ESTIMATING THE PARAMETERS OF A TRUNCATED GAMMA DISTRIBUTION[1]

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1. Summary. A table is given to simplify the estimation of the parameters of an incomplete gamma or Type III distribution. A new procedure is also suggested for estimating the parameters of a truncated gamma distribution. This method is also applicable to a number of other truncated distributions, whether the truncation is in the tails or the center of the distribution.

2. Introduction. Several examples have been given recently, employing the incomplete gamma or Type III distribution in fitting rainfall data; for instance, see [1, 2]. In an animal population study [3], it was found that the migration pattern could be fitted by this type of distribution. Frequently in such migration studies the data are truncated, that is, observations begin after migration has commenced or conclude before it has stopped.

The parameters of the gamma distribution are often estimated by the method of moments in such cases (for example, see [4], pp. 121, 125), despite the fact that Fisher [5] showed the method to be inefficient. To facilitate solution of the maximum likelihood equations for estimation of the parameters in the untruncated case, a simple table is given.

The estimation of the parameters of a truncated gamma distribution by the method of moments has been studied by Cohen [6]. Since the integral of the probability density cannot be expressed in closed form, even the moment estimates are tedious to obtain; no attempt has been made to evaluate their variances or to study their efficiencies. After this paper was completed, a new study of the problem was published by Des Raj [7]. He gives the maximum likelihood equations for a number of cases of truncated and censored samples, mainly, however, under the assumption that the third standard moment is known. These equations can be solved only by iterative methods. In this paper a new method of estimation of these parameters is introduced which is easier to apply. The asymptotic variance-covariance matrix of the estimates is determined.

3. Estimation with origin known. The density function of the gamma distribution may be written in the form

$$f(x) = \frac{a^b}{\Gamma(b)} e^{-a(x-c)}(x-c)^{b-1} \qquad x \geqq c$$

$$= 0, \qquad x < c.$$

Received July 16, 1953.

[1] Research sponsored by the Office of Naval Research.

The parameters are frequently transformed so that the distribution is expressed as a function of the mean, variance, and skewness. Since the corresponding sample quantities do not efficiently estimate the parameters, such a transformation appears to be misleading.

The maximum likelihood equations, based on a sample of n observations, have been given by Fisher [5]:

$$\frac{1}{n}\frac{\partial L}{\partial a} = \frac{b}{a} - (\bar{x} - c) = 0; \tag{1}$$

$$\frac{1}{n}\frac{\partial L}{\partial b} = \ln a - \frac{\Gamma'(b)}{\Gamma(b)} + \frac{1}{n}\sum_{i=1}^{n} \ln (x_i - c) = 0; \tag{2}$$

$$\frac{1}{n}\frac{\partial L}{\partial c} = a - \frac{(b-1)}{n}\sum_{i=1}^{n}\left(\frac{1}{(x_i - c)}\right) = 0. \tag{3}$$

Since the parameter c determines the region of positive density, care must be exercised in obtaining its maximum likelihood estimate. If $b > 1$, then it is easy to verify that equation (3) gives the maximum likelihood estimate of c; if, however, $b \leqq 1$, this is no longer true. In this case $f(x)$ is monotone decreasing for $x \geqq c$, and $z = \min_i x_i$ is the maximum likelihood estimate of c.

We consider first the case where the origin is known, so that c may be set equal to zero without loss of generality and equation (3) drops out. Letting

$$\bar{x}_L = \frac{1}{n}\sum_{i=1}^{n} \ln x_i ,$$

(1) and (2) yield

$$\gamma(b) = \ln b - \frac{\Gamma'(b)}{\Gamma(b)} = \ln \bar{x} - \bar{x}_L .$$

Since $\frac{\Gamma'(b)}{\Gamma(b)}$, the digamma function, has been tabulated by Gauss [8] and by Pairman [9], it is easy to construct a table of $\gamma(b)$ and solve for b by inverse interpolation. A small tabulation of $\gamma^{-1}(b)$ is given in Table I; a more complete tabulation of $\gamma(b)$ is available in mimeographed form from the Laboratory of Statistical Research, University of Washington. There $\gamma(b)$ and its first and second differences are tabulated for $b = 0.01(0.01)2, 2(0.02)5, 5(0.1)20, 20(1)100$. The table was checked by summing columns in the basic tables and should be correct to one figure in the fifth decimal.

4. Estimation in the truncated case with known origin. The density function is now written

$$f(x) = K^{-1}e^{-ax}x^{b-1} \qquad 0 \leqq x \leqq T \tag{4}$$

$$= 0 \text{ elsewhere,}$$

where

$$K(a, b) = \int_0^T e^{-ax}x^{b-1}\,dx.$$

TABLE I

$b = \gamma^{-1}(a)$, where $\gamma(b) = \ln b - \Gamma'(b)/\Gamma(b)$ may be used to estimate the parameter b in the density

$f(x) = [a^b/\Gamma(b)]e^{-ax}x^{b-1}$; i.e., $b = \gamma^{-1}[\ln \bar{x} - \overline{\ln x}]$

a	Third Decimal of a									
	0	1	2	3	4	5	6	7	8	9
0.01	50.17	45.63	41.84	38.63	35.88	33.51	31.42	29.59	27.94	26.49
0.02	25.17	23.98	22.90	21.91	21.00	20.17	19.40	18.68	18.02	17.41
0.03	16.83	16.29	15.79	15.32	14.87	14.45	14.05	13.68	13.32	12.99
0.04	12.66	12.36	12.07	11.79	11.53	11.28	11.03	10.80	10.58	10.37
0.05	10.16	9.97	9.78	9.60	9.42	9.25	9.09	8.94	8.78	8.64
0.06	8.50	8.36	8.23	8.10	7.98	7.86	7.74	7.63	7.52	7.41
0.07	7.31	7.20	7.11	7.01	6.92	6.83	6.74	6.66	6.57	6.49
0.08	6.41	6.34	6.26	6.19	6.11	6.04	5.98	5.91	5.84	5.78
0.09	5.72	5.66	5.60	5.54	5.48	5.42	5.37	5.32	5.26	5.21

a	Second and Third Decimals of a									
	00 05	10 15	20 25	30 35	40 45	50 55	60 65	70 75	80 85	90 95
0.1	5.16	4.71	4.33	4.01	3.73	3.49	3.28	3.10	2.93	2.79
	4.92	4.51	4.16	3.86	3.61	3.38	3.19	3.01	2.86	2.72
0.2	2.65	2.54	2.43	2.33	2.24	2.15	2.07	2.00	1.94	1.87
	2.59	2.48	2.38	2.28	2.19	2.11	2.04	1.97	1.90	1.84
0.3	1.82	1.76	1.71	1.66	1.62	1.57	1.53	1.50	1.46	1.43
	1.79	1.74	1.69	1.64	1.60	1.55	1.52	1.48	1.44	1.41
0.4	1.39	1.36	1.33	1.30	1.28	1.25	1.23	1.20	1.18	1.16
	1.38	1.35	1.32	1.29	1.26	1.24	1.22	1.19	1.17	1.15

a	Second Decimal of a									
	0	1	2	3	4	5	6	7	8	9
0.5	1.14	1.12	1.10	1.08	1.06	1.04	1.03	1.01	0.996	0.981
0.6	0.966	0.952	0.938	0.925	0.912	0.900	0.887	0.876	0.864	0.853
0.7	0.842	0.832	0.822	0.812	0.802	0.792	0.783	0.774	0.765	0.757
0.8	0.748	0.740	0.732	0.725	0.717	0.710	0.702	0.695	0.688	0.682
0.9	0.675	0.670	0.662	0.656	0.650	0.644	0.638	0.632	0.627	0.621
1.0	0.616									

The maximum likelihood functions now involve derivatives of K with respect to a and b, respectively; a double-entry table would be necessary to obtain the maximum likelihood estimates of a and b, and even this would involve double inverse interpolation.

In lieu of this, another method of estimation is proposed. Let the n observations be grouped by classes $(\xi_i - h_i, \xi_i + h_i)$ $(i = 1, 2, \cdots r)$, where $\xi_1 - h_1 =$

$0, \xi_r + h_r = T, \xi_i + h_i = \xi_{i+1} - h_{i+1}, i = 1, 2, \cdots r - 1$. Denote by ν_i the number of observations falling in class i, i.e., between $\xi_i - h_i$ and $\xi_i + h_i$.

Define

$$p_i = K^{-1} \int_{\xi_i - h_i}^{\xi_i + h_i} e^{-ax} x^{b-1} \, dx \doteq K^{-1} e^{-a\xi_i} \xi_i^{b-1} (2h_i) \tag{5}$$

$$q_i = \frac{\nu_i}{n}.$$

Now

$$\ln p_i - \ln p_{i+1} = a(\xi_{i+1} - \xi_i) + (b-1) \ln \frac{\xi_i}{\xi_{i+1}} + \ln \frac{h_i}{h_{i+1}} \tag{6}$$

$$i = 1, 2, \cdots r - 1$$

to the degree of approximation indicated by (5).

The form of equation (6) suggests estimating a and b by a least-squares procedure, with q_i replacing p_i. This can be justified as an approximate procedure by the following results. To terms of order $1/n$,

$$E(\ln q_i) = \ln p_i - \frac{1}{2n} \frac{1 - p_i}{p_i}, \tag{7}$$

$$E(\ln q_i - \ln p_i)^2 = \frac{1}{n} \frac{1 - p_i}{p_i}, \tag{8}$$

$$E\left[\left(\ln \frac{q_i}{p_i}\right)\left(\ln \frac{q_j}{p_j}\right)\right] = -\frac{1}{n}, \qquad i \neq j. \tag{9}$$

These results can be obtained by expanding $\ln(q_i / p_i) = \ln(1 + (q_i - p_i) / p_i)$ in a Taylor series (assuming that $\Pr(q_i = 0)$ and $\Pr(q_i > 2p_i)$ may be neglected for large n).

To show that the higher-order terms of the series expansion may be neglected, the following results are needed:

$$E(q - p)^{2s+1} = O\left(\frac{1}{n^{s+1}}\right), \qquad s \geqq 1,$$

$$E(q - p)^{2s} = O\left(\frac{1}{n^s}\right).$$

These may be proven by induction, making use of the recurrence formula for the central moments of the binomial (and hence also of the multinomial) distribution. This recurrence formula is

$$\mu_{s+1} = pq\left(ns\mu_{s-1} + \frac{d\mu_s}{dp}\right),$$

where μ_s is the sth central moment.

Moreover, the limiting distribution of the $\ln q_i$ is easily obtained from the following lemma:

LEMMA. *Let* $\{X_i^{(n)}\}$ $(i = 1, 2, \cdots, r)$ *be a sequence of random variables and* $\mu_i, \sigma_i (i = 1, 2, \cdots, r)$ *be constants such that the joint distribution of*

$$Y_i^{(n)} = \frac{(X_i^{(n)} - \mu_i)}{\sigma_i} \sqrt{n} \qquad (i = 1, 2, \cdots, r)$$

tends to the limiting distribution $F(y_1, y_2, \cdots, y_r)$ *as* $n \to \infty$, *and let* $f(x)$ *be of class* $C^{(1)}$ *in the neighborhood of* $(\mu_1, \mu_2, \cdots, \mu_r)$ *with*

$$t_i = \left(\frac{df}{dx}\right)_{x=\mu_i} \neq 0.$$

Then $Z_i^{(n)} = \sqrt{n}[f(X_i^{(n)}) - f(\mu_i)] / (\sigma_i \cdot t_i)$ $(i = 1, 2, \cdots, r)$ *have the same joint limiting distribution.*

This is a consequence of the general theorems on stochastic limit relationships proved by Mann and Wald [10] (see their Theorems 3 and 5; however, a trivial modification is required, since our $f(x)$ is a function of a single real variable, whereas their corresponding $g(x)$ is a function of a vector-valued random variable).

Finally, writing

$$y_i = \ln q_i - \ln q_{i+1} \qquad (i = 1, 2, \cdots, r - 1), \tag{10}$$

it follows that the y_i are asymptotically multinormal with means

$$a(\xi_{i+1} - \xi_i) + (b - 1) \ln \frac{\xi_i}{\xi_{i+1}} + \ln \frac{h_i}{h_{i+1}}$$

and moment matrix

$$\mathfrak{M} = \begin{Vmatrix} \frac{1}{n}\left(\frac{1}{p_1} + \frac{1}{p_2}\right) & -\frac{1}{n}\left(\frac{1}{p_2}\right) & 0 & \cdots & 0 \\ -\frac{1}{n}\left(\frac{1}{p_2}\right) & \frac{1}{n}\left(\frac{1}{p_2} + \frac{1}{p_3}\right) & -\frac{1}{n}\left(\frac{1}{p_3}\right) & \cdots & 0 \\ 0 & -\frac{1}{n}\left(\frac{1}{p_3}\right) & \frac{1}{n}\left(\frac{1}{p_3} + \frac{1}{p_4}\right) & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots\vdots\vdots & \vdots \\ 0 & 0 & 0 & \cdots & \frac{1}{n}\left(\frac{1}{p_{r-1}} + \frac{1}{p_r}\right) \end{Vmatrix}.$$

Least-squares estimators of a and b are found by minimizing the quadratic form

$$[\mathbf{y} - \mathbf{E}(y)]'\mathfrak{M}^{-1}[\mathbf{y} - \mathbf{E}(y)],$$

where the vector $\mathbf{y}' = (y_1, y_2, \cdots y_{r-1})$.

In view of the asymptotic distribution of the y_i, these estimates are asymptotically efficient relative to the y_i. What information is lost in using the variables y_i rather than the original observations x_i? If the original observations were ungrouped, a slight loss of information would be caused by grouping to form the variables ν_i. Since the ln q_i are monotone functions of the ν_i, no information is lost in this transformation. The y_i are linear combinations of the ln q_i; some further information is lost here in exchange for elimination of the factor K^{-1} from the estimation process.

Since the true values of the p_i are not known, it is necessary to replace the p_i in $\mathfrak{M}$ by their estimates, the q_i. Introducing the notation

$$w_i = y_i - \ln h_i + \ln h_{i+1}, \tag{11}$$

$$u_i = \xi_{i+1} - \xi_i, \tag{12}$$

$$v_i = \ln \xi_i - \ln \xi_{i+1}. \tag{13}$$

the equations for a and $b' = b - 1$ are

$$a\left(\sum_i \sum_j m_0^{ij} u_i u_j\right) + b'\left(\sum_i \sum_j m_0^{ij} u_i v_j\right) = \sum_i \sum_j m_0^{ij} u_i w_j, \tag{14}$$

$$a\left(\sum_i \sum_j m_0^{ij} u_i v_j\right) + b'\left(\sum_i \sum_j m_0^{ij} v_i v_j\right) = \sum_i \sum_j m_0^{ij} v_i w_j, \tag{15}$$

m_0^{ij} denoting the elements of $\mathfrak{M}_0^{-1}$ ($\mathfrak{M}^{-1}$ with p's replaced by q's).

The solutions of these are

$$\hat{a} = \frac{1}{\Delta}\left[(\mathbf{v}'\mathfrak{M}_0^{-1}\mathbf{v})(\mathbf{u}'\mathfrak{M}_0^{-1}\mathbf{w}) - (\mathbf{u}'\mathfrak{M}_0^{-1}\mathbf{v})(\mathbf{v}'\mathfrak{M}_0^{-1}\mathbf{w})\right], \tag{16}$$

$$\hat{b}' = \frac{1}{\Delta}\left[(\mathbf{u}'\mathfrak{M}_0^{-1}\mathbf{u})(\mathbf{v}'\mathfrak{M}_0^{-1}\mathbf{w}) - (\mathbf{u}'\mathfrak{M}_0^{-1}\mathbf{v})(\mathbf{u}'\mathfrak{M}_0^{-1}\mathbf{w})\right], \tag{17}$$

where

$$\Delta = (\mathbf{u}'\mathfrak{M}_0^{-1}\mathbf{u})(\mathbf{v}'\mathfrak{M}_0^{-1}\mathbf{v}) - (\mathbf{u}'\mathfrak{M}_0^{-1}\mathbf{v})^2,$$

and the covariance matrix of (a, b') is

$$\begin{pmatrix} \frac{1}{\Delta}(\mathbf{v}'\mathfrak{M}_0^{-1}\mathbf{v}) & -\frac{1}{\Delta}(\mathbf{u}'\mathfrak{M}_0^{-1}\mathbf{v}) \\ -\frac{1}{\Delta}(\mathbf{u}'\mathfrak{M}_0^{-1}\mathbf{v}) & \frac{1}{\Delta}(\mathbf{u}'\mathfrak{M}_0^{-1}\mathbf{u}) \end{pmatrix}. \tag{18}$$

The estimates $\hat{a}$ and $\hat{b}'$ are found by direct simple routine calculations except for the determination of $\mathfrak{M}_0^{-1}$ from $\mathfrak{M}_0$. This may be a tedious process unless r is small. However, if all p_i are equal to $1/r$, then

$$\frac{1}{n}\mathfrak{M}^{-1} = \begin{pmatrix} r-1 & r-2 & r-3 & \cdots & 3 & 2 & 1 \\ r-2 & 2(r-2) & 2(r-3) & \cdots & 6 & 4 & 2 \\ r-3 & 2(r-3) & 3(r-3) & \cdots & 9 & 6 & 3 \\ \vdots & \vdots & \vdots & \vdots\vdots\vdots & \vdots & \vdots & \vdots \\ 3 & 6 & 9 & & 3(r-3) & 2(r-3) & r-3 \\ 2 & 4 & 6 & & 2(r-3) & 2(r-2) & r-2 \\ 1 & 2 & 3 & & (r-3) & r-2 & r-1 \end{pmatrix}.$$

This is easily verified by direct multiplication.

It was pointed out that if $\mathfrak{M}$ were known, least-squares estimates would be in some sense best among estimates which are functions of the y_i. What does the substitution of $\mathfrak{M}_0^{-1}$ for $\mathfrak{M}^{-1}$ do to the estimates? Denoting by m_{ij}, m_{ij}^0 representative elements of $\mathfrak{M}$, $\mathfrak{M}_0$, it is known that m_{ij}^0 converges in probability to m_{ij} as $n \to \infty$. Since $\hat{a}$, $\hat{b}'$ are continuous functions of m_{ij}^0, the results of Mann and Wald cited above [10] are sufficient to conclude that $\hat{a}(\mathfrak{M}_0^{-1})$, $\hat{b}'(\mathfrak{M}_0^{-1})$ have the same limiting distribution as $\hat{a}(\mathfrak{M}^{-1})$, $\hat{b}'(\mathfrak{M}^{-1})$. Consequently, to terms of $1/n$ the variance-covariance matrix is given by (18). This result is, of course, closely related to similar ones in modified minimum χ^2 estimation; e.g., see Neyman [11].

For these asymptotic results to hold rigorously, it is necessary not only for $n \to \infty$, but also for $\max_{1 \leqq i \leqq r} h_i \to 0$ (so that (5) holds exactly), whch in turn implies $r \to \infty$. Furthermore, for the multinormality of the q_i we need $\max_{1 \leqq i \leqq r} h_i \to 0$, $r \to \infty$ in such a way that $\min_{1 \leqq i \leqq r} np_i \to \infty$.

However, these considerations leave open the question of determining r and the ξ_i in any practical situation. While some studies have been made of the optimum allocation in linear regression problems (e.g., Elfving [12]), these refer to situations where the observations are independent. Moreover, our choice of the ξ_i is limited by the requirement that the classes should not be so broad that (5) is seriously invalidated. Since $\mathfrak{M}_0^{-1}$ is quite simple if the q_i are all equal to $1/r$, it seems to be reasonable to choose the ξ_i so that this is so. The device is analogous to that suggested by Gumbel [13] and by Mann and Wald [14] in applying the χ^2 "goodness of fit" test.

The fact that $\mathfrak{M}$ involves reciprocals of the np_i makes it seem desirable that no nq_i should fall below 10. This will set an upper bound for r, namely, $r \leqq n/10$. The lower bound should be determined so that (5) is a reasonable approximation, though more usually it will be determined by considerations of the labor involved in calculating (16), (17), and (18).

Of course, it will often happen that the data will be grouped to begin with, so that the statistician is not free to choose the ξ_i or r. It should be noted that in any case simpler but less efficient estimates can be obtained by utilizing only the odd (or even) w_i's. The odd w_i's are mutually independent among themselves and consequently $\mathfrak{M}_0$ and $\mathfrak{M}_0^{-1}$ reduce to diagonal matrices.

5. Estimation with unknown origin. If the parameter c, the origin, is unknown, then the estimation problem is more difficult whether or not the distribution is truncated. Iterative methods are of course possible in solving (1) (2) and (3) with the aid of Table I, i.e., for the untruncated case. In the truncated case this method is too tedious to have much practical value.

If, in the truncated case, there is available supplementary information so that the restriction $0 < c \ll \xi_1$ may be utilized, then a procedure similar to that outlined above may be followed. In this case

$$\ln p_i - \ln p_{i+1} = a(\xi_{i+1} - \xi_i) + (b - 1) \ln \frac{\xi_i - c}{\xi_{i+1} - c} + \ln \frac{h_i}{h_{i+1}} \qquad (i = 1, 2, \cdots r - 1) \tag{19}$$

again to the degree of approximation indicated by (5). With the restriction noted above, it is adequate to write

$$\ln p_i - \ln p_{i+1} = a(\xi_{i+1} - \xi_i) + (b - 1) \ln \frac{\xi_i}{\xi_{i+1}} + (b - 1)c\left(\frac{1}{\xi_{i+1}} - \frac{1}{\xi_i}\right) + \ln \frac{h_i}{h_{i+1}}. \tag{20}$$

Defining y_i, w_i as above, least-squares estimates of a, b, and c may be found in a procedure exactly analogous to that of Section 4.

6. Conclusion. The method used to estimate the parameters a and b in Section 4 may also be applied if the sample is drawn from a doubly truncated gamma distribution; from a singly or doubly truncated normal distribution; or from a beta distribution with known range, either truncated or not. Methods of obtaining the maximum likelihood estimates of the parameters of a truncated normal distribution are, of course, well known, and extensive tabulations have been made to facilitate the determination of such solutions (e.g., compare particularly Hald [15]).

The method outlined above would also be useful in estimating the parameters of the normal curve where there are systematic gaps in the observations. This may occur particularly in time distributions—an example may be found in [16]. For distributions with finite but unknown range, however, the method does not appear to be satisfactory.

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