2\int\omega^2(u)\,du+O\left(\frac{n^{\alpha/2}}{n_1}\right)+O\left(\frac{n^\alpha}{n_1^2}\right)\right\}.
\end{aligned}
$$

We prove (13) below in a series of steps, but first we derive the second result of Lemma A4. Note that note by substituting n for n_1 we get the following expression for $\operatorname{var}[u,u]^n$,

$$
\operatorname{var}[u,u]^n=\frac{n}{n^{2\alpha}}\left\{12\delta^4+4E\epsilon^4\int\omega^4(u)\,du+24\delta^2\int\omega^2(u)\,du+O\left(\frac{n^{\alpha/2}}{n}\right)+O\left(\frac{n^\alpha}{n^2}\right)\right\}.
$$

From this, we have

$$
\begin{aligned}
& \left(\frac{\overline{n}^{3/2}}{n}\right)^2\operatorname{var}[u,u]^n \\
= & \left(\frac{\overline{n}^{3/2}}{n}\right)^2\frac{n}{n^{2\alpha}}\left\{12\delta^4+4E\epsilon^4\int\omega^4(u)\,du+24\delta^2\int\omega^2(u)\,du+O\left(\frac{n^{\alpha/2}}{n}\right)+O\left(\frac{n^\alpha}{n^2}\right)\right\} \\
= & \frac{\overline{n}^3}{nn^{2\alpha}}\left\{12\delta^4+4E\epsilon^4\int\omega^4(u)\,du+24\delta^2\int\omega^2(u)\,du+o(1)\right\} \\
= & c^{-3}\left\{12\delta^4+4E\epsilon^4\int\omega^4(u)\,du+24\delta^2\int\omega^2(u)\,du\right\}+o(1),
\end{aligned}
$$

since $\frac{\overline{n}^3}{nn^{2\alpha}}=c^{-3}n^{3(1-\beta)-1-2\alpha}=c^{-3}n^{2-2\alpha-2\frac{2}{3}(1-\alpha)}=c^{-3}$. We get the second result of Lemma A4 by noting that $[u,u]^n-[u,u]^{\{n\}}$ is of smaller order than $[u,u]^{\{n\}}$, so $\left(\frac{\overline{n}^{3/2}}{n}\right)^2\operatorname{var}[u,u]^n$ has the same leading term as $\left(\frac{\overline{n}^{3/2}}{n}\right)^2\operatorname{var}[u,u]^{\{n\}}$.

Now we prove (13) by calculating separately each of the three components of $\mathrm{var}[u,u]^{n_1}$. The first component of $[u,u]^{n_1}$ is

$$\begin{aligned}
[v,v]^{n_1} &= \sum_{i=1}^{n_1-1}\left(v_{t_{iK+1}} - v_{t_{(i-1)K+1}}\right)^2 \\
&= \delta^2\gamma_n^2 \sum_{i=1}^{n_1-1}\left((W_{t_{iK+1}} - W_{t_{iK}}) - \left(W_{t_{(i-1)K+1}} - W_{t_{(i-1)K}}\right)\right)^2 \\
&\overset{d}{=} \delta^2\gamma_n^2 \sum_{i=1}^{n_1-1} Z_i^2, \text{ where } Z_i \text{ are 1-dependent } N\left(0,\frac{2}{n}\right), \mathrm{cov}(Z_i^2, Z_{i-1}^2) = \frac{2}{n^2},
\end{aligned}$$

and hence,

$$\begin{aligned}
\mathrm{var}[v,v]^{n_1} &= \delta^4\gamma_n^4\,(n_1-1)\times 2\left(\frac{2}{n}\right)^2 + 2\delta^4\gamma_n^4\,(n_1-2)\,\frac{2}{n^2} \\
&= 12\delta^4\gamma_n^4\,(n_1-1)\,\frac{1}{n^2} + O\left(n^{-2\alpha}\right) \\
&= 12\delta^4 n^{-2\alpha} n_1 + O\left(n^{-2\alpha}\right).
\end{aligned}$$

The second component of $\mathrm{var}[u,u]^{n_1}$ is

$$\begin{aligned}
\mathrm{var}[e,e]^{n_1} &= \mathrm{var}\sum_{i=1}^{n_1-1}\left(\varepsilon_{t_{iK+1}} - \varepsilon_{t_{(i-1)K+1}}\right)^2 \\
&= \sum_{i=1}^{n_1-1}\mathrm{var}\left(\varepsilon_{t_{iK+1}} - \varepsilon_{t_{(i-1)K+1}}\right)^2 + 2\sum_{i=2}^{n_1-1}\mathrm{cov}_{i,i-1} \qquad (14) \\
&= \sum_{i=1}^{n_1-1}\left\{2\omega_{t_{iK+1}}^4 n^{-2\alpha}\left(E\epsilon^4+1\right) + O\left(n^{-\alpha}\frac{K^2}{n^2}\right) + O\left(n^{-\frac{3}{2}\alpha}\frac{K}{n}\right)E\epsilon^3\right\} \\
&\quad +2\sum_{i=2}^{n_1-1} n^{-2\alpha}\omega_{t_{iK+1}}^4\left(E\epsilon^4-1\right) + O\left(n^{-3\alpha/2}\right) + O\left(n^{-\alpha}\overline{n}^{-1}\right) \\
&= 4n^{-2\alpha}E\epsilon^4\sum_{i=1}^{n_1}\omega_{t_{iK+1}}^4 + O\left(n^{-3\alpha/2}\right) + O\left(n^{-\alpha}\overline{n}^{-1}\right) \\
&= 4n_1 n^{-2\alpha}E\epsilon^4\int\omega^4(u)\,du + O\left(n^{-3\alpha/2}\right) + O\left(n^{-\alpha}\overline{n}^{-1}\right),
\end{aligned}$$

where we denote by $\mathrm{cov}_{i,i-1}$ the terms that appear because the sum in $var[e,e]^{n_1}$ involves 1-dependent terms. For exact expression and calculation of the second term in (14) see below. Before that, the first term in (14) is

$$
\begin{aligned}
& \operatorname{var}\left(\varepsilon_{t_{iK+1}}-\varepsilon_{t_{(i-1)K+1}}\right)^2 \\
= & \operatorname{var}\left(\varepsilon_i-\varepsilon_{i-1}\right)^2 \text{ by an obvious change of notation} \\
= & \operatorname{var}\left(m_i-m_{i-1}+n^{-\alpha/2}\left(\omega_i\epsilon_i-\omega_{i-1}\epsilon_{i-1}\right)\right)^2 \\
= & \operatorname{var}\left(2n^{-\alpha/2}\left(m_i-m_{i-1}\right)\left(\omega_i\epsilon_i-\omega_{i-1}\epsilon_{i-1}\right)+n^{-\alpha}\left(\omega_i\epsilon_i-\omega_{i-1}\epsilon_{i-1}\right)^2\right) \\
= & 4n^{-\alpha}\left(m_i-m_{i-1}\right)^2\operatorname{var}\left(\omega_i\epsilon_i-\omega_{i-1}\epsilon_{i-1}\right)+n^{-2\alpha}\operatorname{var}\left(\omega_i\epsilon_i-\omega_{i-1}\epsilon_{i-1}\right)^2+ \\
& +4n^{-\frac{3}{2}\alpha}\left(m_i-m_{i-1}\right)\operatorname{cov}\left(\omega_i\epsilon_i-\omega_{i-1}\epsilon_{i-1},\left(\omega_i\epsilon_i-\omega_{i-1}\epsilon_{i-1}\right)^2\right) \\
= & 4n^{-\alpha}\left(m_i-m_{i-1}\right)^2\left(\omega_i^2+\omega_{i-1}^2\right)+n^{-2\alpha}\left\{\left(\omega_i^4+\omega_{i-1}^4\right)\left(E\epsilon^4-1\right)+4\omega_i^2\omega_{i-1}^2\right\}+ \\
& +4n^{-\frac{3}{2}\alpha}\left(m_i-m_{i-1}\right)\left(\omega_i^3+\omega_{i-1}^3\right)E\epsilon^3 \\
= & O\left(n^{-\alpha}\frac{K^2}{n^2}\right)+n^{-2\alpha}\left\{\left(\omega_i^4+\omega_{i-1}^4\right)\left(E\epsilon^4-1\right)+4\omega_i^2\omega_{i-1}^2\right\}+O\left(n^{-\frac{3}{2}\alpha}\frac{K}{n}\right)E\epsilon^3 \\
= & 2\omega_{t_{iK+1}}^4 n^{-2\alpha}\left(E\epsilon^4+1\right)+O\left(n^{-\alpha}\frac{K^2}{n^2}\right)+O\left(n^{-\frac{3}{2}\alpha}\frac{K}{n}\right)E\epsilon^3.
\end{aligned}
$$

The second term in (14) is

$$
\begin{aligned}
\operatorname{cov}_{i,i-1} = & \operatorname{cov}\left[\left(\varepsilon_{t_{iK+1}}-\varepsilon_{t_{(i-1)K+1}}\right)^2,\left(\varepsilon_{t_{(i-1)K+1}}-\varepsilon_{t_{(i-2)K+1}}\right)^2\right] \\
= & \operatorname{cov}\left[\left(\varepsilon_3-\varepsilon_2\right)^2,\left(\varepsilon_2-\varepsilon_1\right)^2\right] \text{ (change of notation)} \\
= & \operatorname{var}\left(\varepsilon_2^2\right)-2\operatorname{cov}\left(\varepsilon_2^2,\varepsilon_1\varepsilon_2\right)-2\operatorname{cov}\left(\varepsilon_2\varepsilon_3,\varepsilon_2^2\right)+4\operatorname{cov}\left(\varepsilon_1\varepsilon_2,\varepsilon_2\varepsilon_3\right) \\
= & 4m_2^2n^{-\alpha}\omega_2^2+n^{-2\alpha}\omega_2^4\left(E\epsilon^4-1\right)+4m_2n^{-3\alpha/2}\omega_2^3E\epsilon^3 \\
& -2\left\{2m_2m_1n^{-\alpha}\omega_2^2+m_1n^{-3\alpha/2}\omega_2^3E\epsilon^3\right\} \\
& -2\left\{2m_2m_3n^{-\alpha}\omega_2^2+m_3n^{-3\alpha/2}\omega_2^3E\epsilon^3\right\}+4\left\{n^{-\alpha}m_1m_3\omega_2^2\right\} \\
= & n^{-2\alpha}\omega_2^4\left(E\epsilon^4-1\right)+2n^{-3\alpha/2}\omega_2^3E\epsilon^3\left\{2m_2-m_1-m_3\right\} \\
& +4n^{-\alpha}\omega_2^2\left(m_2-m_3\right)\left(m_2-m_1\right) \\
= & n^{-2\alpha}\omega_{t_{iK+1}}^4\left(E\epsilon^4-1\right)+O\left(n^{-3\alpha/2}\overline{n}^{-1}\right)+O\left(n^{-\alpha}\overline{n}^{-2}\right),
\end{aligned}
$$

where we have used $\operatorname{var}\left(\varepsilon_i^2\right)=4m_i^2n^{-\alpha}\omega_i^2+n^{-2\alpha}\omega_i^4\left(E\epsilon^4-1\right)+4m_in^{-3\alpha/2}\omega_i^3E\epsilon^3$, $\operatorname{cov}\left(\varepsilon_{t_i}^2,\varepsilon_{t_{i+1}}\varepsilon_{t_i}\right)=2m_{t_i}m_{t_{i+1}}n^{-\alpha}\omega_{t_i}^2+m_{t_{i+1}}n^{-3\alpha/2}\omega_{t_i}^3E\epsilon^3$ and $\operatorname{cov}\left(\varepsilon_1\varepsilon_2,\varepsilon_2\varepsilon_3\right)=n^{-\alpha}m_1m_3\omega_2^2$.

The third component of $\text{var}[u,u]^{n_1}$ is

$$\begin{aligned}
& 4\text{var}[v,e]^{n_1} \\
= & 4\text{var}\sum_{i=1}^{n_1-1}\left(v_{t_{iK+1}}-v_{t_{(i-1)K+1}}\right)\left(\varepsilon_{t_{iK+1}}-\varepsilon_{t_{(i-1)K+1}}\right) \\
= & 4\delta^2\gamma_n^2\text{var}\sum_{i=1}^{n_1-1}Z_i\left(\varepsilon_{t_{iK+1}}-\varepsilon_{t_{(i-1)K+1}}\right), \\
& \text{where } Z_i \text{ are 1-dependent } N\left(0,\frac{2T}{n}\right) \text{ r.v.'s with autocovariance} -\frac{1}{n}, Z\perp\varepsilon \\
= & 4\delta^2\gamma_n^2\sum_{i=1}^{n_1-1}\text{var}\left[Z_i\Delta_K\varepsilon_{t_{iK+1}}\right]+8\delta^2\gamma_n^2\sum_{i=2}^{n_1-1}\text{cov}\left\{Z_i\Delta_K\varepsilon_{t_{iK+1}},Z_{i-1}\Delta_K\varepsilon_{t_{(i-1)K+1}}\right\} \\
= & 4\delta^2\gamma_n^2\sum_{i=1}^{n_1-1}EZ_i^2E\left(\varepsilon_{t_{iK+1}}-\varepsilon_{t_{(i-1)K+1}}\right)^2+8\delta^2\gamma_n^2\sum_{i=2}^{n_1-1}\left\{O\left(\frac{1}{n}\frac{K^2}{n^2}\right)+Tn^{-1-\alpha}\omega^2_{t_{(i-1)K+1}}\right\} \\
= & 24\delta^2n^{-2\alpha}n_1\int\omega^2(u)\,du+O\left(n^{-2\alpha}\right)+O\left(n^{-\alpha}\frac{K}{n}\right).
\end{aligned}$$

■

Proof of Lemma A5. We have

$$\begin{aligned}
E[X,u]^{avg} &= \frac{1}{K}\sum_{j=1}^{K}E[X,u]^{n_j}=\frac{1}{K}\sum_{j=1}^{K}\delta\gamma_n\sum_{i=1}^{n_j-1}\int_{t_{iK+j-1}}^{t_{iK+j}}\sigma_t dt \\
&= \frac{\delta\gamma_n}{K}\sum_{i=1}^{n_j-1}\sum_{j=1}^{K}\int_{t_{iK+j-1}}^{t_{iK+j}}\sigma_t dt=\frac{\delta\gamma_n}{K}\sum_{i=1}^{n_j-1}\int_{t_{iK}}^{t_{(i+1)K}}\sigma_t dt \\
&= \frac{\delta\gamma_n}{K}\int_{t_K}^{t_n}\sigma_t dt=\frac{\delta\gamma_n}{K}\int_0^1\sigma_t dt+O\left(\frac{\gamma_n}{K}\frac{K}{n}\right) \\
&= \frac{\delta\gamma_n}{K}\int_0^1\sigma_t dt+O\left(n^{-\frac{1+\alpha}{2}}\right).
\end{aligned}$$

As for the second part of Lemma A5, we first calculate $E[X,u]^{\{n\}}$,

$$\begin{aligned}
E[X,u]^{\{n\}} &= \sum_{i=K}^{n-K} E\left[\Delta X_{t_i}\Delta u_{t_i}\right] + \frac{1}{2}\left(\sum_{i=n-K+1}^{n} E\left[\Delta X_{t_i}\Delta u_{t_i}\right] + \sum_{i=1}^{K-1} E\left[\Delta X_{t_i}\Delta u_{t_i}\right]\right) \\
&= c\gamma_n \sum_{i=K}^{n-K} \int_{t_{i-1}}^{t_i} \sigma_t dt + \frac{1}{2}c\gamma_n \sum_{i=n-K+1}^{n} \int_{t_{i-1}}^{t_i} \sigma_t dt + \frac{1}{2}c\gamma_n \sum_{i=1}^{K-1} \int_{t_{i-1}}^{t_i} \sigma_t dt \\
&= c\gamma_n \int_{t_{K-1}}^{t_{n-K}} \sigma_t dt + \frac{1}{2}c\gamma_n \left(\int_{t_{n-K}}^{t_n} \sigma_t dt + \int_{t_0}^{t_{K-1}} \sigma_t dt\right) \\
&= c\gamma_n \int_0^1 \sigma_t dt - \frac{1}{2}c\gamma_n O\left(\overline{n}^{-1}\right).
\end{aligned}$$

Then,

$$\begin{aligned}
&\overline{n}^{1/2}E[X,u]^{avg} - \frac{\overline{n}^{3/2}}{n}E[X,u]^{\{n\}} \\
=\ & \overline{n}^{1/2}\frac{\delta\gamma_n}{K}\int_0^1 \sigma_t dt + \overline{n}^{1/2}O\left(n^{-\frac{1+\alpha}{2}}\right) - \frac{\overline{n}^{3/2}}{n}\delta\gamma_n \int_0^1 \sigma_t dt + \frac{\overline{n}^{3/2}}{n}O\left(\gamma_n \overline{n}^{-1}\right) \\
=\ & O\left(n^{-\frac{\alpha+\beta}{2}}\right) = o(1).
\end{aligned}$$

■

Proof of Lemma A6. We have

$$\begin{aligned}
&\operatorname{var}[X,u]^{avg} \\
=\ & \operatorname{var}\left\{\frac{1}{K}\sum_{j=1}^{K}[X,u]^{n_j}\right\} \\
=\ & \frac{1}{K^2}\sum_{j=1}^{K}\operatorname{var}[X,u]^{n_j} + \frac{1}{K^2}\sum_{j\neq m}^{K}\sum_{m=1}^{K}\operatorname{cov}\left\{[X,u]^{n_j},[X,u]^{n_m}\right\} \\
=\ & \frac{1}{K^2}\sum_{j=1}^{K}\operatorname{var}[X,u]^{n_j} + O\left(n^{-1-\alpha}\overline{n}\right) + O\left(\overline{n}^{-1}n^{-1}\right) \qquad (15) \\
=\ & \frac{1}{K}\left(O\left(n^{-\alpha}\right) + O\left(\frac{1}{\overline{n}^2}\right)\right) + O\left(n^{-1-\alpha}\overline{n}\right) + O\left(\overline{n}^{-1}n^{-1}\right) \\
\sim\ & n^{-\beta-\alpha} + n^{-\beta-2+2\beta} + n^{-1-\alpha}n^{1-\beta} + n^{-1+\beta}n^{-1} \sim n^{\beta-2}.
\end{aligned}$$

The above (15) follows by noticing that all covariance terms are of the same order, so we can explore the magnitude of one of them. Since we are looking at the magnitudes only, assume without loss of generality that $\delta = 1$. Then,

$$
\begin{aligned}
& \operatorname{cov}\left\{[X,u]^{\overline{n}}_{(1)}, [X,u]^{\overline{n}}_{(2)}\right\} \\
= \; & \operatorname{cov}\left[\sum_{i=1}^{\overline{n}-1}\left(u_{t_{iK+1}} - u_{t_{(i-1)K+1}}\right)\left(X_{t_{iK+1}} - X_{t_{(i-1)K+1}}\right), \sum_{i=1}^{\overline{n}-1}\left(u_{t_{iK+2}} - u_{t_{(i-1)K+2}}\right)\left(X_{t_{iK+2}} - X_{t_{(i-1)K+2}}\right)\right].
\end{aligned}
$$

The latter can be easily shown to be $O\left(n^{-1-\alpha}\overline{n}\right) + O\left(\overline{n}^{-1}n^{-1}\right).$

■

Proof of Lemma A7. We have

$$
\begin{aligned}
& \overline{n}^{1/2}E[u,u]^{avg} - \frac{\overline{n}^{3/2}}{n}E\left[u,u\right]^{\{n\}} \\
= \; & \overline{n}^{1/2}\frac{1}{K}\sum_{j=1}^{K}\sum_{i=1}^{n_j-1}E\left[\left(u_{t_{iK+1}} - u_{t_{(i-1)K+1}}\right)^2\right] - \frac{\overline{n}^{3/2}}{n}E\left[u,u\right]^{\{n\}} \\
= \; & \frac{\overline{n}^{1/2}}{K}\sum_{i=1}^{n-K}E\left[\left(\varepsilon_{t_{i+K}} - \varepsilon_{t_i}\right)^2\right] - \frac{\overline{n}^{3/2}}{n}\frac{1}{2}\left\{\sum_{i=1}^{n-K}E\left[\left(\varepsilon_{t_{i+1}} - \varepsilon_{t_i}\right)^2\right] + \sum_{i=K}^{n-1}E\left[\left(\varepsilon_{t_{i+1}} - \varepsilon_{t_i}\right)^2\right]\right\} \quad (16) \\
= \; & n^{-\alpha}\frac{\overline{n}^{1/2}}{K}\sum_{i=1}^{n-K}\left(\omega^2_{t_{i+K}} + \omega^2_{t_i}\right) - n^{-\alpha}\frac{\overline{n}^{3/2}}{2n}\left\{\sum_{i=1}^{n-K}\left(\omega^2_{t_{i+1}} + \omega^2_{t_i}\right) + \sum_{i=K}^{n-1}\left(\omega^2_{t_{i+1}} + \omega^2_{t_i}\right)\right\} \quad (17) \\
= \; & n^{-\alpha}\frac{\overline{n}^{1/2}}{K}\left\{\sum_{i=K+1}^{n}\omega^2_{t_i} + \sum_{i=1}^{n-K}\omega^2_{t_i} - \frac{1}{2}\sum_{i=2}^{n-K+1}\omega^2_{t_i} - \frac{1}{2}\sum_{i=1}^{n-K}\omega^2_{t_i} - \frac{1}{2}\sum_{i=K+1}^{n}\omega^2_{t_i} - \frac{1}{2}\sum_{i=K}^{n-1}\omega^2_{t_i}\right\} \\
\sim \; & n^{-\alpha}\frac{\overline{n}^{1/2}}{K} = o\left(1\right),
\end{aligned}
$$

where (16) follows because contributions from v are zero,

$$
\begin{aligned}
& \frac{\overline{n}^{1/2}}{K}\left(n-K\right)\delta^2\gamma_n^2\frac{1}{n} - \frac{1}{2}\left\{\frac{\overline{n}^{3/2}}{n}\left(n-K\right)\delta^2\gamma_n^2\frac{1}{n} + \frac{\overline{n}^{3/2}}{n}\left(n-K\right)\delta^2\gamma_n^2\frac{1}{n}\right\} \\
= \; & \frac{\overline{n}^{1/2}}{K}\left(n-K\right)\delta^2\gamma_n^2\frac{1}{n} - \frac{\overline{n}^{3/2}}{n}\left(n-K\right)\delta^2\gamma_n^2\frac{1}{n} = 0,
\end{aligned}
$$

and (17) follows because contributions from $m\left(.\right)$ are negligible,

$$
\begin{aligned}
& \frac{\overline{n}^{1/2}}{K}\sum_{i=1}^{n-K}O\left(\frac{1}{\overline{n}^2}\right) - \frac{\overline{n}^{3/2}}{n}\frac{1}{2}\left\{\sum_{i=1}^{n-K}O\left(\frac{1}{n^2}\right) + \sum_{i=K+1}^{n}O\left(\frac{1}{n^2}\right)\right\} \\
\sim \; & \frac{\overline{n}^{1/2}}{K}n\frac{1}{\overline{n}^2} + \frac{\overline{n}^{3/2}}{n}n\frac{1}{n^2} = \overline{n}^{-1/2} + \frac{\overline{n}^{3/2}}{n^2} = o\left(1\right).
\end{aligned}
$$

■

PROOF OF LEMMA A8. Using Lemma 4,

$$
\begin{aligned}
& \operatorname{var}[u,u]^{avg} \\
= & \operatorname{var}\left\{\frac{1}{K}\sum_{j=1}^{K}[u,u]^{n_j}\right\} = \frac{1}{K^2}\sum_{j=1}^{K}\operatorname{var}[u,u]^{n_j} \qquad (18)\\
= & \frac{1}{K}\left[\frac{n_1}{n^{2\alpha}}\left\{12\delta^4 + 4E\epsilon^4\int\omega^4(u)\,du + 24\delta^2\int\omega^2(u)\,du + O\left(\frac{n^{\alpha/2}}{n_1}\right) + O\left(\frac{n^{\alpha}}{n_1^2}\right)\right\}\right] \\
= & \frac{n_1}{Kn^{2\alpha}}\left\{12\delta^4 + 4E\epsilon^4\int\omega^4(u)\,du + 24\delta^2\int\omega^2(u)\,du + O\left(\frac{n^{\alpha/2}}{n_1}\right) + O\left(\frac{n^{\alpha}}{n_1^2}\right)\right\} \\
= & c^{-3}\overline{n}^{-1}\left\{12\delta^4 + 4E\epsilon^4\int\omega^4(u)\,du + 24\delta^2\int\omega^2(u)\,du + o(1)\right\} \text{ if } \beta = \frac{2}{3}(1-\alpha).
\end{aligned}
$$

In above, (18) follows because all covariance terms are zero. For example,

$$
\operatorname{cov}\{[u,u]^{n_1},[u,u]^{n_2}\} = \sum_{i=1}^{n_1-1}\sum_{j=1}^{n_2-1}\operatorname{cov}\left\{\left(u_{t_{iK+1}} - u_{t_{(i-1)K+1}}\right)^2, \left(u_{t_{jK+2}} - u_{t_{(j-1)K+2}}\right)^2\right\}.
$$

To show all terms in the summation above are zero, we do the calculation for the term with indices $(i=1, j=1)$

$$
\begin{aligned}
& \operatorname{cov}\left\{\left(u_{t_{K+1}} - u_{t_1}\right)^2, \left(u_{t_{K+2}} - u_{t_2}\right)^2\right\} \\
= & \operatorname{cov}\Big(\left[\delta\gamma_n\left(W_{t_{K+1}} - W_{t_K}\right) - \delta\gamma_n\left(W_{t_1} - W_{t_0}\right) + \left(\varepsilon_{t_{K+1}} - \varepsilon_{t_1}\right)\right]^2, \\
& \left[\delta\gamma_n\left(W_{t_{K+2}} - W_{t_{K+1}}\right) - \delta\gamma_n\left(W_{t_2} - W_{t_1}\right) + \left(\varepsilon_{t_{K+2}} - \varepsilon_{t_2}\right)\right]^2\Big) = 0
\end{aligned}
$$

as well as for the term with indices $(i=2, j=1)$

$$
\begin{aligned}
& \operatorname{cov}\left\{\left(u_{t_{2K+1}} - u_{t_{K+1}}\right)^2, \left(u_{t_{K+2}} - u_{t_2}\right)^2\right\} \\
= & \operatorname{cov}\Big(\left[\delta\gamma_n\left(W_{t_{2K+1}} - W_{t_{2K}}\right) - \delta\gamma_n\left(W_{t_{K+1}} - W_{t_K}\right) + \left(\varepsilon_{t_{2K+1}} - \varepsilon_{t_{K+1}}\right)\right]^2, \\
& \left[\delta\gamma_n\left(W_{t_{K+2}} - W_{t_{K+1}}\right) - \delta\gamma_n\left(W_{t_2} - W_{t_1}\right) + \left(\varepsilon_{t_{K+2}} - \varepsilon_{t_2}\right)\right]^2\Big) = 0.
\end{aligned}
$$

■

Proof Lemma B1. We have

$$
\begin{aligned}
&\operatorname{cov}\left([X,X]^{\overline{n}}_{avg},[u,u]^{\overline{n}}_{avg}\right)\\
=\;& \operatorname{cov}\left(\frac{1}{K}\sum_{j=1}^{K}[X,X]^{n_j},\frac{1}{K}\sum_{j=1}^{K}[u,u]^{n_j}\right)\\
=\;& \operatorname{cov}\left(\frac{1}{K}\sum_{j=1}^{K}\sum_{i=1}^{n_j-1}\left(X_{t_{iK+j}}-X_{t_{(i-1)K+j}}\right)^2,\frac{1}{K}\sum_{j=1}^{K}\sum_{i=1}^{n_j-1}\left(u_{t_{iK+j}}-u_{t_{(i-1)K+j}}\right)^2\right)\\
=\;& \operatorname{cov}\left(\frac{1}{K}\sum_{j=1}^{K}\sum_{i=1}^{n_j-1}a_{ij},\frac{1}{K}\sum_{j=1}^{K}\sum_{i=1}^{n_j-1}\left(\gamma_n b_{1,ij}-\gamma_n b_{2,ij}+b_{3,ij}\right)^2\right)\\
=\;& \operatorname{cov}\left(\frac{1}{K}\sum_{j=1}^{K}\sum_{i=1}^{n_j-1}a_{ij},\frac{1}{K}\sum_{j=1}^{K}\sum_{i=1}^{n_j-1}\left(\gamma_n^2 b_{1,ij}^2+\gamma_n^2 b_{2,ij}^2+b_{3,ij}^2-2\gamma_n^2 b_{1,ij}b_{2,ij}+2\gamma_n b_{1,ij}b_{3,ij}-2\gamma_n b_{2,ij}b_{3,ij}\right)\right)\\
=\;& c_1+c_2+c_3+c_4+c_5+c_6\\
=\;& O\left(K^{-1}n^{-\alpha}\right)=o\left(\overline{n}^{-1}\right)\ \text{ as }\ -\beta-\alpha<-(1-\beta)\ \text{ holds if }\beta=\frac{2}{3}(1-\alpha)
\end{aligned}
$$

where $b_{1,ij}=W_{t_{iK+j}}-W_{t_{iK+j-1}}$, $b_{2,ij}=W_{t_{(i-1)K+j}}-W_{t_{(i-1)K+j-1}}$, $b_{3,ij}=\varepsilon_{t_{iK+j}}-\varepsilon_{t_{(i-1)K+j}}$.

The last line follows because $c1\sim c2\sim K^{-2}$ and $c_3=c_4=c_5=c_6=0$ by properties of normal random variables.

■

Proof of Lemma B2. First, note that $\operatorname{cov}\left([X,X]^{\overline{n}}_{avg},[u,u]^{n}\right)$ is of the same order as $\operatorname{cov}\left([X,X]^{\overline{n}}_{avg},[u,u]^{\{n\}}\right)$ since $[u,u]^n-[u,u]^{\{n\}}$ is of smaller order than $[u,u]^{\{n\}}$. Also, notice that $\operatorname{cov}\left([X,X]^{\overline{n}}_{avg},[u,u]^{n}\right)$ has to be of the same order as $\operatorname{cov}\left([X,X]^{\overline{n}}_{avg},[u,u]^{\overline{n}}_{avg}\right)$ by similarity in construction of $[u,u]^n$ and $[u,u]^{\overline{n}}_{avg}$. Hence, $\operatorname{cov}\left([X,X]^{\overline{n}}_{avg},[u,u]^{\{n\}}\right)=o\left(\overline{n}^{-1}\right)$ by Lemma B1.

■

Proof of Lemma B3. We need to prove here that

$$\operatorname{cov}\left([u,u]^{\{n\}},[u,u]^{avg}\right)=K^{-1}n^{1-2\alpha}\left\{8\delta^4+4\left(E\epsilon^4-1\right)\sigma_\epsilon^4+o(1)+O\left(n^{\alpha+\beta-1}\right)\right\}.$$

We will calculate the expression for $\operatorname{cov}\left([u,u]^{avg},[u,u]^{n}\right)$. It has the same leading term as

$\text{cov}\left([u,u]^{avg},[u,u]^{\{n\}}\right)$ since $[u,u]^n-[u,u]^{\{n\}}$ is of smaller order than $[u,u]^{\{n\}}$.

$$\begin{aligned}
\text{cov}\left([u,u]^{avg},[u,u]^n\right) &= \text{cov}\left(\frac{1}{K}\sum_{j=1}^{K}\sum_{i=1}^{n_j-1}\left(u_{t_{iK+j}}-u_{t_{(i-1)K+j}}\right)^2,\sum_{i=1}^{n-1}\left(u_{t_{i+1}}-u_{t_i}\right)^2\right)\\
&= \text{cov}\left(\frac{1}{K}\sum_{i=1}^{n-K}\left(u_{t_{i+K}}-u_{t_i}\right)^2,\sum_{i=1}^{n-1}\left(u_{t_{i+1}}-u_{t_i}\right)^2\right)\\
&= \frac{1}{K}\text{cov}\left(a_1^2+a_2^2+a_3^2+2a_1a_2+2a_1a_3+2a_2a_3,\right.\\
&\quad \left.b_1^2+b_2^2+b_3^2+2b_1b_2+2b_1b_3+2b_2b_3\right)\\
&= \frac{1}{K}\text{cov}\left(a_1^2+a_2^2+a_3^2,b_1^2+b_2^2+b_3^2\right)\\
&\quad +\frac{2}{K}\text{cov}\left(a_1^2+a_2^2+a_3^2,b_1b_2+b_1b_3+b_2b_3\right)\\
&\quad +\frac{2}{K}\text{cov}\left(a_1a_2+a_1a_3+a_2a_3,b_1^2+b_2^2+b_3^2\right)\\
&\quad +\frac{4}{K}\text{cov}\left(a_1a_2+a_1a_3+a_2a_3,b_1b_2+b_1b_3+b_2b_3\right)
\end{aligned}$$

Denote the terms in last four lines by

$$\begin{aligned}
&\text{cov}\left([u,u]^{avg},[u,u]^n\right)\\
=\ & B3_1+B3_2+B3_3+B3_4\\
=\ & B3_1+0+0+\left\{O\left(n^{\beta-1-\alpha}\right)+O\left(n^{-2\alpha-\beta}\right)\right\}\\
=\ & 8\delta^4K^{-1}n^{1-2\alpha}+4\left(E\epsilon^4-1\right)K^{-1}n^{1-2\alpha}\int_0^1\omega_u^4du+o\left(K^{-1}n^{1-2\alpha}\right)+O\left(n^{-\alpha}\right)\\
&+\left\{O\left(n^{\beta-1-\alpha}\right)+O\left(n^{-2\alpha-\beta}\right)\right\}\\
=\ & K^{-1}n^{1-2\alpha}\left\{8\delta^4+4\left(E\epsilon^4-1\right)\sigma_\epsilon^4+o\left(1\right)+O\left(n^{\alpha+\beta-1}\right)\right\}\\
=\ & \frac{n}{\overline{n}^2}\left\{8\delta^4+4\left(E\epsilon^4-1\right)\sigma_\epsilon^4+o\left(1\right)\right\}\ \text{if}\ \beta=\frac{2}{3}\left(1-\alpha\right).
\end{aligned}$$

This result is similar to Zhang et al. (2005) paper, where the covariance is, apart from normalisation factor, $\text{cov}\left([\epsilon,\epsilon]^{avg},[\epsilon,\epsilon]^n\right)=4\text{var}\left(\epsilon^2\right)=4\left(E\epsilon^4-1\right)$.

To obtain the expression for the $B3_1$ term, note that the terms $\text{cov}\left(a_1^2,b_1^2\right)$, $\text{cov}\left(a_1^2,b_2^2\right)$, $\text{cov}\left(a_2^2,b_1^2\right)$, and $\text{cov}\left(a_2^2,b_2^2\right)$ are all equal to $2\delta^4\gamma_n^4n^{-1}+o\left(n^{1-2\alpha}\right)$, and also $\text{cov}\left(a_1^2,b_3^2\right)=\text{cov}\left(a_2^2,b_3^2\right)=\text{cov}\left(a_3^2,b_1^2\right)=$

$\text{cov}\left(a_3^2, b_2^2\right) = 0$. The final term in $B3_1$ is

$$\begin{aligned} \text{cov}\left(a_3^2, b_3^2\right) &= \text{cov}\left(\sum_{i=1}^{n-K}\left(\varepsilon_{t_{i+K}} - \varepsilon_{t_i}\right)^2, \sum_{i=1}^{n-1}\left(\varepsilon_{t_{i+1}} - \varepsilon_{t_i}\right)^2\right) \\ &= 4\left(E\epsilon^4 - 1\right)n^{1-2\alpha}\int_0^1 \omega_u^4 du + o\left(n^{1-2\alpha}\right) + O\left(Kn^{-\alpha}\right), \end{aligned}$$

using similar steps as in Lemma A4. Hence,

$$\begin{aligned} B3_1 &= \frac{1}{K}\text{cov}\left(a_1^2 + a_2^2 + a_3^2, b_1^2 + b_2^2 + b_3^2\right) \\ &= 8\delta^4 K^{-1}n^{1-2\alpha} + 4\left(E\epsilon^4 - 1\right)K^{-1}n^{1-2\alpha}\int_0^1 \omega_u^4 du + o\left(K^{-1}n^{1-2\alpha}\right) + O\left(n^{-\alpha}\right). \end{aligned}$$

The next two terms are zero, $B3_2 = B3_3 = 0$.

Finally, we show how to obtain $B3_4 = O\left(n^{2\beta-1-\alpha}\right) + O\left(n^{-2\alpha}\right)$. We have

$$\begin{aligned} & \text{cov}\left(a_1a_2 + a_1a_3 + a_2a_3, b_1b_2 + b_1b_3 + b_2b_3\right) \\ = \; & 0 + 0 + 0 + \\ & 0 + \left\{O\left(\frac{\gamma_n^2}{\overline{n}^2}\right) + \delta^2 n^{-\alpha}\gamma_n^2\frac{1}{n}\sum_{i=K}^{n-1}\omega_{t_{i+1}}^2\right\} + \left\{O\left(\frac{\gamma_n^2}{\overline{n}^2}\right) - \delta^2 n^{-\alpha}\gamma_n^2\frac{1}{n}\sum_{i=K}^{n-2}\omega_{t_{i+1}}^2\right\} + \\ & 0 + \left\{O\left(\frac{\gamma_n^2}{\overline{n}^2}\right) - \delta^2 n^{-\alpha}\gamma_n^2\frac{1}{n}\sum_{i=1}^{n-K-1}\omega_{t_{i+1}}^2\right\} + \left\{O\left(\frac{\gamma_n^2}{\overline{n}^2}\right) + \delta^2 n^{-\alpha}\gamma_n^2\frac{1}{n}\sum_{i=0}^{n-K-1}\omega_{t_{i+1}}^2\right\} \\ = \; & O\left(\frac{\gamma_n^2}{\overline{n}^2}\right) + O\left(n^{-\alpha}\gamma_n^2\frac{1}{n}\right) = O\left(n^{1-\alpha}n^{-2(1-\beta)}\right) + O\left(n^{-\alpha}n^{1-\alpha}n^{-1}\right) \\ = \; & O\left(n^{2\beta-1-\alpha}\right) + O\left(n^{-2\alpha}\right). \end{aligned}$$

The first equality above follows because $\text{cov}\left(a_1a_2, b_1b_2\right) = \text{cov}\left(a_1a_2, b_1b_3\right) = \text{cov}\left(a_1a_2, b_2b_3\right) = \text{cov}\left(a_1a_3, b_1b_2\right) = \text{cov}\left(a_2a_3, b_1b_2\right) = 0$ and we have:

$$
\begin{aligned}
\operatorname{cov}\left(a_1 a_3, b_1 b_3\right) &= \delta^2 \gamma_n^2 \operatorname{cov}\left(\sum_{i=1}^{n-K}\left(W_{t_{i+K}}-W_{t_{i+K-1}}\right)\left(\varepsilon_{t_{i+K}}-\varepsilon_{t_i}\right), \sum_{i=1}^{n-1}\left(W_{t_{i+1}}-W_{t_i}\right)\left(\varepsilon_{t_{i+1}}-\varepsilon_{t_i}\right)\right) \\
&= O\left(\frac{\gamma_n^2}{\overline{n}^2}\right)+\delta^2 n^{-\alpha} \gamma_n^2 \frac{1}{n} \sum_{i=K}^{n-1} \omega_{t_{i+1}}^2, \\
\operatorname{cov}\left(a_1 a_3, b_2 b_3\right) &= \delta^2 \gamma_n^2 \operatorname{cov}\left(\sum_{i=1}^{n-K}\left(W_{t_{i+K}}-W_{t_{i+K-1}}\right)\left(\varepsilon_{t_{i+K}}-\varepsilon_{t_i}\right), \sum_{i=1}^{n-1}\left(W_{t_i}-W_{t_{i-1}}\right)\left(\varepsilon_{t_{i+1}}-\varepsilon_{t_i}\right)\right) \\
&= O\left(\frac{\gamma_n^2}{\overline{n}^2}\right)-\delta^2 n^{-\alpha} \gamma_n^2 \frac{1}{n} \sum_{i=K}^{n-2} \omega_{t_{i+1}}^2, \\
\operatorname{cov}\left(a_2 a_3, b_1 b_3\right) &= \delta^2 \gamma_n^2 \operatorname{cov}\left(\sum_{i=1}^{n-K}\left(W_{t_i}-W_{t_{i-1}}\right)\left(\varepsilon_{t_{i+K}}-\varepsilon_{t_i}\right), \sum_{i=1}^{n-1}\left(W_{t_{i+1}}-W_{t_i}\right)\left(\varepsilon_{t_{i+1}}-\varepsilon_{t_i}\right)\right) \\
&= O\left(\frac{\gamma_n^2}{\overline{n}^2}\right)-\delta^2 n^{-\alpha} \gamma_n^2 \frac{1}{n} \sum_{i=1}^{n-K-1} \omega_{t_{i+1}}^2, \\
\operatorname{cov}\left(a_2 a_3, b_2 b_3\right) &= \delta^2 \gamma_n^2 \operatorname{cov}\left(\sum_{i=1}^{n-K}\left(W_{t_i}-W_{t_{i-1}}\right)\left(\varepsilon_{t_{i+K}}-\varepsilon_{t_i}\right), \sum_{i=1}^{n-1}\left(W_{t_i}-W_{t_{i-1}}\right)\left(\varepsilon_{t_{i+1}}-\varepsilon_{t_i}\right)\right) \\
&= O\left(\frac{\gamma_n^2}{\overline{n}^2}\right)+\delta^2 n^{-\alpha} \gamma_n^2 \frac{1}{n} \sum_{i=0}^{n-K-1} \omega_{t_{i+1}}^2 .
\end{aligned}
$$

■

B Proof of Theorem 2.

To prove Theorem 2, we work though all Lemmas for Theorem 1, and then combine them in the same way to conclude the main result.

Note that Lemmas A1, A3, A5, A6, A7 do not use $\beta = \frac{2}{3}(1-\alpha)$ and and so are valid also for other choices of β.

In Lemma A2, the first part remains exactly the same. The conclusion of the second part also remains the same since

$$\begin{aligned} 4\left(\frac{\overline{n}^{3/2}}{n}\right)^2 \text{var}\,[X,u]^{(n)} &\sim \frac{\overline{n}^3}{n^2}\left(n^{-\alpha}+\frac{1}{n^2}\right) \sim \frac{\overline{n}^3}{n^2}n^{-\alpha} \\ &= n^{3(1-\beta)-2-\alpha} = n^{1-3\beta-\alpha} = o(1) \text{ because } \beta > 1/3. \end{aligned}$$

In Lemma A4, the first part remains exactly the same. The second part now becomes $\left(\frac{\overline{n}^{3/2}}{n}\right)^2 \text{var}[u,u]^{\{n\}} = o(1)$:

$$\begin{aligned} &\left(\frac{\overline{n}^{3/2}}{n}\right)^2 \text{var}[u,u]^{n} \\ &= \frac{\overline{n}^3}{n^2}\frac{n_1}{n^{2\alpha}}\left\{12\delta^4 + 4E\epsilon^4\int\omega^4(u)\,du + 24\delta^2\int\omega^2(u)\,du + O\left(\frac{n^{\alpha/2}}{n}\right) + O\left(\frac{n^{\alpha}}{n^2}\right)\right\} \\ &= o(1), \end{aligned}$$

by straightforward calculations.

In Lemma A8, the first part remains exactly the same, and the second changes to $\overline{n}\text{var}[u,u]^{avg} = o(1)$.

$$\overline{n}\text{var}[u,u]^{avg} = \frac{n_1^2}{Kn^{2\alpha}}\left\{12\delta^4 + 4E\epsilon^4\int\omega^4(u)\,du + 24\delta^2\int\omega^2(u)\,du + O\left(\frac{n^{\alpha/2}}{n_1}\right) + O\left(\frac{n^{\alpha}}{n_1^2}\right)\right\} = o(1)$$

by straightforward calculations.

From Lemmas A4 and A8 we know that all the correlation terms as in Lemmas B1 - B3 are $o(1)$. Theorem 2 follows.

■

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C Tables and Figures

Table 2. IQR percentage error with $K = (2V_2/V_1)^{1/3} n^{\frac{2}{3}(1-\alpha)}$

$n \diagdown \alpha$	0	0.05	0.1	0.15	0.2	0.25	0.3	0.35	0.4	0.45	0.5
195	96	186	145	120	145	114	95	78	65	54	N/A
390	94	135	110	200	156	128	143	111	89	71	59
780	67	90	108	137	107	181	151	162	119	100	76
1560	55	74	67	86	94	125	205	161	119	125	92
4680	48	47	56	58	74	96	99	117	201	144	151
5850	44	51	57	57	66	81	76	135	98	160	163
7800	45	46	52	53	68	70	90	94	109	175	134
11700	40	44	45	52	53	59	81	78	141	208	148
23400	36	40	43	46	49	58	61	79	106	123	196

Table 3. $K = (2V_2/V_1)^{1/3} n^{\frac{2}{3}(1-\alpha)}$

$n \diagdown \alpha$	0	0.05	0.1	0.15	0.2	0.25	0.3	0.35	0.4	0.45	0.5
195	3	2	2	2	1	1	1	1	1	1	0
390	4	3	3	2	2	2	1	1	1	1	1
780	7	5	4	3	3	2	2	1	1	1	1
1560	11	8	7	5	4	3	2	2	2	1	1
4680	22	17	13	10	7	5	4	3	2	2	1
5850	26	19	14	11	8	6	5	3	3	2	1
7800	31	23	17	13	9	7	5	4	3	2	2
11700	41	30	22	16	12	9	6	5	3	2	2
23400	65	47	33	24	17	12	9	6	4	3	2

Table 4. IQR percentage error with $K = n^{\frac{2}{3}(1-\alpha)}$

$n \diagdown \alpha$	0	0.05	0.1	0.15	0.2	0.25	0.3	0.35	0.4	0.45	0.5
195	-21	-16	-13	-7	-7	-3	-1	4	8	13	13
390	-15	-12	-7	-3	-3	1	3	6	7	12	14
780	-13	-11	-4	-2	0	0	4	5	6	11	14
1560	-9	-7	-2	-1	1	3	5	7	8	13	12
4680	-5	-3	-1	-2	1	0	3	5	6	7	11
5850	-4	-3	1	3	5	5	2	4	8	8	8
7800	-2	-2	0	1	3	2	5	3	6	8	10
11700	-3	0	0	2	2	5	4	2	6	3	8
23400	-2	1	2	1	3	4	2	6	6	6	8

Table 5. $K = n^{\frac{2}{3}(1-\alpha)}$

$n \diagdown \alpha$	0	0.05	0.1	0.15	0.2	0.25	0.3	0.35	0.4	0.45	0.5
195	34	28	24	20	17	14	12	10	8	7	6
390	53	44	36	29	24	20	16	13	11	9	7
780	85	68	54	44	35	28	22	18	14	11	9
1560	135	105	82	64	50	39	31	24	19	15	12
4680	280	211	159	120	91	68	52	39	29	22	17
5850	325	243	182	136	102	76	57	43	32	24	18
7800	393	292	216	161	119	88	66	49	36	27	20
11700	515	377	276	202	148	108	79	58	42	31	23
23400	818	585	418	299	214	153	109	78	56	40	29

Table 6. IQR percentage error with $K = n^{\frac{2}{3}}$

$n \diagdown \alpha$	0	0.05	0.1	0.15	0.2	0.25	0.3	0.35	0.4	0.45	0.5	K
195	-23	-23	-24	-23	-23	-21	-23	-24	-23	-24	-23	34
390	-17	-19	-19	-17	-19	-20	-18	-16	-16	-18	-18	53
780	-14	-15	-12	-15	-14	-12	-15	-15	-16	-14	-13	85
1560	-12	-9	-10	-10	-12	-11	-11	-9	-11	-12	-9	135
4680	-7	-2	-7	-5	-5	-7	-6	-5	-5	-6	-5	280
5850	-6	-6	-6	-6	-6	-6	-5	-7	-6	-5	-4	325
7800	-5	-6	-4	-4	-3	-4	-5	-4	-5	-6	-5	393
11700	-2	-6	-3	-3	-3	-4	-2	-5	-6	-2	-3	515
23400	-2	-2	-3	-2	-1	-2	-1	-3	-4	-2	-4	818

Table 7. IQR percentage error with $K^{BR} = \lfloor \phi n \rfloor, \phi = \left(\frac{3 \frac{RV^2}{n^2}}{2RQ} \right)^{1/3}$

$n \diagdown \alpha$	0	0.05	0.1	0.15	0.2	0.25	0.3	0.35	0.4	0.45	0.5	K^{BR}
195	55	46	34	29	27	21	22	19	16	18	15	6
390	67	49	37	28	23	20	17	18	15	15	14	8
780	94	65	48	32	26	22	19	16	16	14	12	10
1560	124	81	54	36	27	24	15	14	14	13	13	13
4680	243	146	91	54	34	24	18	16	12	14	8	18
5850	263	155	92	53	35	24	18	11	11	11	12	20
7800	300	182	97	60	33	26	15	13	10	11	9	22
11700	381	223	125	68	39	24	17	11	12	9	8	25
23400	539	305	163	86	47	28	15	13	8	8	8	32

Table 8. Effect of δ^2 on the estimates

δ^2/δ_b^2	$\mathrm{corr}(\Delta X_{t_i}, \Delta u_{t_i})$	IQR error
0.01	-0.0010	0.0133
0.05	-0.0051	0.0128
0.1	-0.0102	0.0049
0.25	-0.0254	0.0182
0.5	-0.0506	0.0037
1	-0.1000	0.0136
2	-0.1909	0.0100
4	-0.3280	0.0090
10	-0.4869	0.0130
20	-0.5351	0.0105