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Estimating the Probability Distribution of von Mises Stress for Structures Undergoing Random Excitation, Part 1: Derivation

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Abstract

The von Mises stress is often used as the metric for evaluating design margins, particularly for structures made of ductile materials. For deterministic loads, both static and dynamic, the calculation of von Mises stress is straightforward, as is the resulting calculation of reliability. For loads modeled as random processes, the task is different; the response to such loads is itself a random process and its properties must be determined in terms of those of both the loads and the system. This has been done in the past by Monte Carlo sampling of numerical realizations that reproduce the second order statistics of the problem. Here, we present a method that provides analytic expressions for the probability distributions of von Mises stress which can be evaluated efficiently and with good precision numerically. Further, this new approach has the important advantage of providing the asymptotic properties of the probability distribution.

Introduction

The primary purpose of finite element stress analysis is to estimate the reliability of engineering designs. In structural applications, the von Mises stress due to a given load is often used as the metric for evaluating design margins. For deterministic loads, both static and dynamic, the calculation of von Mises stress is straightforward, e.g. Shigley, 1972. For loads modeled as random processes, the task is different; the response to such loads is itself a random process and its properties must be determined in terms of those of both the loads and the system. There are many ways to analyze such systems (see for example Lin, 1967, Soong, 1993 or Jazwinski, 1970). In a previous paper

Nomenclature

$F(\omega, T)$	FFT of imposed load sampled over period T
$E[\bullet]$	expected value operator
$S_{FF}(\omega)$	cross-spectral density matrix of imposed loads
$q_n(t)$	modal coordinate of n'th mode
$q(t)$	array of all modal coordinates
$\sigma(t, x)$	stress vector at location x and time t
$\sigma_n(x)$	stress vector at location x associated with mode n
$p(t, x)$	von Mises stress at location x and time t
S_{qq}	covariance matrix of modal coordinates
C	covariance matrix defined in equation 12
D^2	diagonal intrinsic covariance matrix defined in equation 13
N	rank of D
$E(\{D\}, Y)$	N dimensional ellipse about origin whose semi axes are the diagonals of D
$V_U(\{D\}, Y, \alpha)$	collection of N dimensional boxes that contain the ellipse $E(\{D\}, Y)$, indexed by parameter α.
$V_L(\{D\}, Y, \alpha)$	collection of N dimensional boxes that are contained in the ellipse $E(\{D\}, Y)$, indexed by parameter α.

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(Segalman *et al*, 1998), a computationally efficient method of estimating the RMS value of von Mises stress for the case of input force of Gaussian distribution with zero mean was presented.

The reliability calculations for a structure of ductile material require a linear model for the structure and a statistical specification of the input forces. In principle, from the linear model one can deduce all required transfer functions. Input forces are specified by their auto spectral densities. In the case of multiple force inputs, the forces may be specified by a cross spectral density matrix. It is demonstrated here how that information can be used to calculate the probability distributions for the von Mises stress at different locations on the body. An integral formulation is presented for cumulative probabilities and a method for approximating those integrals is also presented.

These results may be compared to a sampling of many realizations of random input and corresponding output quantities (see for example Chen and Harichandran, 1998). This Monte Carlo simulation requires computation of long series of values of von Mises stress and determination of probability distributions from histograms of that data. This method was used to check and compare results generated by the core method of this paper. One notes that there are two serious deficiencies of this sampling based approach:

- The method is computationally expensive especially when output is required at a great many response locations in a large model.
- The asymptotic properties of the distribution can only be determined with extremely large sample sizes. It is these asymptotic properties that are important in reliability estimation.

Problem Formulation

Where the applied random load involves either forces applied at several locations or forces applied at one location but in more than one direction, the loads are usually represented by the cross spectral density matrix: (Bendat and Piersol, 1986),

$$S_{FF}(\omega) = \lim_{T\to\infty}\frac{1}{2\pi T}E[\bar{F}(\omega,T)F(\omega,T)^T] \qquad \text{Eq 1}$$

where $F(\omega, T)$ is the finite Fourier transform of the vector of force components sampled over at period T; $(\bullet)^T$ denotes the matrix transpose; $(\bar{\bullet})$ denotes the complex conjugate; and $E[\bullet]$ is the operator of mathematical expectation. In the case of a single scalar input force, this reduces to the auto spectral density. The above assumes that the loads constitute stationary, continuous processes.

The stress at the point in question can be assembled from the contributions of each mode:

$$\sigma(t,x) = \sum_n q_n(t)\sigma_n(x) \qquad \text{Eq 2}$$

where q_n is the n^{th} modal coordinate and $\sigma_n(x)$ is the stress vector at location x associated with that mode, comprised of the six non-redundant terms for the stress tensor. In what follows, we use the vector $q(t) = \{q_n(t)\}$ of modal coordinates.

The square of the von Mises stress can be expressed as a quadratic operator on the stress vector

$$p^2(t,x) = \sigma(t,x)^T A\,\sigma(t,x) \qquad \text{Eq 3}$$

where

$$A = \begin{bmatrix} 1 & -\frac{1}{2} & -\frac{1}{2} & & & \\ -\frac{1}{2} & 1 & -\frac{1}{2} & & & \\ -\frac{1}{2} & -\frac{1}{2} & 1 & & & \\ & & & 3 & & \\ & & & & 3 & \\ & & & & & 3 \end{bmatrix} \qquad \text{Eq 4}$$

To obtain the probability distribution of von Mises stress we begin with the covariance matrix of modal coordinates $S_{qq} = E[q(t)q(t)^T]$, which may be obtained directly from $S_{FF}(\omega)$ and the modal response of the structure (Soong and Grigoriu, 1993).

We use the standard methods to decompose S_{qq} and to map the modal coordinates into uncorrelated variables. Observing that S_{qq} is symmetric and positive semi-definite, its singular value decomposition is (Strang 1988),

$$S_{qq} = QX^2Q^T \qquad \text{Eq 5}$$

where X is a diagonal matrix whose dimension is the rank of S_{qq} and Q is a rectangular matrix having the property that $Q^TQ = I$ is the identity whose dimension is the rank of S_{qq}. (Here we retain only the nonzero terms of the diagonal matrix and the corresponding columns of the rotation matrix. For a symmetric, positive semi-definite matrix eigen analysis and singular value decomposition are the same.) Defining

$$\beta = X^{-1}Q^Tq, \qquad \text{Eq 6}$$

we find that components of β are independent, identically distributed (IID) Gaussian processes, each with unit variance.

$$E[\beta\beta^T] = I \qquad \text{Eq 7}$$

We define another set of random variables by

$$q' = QX\beta = QQ^Tq. \qquad \text{Eq 8}$$

A little algebra shows that

$$E[(q-q')(q-q')^T] = 0 \qquad \text{Eq 9}$$

from which we conclude that

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$$q = q' = QX\beta \qquad \text{Eq 10}$$

This indirect derivation is necessary because Q is generally a rectangular, non-invertible matrix. In our new coordinates, β, the square of the von Mises stress is

$$p^2 = \beta^T C \beta \qquad \text{Eq 11}$$

where

$$C_{kl} = \sum_{m,n} (\sigma_m^T A \sigma_n) K_{mk} K_{nl} \qquad \text{Eq 12}$$

and $K = QX$. Matrix C is square having dimensionality equal to the rank of S_{qq} but possibly much lower rank. Note that the rank of C is the minimum of the rank of A, the rank of S_{qq} and the dimensionality of the stress vectors. Because the rank of A is five, the rank of C can be at most five.

We exploit the symmetry and the positive semi-definiteness of C in doing its singular value decomposition:

$$C = RD^2R^T \qquad \text{Eq 13}$$

where the matrix D is diagonal and has dimension equal to the rank of C and R is a rectangular matrix having property that $R^T R = I$ is the identity matrix whose dimension is the rank of C. The von Mises stress is now

$$p^2 = \beta^T R D^2 R^T \beta \qquad \text{Eq 14}$$

This suggests yet another change of variables:

$$y = R^T \beta \qquad \text{Eq 15}$$

It is easily shown that the elements of y are IID, Gaussian processes with unit variance. The advantages of the above transformation are first that it reduces the number of random variables of this problem to the rank of C (at most five) and second that it aligns the random variables in the directions of the axes of the ellipsoids of constant von Mises stress.

$$p^2 = y^T D^2 y = \sum_n y_n^2 D_n^2 \qquad \text{Eq 16}$$

The mean square of the von Mises stress is

$$E[p^2] = \int_{-\infty}^{\infty} \dots \int_{-\infty}^{\infty} p^2 \prod \rho_r(y_r) \prod dy_r$$

$$= \int_{-\infty}^{\infty} \dots \int_{-\infty}^{\infty} y^T D^2 y \prod \rho_r(y_r) \prod dy_r \qquad \text{Eq 17}$$

Noting that $\int_{-\infty}^{\infty} y_r^2 \rho_r(y_r) dy_r = 1$, the above becomes

$$E[p^2] = \sum_r D_r^2 \qquad \text{Eq 18}$$

We see that D_r^2 is the contribution of the r^{th} random process to $E[p^2]$ and the rank of D is the number of independent random processes taking place at that location. For convenience, we refer to $\sqrt{E[p^2]}$ as p_{RMS}.

We now calculate the probability of the von Mises stress being less than some value Y:

$$P(p < Y) = \int_{E(\{D\}, Y)} \prod \rho_r(y_r) \prod dy_r \qquad \text{Eq 19}$$

where $E(\{D\}, Y)$ is the N-dimensional ellipsoid containing points y associated with von Mises stress less than Y:

$$E(\{D\}, Y) = \{y : (y^T D^2 y) \le Y\} \qquad \text{Eq 20}$$

and N is the rank of D. The integral of Equation 19 is difficult to evaluate.

Quadrature by Boxes

We discuss here how to achieve upper and lower bounds for the integral in Equation 19. This discussion then leads to reasonably good approximations for that integral.

We first note that the integral of $\prod \rho_r(y_r) dy_r$ over an N dimensional box, B, having faces normal to each of the coordinates y_r, can be calculated analytically:

$$P_B = \int_B \prod_{k=1}^{N} \rho_r(y_r) dy_r$$

$$= \prod_{k=1}^{N} [\Phi(y_{r,max}) - \Phi(-y_{r,min})] \qquad \text{Eq 21}$$

where $y_{r,max}$ and $y_{r,min}$ define the boundaries of B and

$$\Phi(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{x} \exp(-s^2/2) ds \qquad \text{Eq 22}$$

is the cumulative distribution function for a standard normal distribution (Wirsching *et al.* 1995).

We next consider volumes $V_L(\{D\}, Y, \alpha)$ and $V_U(\{D\}, Y, \alpha)$ each of which is a union of N dimensional boxes selected so that

$$V_L(\{D\}, Y, \alpha) \subseteq E(\{D\}, Y) \subseteq V_U(\{D\}, Y, \alpha) \qquad \text{Eq 23}$$

The parameter α is an indicator of the level of refinement so that

$V_L(\{D\}, Y, \alpha), V_U(\{D\}, Y, \alpha) \to E(\{D\}, Y)$ as $\alpha \to \infty$.

These contained and containing volumes are illustrated for a problem of two processes ($N = 2$) in Figure 1.

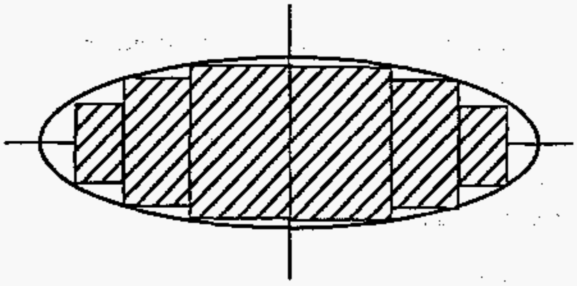

Figure 1. A collection of boxes entirely contained in the ellipsoid, is an admissible $V_L(\{D\}, Y, \alpha)$

Expressing each of these volumes in terms of its component boxes:

$$V_L(\{D\}, Y, \alpha) = \bigcup_k B_{L,k}(\{D\}, Y, \alpha) \qquad \text{Eq 24}$$

and

$$V_U(\{D\}, Y, \alpha) = \bigcup_k B_{U,k}(\{D\}, Y, \alpha) \qquad \text{Eq 25}$$

The integral is now approximated by:

$$\int_{V_L(\{D\}, Y, \alpha)} \prod \rho_r(y_r) dy_r = \int_{\bigcup_k B_{L,k}(\{D\}, Y, \alpha)} \prod \rho_r(y_r) dy_r$$

$$= \sum_k P_{B_{L,k}(\{D\}, Y, \alpha)} \qquad \text{Eq 26}$$

and by

$$\int_{V_U(\{D\}, Y, \alpha)} \prod \rho_r(y_r) dy_r = \int_{\bigcup_k B_{U,k}(\{D\}, Y, \alpha)} \prod \rho_r(y_r) dy_r$$

$$= \sum_k P_{B_{U,k}(\{D\}, Y, \alpha)} \qquad \text{Eq 27}$$

Recalling Equation 23 and observing that the integrand is positive, we have upper and lower bounds for $P(p < Y)$:

$$\sum_k P_{B_{L,k}(\{D\}, Y, \alpha)} \le P(p < Y) = \int_{E(\{D\}, Y)} \prod \rho_r(y_r) \prod dy_r$$

$$\le \sum_k P_{B_{U,k}(\{D\}, Y, \alpha)} \qquad \text{Eq 28}$$

We also note that

$$\sum_k P_{B_{L,k}(\{D\}, Y, \alpha)} , \sum_k P_{B_{U,k}(\{D\}, Y, \alpha)} \to P(p < Y) \qquad \text{Eq 29}$$

as $\alpha \to \infty$ and that convergence is assessed by the difference of the upper and lower bound quadrature.

The mathematics discussed above has been implemented in a simple recursive C language procedure which is listed in the Appendix.

Numerical Comparison

To evaluate the algorithm, we consider a case for which two independent random processes contribute equally to the von Mises stress, $D_1 = D_2 = 1$. This occurs on a surface with independent x and y components of normal stress and no shear. The resulting probability density can be computed analytically and is found to be a Rayleigh distribution. Figure 2 compares the Rayleigh distribution with approximate results obtained using the algorithm above.

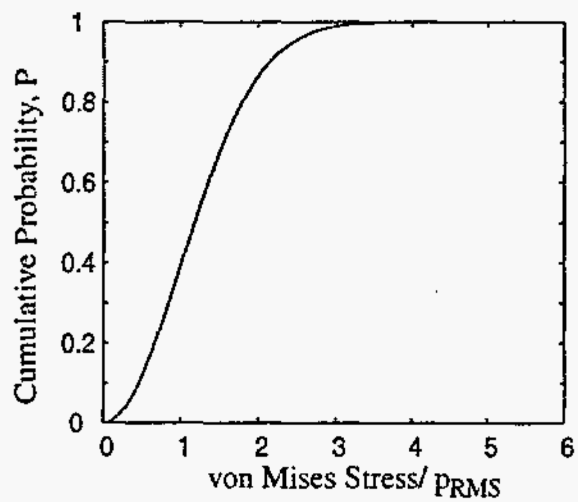

Figure 2. Comparison of exact ($[1 - \exp(-Y^2/2)]$) cumulative distribution function for $D_1 = D_2 = 1$ and numerical quadrature. Quadrature generates upper and lower bounds which almost overly the analytic curve.

The numerical quadrature used here employed 128^2 boxes in the calculation of the lower bound and 129^2 in the calculation of the upper bound. The error is shown in Figure 3. The maximum error in this case was $2.0x10^{-3}$ and occurred near the RMS value of von Mises stress. In the quadrature employed, the magnitude of the upper-bound error was almost exactly the magnitude of the lower-bound error. Also interesting is the comparison of the magnitude of the error and the function 1-*P*, the difference between the cumulative probability and 1.0. It is seen that the error stays substantially below $1 - P$, indicating that the quadrature remains accurate even out to high values of von Mises stress.

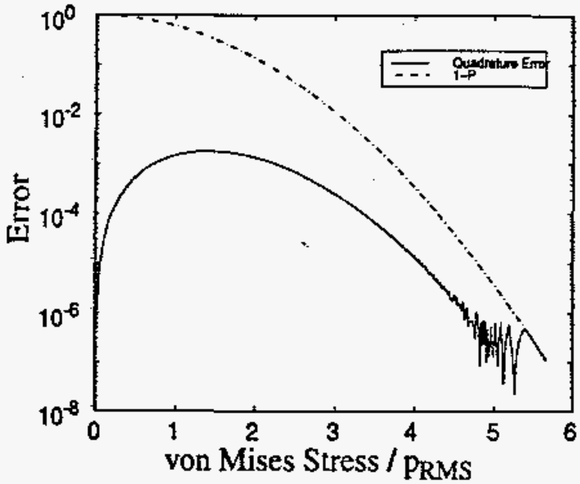

Figure 3. The quadrature error and $1-P$ for the cumulative distribution function for $D_1 = D_2 = 1$.

Summary

The authors have derived and presented an expression for the cumulative probability distribution for the von Mises stress resulting from random loadings that are Gaussian and of zero mean. This is an important result for reliability of structures subject to such loads.

Additionally, a convenient set of expressions were derived for upper and lower bounds to the cumulative probability.

Finally, it should be noted that the derivation of the cumulative probability integral and of the approximations for it could also be applied to any other quadratic function of the load.

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Appendix: Code Fragment for Recursive Calculation of Lower-Bound Quadrature

```
// recursive routine to calculate a lower
//   bound for the integral
double root2 = sqrt(2.0);
double slabL(double *D,     int generation,
             double remain, double *xi,
             int Inner)
{
   double ymax=sqrt(remain)/D[generation];
   if(generation==4)
      return(erf(ymax/root2));
   if(D[generation+1] < D[0]*0.01)
      return(erf(ymax/root2));
   double sum=0;
   double y1, y2;
   y1 = 0;
   int i;
   // in the following, it is assumed that
   //xi[Inner] < 1;
   for(i=0; i<Inner; i++){
      y1 = xi[i]  *ymax;
      y2 = xi[i+1]*ymax;
      double remain2 = remain-
         (y2*D[generation])
         *(y2*D[generation]);
      sum += (erf(y2/root2) -
         erf(y1/root2))*
      slabL( D,     generation+1,
           remain2, xi, Inner);
   }
 return(sum);
}
```