

ESTIMATING SPARSE MIMO CHANNELS HAVING COMMON SUPPORT

Yann Barbotin[†]

Ali Hormati[†]

Sundeep Rangan[‡]

Martin Vetterli[†]

[†]School of Computer and Communication Sciences, Ecole Polytechnique Fédérale de Lausanne, 1015 Lausanne, Switzerland

[‡]Polytechnic Institute of New York University, Brooklyn, NY

email : yann.barbotin@epfl.ch ali.hormati@epfl.ch srangan@poly.edu martin.vetterli@epfl.ch

web : http://lcavwww.epfl.ch/~{barbotin, hormati, vetterli}, http://eeweb.poly.edu/~srangan

ABSTRACT

We propose an algorithm (SCS-FRI) to estimate multipath channels with Sparse Common Support (SCS) based on Finite Rate of Innovation (FRI) sampling. In this setup, theoretical lower-bounds are derived, and simulation in a Rayleigh fading environment shows that SCS-FRI gets very close to these bounds. We show how to apply SCS-FRI to OFDM and CDMA downlinks. Recovery of a sparse common support is, among other, especially relevant for channel estimation in a multiple output system or beam-forming from multiple input. The present algorithm is based on a multi-output extension of the Cadzow denoising/annihilating filter method [1, 2].

Index Terms— Channel estimation, MIMO, OFDM, CDMA, Finite Rate of Innovation

1. INTRODUCTION

We consider the problem of estimating P multipath channels with common support, which we call Sparse Common Support (SCS) channels. SCS channels have a small number of paths (i.e. they are sparse) with the same time of arrival (ToA) across the different channels, up to a delay $\pm\epsilon$ as shown in Figure 1. The idealized case, $\epsilon = 0$, is called exact SCS channel.

This paper considers the use of sparse common support for channel estimation with multiple receive antennas. Multiple receive antennas can be used for either spatial diversity [3] or spatial multiplexing, but generally require separate channel estimates for each transmit-receive antenna pair. Most receivers estimate these channels separately. However, using sparse common support, the total number of parameters to be estimated can be reduced, thereby improving the estimate or reducing the pilot overhead.

The common support assumption is physically relevant [4, 5] if the receiver's antennas are separated by a fraction of the distance an electromagnetic wave travels in a time corresponding to the inverse-bandwidth of the channel — e.g. Table 1. Under this assumption, the channels' supports differ only by a quantity ϵ unresolvable in practical operating conditions. Discrete-time SCS signals were studied in [6, 7] within the compressed sensing framework. The key difference with our approach is the discrete nature of the model and the randomization of the measurements instead of uniform sampling.

We will first define the SCS channel model and outline the theory behind the proposed algorithm. It uses an annihilating filter shared among all channels and denoises the measurements with a blocked extension of Cadzow's algorithm [2]. It can be seen as the common

Table 1. Channel bandwidth in popular wireless systems

System (downlink)	Code	Bandwidth B	Resolvable distance c/B
DVB-T [11]	DFT	5–8 MHz	38–60 m
IS-95 [12]	WHT	1.25 MHz	240 m
3GPP LTE [13]	DFT	1.4–20 MHz	15–215 m
UWB	—	> 500 MHz	< 60 cm

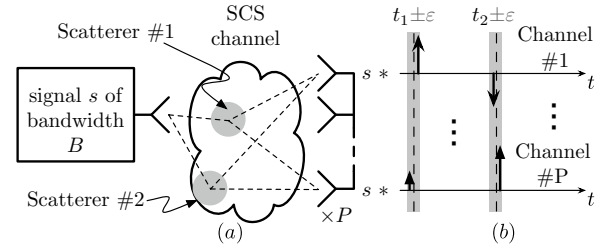


Fig. 1. (a) Transmission over a medium with two scatterers and P receiving antennas. (b) The P channels contain two paths arriving at the same time up to $\pm\epsilon$. The amplitudes of paths from a scatterer are (possibly correlated) Rayleigh variates [9].

support generalization of classical FRI sampling [1, 8]. To assess the performance of said algorithm, we compute the Cramér-Rao bounds (CRB) on the support parameters [2, 9, 10] and perform simulation in a Rayleigh fading environment. Finally, it is shown that SCS-FRI is directly applicable to pilots uniformly laid out in the DFT domain (OFDM scenario), and more surprisingly in the Walsh-Hadamard Transform (WHT) domain as well (CDMA scenario). Those two cases cover many of today's wireless communication systems.

2. PROBLEM FORMULATION

Let $\mathbf{h} = [h_1 \cdots h_P]^T$ be a vector of P exact SCS channels shaped by a function φ , the complex baseband equivalent channels are

$$h_p(t) = \sum_{k=1}^K c_{k,p} \varphi(t - t_k), \quad c_{k,p} \in \mathbb{C}, t_k \in [0, \tau]. \quad (1)$$

The paths coefficients $c_{k,p}$ are treated as complex random variables. N measurements $y_p[n]$ are acquired at a rate $1/T = N/\tau$ (with τ the signal period) and corrupted by AWGN

$$y_p[n] = h_p[n] + q_p[n] \quad n \in \{0, \dots, N-1\}, \quad (2)$$

M. Vetterli is also with the Department of Electrical Engineering and Computer Sciences, University of California, Berkeley, CA 94720, USA. This research is supported by *Qualcomm Inc., ERC Advanced Grant Support for Frontier Research - SPARSAM Nr : 247006* and *SNF Grant - New Sampling Methods for Processing and Communication Nr : 200021-121935/1*.

Algorithm 1 Block-Cadzow denoising

Require: A block-Toeplitz matrix $\mathbf{H}^{(L)}$ and a target rank K .
Ensure: A block-Toeplitz matrix $\mathbf{H}^{(L)}$ with rank $\leq K$.
1: **repeat**
2: Reduce $\mathbf{H}^{(L)}$ to rank K by a truncated SVD.
3: Make $\mathbf{H}_p^{(L)}$ $p = 1 \dots P$, Toeplitz by averaging diagonals.
4: **until** convergence

where $\mathbf{q}_p \sim \mathcal{N}_{\mathbb{C}}(\mathbf{0}, \sigma^2 \mathbb{I})$. In the DFT domain, the received signal is:

$$\hat{y}_p[m] = \hat{\varphi}[m] \cdot \sum_{k=1}^K c_{k,p} W_N^{mt_k} + \hat{q}_p[m]. \quad (3)$$

where $W = e^{-2\pi j/\tau}$. The goal is to estimate the support $\{t_k\}_{k=1 \dots K}$ and the paths amplitudes $\{c_{k,p}\}_{k=1 \dots K, p=1 \dots P}$ from the NP samples collected in (2). Once the support is known, estimation of the path amplitudes is simple linear algebra as seen in (3).

3. SPARSE COMMON SUPPORT FRI (SCS-FRI)

We start from (3). Assuming the spectral mask $\hat{\varphi}$ is flat in the base-band region $|m| \leq M$, the DFT samples $\hat{y}_p[m]$ in this region are the DFT coefficients of the channel corrupted by some Gaussian noise. The coefficients $\hat{y}_p[m]$ maybe arranged in a tall block-toeplitz matrix $\mathbf{H}^{(L)} = [\mathbf{H}_1^{(L)}; \mathbf{H}_2^{(L)}; \dots; \mathbf{H}_P^{(L)}]^T$ such that

$$\mathbf{H}_p^{(L)} = \begin{bmatrix} \hat{y}_{p,L-M-1} & \hat{y}_{p,L-M-2} & \cdots & \hat{y}_{p,-M} \\ \hat{y}_{p,L-M} & \hat{y}_{p,L-M-1} & \cdots & \hat{y}_{p,1-M} \\ \vdots & \vdots & \ddots & \vdots \\ \hat{y}_{p,M} & \hat{y}_{p,M-1} & \cdots & \hat{y}_{p,M-L+1} \end{bmatrix}, \quad (4)$$

where $\hat{y}_{i,j} = \hat{y}_i[j]$. The matrix $\mathbf{H}^{(L)}$ is made of P Toeplitz blocks of size $(2M+2-L) \times L$, with both block dimensions $\geq K$. It posses interesting algebraic properties. The first property is called the *annihilating filter* property [14, 15], and allows us to retrieve the channel support as the solution of a linear system of equations:

Proposition 1. *In the absence of noise, a set of exact SCS channels with K distinct paths verifies*

$$\mathbf{H}^{(K+1)} \mathbf{f} = \mathbf{0}, \quad (5)$$

where $\mathbf{f} = [f_1 \dots f_K]^T$ are the annihilating filter coefficients such that the polynomial $p_f(z) = 1 + \sum_{k=1}^K f_k z^k$ has K roots $\{e^{-2\pi j t_k / \tau}\}_{k=1 \dots K}$. The matrix $\mathbf{H}^{(K+1)}$ is built with blocks as in (4) (with $L = K+1$).

The second property is on the rank of $\mathbf{H}^{(L)}$ and is useful to denoise the measurements:

Proposition 2. *For a set of exact SCS channels with K distinct paths and in the absence of noise, $\mathbf{H}^{(L)}$ satisfies*

$$\text{rank } \mathbf{H}^{(L)} = K.$$

For the proofs of Propositions 1 and 2 see [2]. Proposition 2 used together with the block-toeplitz structure property yields the “lift-and-project” denoising Algorithm 1, which we call *Block-Cadzow* denoising [2]. Using the same argument as in [16], the block-Cadzow algorithm provably converges. The processing chain at the receiver is listed in Algorithm 2.

Algorithm 2 SCS-FRI channel estimation

Require: An estimate on the number of effective paths K^{est} , $2M+1$ ($M \geq K$) noisy channel DFT coefficients $\hat{y}_p[m] = \sum_{k=1}^K c_{k,p} W_N^{mt_k} + \hat{q}_p[m]$ for $|m| \leq M$, $p = 1 \dots P$.
Ensure: Support estimate $\{t_k^{\text{est}}\}_{k=1 \dots K^{\text{est}}}$
1: Build $\mathbf{H}^{(M)}$ according to (4).
2: $\mathbf{H}^{(M)} \leftarrow \text{Block-Cadzow}(\mathbf{H}^{(M)}, K^{\text{est}})$.
3: Update $\hat{y}_p[m]$ with the first row and column of the denoised block $\mathbf{H}_p^{(M)}$.
4: Build $\mathbf{H}^{(K^{\text{est}}+1)}$ according to (4).
5: Solve the annihilating filter equation (5) to get \mathbf{f} .
6: $\{t_k^{\text{est}}\}_{k=1 \dots K^{\text{est}}} \leftarrow -\frac{\tau}{2\pi} \angle \text{roots}(\mathbf{f})$.
7: Estimate $\{c_{k,p}\}$ solving P linear Vandermonde systems (3).

4. APPLICATION TO OFDM AND CDMA DOWNLINK**4.1. SCS-FRI with uniformly scattered DFT pilots (OFDM)**

The theory in Section 3 is developed for contiguous DFT coefficients. In OFDM communications, pilots are often uniformly laid out in frequency (ETSI DVB-T [11], 3GPP LTE [13], ...). The period of pilot insertion D is upper-bounded by Δ the delay-spread of the channel impulse response (CIR): $D < \tau/\Delta$. If not, the CIR cannot be unambiguously recovered from the pilots because of aliasing. For a fixed number of pilots, D is chosen as large as possible ($D = \lfloor \tau/\Delta \rfloor$), as interpolation of the CIR spectrum is more robust than extrapolation.

SCS-FRI can take advantage of uniformly scattered pilot layouts [17, 9]. For $\hat{\varphi}$ flat in $\{-MD, \dots, MD\}$, equation (3) becomes:

$$\hat{y}_p[mD] = \sum_{k=1}^K c_{k,p} W_N^{mD t_k} + \hat{q}_p[mD], \quad (6)$$

which corresponds to a dilation by D of the support parameters $\{t_k\}$. By definition $0 \leq t_k < \Delta$, and so the bound on D prohibits aliasing of $D t_k$. Therefore, SCS-FRI is applicable without other modification than division of the recovered support parameters by D .

4.2. Extension to Walsh-Hadamard coded schemes (CDMA)

Numerous applications use the 2^n -WHT to code the channel into 2^n subchannels ($N = 2^n$). Among others, IS-95 uses a 64-WHT to code the downlink channel. The straightforward way to insert pilots is to use one of these subchannels as a pilot itself and use correlation based channel estimation methods as the Rake-receiver for example [18]. The SCS-FRI algorithm works in the DFT domain but can nevertheless be applied in the WHT domain with pilots uniformly scattered by D a power of 2. This is consequence of the 2^n -WHT being itself the Fourier transform on the finite group $(\mathbb{Z}/2\mathbb{Z})^n$ instead of $\mathbb{Z}/2^n\mathbb{Z}$ for the classical 2^n -points DFT [9, 19].

Proposition 3. *Let \mathbf{F}_n and \mathbf{W}_n be respectively the 2^n -points DFT and WHT matrix obtained by Sylvester's construction:*

$$\mathbf{W}_1 = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}, \quad \mathbf{W}_{i+1} = \mathbf{W}_1 \otimes \mathbf{W}_i.$$

Then, for $l \in \{1, \dots, n-1\}$ the set of \mathbf{W}_n 's columns with indices in $\{2^l + i\}_{i=1 \dots 2^l}$ and the set of \mathbf{F}_n 's columns with indices in $\{(i-1/2) \cdot 2^{n-l} + 1\}_{i=1 \dots 2^l}$ span the same subspace.

For a proof see [9]. Proposition 3 states that one can choose 2^l contiguous Walsh-Hadamard codeword for pilots and get 2^l uniformly spread DFT pilots with layout gap $D = 2^{n-l}$.

5. CRAMÉR-RAO BOUNDS WITH SCS-FRI MEASUREMENTS

We derive bounds on the support estimation accuracy with measurements taken according to (2).

The paths coefficients $c_{k,p}$ are assumed to be jointly Gaussian, and modeled as the product of $a_{k,p} = \mathbb{E}[c_{k,p}c_{k,p}^*]^{1/2}$ by a standard normal random variable $Z_{k,p}$ having the following properties, consistent with the well-known Rayleigh-fading model:

- $Z_{k,p} \sim \mathcal{N}_\mathbb{C}(0, \sqrt{1/2}\mathbb{I})$.
- Independence between paths: $\mathbb{E}[Z_{k,p}Z_{k',p'}^*] = 0, k \neq k'$.
- The random vector $\mathbf{Z}_k = [Z_{k,1} \cdots Z_{k,P}]^T$ is defined as $\mathbf{Z}_k = \mathbf{L}_k \mathbf{r}$, where \mathbf{L}_k is the Cholesky factor of the covariance matrix \mathbf{R}_k and \mathbf{r} is a vector of iid standard complex Gaussian random variables.

The Rayleigh-fading case can be seen as deterministic if conditioned on the path amplitudes. Thus, the Cramér-Rao bounds for random paths coefficients are random variables for which we can compute statistics. Expectation and standard deviation will respectively give the expected accuracy of the estimator and its volatility. For a single path, and a symmetric or antisymmetric φ , the Crámer-Rao bound has a concise closed form formula:

Proposition 4. *With measurements according to (2), $K = 1$, and \mathbf{Z}_1 be a random Gaussian vector, then*

$$\mathbb{E}[(t_1 - t_1^{\text{est}})^2 / \tau^2] = \frac{\mathbb{E}[(\mathbf{Z}_1^* \mathbf{Z}_1)^{-1}]}{2N \cdot \text{dSNR}}, \quad (7)$$

where $\text{dSNR} = |a_1|^2 \|\varphi'(\mathbf{n}T - t_1)\|^2 / (N\sigma^2)$ is the differential SNR. Let $P > 1$ and $\{\lambda_p\}_{p=1,\dots,P}$ be the eigenvalues of $\mathbf{R}_1 = \mathbf{L}_1^* \mathbf{L}_1$:

- *Uncorrelated paths coefficients*, $\lambda_1 = \cdots = \lambda_P = 1$:

$$\mathbb{E}[(\mathbf{Z}_1^* \mathbf{Z}_1)^{-1}] = (P - 1)^{-1}.$$

- *Correlated path coefficients*, such that $\lambda_1 \neq \cdots \neq \lambda_P$:

$$\mathbb{E}[(\mathbf{Z}_1^* \mathbf{Z}_1)^{-1}] = \sum_{p=1}^P (-\lambda_p)^{P-1} \frac{\ln \lambda_p}{\lambda_p} \prod_{p' \neq p} (\lambda_{p'} - \lambda_p)^{-1}.$$

Proof. See [9]. The uncorrelated case is found in various statistical handbook as the moments of an inverse- χ^2 distributed random variable. For the correlated case, see [20]. \square

This expression is a suitable approximation for multipaths scenario with distant paths (separated by more than twice the inverse bandwidth [2]). It gives an important insight on the evolution of the estimation performance when uncorrelated antennas are added to the system. Namely, the RMSE decays as $1/\sqrt{P-1}$.

In general, multiple paths are interacting with each other and the information matrix cannot be considered diagonal, in this case Yau and Bresler derived the following expression:

Proposition 5. [10] *Let Φ and Φ' be $N \times K$ matrices such that*

$$\Phi_{n,k} = \varphi((n-1)T - t_k), \quad \Phi'_{n,k} = \varphi'((n-1)T - t_k),$$

$n \in \{1, \dots, N\}$, $k \in \{1, \dots, K\}$. Given the stochastic matrix

$$\mathbf{C} = \text{diag}(a_1, \dots, a_K) \left(\sum_{p=1}^P \mathbf{Z}_p \mathbf{Z}_p'^* \right) \text{diag}(a_1^*, \dots, a_K^*),$$

Table 2. Simulation parameters

Sim	System param.	Channel parameters ($\Delta = 1.6\mu\text{s}$)
1	$T = 50\text{ns}$, $B = 20\text{MHz}$,	$K = 1, a_1 = 1, t_1 = 39.22\mu\text{s}, \mathbf{R}_1 = \mathbb{I}, \varepsilon = \text{var.}$
2	$f_c = 2.6\text{GHz}$, $\tau = 25.55\mu\text{s}$, $N_{\text{frame}} = 511$,	$K = 2, a_1 = 1, t_1 = 39.22\mu\text{s}, \varepsilon = 0$, $a_2 = \text{var.}, t_2 = \text{var.}, \mathbf{R}_1 = \mathbf{R}_2 = \mathbb{I}$
3	$N = 31$, $P = 4$, $D = 16$	$K = 4, \mathbf{a} = [.86 \ 1.13 \ -1.85 \ 0.81]^T$, $\mathbf{t} = [22.9 \ 72.5 \ 354.1 \ 470.2]^T, \varepsilon = 0$, $\mathbf{R}_1 = \text{toeplitz}(1, .47-.69j, -.19-.45j, -.19-.03j)$, $\mathbf{R}_2 = \text{toeplitz}(1, -.53+.77j, -.27-.72j, .52+.13j)$, $\mathbf{R}_3 = \text{toeplitz}(1, .6-.75j, -.2-.83j, -.64-.3j)$, $\mathbf{R}_4 = \text{toeplitz}(1, -.62+.73j, -.14-.83j, .58+.34j)$

with $\mathbf{Z}'_p = [Z_{1,p} \cdots Z_{K,p}]^T$, the Fisher information matrix \mathbf{J} conditioned on the path amplitudes is given by

$$\mathbf{J} = 2\sigma^{-2} \Phi'^* P_{\ker \Phi} \Phi' \odot \mathbf{C}. \quad (8)$$

such that $P_{\ker \Phi} = \mathbb{I} - \Phi \Phi^\dagger$ is the projection into the nullspace of Φ and “ \odot ” denotes the entrywise matrix product.

See [10] for the proof. The Cramér-Rao bounds for the estimation of the normalized times of arrival are on the diagonal of the expectation of \mathbf{J}^{-1} an inverse complex Wishart matrix. Computing its moments analytically is hard, so we turn to Monte-Carlo simulations.

6. NUMERICAL RESULTS

The SCS-FRI algorithm is tested in three different Rayleigh fading channel scenarios listed in Table 2. Simulations are performed in an OFDM-like setup with $N_{\text{frame}} = 511$ samples per frames and $N = 31$ uniformly laid out DFT pilots. The receiver has 4 antennas.

Simulation 1 (Figure 2.a) shows that the effect of mismatched ToA between antennas limits accuracy on the order of ε as intuitively expected. The quantity ε is related to the maximum distance between antennas, which we chose to be about 1 meter.

Simulation 2 (Figure 2.b) shows that SCS-FRI is able to discriminate paths (with equal power) distant by less than the inverse-bandwidth, which is problematic for algorithms estimating ToAs sequentially. Also, it shows that joint estimation with SCS-FRI performs significantly better than independent estimations with the same core algorithm and a sensibly chosen statistics¹ for selection. The SNR is measured with respect to one path.

Simulation 3 (Figure 3) synthetize previous results in a more realistic setup. The multipath channel and antennas correlation figures use a physically motivated model [9] based on [21]. We compare SCS-FRI to lowpass interpolation of the spectrum, which is a widely used technique for channel estimation from scattered DFT pilots. We used MATLAB signal toolbox function `interp`. The input SNR is the ratio between the total expected signal power (all paths and all channels) and the total expected noise power (all channels).

The studied channels have three main properties: the CIR is smaller than the signal period τ , it is sparse and they share a common support. The first property is assumed as soon as pilots are uniformly laid out in frequency. Taking into account this property only, low-pass interpolation is energy-wise optimal since it is the orthogonal projection into the signal space. If sparsity is additionally taken into account, independent estimations with FRI are a good choice since they estimate efficiently the ToAs. With the addition of common support, SCS-FRI is the proper tool. It is shown in [9] that sparsity alone (i.e. independent FRI) provides approximately 2/3 of the SNR gain reported in Figure 3 for SCS-FRI, i.e. compared to the gain obtained with addition of the common support assumption.

¹The median was picked as the best performing statistics among the mean and selection of antenna experiencing the least fading.

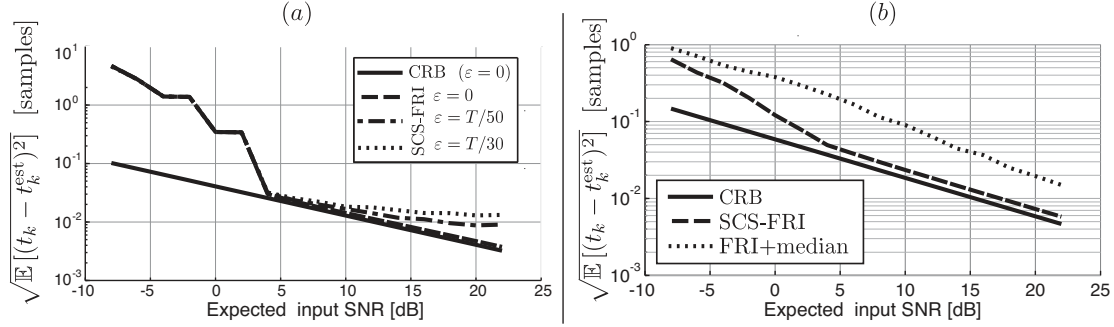


Fig. 2. Figure (a), verifies that the algorithm behaves stably on non-exact SCS channels: as expected, the estimation error stalls at a magnitude similar to ε . In (b), two paths of unit average amplitudes are separated by one sampling step which is equal to the inverse bandwidth of the channel. SCS-FRI is able to discriminate the two paths near optimally. The improved performance of SCS-FRI compared to independent FRI followed by selection of the median ToA indicates that joint estimation of the ToA is crucial.

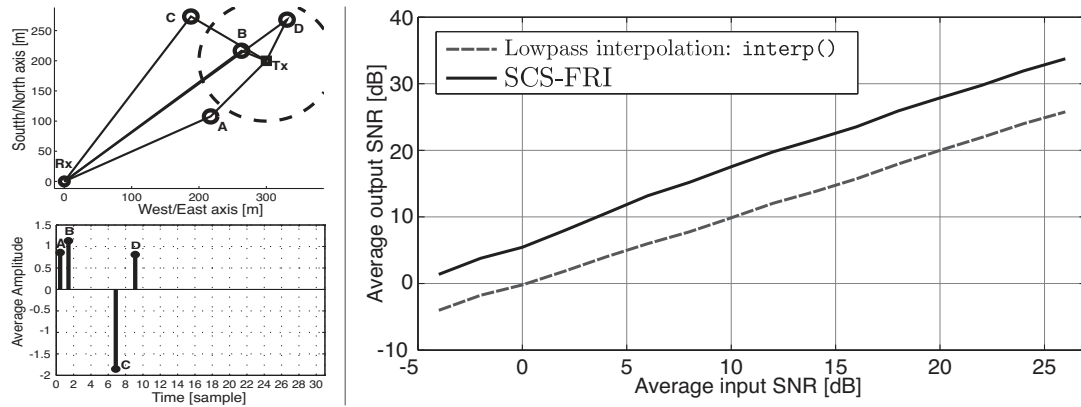


Fig. 3. SCS-FRI provides a SNR gain of 6–8dB compared to conventional lowpass interpolation. The medium contains four scatterers (upper-left panel) labeled A, B, C and D. Their positions and widths relative to the receiver's (Rx) antenna array (4 aligned antennas spaced by 30cm) correlates the paths' amplitudes as listed in Table 2. The expected CIR (same for all four antennas) is shown on the bottom-left.

7. CONCLUSIONS AND FUTURE WORK

We outlined the SCS-FRI algorithm, and showed its adequacy to tackle the multipath channel estimation problem with a multiple output receiver. The assumed SCS channel model is particularly relevant for medium-band communications, and SCS-FRI may be used on DFT or WHT multiplexed channels as in OFDM or CDMA downlinks. Future work is necessary to study how transmit spatial diversity may be taken advantage of as well.

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