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ESTIMATING THE DOMAIN OF ATTRACTION VIA MOMENT MATRICES

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Abstract. The domain of attraction of a nonlinear differential equations is the region of initial points of solution tending to the equilibrium points of the systems as the time going. Determining the domain of attraction is one of the most important problems to investigate nonlinear dynamical systems. In this article, we first present two algorithms to determine the domain of attraction by using the moment matrices. In addition, as an application we consider a class of SIRS infection model and discuss asymptotical stability by Lyapunov method, and also estimate the domain of attraction by using the algorithms.

1. Introduction

Estimating the domain of attraction (DOA) of a dynamical system is an important subject of the theory of stability and is well known in the area of nonlinear system analysis and control. The DOA of the system makes an important role in applied mathematics such as electric systems, chemical reactors, and many non-linear dynamical systems, for the work in security. Note that infectious disease is a very common phenomenon. The DOA of this system can be applied to determine and forecast the development trend of infection.

Given the autonomous system

$$\dot{x} = f(x), \qquad x \in \mathbb{R}^n, \tag{1.1}$$

with $f(0) = 0$, the domain of attraction (DOA) of $x = 0$ is

$$S = \{x^0 \in \mathbb{R}^n | \lim_{t\to\infty} x(t, x^0) = 0\}, \tag{1.2}$$

where $x(\cdot, x^0)$ denotes the solution of (1.1) corresponding to the initial condition $x(0) = x^0$. Let $V(x)$ be a continuously differentiable real-valued function defined on a domain $D \subset \mathbb{R}^n$ containing the origin. The function $V(x)$ is called *a Lyapunov function* for the system (1.1) if $V(x)$ is positive definite on D and

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$\dot{V}(x) = \left(\frac{\partial V}{\partial x}\right)^{\mathrm{T}} f(x)$ is negative semidefinite on D. In [2] O. Hachicho showed that if the domain

$$\Omega_c = \{x \in \mathbb{R}^n | V(x) \leq c\}, \qquad c > 0, \tag{1.3}$$

is bounded, $0 \in \Omega_c$ and $\dot{V}(x)$ is negative definite in Ω_c, then $\Omega_c \subset S$. Let

$$V(x) = x^{\mathrm{T}} P x, \qquad P = P^{\mathrm{T}} \in \mathbb{R}^{n \times n}, \tag{1.4}$$

where P is a positive definite matrix. The hypersurfaces given by $\dot{V}(x) = 0, x \neq 0$ define the boundary of the region of negative definiteness of $\dot{V}(x)$ in which we seek the guaranteed estimation Ω_c. In the case of quadratic Lyapunov functions such an estimation is the interior of the ellipsoid defined by (1.3). Our objective is to find the maximum value c^* of c such that $\dot{V}(x)$ is negative definite in Ω_c. Note that this c^* is defined by the following optimization problem

$$\begin{cases} \text{find } c^* = \min V(x) \\ \text{subject to the constraints:} \\ \dot{V}(x) = 0, \qquad x \neq 0. \end{cases} \tag{1.5}$$

This paper is organized as follows. In Section 2 we recall some results from the mathematical theory of moments proved by J. Lasserre in [7], which will be used frequently in this note, and obtain the algorithms that can be used to estimate the DOA for polynomial dynamical systems using polynomial Lyapunov function. As an application, in Section 3, we estimate the domain of attraction of a class of SIRS infection model by using the algorithms. All of the calculations in this paper were obtained throughout computer experiments using the YALMIP-an LMI package of MATLAB ([10]).

2. The estimation of the DOA via moment matrices

2.1. Minimization of polynomials and the problem of moments

In [7] J. Lasserre considered the following two classical problems.

The problem of global minimization

$$\mathbb{P} \mapsto p^* := \min_{x \in \mathbb{R}^n} p(x). \tag{2.1}$$

The problem of constrained minimization

$$\mathbb{P}_K \mapsto p_K^* := \min_{x \in K} p(x), \tag{2.2}$$

where $p(x)$ is a real-valued polynomial and K is a compact set defined by polynomial inequalities

$$g_i(x) \geq 0, \quad i = 1, \ldots, r.$$

Let

$$1, x_1, x_2, \ldots, x_n, x_1^2, x_1 x_2, \ldots, x_1 x_n, x_2^2, x_2 x_3, \ldots, x_1^m, \ldots, x_n^m, \tag{2.3}$$

be a basis for the m-degree real-valued polynomials $p(x)$ and let $s\,(2m)$ be its dimension, where $s\,(m) := \binom{n+m}{n} = \frac{(n+m)!}{n!m!}$. Let

$$p(x) = \sum_{\alpha} p_\alpha x^\alpha, \text{ with } x^\alpha := x_1^{\alpha_1} x_2^{\alpha_2} \cdots x_n^{\alpha_n} \quad \text{and} \quad \sum_{i=1}^{n} \alpha_i \leq m, \tag{2.4}$$

where $p = \{p_\alpha\} \in \mathbb{R}^{s(m)}$ is the coefficient vector of $p(x)$ in the basis (2.3).

Given an $s\,(2m)$-vector $y := \{y_\alpha\}$ with first element $y_{0,\ldots,0} = 1$, let $M_m(y)$ be the moment matrix of dimension $s\,(m)$. To illustrate $M_m(y)$, let us consider a simple example where $n = 2$. In this case the matrix $M_m(y)$ is a block matrix

$$\{M_{i,j}(y)\}_{0 \leq i,j \leq m}$$

with

$$M_{i,j}(y) = \begin{pmatrix} y_{i+j,0} & y_{i+j-1,1} & \cdots & y_{i,j} \\ y_{i+j-1,1} & y_{i+j-2,2} & \cdots & y_{i-1,j+1} \\ \vdots & \vdots & \ddots & \vdots \\ y_{j,i} & y_{j-1,i+1} & \cdots & y_{0,i+j} \end{pmatrix}$$

such that $y_{i,j}$ represents the $(i+j)$ order moment

$$y_{i,j} = \int x^i y^j \mu\,(d\,(x,y))$$

for some probability measure μ (cf. [1], [5], [8], [9]). In particular, for the case $n = 2, m = 2$, one obtains

$$\begin{aligned} M_2(y) \quad &= \quad \begin{pmatrix} M_{0,0}(y) & M_{0,1}(y) & M_{0,2}(y) \\ M_{1,0}(y) & M_{1,1}(y) & M_{1,2}(y) \\ M_{2,0}(y) & M_{2,1}(y) & M_{2,2}(y) \end{pmatrix} \\ &= \left(\begin{array}{c|cc|ccc} 1 & y_{1,0} & y_{0,1} & y_{2,0} & y_{1,1} & y_{0,2} \\ - & - & - & - & - & - \\ y_{1,0} & y_{2,0} & y_{1,1} & y_{3,0} & y_{2,1} & y_{1,2} \\ y_{0,1} & y_{1,1} & y_{0,2} & y_{2,1} & y_{1,2} & y_{0,3} \\ - & - & - & - & - & - \\ y_{2,0} & y_{3,0} & y_{2,1} & y_{4,0} & y_{3,1} & y_{2,2} \\ y_{1,1} & y_{2,1} & y_{1,2} & y_{3,1} & y_{2,2} & y_{1,3} \\ y_{0,2} & y_{1,2} & y_{0,3} & y_{2,2} & y_{1,3} & y_{0,4} \end{array} \right). \end{aligned}$$

It is not difficult to conclude that for any $N \in \mathbb{N}$ we have

$$M_N(y) = M_N^{\mathrm{T}}(y) \quad \text{and} \quad M_N(y) \in \mathbb{R}^{s(N) \times s(N)}.$$

Let $g(x) : \mathbb{R}^n \to \mathbb{R}$ be a real valued polynomial of degree w with coefficient vector $g \in \mathbb{R}^{s(w)}$. For the (i,j) entry y_β of the matrix $M_m\,(y)$, we write $\beta\,(i,j)$ for the subscript β of y_β. Then $M_m\,(gy)$ is defined by

$$M_m\,(gy)\,(i,j) = \sum_{\alpha} g_\alpha y_{\{\beta(i,j)+\alpha\}}. \tag{2.5}$$

Let $\deg g_i(x) = w_i$ and define

$$\tilde{w_i} = \left\lceil \frac{w_i}{2} \right\rceil$$

which is the smallest integer larger than $\frac{w_i}{2}$. Then optimization problem (2.2) is equivalent to the following problem

$$\mathbb{Q}_K^N \mapsto \begin{cases} \inf_y \sum_\alpha p_\alpha y_\alpha \\ \text{subject to the constraints:} \\ M_N(y) \geq 0, \\ M_{N-\tilde{w}_i}(g_i y) \geq 0, \ i = 1, \ldots r. \end{cases} \tag{2.6}$$

The number N has to be chosen according to the following conditions

$$N \geq \left\lceil \frac{m}{2} \right\rceil \quad \text{and} \quad N \geq \max_i \tilde{w}_i.$$

J. Lasserre also formulated that $(\mathbb{Q}_K^N)^*$, the *dual* of $\mathbb{Q}_K^N$,

$$(\mathbb{Q}_K^N)^* \mapsto \begin{cases} \inf_{X_i, Z_i} X(1,1) + \sum\limits_{i=1}^{r} g_i(0) Z_i(1,1), \\ \text{subject to the constraints:} \\ \langle X, B_\alpha \rangle + \sum\limits_{i=1}^{r} \langle Z_i, C_{i\alpha} \rangle = p_\alpha, \alpha \neq 0, \\ X, Z_i \geq 0, \ i = 1, \ldots, r, \end{cases} \tag{2.7}$$

where $\langle \cdot, \cdot \rangle$ means the trace inner product, that is, $\langle A, B \rangle = \operatorname{tr}(AB)$. The matrices B_α and $C_{i\alpha}$ are derived from $M_N(y)$ and $M_{N-\tilde{w}}(g_i y)$ as follow

$$M_N(y) = B_0 + \sum_{\alpha \neq 0}^{\alpha} B_\alpha y_\alpha, \qquad M_{N-\tilde{w}}(g_i y) = \sum_\alpha C_{i\alpha} y_\alpha. \tag{2.8}$$

The following theorem will be used crucially for our work.

Theorem 2.1 ([7]). *Let $K \subseteq \{x : \|x\| \leq a\}$ for sufficiently large $a > 0$. Then $\inf \mathbb{Q}_K^N \uparrow p_K^*$ as $N \to \infty$. If K has nonempty interior for N sufficient large, then there is no duality gap between $\mathbb{Q}_K^N$ and its dual $(\mathbb{Q}_K^N)^*$.*

2.2. Algorithms for estimations

We first modify algorithms in [3] as following.

Algorithm 2.2 (The primal problem).
I. Rewrite Ω_c in (1.3) as $\Omega_c = \{x | V(x) \leq c_0\} \cup \{x | c_0 \leq V(x) \leq c\}$ such that $\dot{V}(x) < 0$ in $\Omega_{c_0} \backslash \{0\}$.
II. Rewrite (1.5) as

$$\begin{cases} \text{find } c^* = \min V(x) \\ \text{subject to the constraints:} \\ g_1(x) = \dot{V}(x) \geq 0, \\ g_2(x) = x^{\mathrm{T}} x - c_0 \geq 0, \\ g_3(x) = -x^{\mathrm{T}} x + R \geq 0. \end{cases} \tag{2.9}$$

III. Translation (2.9) to (2.6).
IV. Use an LMI-solver to compute c^*.

Algorithm 2.3 (The dual problem).
I. Define block diagonal matrix $T_S := \mathrm{Diag}\{(T_S)_{1,1}, (T_S)_{2,2}, \dots, (T_S)_{r+1,r+1}\}$, where

$$(T_S)_{1,1} = \begin{pmatrix} 1 & \\ & 0 \end{pmatrix}, \quad \text{and} \quad (T_S)_{i,i} = \begin{pmatrix} g_i(0) & \\ & 0 \end{pmatrix}, \quad i = 2, \dots, r+1.$$

II. Define $X_S := \mathrm{Diag}\{X, Z_1, \dots, Z_r\}$.
III. Provide all $\alpha \neq 0, \dots, 0$ with an index k, and define $b_k = p_{\alpha_k}, (\alpha_k \neq 0, \dots, 0)$.
IV. Define $(A_S)_k := \mathrm{Diag}\{B_{\alpha_k}, C_{1\alpha_k}, C_{2\alpha_k}, \dots, C_{r\alpha_k}\}$.
V. Transform the dual problem (2.7) into the following semidefinite problem:

$$\left\{ \begin{array}{l} \min\langle T_S, X_S\rangle \\ \text{subject to the constraints:} \\ \langle (A_S)_k, X_S\rangle = b_k, k = 1, \dots, l, \\ X_S \geq 0. \end{array} \right. \tag{2.10}$$

VI. Solve optimization problem (2.10), by using the LMI-solver toolbox, such as YALMIP ([10]).

By using the algorithms, we can obtain the following result.

Theorem 2.4. *Let $V = x_1^2 + x_2^2$ be the Lyapunov function for the following 2-dimensional nonlinear system*

$$\left\{ \begin{array}{l} \frac{dx_1}{dt} = -x_1, \\ \frac{dx_2}{dt} = -x_2 + x_1^2 x_2. \end{array} \right. \tag{2.11}$$

Then $\Omega_c = \left\{(x_1, x_2) \in \mathbb{R}^2 | x_1^2 + x_2^2 \leq 4\right\}$ is a subset of the domain of attraction for system (2.11).

Proof. By direct calculation, we have $\dot{V} = -2x_1^2 - x_2^2 + 2x_1^2 x_2^2$. First, by Algorithm 2.2, we need to solve the following optimization problem

$$\left\{ \begin{array}{l} \text{find } c^* = \min\left(x_1^2 + x_2^2\right) \\ \text{subject to the constraints:} \\ g_1(x) = -2x_1^2 - x_2^2 + 2x_1^2 x_2^2 \geq 0, \\ g_2(x) = x_1^2 + x_2^2 - c_0 \geq 0, \\ g_3(x) = -x_1^2 - x_2^2 + R \geq 0. \end{array} \right. \tag{2.12}$$

And rewrite (2.12) as

$$\left\{ \begin{array}{l} \inf_y \sum_\alpha p_\alpha y_\alpha = y_{2,0} + y_{0,2} \\ \text{subject to the constraints:} \\ M_N(y) \geq 0, \\ M_{N-\tilde{w}_i}(g_i y) \geq 0, \ i = 1, 2, 3. \end{array} \right. \tag{2.13}$$

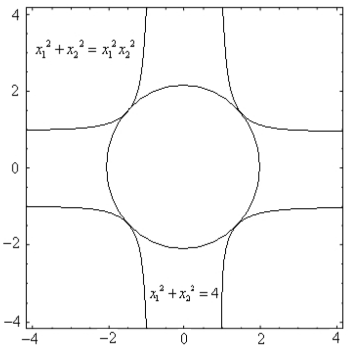

FIGURE 1. The subset of the domain of attraction for system (2.11)

If we choose $N = 4, c_0 = 1, R = 5$ and use an LMI-solver, then we can obtain $c^* = 4$. Second, by Algorithm 2.3, we need to solve the optimization problem (2.10). Here $b_0 = b_1 = 1$, and

$$T_S = \text{Diag}\left((T_S)_{1,1}, (T_S)_{2,2}, (T_S)_{3,3}, (T_S)_{4,4}\right),$$

where

$$\begin{aligned}(T_S)_{1,1} &= \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, (T_S)_{2,2} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}, \\ (T_S)_{3,3} &= \begin{pmatrix} -c_0 & 0 \\ 0 & 0 \end{pmatrix}, (T_S)_{4,4} = \begin{pmatrix} R & 0 \\ 0 & 0 \end{pmatrix},\end{aligned}$$

and

$$\begin{aligned}X_S &= \text{Diag}\left(X, Z_1, Z_2, Z_3\right), \\ (A_S)_1 &= \text{Diag}\left(B_{2,0}, C_{1,2,0}, C_{2,2,0}, C_{3,2,0}\right), \\ (A_S)_2 &= \text{Diag}\left(B_{0,2}, C_{1,0,2}, C_{2,0,2}, C_{3,0,2}\right).\end{aligned}$$

If we choose $N = 4, c_0 = 1, R = 5$, and use an LMI-solver, we can also obtain $c^* = 4$. □

Remark. Figure 1 represents the subset of the domain of attraction for system (2.11).

3. The DOA of SIRS epidemic model

In recent years, many researches studied SIRS epidemic model (see [4], [6]). In this section, we consider the following SIRS model

$$\left\{\begin{array}{l} \frac{dS}{dt} = A - \beta SI - dS + cI + \delta R = f, \\ \frac{dI}{dt} = \beta SI - rI - dI - \alpha I - cI = g, \\ \frac{dR}{dt} = rI - dR - \delta R = h, \end{array}\right. \tag{3.1}$$

where $S(t)$ is the number of susceptible individuals at time t, $I(t)$ is the number of infective individuals at time t, $R(t)$ is the number of recovered individuals at time t, β is the infection rate, A is the recruitment rate of the population, c is the sensible rate without immunity, d is the natural mortality rate of the population, r is the recovery rate of infective individuals, α is the death rate due to disease, and δ is the rate that removed return to the susceptible class. $N = S + I + R$ is the number of the total population. And $R_0 = \frac{\beta A}{d(d+\alpha+r+c)}$ denotes the basic reproduction number.

Let $P_1\left(\frac{A}{d}, 0, 0\right)$ and $P_2\left(\frac{d+\alpha+r+c}{\beta}, \frac{\left(A\beta-cd-d\alpha-dr-d^2\right)(d+\delta)}{\beta(d\alpha+dr+d\delta+\alpha\delta+d^2)}, \frac{\left(A\beta-cd-d\alpha-dr-d^2\right)r}{\beta(d\alpha+dr+d\delta+\alpha\delta+d^2)}\right)$. First we give the following theorem.

Theorem 3.1. *If* $R_0 < 1$, *then* P_1 *is the unique equilibrium point of* (3.1), *and it is globally asymptotically stable*; *if* $R_0 > 1$, *then* P_1 *and* P_2 *are two equilibrium points of* (3.1), *which* P_1 *is unstable, but* P_2 *is locally asymptotically stable.*

Proof. Let

$$M := \begin{pmatrix} \frac{\partial f}{\partial S} & \frac{\partial f}{\partial I} & \frac{\partial f}{\partial R} \\ \frac{\partial g}{\partial S} & \frac{\partial g}{\partial I} & \frac{\partial g}{\partial I} \\ \frac{\partial h}{\partial S} & \frac{\partial h}{\partial I} & \frac{\partial h}{\partial R} \end{pmatrix} = \begin{pmatrix} -d - I\beta & c - S\beta & \delta \\ I\beta & S\beta - d - r - \alpha - c & 0 \\ 0 & r & -d - \delta \end{pmatrix}.$$

For P_1, we have

$$M|_{P_1} = M_1 = \begin{pmatrix} -d & c - \frac{A}{d}\beta & \delta \\ 0 & \frac{A}{d}\beta - d - r - \alpha - c & 0 \\ 0 & r & -d - \delta \end{pmatrix},$$

the eigenvalues of M_1 are

$$\begin{aligned} \lambda_1 &= -d < 0, \\ \lambda_2 &= -d - \delta < 0, \\ \lambda_3 &= \frac{1}{d}\left(-cd - dr + A\beta - d\alpha - d^2\right). \end{aligned}$$

So, when $R_0 = \frac{\beta A}{d(d+\alpha+r+c)} < 1$, we know that $\lambda_3 < 0$, and thus P_1 is locally asymptotically stable. Since P_1 is unique, we can also know that P_1 is globally asymptotically stable. When $R_0 = \frac{\beta A}{d(d+\alpha+r+c)} > 1$, we know that $\lambda_3 > 0$, and thus P_1 is unstable.

For P_2, we have

$$M|_{P_2} = M_2 = \begin{pmatrix} -d + \frac{\left(cd-A\beta+d\alpha+dr+d^2\right)(d+\delta)}{(d\alpha+dr+d\delta+\alpha\delta+d^2)} & -d - \alpha - r & \delta \\ -\frac{\left(cd-A\beta+d\alpha+dr+d^2\right)(d+\delta)}{(d\alpha+dr+d\delta+\alpha\delta+d^2)} & 0 & 0 \\ 0 & r & -d - \delta \end{pmatrix},$$

and the characteristic polynomial is

$$X^3 + a_1X^2 + a_2X + a_3$$

with

$$\begin{aligned} a_1 &= 2d+\delta+\frac{(d+\delta)\left(A\beta-cd-dr-d\alpha-d^2\right)}{dr+d\alpha+d\delta+\alpha\delta+d^2}, \\ a_2 &= -\frac{\Theta\,(d+\delta)}{dr+d\alpha+d\delta+\alpha\delta+d^2}, \\ a_3 &= \left(A\beta-cd-dr-d\alpha-d^2\right)(d+\delta)>0, \end{aligned}$$

where

$$\begin{aligned} \Theta &= d^3+2d^2\,(c+r+\alpha)-\beta A\,(r+\alpha+\delta) \\ &\quad +d\left(cr-2A\beta+c\alpha+c\delta+2r\alpha+r\delta+r^2+\alpha^2\right). \end{aligned}$$

It is easy to show that $a_1>0$. Furthermore, $\det\Delta_2=\left|\begin{smallmatrix} a_1 & 1 \\ a_3 & a_2 \end{smallmatrix}\right|>0$. In fact, let $\Omega=A\beta-\left(dr+d\alpha+dc+d^2\right)>0$. Then we have

$$\det\Delta_2=\frac{\Lambda}{\left(dr+d\alpha+d\delta+\alpha\delta+d^2\right)^2},$$

where

$$\begin{aligned} \Lambda &= \Omega^2\,(d+\delta)^2\,(2d+r+\alpha+\delta) \\ &\quad +\Omega\,(d+\delta)\left(dr+d\alpha+4d\delta+r\delta+4d^2+\delta^2\right)\left(dr+d\alpha+d\delta+\alpha\delta+d^2\right) \\ &\quad +d\,(d+\delta)\,(2d+\delta)\left(dr+d\alpha+d\delta+\alpha\delta+d^2\right)^2. \end{aligned}$$

Hence P_2 is locally asymptotically stable, when $R_0>1$. □

Notice that P_2 is local asymptotically stable, when $R_0>1$. We take $A=4,\alpha=\beta=r=d=\delta=c=\frac{1}{2}$. Then $R_0=\frac{\beta A}{d(d+\alpha+r+c)}=2>1$. And the coordinate of the point P_2 is $\left(4,\frac{8}{5},\frac{4}{5}\right)$. Let $S=x+4, I=y+\frac{8}{5}, R=z+\frac{4}{5}$. Then we obtain

$$\begin{pmatrix} \frac{dx}{dt} \\ \frac{dy}{dt} \\ \frac{dz}{dt} \end{pmatrix}=\begin{pmatrix} -\frac{13}{10} & -\frac{3}{2} & \frac{1}{2} \\ \frac{4}{5} & 0 & 0 \\ 0 & \frac{1}{2} & -1 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix}+\begin{pmatrix} -\frac{1}{2}xy \\ \frac{1}{2}xy \\ 0 \end{pmatrix}. \tag{3.2}$$

The linearized system of (3.2) is

$$\begin{pmatrix} \frac{dx}{dt} \\ \frac{dy}{dt} \\ \frac{dz}{dt} \end{pmatrix}=\begin{pmatrix} -\frac{13}{10} & -\frac{3}{2} & \frac{1}{2} \\ \frac{4}{5} & 0 & 0 \\ 0 & \frac{1}{2} & -1 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix}. \tag{3.3}$$

Proposition 3.2. *The function*

$$V(x,y,z)=\frac{299}{475}x^2+\frac{6457}{3800}y^2+\frac{286}{475}z^2+\frac{378}{475}xy+\frac{184}{475}yz+\frac{194}{475}xz$$

is a Lyapunov function for system (3.3).

Proof. Notice that

$$V(x,y,z) = (x,y,z)\, M \begin{pmatrix} x \\ y \\ z \end{pmatrix} \quad \text{and} \quad M = \begin{pmatrix} \frac{299}{475} & \frac{189}{475} & \frac{97}{475} \\ \frac{189}{475} & \frac{6457}{3800} & \frac{92}{475} \\ \frac{97}{475} & \frac{92}{475} & \frac{286}{475} \end{pmatrix}.$$

Since $\det [M]_1 = \frac{299}{475}$, $\det [M]_2 = \frac{13\,159}{14\,440}$, and $\det M = \frac{35\,067}{72\,200}$, we obtain that matrix M is positive definite. And by a simple computation we obtain that

$$\begin{aligned} \left.\frac{dV}{dt}\right|_{(3.3)} &= \frac{\partial V}{\partial x}\frac{dx}{dt} + \frac{\partial V}{\partial y}\frac{dy}{dt} + \frac{\partial V}{\partial z}\frac{dz}{dt} \\ &= -x^2 - y^2 - z^2. \end{aligned}$$

Thus we have our conclusion. □

Remark. For the Lyapunov function in Proposition 3.2, we have

$$\left.\frac{dV}{dt}\right|_{(3.2)} = -x^2 - y^2 - z^2 + \frac{989}{760}xy^2 - \frac{1}{95}xyz - \frac{22}{95}x^2y.$$

Theorem 3.3. *The region* $\Omega_c = S \cap E$ *is a subset of the domain of attraction for system* (3.2) *with*

$$\begin{aligned} S &= \{(x,y,z) \in \mathbb{R}^3 | x \geq -4, y \geq -\frac{8}{5}, z \geq -\frac{4}{5}\}, \\ E &= \{(x,y,z) \in \mathbb{R}^3 | \varphi(x,y,z) \leq 2.6094\}, \end{aligned}$$

where $\varphi(x,y,z) = \frac{299}{475}x^2 + \frac{6457}{3800}y^2 + \frac{286}{475}z^2 + \frac{378}{475}xy + \frac{184}{475}yz + \frac{194}{475}xz.$

Proof. By Algorithm 2.2, we need to solve the following optimization problem

$$\begin{cases} \min\left(\frac{299}{475}x^2 + \frac{6457}{3800}y^2 + \frac{286}{475}z^2 + \frac{378}{475}xy + \frac{184}{475}yz + \frac{194}{475}xz\right) \\ \text{such that} \begin{cases} g_1(x,y,z) = -x^2 - y^2 - z^2 + \frac{989}{760}xy^2 - \frac{1}{95}xyz - \frac{22}{95}x^2y \geq 0 \\ g_2(x,y,z) = \left(x^2+y^2+z^2\right) - 1 \geq 0 \\ g_3(x,y,z) = 4 - \left(x^2+y^2+z^2\right) \geq 0. \end{cases} \end{cases}$$

It is equivalent to solve the following optimization problem

(3.4)

$$\begin{cases} c^* = \min\left(\frac{299}{475}y_{2,0,0} + \frac{6457}{3800}y_{0,2,0} + \frac{286}{475}y_{0,0,2} + \frac{378}{475}y_{1,1,0} + \frac{184}{475}y_{0,1,1} + \frac{194}{475}y_{1,0,1}\right) \\ \text{such that} \begin{cases} M_N(y) \geq 0, \\ M_{N-\tilde{w}_i}(g_i y) \geq 0, \ i = 1,2,3. \end{cases} \end{cases}$$

Since, $w_1 = 3, w_2 = 2, w_3 = 2, m = 2$, and $N \geq \max \tilde{w}_i = \max \lceil \frac{w_i}{2} \rceil = 2$ and $N \geq \lceil \frac{m}{2} \rceil = 2$, we first take $N = 3$. Then

$$M_3(y) = \begin{pmatrix} M_{0,0,0} & M_{0,0,1} & M_{0,0,2} & M_{0,0,3} \\ M_{1,0,0} & M_{1,0,1} & M_{1,0,2} & M_{1,0,3} \\ M_{2,0,0} & M_{2,0,1} & M_{2,0,2} & M_{2,0,3} \\ M_{3,0,0} & M_{3,0,1} & M_{3,0,2} & M_{3,0,3} \end{pmatrix},$$

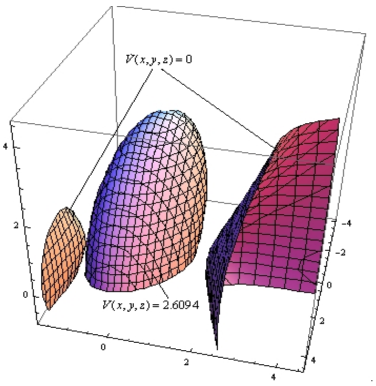

FIGURE 2. The subset of the domain of attraction for SIRS system (3.2)

where

$$M_{i,j,k}=\begin{pmatrix} y_{i+j+k,0,0} & y_{i+j+k-1,1,0} & y_{i+j+k-1,0,1} & y_{i+j+k-2,2,0} & \cdots \\ y_{i+j+k-1,1,0} & y_{i+j+k-2,2,0} & y_{i+j+k-2,1,1} & y_{i+j+k-3,3,0} & \cdots \\ y_{i+j+k-1,0,1} & y_{i+j+k-2,1,1} & y_{i+j+k-2,0,2} & y_{i+j+k-3,2,1} & \cdots \\ y_{i+j+k-2,2,0} & y_{i+j+k-3,3,0} & y_{i+j+k-3,2,1} & y_{i+j+k-4,4,0} & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix},$$

and

$$M_1\left(g_r y\right)=\begin{pmatrix} G_r\left(0,0,0\right) & G_r\left(1,0,0\right) & G_r\left(0,1,0\right) & G_r\left(0,0,1\right) \\ G_r\left(1,0,0\right) & G_r\left(2,0,0\right) & G_r\left(1,1,0\right) & G_r\left(1,0,1\right) \\ G_r\left(0,1,0\right) & G_r\left(1,1,0\right) & G_r\left(0,2,0\right) & G_r\left(0,1,1\right) \\ G_r\left(0,0,1\right) & G_r\left(1,0,1\right) & G_r\left(0,1,1\right) & G_r\left(0,0,2\right) \end{pmatrix},\ r=1,2,3,$$

with

$$\begin{aligned} G_1\left(i,j,k\right) &= \frac{989}{760}y_{1+i,2+j,k}-y_{2+i,j,k}-y_{i,2+j,k}-y_{i,j,2+k} \\ &\quad -\frac{1}{95}y_{1+i,1+j,1+k}-\frac{22}{95}y_{2+i,1+j,k}, \\ G_2\left(i,j,k\right) &= y_{2+i,j,k}+y_{i,2+j,k}+y_{i,j,2+k}-y_{i,j,k}, \\ G_3\left(i,j,k\right) &= -y_{2+i,j,k}-y_{i,2+j,k}-y_{i,j,2+k}+4y_{i,j,k}. \end{aligned}$$

By using the YALMIP-yet another LMI package of Matlab (see [10]), we can solve the optimization problem (3.4), and obtain $c^*\approx 1.5590$. Similarly, we can consider the cases $N=4$ and $N=5$. Thus we obtain same values $c^*\approx 2.6094$. Next, by Algorithm 2.3, we can obtain $c^*\approx 1.5843$, and 2.6094, when $N=3$ and 4, respectively. Therefore, we have our conclusion. □

Remark. Figure 2 shows the subset of the domain of attraction for system (3.2).

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