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Estimating the scaling function of multifractal measures and multifractal random walks using ratios

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In this paper, we prove central limit theorems for bias reduced estimators of the structure function of several multifractal processes, namely mutiplicative cascades, multifractal random measures, multifractal random walk and multifractal fractional random walk as defined by Ludeña [*Ann. Appl. Probab.* **18** (2008) 1138–1163]. Previous estimators of the structure functions considered in the literature were severely biased with a logarithmic rate of convergence, whereas the estimators considered here have a polynomial rate of convergence.

Keywords: multifractal random measure; multifractal random walk; p-variations; scaling function

1. Introduction

A random process $X = \{X(s), s \in [0, T]\}$ $(T > 0)$ with stationary increments will be called multifractal if its scaling behaviour is characterized by a strictly concave function ζ, called the scaling function, such that for a certain range of real numbers q

$$\mathbb{E}\big[\big|X(t) - X(s)\big|^q\big] = c(q)|t - s|^{\zeta(q)}.$$

If the function ζ is linear, then the process is said to be monofractal, as is the case, for instance, for the fractional Brownian motion (FBM) B_H, $0 < H < 1$, which is defined as a continuous centered Gaussian process such that $B_H(0) = 0$ and for all $s, t \geq 0$,

$$\operatorname{var}\big(B_H(t) - B_H(s)\big) = |t - s|^{2H}.$$

Then, for all $q > -1$, $\mathbb{E}[|B_H(t) - B_H(s)|^q] = c(q)|t - s|^{qH}$, with $c(q) = \mathbb{E}[|B_H(1)|^q]$.

Several truly multifractal processes with stationary increments have been defined. The earliest one is the multiplicative cascade introduced by Mandelbrot [11] and rigorously studied by Kahane and Peyrière [9]. These processes were generalized by Barral and Mandelbrot [6], Muzy and Bacry [12] and Bacry and Muzy [5]. The latter authors introduced multifractal random measures (MRM) and multifractal random walks (MRW) as time changed Brownian motion. Ludeña [10] and Abry *et al.* [1] introduced multifractal (fractional) random walks which are conditionally fractional Gaussian processes.

1350-7265

For these processes, multifractality results from a distributional scaling property which can be written as

$$\{X(\lambda t), 0 \le t \le T\} \stackrel{\text{law}}{=} \{U_\lambda X(t), 0 \le t \le T\}$$

for $0 < \lambda < 1$, U_λ is a positive random variable independent of the process X such that $\mathbb{E}[U_\lambda^q] = \lambda^{\zeta(q)}$ for $q < q_{\max}$ a certain parameter depending on the process under consideration (and with certain additional restrictions on the values of λ for which this identiy holds in the case of multifractal cascades, see Section 2). For the models, we will formally introduce in the sequel, it is defined as

$$q_{\max} = \sup\{q \colon \zeta(q) \ge 1\}.$$

It is also important to note that the fixed time horizon T beyond which this scaling property need not be true is finite, except for monofractal processes such as the FBM.

Given a multifractal process observed discretely on $[0, T]$, it is of obvious interest to be able to identify the scaling function ζ.

Let $t_1, \dots, t_N$, with $t_i - t_{i-1} = \Delta = T/N$ be a regular partition of $[0, T]$ (typically on a dyadic scale). Typically, for $q < q_{\max}$, $\zeta(q)$ is estimated by calculating logarithms of the empirical structure function

$$S_N(X, q) := \sum_{j=0}^{N-1} |\Delta X_j|^q,$$

where $\Delta X_j = X((j+1)\Delta) - X(j\Delta)$. Estimators of ζ can then be defined by

$$\hat{\zeta}_N(q) := 1 + \frac{\log_2(S_N(X, q))}{\log_2(\Delta)},$$

$$\tilde{\zeta}_N(q) := 1 + \log_2\left(\frac{S_N(X, q)}{S_{2N}(X, q)}\right).$$

These estimators have been thoroughly dealt with for multiplicative cascades in Ossiander and Waymire [14]. The authors show that $\hat{\zeta}_N(q)$ and $\tilde{\zeta}_N(q)$ are consistent estimators of $\zeta(q)$ for $q < q_0$, where $q_0 < q_{\max}$ is the largest value of q such that

$$\zeta(q) - q\zeta'(q) < 1.$$

For $q > q_0$, $\hat{\zeta}_N(q)$ is seen to converge almost surely to a linear function of q. Moreover, conditional central limit theorems (where the limiting distribution is a mixture of normal laws) are seen to hold for suitably normalized versions of both estimators if $2q < q_0$. However, as shown in Ossiander and Waymire [14], the convergence rates for these estimators are very different. The rate of convergence of $\hat{\zeta}_N(q)$ is of order $\log_2(N)$ because of the existence of a bias term, whereas we will show that of $\tilde{\zeta}_N(q)$ is a power of N which depends on ζ.

In order to enlarge the domain of consistency of the estimators and obtain unconditional central limit theorems, the so-called mixed asymptotic framework has been introduced by allowing the number L of basic observations intervals to increase with N. In the case of multiplicative cascades and MRM, the processes over different intervals are independent. The observations are $X((jL+k)\Delta)$, $0 \le j \le L-1$, $0 \le k \le N-1$, and the estimators are now modified as follows

$$\hat{\zeta}_{L,N}(X,q) := 1 + \frac{\log_2(S_{L,N}(X,q))}{\log_2(\Delta)},$$
$$\tilde{\zeta}_{L,N}(X,q) := 1 + \log_2\left(\frac{S_{L,N}(X,q)}{S_{L,2N}(X,q)}\right),$$

with

$$S_{N,L}(X,q) := \sum_{j=0}^{L-1}\sum_{k=0}^{N-1} |\Delta X_{jL+k}|^q.$$

The mixed asymptotic framework for multiplicative cascades has been recently developed in Bacry *et al.* [4]. The authors show that if $L = [N^\chi]$, where $[x]$ stands for the greatest integer $m \le x$ with $\chi > 0$, then $\hat{\zeta}_{N,L}(X,q)$ is consistent for $q < q_\chi$ where q_χ is the largest value of q such that

$$\zeta(q) - q\zeta'(q) < \chi + 1.$$

Note that as χ tends to infinity, q_χ might become greater than $q_{\max}$, so we will only consider values of χ such that $q_\chi < q_{\max}$.

However, once again, there exists a bias term $b_N := \mathbb{E}[M_1^q]/\log_2(N)$, which entails slow convergence of the estimator. In analogy to the nonmixed asymptotic framework it is reasonable to consider ratio based estimators such as $\tilde{\zeta}_{N,L}(X,q)$ in order to improve convergence rates. It turns out, as follows quite straightforwardly from the results of Bacry *et al.* [4], that $\tilde{\zeta}_{N,L}(X,q) \to \zeta(q)$, a.s., for a dyadic partition, but the authors failed to prove a central limit theorem, although they hint at it at the end of their Section 3. Almost sure convergence for dyadic partitions, or in probability for general partitions, of $\hat{\zeta}_{N,L}(X,q)$ has also been recently considered by Duvernet [7] for $\chi \ge 0$ and X a Brownian MRW or a MRM. However, the author does not prove central limit theorems nor establish convergence rates in either case. An interesting application for testing whether a process is a semimartigale or a multifractal process is developed in Duvernet, Robert and Rosenbaum [8] which is based on the limiting behaviour of variation ratios, but the authors restrict their attention to log-normal multifractal random walks and $q = 2$.

The main goal of this paper is to obtain central limit theorems for the estimator $\tilde{\zeta}_{N,L}$ in the mixed asymptotic setting, for multiplicative cascades, multifractal random measures (MRM) and multifractal random walks (MRW) that are either a time changed Brownian motion or a more general process related to a fractional Brownian motion with Hurst index $H > 1/2$. Our main results in all these cases state unconditional central limit theorems with polynomial rates of convergence, contrary to $\hat{\zeta}_{L,N}$ which can only achieve logarithmic rates of convergence, and to the case $L = 1$ where only conditional central limit theorems can be obtained.

For multiplicative cascades, Ossiander and Waymire [14] also considered negative values of q such that $\mathbb{E}[M^q([0,1])] < \infty$ and $0 > q > \inf_{h \le 0}\{h\psi'(h) - \psi(h) < 1\}$. However, we cannot extend such a result in full generality in the present context, since for certain MRM which are considered here, $\mathbb{E}[M^q([0,1])] = \infty$ for all $q < 0$. Moreover, negative moments of the Gaussian law are infinite for $q \leq -1$, thus even if the MRM considered has finite negative moments, that might not be the case for the MRW. For these reasons, and not to increase the length of the paper, we do not consider the case $q < 0$.

The rest of the paper is organized as follows. We will consider multiplicative cascades in Section 2, MRM in Section 3, and MRW in Section 4. Section 5 contains the main ideas of the proofs and technical lemmas are relegated to the Appendix. To the best of our knowledge, our results are the first to deal with the MRW in the case $H > 1/2$.

2. Multiplicative cascades

In this section, we give a precise formulation of consistency results for $\tilde{\zeta}(q)$, whenever $q < q_\chi$, and a central limit theorem whenever $2q < q_\chi$, in the case of multiplicative cascades. The results are a straightforward application of previous results of Bacry *et al.* [4] and Ossiander and Waymire [14]. However, they provide the framework for dealing with both MRM and MRW so will be dealt with in some detail. Before we state the main results, we shall introduce the mixed asymptotic setting, following Bacry *et al.* [4].

For any given n-tuple r and $i < n$ set $r|i = (r_1, \dots, r_i)$ and if s is an i-tuple and v an $(n-i)$-tuple set $r = s * v$ to be the resulting n-tuple obtained by concatenation. For each $j \in \mathbb{Z}$ and fixed T, set $I^{(j)} := [jT, (j+1)T]$. Over each $I^{(j)}$ we will construct an independent *m*ultiplicative cascade as defined in Mandelbrot [11]. For this, consider a collection $\{W_r^{(j)}, r \in \{0,1\}^n, n \geq 1, j \in \mathbb{Z}\}$ of independent random variables with common law W such that $\mathbb{E}[W] = 1$ and $\mathbb{E}[W \log_2 W] < 1$ and for each $n \geq 1$ and $j \in \mathbb{Z}$, consider the random measure defined by

$$\lambda_n^{(j)}(I) = T2^{-n} \sum_{\{r \in \{0,1\}^n : (j-1+r)T \in I^{(j)}\}} \prod_{i=1}^{n} W_{r|i}^{(j)}$$

for any Borel subset I of $I^{(j)}$, and each $r = (r_1, \dots, r_n) \in \{0,1\}^n$ is associated to the real number $\sum_{i=1}^n r_i 2^{n-k}$. It can be seen (see Kahane and Peyrière [9], Ossiander and Waymire [14] for details on the construction and main results) that there exists a random measure $\lambda_\infty^{(j)}$, such that

$$\mathbb{P}\big(\lambda_n^{(j)} \Rightarrow \lambda_\infty^{(j)} \text{ as } n \to \infty\big) = 1,$$

where $\Rightarrow$ stands for vague convergence. The limiting measure verifies $\mathbb{E}[\lambda_\infty^{(j)}([0,T])] = T$. By construction $\lambda_\infty^{(j)}$ are independent random measures, defined over the disjoint intervals $I^{(j)}$. Set $\lambda_\infty := \sum_{j \in \mathbb{Z}} \lambda_\infty^{(j)}$.

Set $\mathcal{F}_n = \sigma\{W_r^{(j)}, r \in \{0,1\}^n, j \in \mathbb{Z}\}$ and let $\Delta_{k,n}^{(j)} := [(j + k2^{-n})T, (j + (k+1)2^{-n})T]$, $k = 0, \dots, 2^n - 1$, be the kth diadic interval at level n, of the interval $I^{(j)}$. Then,

$$\lambda_\infty\big(\Delta_{k,n}^{(j)}\big) = 2^{-n} Z_{j,k,n} \prod_{i=1}^{n} W_{r_n(k)|i}^{(j)},$$

where for each n, $Z_{j,k,n}$, $0 \le k < 2^n$, $j \in \mathbb{Z}$, are i.i.d. random variables with the same distribution as $\lambda_\infty([0,T])$ and independent of $\mathcal{F}_n$, and $r_n(k)$ is the dyadic representation of k, that is, $k = \sum_{i=1}^n r_{n,i}(k) 2^{n-i}$ for $k < 2^n$. Moreover, $Z_{j,2k,n+1}$ and $Z_{j,2k+1,n+1}$ are independent of $Z_{j,k',n}$ for $k' \neq k$. The above identity straightforwardly yields the scaling property:

$$\mathbb{E}\big[\lambda_\infty^q\big(\Delta_{k,n}^{(j)}\big)\big] = 2^{-n\zeta(q)}\mathbb{E}\big[\lambda_\infty^q\big([0,T]\big)\big],$$

with

$$\zeta(q) = q - \log_2\big(\mathbb{E}\big[W^q\big]\big).$$

It is shown in Kahane and Peyrière [9] that for $q > 1$, the condition $\zeta(q) > 1$ implies $\mathbb{E}[\lambda_\infty^q([0,T])] < \infty$.

Example 2.1. Consider the log-normal cascade, where $\log W = \mu + \sigma Z$ and Z is a standard Gaussian random variable. The condition $\mathbb{E}[W] = 1$ implies that $\mu = -\sigma^2/2$. Then it is easily obtained that

$$\zeta(q) = q - \frac{q(q-1)\sigma^2}{2\log 2}, \qquad q_{\max} = \left(\frac{2\log 2}{\sigma^2}\right) \vee 1, \qquad q_0 = \frac{\sqrt{2\log 2}}{\sigma},$$
$$q_\chi = \frac{\sqrt{2(1+\chi)\log 2}}{\sigma}.$$

Denote

$$S_{L,n}(q) = \sum_{j=0}^{L-1}\sum_{k=0}^{2^n-1} \lambda_\infty^q\big(\Delta_{k,n}^{(j)}\big)$$

and

$$\hat{\zeta}(q) := 1 - \frac{\log_2(S_{L,n}(q))}{n}, \qquad \tilde{\zeta}(q) = 1 + \log_2\left(\frac{S_{L,n}(q)}{S_{L,n+1}(q)}\right).$$

Note that although in the asymptotics L will eventually depend on n, its value is the same in the quantities $S_{L,n}$ and $S_{L+1,n}$.

Consistency. For each $n \ge 1$, let $\{\xi, \xi_{j,k,n}, 0 \le j \le L-1, 0 \le k \le 2^n - 1\}$ be a collection of i.i.d. random variables, independent of $\mathcal{F}_n$. Define

$$\tilde{S}_{n,q} = 2^{-nq}\sum_{j=0}^{L-1}\sum_{k=0}^{2^n-1}\prod_{i=1}^{n}\big(W_{r_n(k)|i}^{(j)}\big)^q \xi_{j,k,n}.$$

In Bacry *et al.* [4], the following general result is shown to hold.

Proposition 2.1. *For* $\chi > 0$, *assume that* $L = [2^{n\chi}]$, $q < q_\chi$ *and there exists* $\varepsilon > 0$ *such that* $\mathbb{E}[\xi^{1+\varepsilon}] < \infty$. *If* ξ *is nonnegative, then*

$$L^{-1}2^{-n}2^{n\zeta(q)}\big(\tilde{S}_{n,q} - \mathbb{E}[\tilde{S}_{n,q}]\big) \to 0 \qquad a.s.$$

Note that by construction $\mathbb{E}[\tilde{S}_{n,q}] = L2^n 2^{-n\zeta(q)}\mathbb{E}[\xi]$, so that the above result yields the almost sure convergence $L^{-1}2^{-n}2^{n\zeta(q)}\tilde{S}_{n,q} \to \mathbb{E}[\xi]$ under the stated conditions. As a consequence, by the definition of $S_{L,n}(q)$, Proposition 2.1 yields

$$L^{-1}2^{-n}2^{n\zeta(q)}S_{L,n}(q) \to \mathbb{E}\big[\lambda_\infty^q\big([0,T]\big)\big] \qquad \text{a.s.} \tag{2.1}$$

for $q < q_\chi$. Then, clearly,

$$\hat{\zeta}(q) - \zeta(q) + \chi + \frac{\log_2 \mathbb{E}[\lambda_\infty^q([0,T])]}{n} \to 0 \qquad \text{a.s.,}$$

and (2.1) also implies that $\tilde{\zeta}(q) \to \zeta(q)$ a.s. On the other hand, if $q > q_\chi$, then Bacry *et al.* [4] show that $\hat{\zeta}(q) \to \zeta'(q_\chi)q$, which is a linear function of q. In this case, $\tilde{\zeta}(q)$ is also not consistent as the normalized structure function tends to zero (Ossiander and Waymire [14]).

Central limit theorem. Based on Proposition 2.1, it is also possible to obtain a central limit theorem for $\tilde{\zeta}(q)$. We remark that in the mixed asymptotic framework the limiting variance is deterministic. The proof of the central limit theorem follows from a series of corollaries of the following general result for the mixed framework which is a direct generalization of Proposition 4.1 in Ossiander and Waymire [14] and Proposition 2.1. We first state some general notation. Let $\{\xi, \xi_{j,k,n}, 0 \le j \le L-1, 0 \le k \le 2^{n-1}, n \ge 0\}$ be as above and define

$$V_{n,q} = 2^{-2nq}\sum_{j=0}^{L-1}\sum_{k=0}^{2^n-1}\prod_{i=1}^{n}\big(W_{r_n(k)|i}^{(j)}\big)^{2q}, \qquad R_{n,q} = \tilde{S}_{n,q}/V_{n,q}^{1/2}.$$

The following proposition is seen to hold true as a direct generalization of Proposition 4.1 in Ossiander and Waymire [14], whenever $2q < q_\chi$.

Proposition 2.2. *If* $2q < q_\chi$, $\mathbb{E}[\xi_{j,k,n}] = 0$, $\mathbb{E}[\xi_{j,k,n]}^2] = \sigma^2$ *and if*

$$\sup_n \sup_{j,k} \mathbb{E}\big[|\xi_{j,k,n}|^{2(1+\delta)}\big] < \infty$$

for some $\delta > 0$, *then*

$$\lim_{n\to\infty} \mathbb{E}\big[\mathrm{e}^{\mathrm{i}zR_{n,q}}|\mathcal{F}_n\big] = \mathrm{e}^{-\sigma^2 z^2/2}$$

and $R_{n,q}$ *converges weakly to the centered Gaussian law with variance* σ^2.

The proof follows exactly as that of Proposition 4.1 in Ossiander and Waymire [14], using Proposition 2.1. The latter also yields that $L^{-1}2^{-n}2^{n\zeta(2q)}V_{n,q}$ converges to 1 a.s. We can now state a central limit theorem for the empirical structure function.

Proposition 2.3. *If* $2q < q_\chi$, *then*

$$L^{-1/2}2^{-n/2}2^{n\zeta(2q)/2}\big\{S_{L,n}(q) - 2^{\zeta(q)-1}S_{L,n+1}(q)\big\} \to_d \mathbf{N}\big(0, V(q)\big),$$

with

$$V(q) = \mathrm{var}\big(Z_0^q - 2^{\zeta(q)-1-q}\{Z_1^q W_1^q + Z_2^q W_2^q\}\big)$$

and Z_1, Z_2 *are i.i.d. with the same distribution as* $\lambda_\infty([0,1])$ *and independent of* W_1, W_2, *which are i.i.d. with the same distribution as* W, *and* $Z_0 = (Z_1 W_1 + Z_2 W_2)/2$ *has the same distribution as* $\lambda_\infty([0,1])$.

Proof. The proof follows from Proposition 2.2, by noting that $S_{L,n}(q) - 2^{\zeta(q)-1} S_{L,n+1}(q)$ can be expressed as

$$S_{L,n}(q) - 2^{\zeta(q)-1} S_{L,n+1}(q) = 2^{-nq} \sum_{j=0}^{L-1} \sum_{k=0}^{2^n-1} \prod_{i=1}^{n} \big(W_{r_n(k)|i}^{(j)}\big)^q \xi_{j,k,n}$$

with

$$\xi_{j,k,n} = Z_{j,k,n}^q - 2^{\zeta(q)-1-q}\{Z_{j,2k,n+1}^q W_{r_n(k)*0}^q + Z_{j,2k,n+1}^q W_{r_n(k)*1}^q\},$$

since $r_n(k) * 0 = r_{n+1}(2k)$ and $r_n(k) * 1 = r_{n+1}(2k+1)$. Indeed, the random variables $\xi_{j,k,n}$, $j \in \mathbb{Z}$, $0 \le k < 2^n$, are i.i.d. (for each fixed n) and it clearly holds that $\mathbb{E}[\xi_{j,k,n}] = 0$, $\mathbb{E}[\xi_{j,k,n}^2] = V(q)$ and $\mathbb{E}[|\xi_{j,k,n}|^{2+\delta}] < \infty$, whenever $2q < q_{max}$ for small enough $\delta > 0$. □

Thus we obtain a central limit theorem for $\tilde{\zeta}(q)$.

Theorem 2.4. *Assume* $2q < q_\chi$. *Then*

$$2^{n(1+\chi+2\psi(q)-\psi(2q))/2}\{\tilde{\zeta}(q) - \zeta(q)\} \to_d \mathbf{N}\big(0, V(q)/\big(\mathbb{E}\big[\lambda_\infty^q([0,T])\big]\big)^2\big).$$

Proof. By Proposition 2.1 and (2.1), $S_{L,n+1}(q) 2^{\zeta(q)-1}/S_{n,L}(q) \to 1$ a.s. so

$$\begin{aligned}
\tilde{\zeta}(q) - \zeta(q) &= \log_2\left(\frac{S_{L,n}(q)}{2^{\zeta(q)-1} S_{L,n+1}(q)}\right) = -\log_2\left(1 - \frac{S_{L,n}(q) - 2^{\zeta(q)-1} S_{L,n+1}(q)}{S_{L,n}(q)}\right) \\
&= \frac{S_{L,n}(q) - 2^{\zeta(q)-1} S_{L,n+1}(q)}{S_{L,n}(q)} \times \{1 + o_P(1)\}.
\end{aligned}$$

The proof is concluded by applying Proposition 2.3 and noting that $2^{-n\chi} L \to 1$. □

3. Multifractal random measures

Once again we are interested in the mixed asymptotic framework defined by the parameter χ. The main ideas dealt with in this section are very similar in spirit to those in Duvernet [7]. We include the proofs for completeness' sake, since they are very similar to those which will be developed to study multifractal random walks. We recall the main definition and properties of

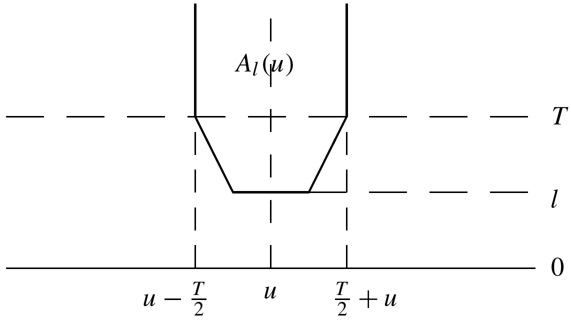

Figure 1. The set $A_l(u)$.

multifractal random measures, hereafter MRM, following Bacry and Muzy [5]. Start by defining for $l > 0$, $w_l(u) = P(A_l(u))$ and set

$$M(I) = \lim_{l \to 0} \int_I \mathrm{e}^{w_l(u)} \, \mathrm{d}u,$$

where I is any Borel set in $\mathbb{R}$. Here P is an independently scattered random measure on $\mathcal{S}^+ = \{(s,t), t > 0\}$ such that $P(\bigcup_{i=1}^{\infty} A_i) = \sum_{i=1}^{\infty} P(A_i)$ if the Borel measurable sets A_i are pairwise disjoint and then the random variables $P(A_i)$, $i \geq 1$, are independent, and

$$\mathbb{E}\bigl[\mathrm{e}^{qP(A)}\bigr] = \mathrm{e}^{\psi(q)\mu(A)}, \tag{3.1}$$

with $\mu(A) = \int_A t^{-2} \, \mathrm{d}s \, \mathrm{d}t$ and

$$A_l(u) = \bigl\{(s,t), u - (t/2 \wedge T/2) < s < u + (t/2 \wedge T/2), t > l\bigr\}.$$

It is readily checked that $\mu(A_l(t)) = T + \log(T/l)$, which implies, with (3.1), that

$$\mathbb{E}\bigl[\mathrm{e}^{q w_l(t)}\bigr] = \mathrm{e}^{(T+\log T)\psi(q)} l^{-\psi(q)}. \tag{3.2}$$

The function ψ is the log-Laplace transform of the infinitely divisible random measure P, assumed to exist for $q < q^*$, for some $q^* > 1$. It is convex and satisfies $\psi(0) = \psi(1) = 0$. By the Lévy–Khinchine representation theorem, it can be expressed as

$$\psi(q) = \frac{\sigma^2}{2} + mq + \int_{-\infty}^{\infty} \bigl\{\mathrm{e}^{qx} - 1 - x\mathbf{1}_{\{|x| \leq 1\}}\bigr\} \nu(\mathrm{d}x),$$

where ν is the Lévy measure of P and satisfies

$$\int_{-\infty}^{\infty} \bigl(x^2 \wedge 1\bigr) \nu(\mathrm{d}x) < \infty.$$

The assumption that $\psi(q)$ is finite for $q < q^*$ entails the following condition. For all $q < q^*$,

$$\int_1^\infty \mathrm{e}^{qx} \nu(\mathrm{d}x) < \infty.$$

By Bacry and Muzy [5], Theorem 4, there exists a certain infinitely divisible random variable Ω_λ, which is independent of $M([0,T])$, such that $\mathbb{E}[\mathrm{e}^{q\Omega_\lambda}] = \lambda^{-\psi(q)}$ and for $\lambda, l \in (0,1)$,

$$\bigl\{w_{\lambda l}(\lambda u), 0 \le u \le T\bigr\} \overset{\text{law}}{=} \bigl\{w_l(u) + \Omega_\lambda, 0 \le u \le T\bigr\}. \tag{3.3}$$

The latter is known as the scaling property. This implies that

$$M\bigl([0,\lambda T]\bigr) \overset{d}{=} \lambda \mathrm{e}^{\Omega_\lambda} M\bigl([0,T]\bigr) \tag{3.4}$$

for $\lambda \in [0,1]$, so that

$$\mathbb{E}\bigl[M^q\bigl([0,\lambda T]\bigr)\bigr] = \lambda^{\zeta(q)} m(q) \tag{3.5}$$

with $\zeta(q) = q - \psi(q)$ and $m(q) = \mathbb{E}[M^q([0,T])]$. It is shown in Bacry and Muzy [5], Theorem 3, that if $\zeta(q) > 1$, then $\mathbb{E}[M^q([0,T])] < \infty$. As previously, set $q_{\max}$ to be the greatest value of q such that $\zeta(q) \ge 1$ and for $\chi \ge 0$, define q_χ as

$$q_\chi = \max\bigl\{q\colon\ q\psi'(q) < \psi(q) + 1 + \chi\bigr\}.$$

Assume moreover that χ is such that $q_\chi < q_{\max}$. Then, for all p such that $pq < q_\chi$, it holds that

$$0 < \psi(pq) - p\psi(q) < (p-1)(1+\chi). \tag{3.6}$$

See Section 5 for a proof.

Example 3.1. Consider the Poisson cascade introduced by Barral and Mandelbrot [6]. Let N be a Poisson point process with intensity measure μ on $(-\infty,\infty) \times (0,\infty]$. Let Γ_i, $i \in \mathbb{Z}$ denote the points of N and let $\{W, W_i\}$ be a collection of i.i.d. positive random variables such that $\mathbb{E}[W] = 1$. Define the random measure P by

$$P(A) = \sum \log(W_i) \mathbf{1}_{\{\Gamma_i \in A\}}$$

for all relatively compact Borel sets $A \in (-\infty,\infty) \times (0,\infty]$. Then (3.1) holds with $\psi(q) = \mathbb{E}[W^q] - 1$ and

$$q_{\max} = \max\bigl\{q\colon\ \mathbb{E}\bigl[W^q\bigr] \le q\bigr\}, \qquad q_\chi = \max\bigl\{q\colon\ q\mathbb{E}\bigl[W^q\bigl(\log(W) - 1\bigr)\bigr] \le 1 + \chi\bigr\}.$$

Example 3.2. The random measure P can be a Gaussian random measure. Then $P(A) \sim \mathbf{N}(-\sigma^2\mu(A)/2, \sigma^2\mu(A))$ and $\psi(q) = \sigma^2 q(q-1)/2$ so that we get the same values of $q_{\max}$, q_0 and q_χ as for the multiplicative cascade of the previous section, up to the $\log 2$ term. Note that in this case, $\operatorname{var}(P(A)) = \psi''(0)\mu(A)$ is finite if and only if $\mu(A) < \infty$.

Example 3.3. Let $\alpha \in (0,1)$ and P be a totally skewed to the left α-stable random measure, that is, $\psi(q) = \sigma^\alpha(q - q^\alpha)$. Then $q_{\max} > 1$ if and only if $\sigma^\alpha(1-\alpha) < 1$ and then $q_{\max} = \infty$ and for $\chi \geq 0$, $q_\chi = \sigma^{-1}((1+\chi)/(1-\alpha))^{1/\alpha}$. It is noteworthy that, contrary to the previous case, we have here that $\mathbb{E}[|P(A)|] = \infty$ and $\mathbb{E}[\mathrm{e}^{qP(A)}] = \infty$ for all A such that $\mu(A) > 0$ and for all $q < 0$, though $\mathbb{E}[|P(A)|^p] = c_{p,\alpha}\sigma^p\mu(A)^{p/\alpha}$ if $p < \alpha$ and $\mu(A) < \infty$.

Example 3.4. Let $\alpha \in (1,2)$ and P be a totally skewed to the left α-stable random measure, that is, $\psi(q) = \sigma^\alpha(q^\alpha - q)$. Then $q_{\max} > 1$ if and only if $\sigma^\alpha(\alpha - 1) < 1$ and then $q_{\max} < \infty$. For $\chi \geq 0$, $q_\chi = \sigma^{-1}((1+\chi)/(\alpha-1))^{1/\alpha}$.

Define, as in the previous section, $L = [2^{n\chi}]$, $\Delta_{k,n}^{(j)} = [(j + k2^{-n})T, (j + (k+1)2^{-n})T]$ and

$$S_{L,n}(M,q) = \sum_{j=0}^{L-1} \sum_{k=0}^{2^n-1} M^q\bigl(\Delta_{k,n}^{(j)}\bigr),$$

$$\tilde{\zeta}_M(q) = 1 + \log_2\left(\frac{S_{L,n}(M,q)}{S_{L,n+1}(M,q)}\right).$$

Consistency. For convenience, denote $\tau(q) = \zeta(q) - 1$. We have the following result, whose proof is in Section 5.

Proposition 3.1. *For $q < q_\chi$,*

$$L^{-1}2^{n\tau(q)}S_{L,n}(M,q) \to m(q) \qquad \textit{a.s.}$$

Plugging this into the definition of $\tilde{\zeta}_M(q)$ yields the consistency of $\tilde{\zeta}_M(q)$.

Corollary 3.2. *For $q < q_\chi$,*

$$\tilde{\zeta}_M(q) \to \zeta(q) \qquad \textit{a.s.}$$

Central limit theorem. We next give a central limit theorem for $\tilde{\zeta}_M(q)$ in the mixed asymptotic framework. Define the centered random variables

$$D_{j,k,n,q} := M^q\bigl(\Delta_{k,n}^{(j)}\bigr) - 2^{\tau(q)}\bigl(M^q\bigl(\Delta_{2k,n+1}^{(j)}\bigr) + M^q\bigl(\Delta_{2k+1,n+1}^{(j)}\bigr)\bigr) \tag{3.7}$$

and $D_{j,n,q} = \sum_{k=0}^{2^n-1} D_{j,k,n,q}$. By construction, the variables $D_{j,k,n,q}$ are centered, and stationary and 2-dependent with respect to j. We will start by proving a central limit theorem for $(L\mathbb{E}[D_{0,n,q}^2])^{-1/2}\sum_{j=0}^{L-1} D_{j,n,q}$. Since the random variables $D_{j,n,q}$, $0 \leq j \leq L-1$, are 2-dependent, it suffices to show that for some $p > 1$,

$$\lim_{n\to\infty} \frac{L^{1-p}\mathbb{E}[D_{0,n,q}^{2p}]}{(\mathbb{E}[D_{0,n,q}^2])^p} = 0. \tag{3.8}$$

We will need the order of magnitude of $D_{0,n,q}$. Set

$$d_q = \mathbb{E}\big[M^q\big([0,T] - 2^{\tau(q)}\big\{M^q\big([0,T/2]\big) + M^q\big([T/2,T]\big)\big\}\big)^2\big]$$

and $d_{k,q} = 2^{n\zeta(2q)}\mathbb{E}[D_{0,0,n,q}D_{0,k,n,q}]$. By the scaling property, $\mathbb{E}[D^2_{0,0,n,q}] = 2^{-n\zeta(2q)}d_q$ and $d_{k,q}$ does not depend on n. Then,

$$\mathbb{E}\big[D^2_{0,n,q}\big] = 2^{-n\tau(2q)}d_q + 2\cdot 2^{-n\tau(2q)}\sum_{k=1}^{2^n-1}\big(1-k2^{-n}\big)d_{k,q}.$$

By Lemma A.4, we have $d_{k,q} = \mathrm{O}(k^{-\{\psi(2q)-2\psi(q)+1\}})$. Since $\psi(2q) - 2\psi(q) > 0$, this implies that the series $\sum|d_{k,q}|$ is convergent, so the Cesaro mean above has a finite limit and thus $\lim_{n\to\infty} 2^{n\tau(2q)}\mathbb{E}[D^2_{0,n,q}] = d_q + 2\sum_{k=1}^{\infty} d_{k,q}$. By Lemma A.5 we have $\mathbb{E}[D^4_{0,n,q}] = \mathrm{O}(n2^{-n\tau(4q)} + 2^{-2n\tau(2q)})$. If $4q < q_\chi$, then $\psi(4q) - 2\psi(2q) < 1+\chi$, thus (3.8) holds for $p=2$. The above discussion leads to the following result.

Proposition 3.3. *If $4q < q_\chi$, then there exists a constant Θ_q such that*

$$L^{-1/2}2^{-n\tau(2q)/2}\sum_{j=0}^{L-1} D_{j,n,q} \to_d \mathbf{N}(0,\Theta_q).$$

We can now prove the asymptotic normality of $\tilde{\zeta}_M(q)$. Denote

$$\begin{aligned} R_n &= \frac{S_{L,n}(M,q) - 2^{\tau(q)}S_{L,n+1}(M,q)}{S_{L,n}(M,q)} \\ &= 2^{n\{2\psi(q)-\psi(2q)-2\psi(q)-1-\chi\}/2}\frac{L^{-1/2}2^{-n\tau(2q)/2}\sum_{j=0}^{L-1} D_{j,n,q}}{L^{-1}2^{-n\tau(q)}S_{L,n}(M,q)}. \end{aligned}$$

By (3.6) applied with $p=2$ and $2q < q_\chi$, it holds that $1+\chi+2\psi(q)-\psi(2) > 0$. Thus, by Propositions 3.1 and 3.3, we have that $R_n = \mathrm{o}(1)$ a.s., so a second order Taylor expansion yields

$$\tilde{\zeta}_M(q) - \zeta(q) = \log_2\bigg(\frac{S_{L,n}(M,q)}{2^{\tau(q)}S_{L,n+1}(M,q)}\bigg) = -\log(1-R_n) = R_n + \mathrm{O}_P\big(R_n^2\big).$$

Applying Propositions 3.1 and 3.3 yields the next result.

Theorem 3.4. *If $4q < q_\chi$, then*

$$2^{n(1+\chi-\psi(2q)+2\psi(q))/2}\big(\tilde{\zeta}_M(q) - \zeta(q)\big) \to \mathbf{N}\big(0, m^{-1}(q)\Theta_q\big).$$

For $q,q' < 4q_\chi$, it can be shown that $2^{n(1+\chi)}(2^{n\{2\psi(q)-\psi(2q)\}/2}(\tilde{\zeta}_M(q) - \zeta(q)), 2^{n\{2\psi(q')-\psi(2q')\}/2}(\tilde{\zeta}_M(q') - \zeta(q')))$ converges to a bivariate Gaussian distribution with dependent components. The same comment holds for the results of the next section.

4. Multifractal random walk

Throughout this section, the MRM M and the process $\{w_l(u)\}$ will be as defined in the previous section. A multifractal random walk (MRW) is the process X obtained as the L^2 limit as $l \to 0$ of the integral $\int_0^t \mathrm{e}^{w_l(u)}\,\mathrm{d}B_H(u)$ where B_H is a standard fractional Brownian motion independent of M; see Abry *et al.* [1], Bacry, Delour and Muzy [3], Bacry and Muzy [5], Ludeña [10]. Recall that B_H is a continuous centered Gaussian process with $B_H(0) = 0$ and

$$\operatorname{var}\bigl(B_H(t) - B_H(s)\bigr) = |t - s|^{2H}$$

for all $t, s \in [0, 1]$. For $H = 1/2$, $B_{1/2}$ is the standard Brownian motion and will be simply denoted by B. Thus, X is the conditionally (with respect to M) Gaussian process whose covariance function is defined in (4.1) or (4.2) below according to whether the Hurst parameter of the fBm is $H = 1/2$ or $H > 1/2$. Except for the case $H = 1/2$, which is ordinary Brownian motion, it is worthwhile to remark that this conditionally Gaussian process X is not the time changed process $B_H(M[0, t])$.

Throughout this section $\to_M$ will stand for conditional convergence in distribution given M and $\mathbb{E}_M$ and var_M stand for the conditional expectation and variance given M. We consider the following two cases.

- Case $H = 1/2$ Bacry, Delour and Muzy [3], Bacry and Muzy [5]. The MRW X is defined as the centered, conditionally Gaussian process with conditional covariance

$$\Gamma(s, t) = \lim_{l \to 0+} \int_0^{t \wedge s} \mathrm{e}^{w_l(u)}\,\mathrm{d}u = M(s \wedge t). \tag{4.1}$$

  The scaling function is $\zeta_{1/2}(q) = \zeta(q/2)$, since by (3.4) and (3.5), for $\lambda \in (0, 1)$,

$$\begin{aligned}\bigl\{X(\lambda t), 0 \le t \le T\bigr\} &\overset{\text{law}}{=} \lambda^{1/2}\mathrm{e}^{\Omega_\lambda/2}\bigl\{X(t), 0 \le t \le T\bigr\},\\ \mathbb{E}\bigl[\bigl|X(t)\bigr|^q\bigr] &= \mathbb{E}\bigl[\mathbb{E}_M\bigl[\bigl|X(t)\bigr|^q\bigr]\bigr] = c_q\mathbb{E}\bigl[M^{q/2}(t)\bigr] = c_q m(q/2) t^{\zeta(q/2)},\end{aligned}$$

  where $c_q = \mathbb{E}[|\mathbf{N}(0, 1)|^q]$ and $m(q) = \mathbb{E}[M^q([0, 1])]$.
- Case $H > 1/2$ Abry *et al.* [1], Ludeña [10], Muzy and Bacry [12]. The MRW X is defined as the centered, conditionally Gaussian process with conditional covariance

$$\Gamma_H(s, t) = \lim_{l \to 0+} \int_0^t \int_0^s \frac{\mathrm{e}^{w_l(u)}\mathrm{e}^{w_l(v)}}{|u - v|^{2-2H}}\,\mathrm{d}u\,\mathrm{d}v = \int_0^t \int_0^s \frac{M(\mathrm{d}u)M(\mathrm{d}v)}{|u - v|^{2-2H}}. \tag{4.2}$$

  This process is well defined whenever $H - \psi(2)/2 > 1/2$, cf. Ludeña [10]. Convexity of ψ yields $\psi(2) > 0$. The scaling function ζ_H is defined by

$$\zeta_H(q) = qH - \psi(q),$$

  since by (4.2) and (3.4) we have

$$\begin{aligned}\bigl\{X(\lambda t), 0 \le t \le T\bigr\} &\overset{\text{law}}{=} \lambda^H \mathrm{e}^{\Omega_\lambda}\bigl\{X(t), 0 \le t \le T\bigr\},\\ \mathbb{E}\bigl[\bigl|X(t)\bigr|^q\bigr] &= c_q m_H(q)(t/T)^{qH - \psi(q)},\end{aligned}$$

with

$$m_H(q) = \mathbb{E}\left[\left\{\int_0^T \int_0^T |u-v|^{2H-2} M(\mathrm{d}u) M(\mathrm{d}v)\right\}^{q/2}\right]. \tag{4.3}$$

Since we are considering the mixed asymptotic framework, we assume we have a collection of MRM $M^{(j)}$, $j = 0, \dots, L-1$, which are independent, defined over consecutive intervals of length T. For $j = 0, \dots, L-1$ and $k = 0, \dots, 2^{n-1}$, define $\Delta X_{j,k,n} = X_{(j+(k+1)2^{-n})T} - X_{(j+k2^{-n})T}$. As above, we will investigate the asymptotic properties of the estimator $\tilde{\zeta}_X(q)$ defined by

$$\tilde{\zeta}_X(q) = \log_2\left(\frac{S_{L,n}(X,q)}{S_{L,n+1}(X,q)}\right) + 1,$$

where now

$$S_{L,n}(X,q) = \sum_{j=0}^{L-1} \sum_{k=0}^{2^n-1} |\Delta X_{j,k,n}|^q.$$

Denote $\tau_H = \zeta_H(q) - 1$ and $T_n(X,q) = S_{L,n}(X,q) - 2^{\tau_H(q)} S_{L,n+1}(X,q)$. Then

$$\tilde{\zeta}_X(q) - \zeta_H(q) = -\log\left(1 - \frac{T_n(X,q)}{S_{L,n(X,q)}}\right).$$

We will prove that $T_n(X,q)/S_{L,n}(X,q) \to 0$ a.s. so that a Taylor expansion is valid and yields

$$\tilde{\zeta}_X(q) - \zeta_H(q) = \frac{T_n(X,q)}{S_{L,n(X,q)}}\bigl(1 + \mathrm{o}(1)\bigr).$$

In order to study the ratio above, we will first prove that if $H = 1/2$, then

$$L^{-1} 2^{n\tau(q/2)} S_{L,n}(X,q) \to c_q m(q/2),$$

and if $H > 1/2$ then,

$$L^{-1} 2^{n\tau_H(q)} S_{L,n}(X,q) \to c_q m_H(q),$$

with $m_H(q)$ as in (4.3) and $c_q = \mathbb{E}[|\mathbf{N}(0,1)|^q]$ in both cases. To study $T_n(X,q)$, we write

$$T_n(X,q) = T_n(X,q) - \mathbb{E}_M\bigl[T_n(X,q)\bigr] + \mathbb{E}_M\bigl[T_n(X,q)\bigr].$$

We will prove that in both cases, $T_n(X,q) - \mathbb{E}_M[T_n(X,q)]$ and $\mathbb{E}_M[T_n(X,q)]$ converge jointly to independent centered Gaussian distributions with the same normalization. This will yield the asymptotic normality of $\tilde{\zeta}_X(q) - \zeta_H(q)$. Because of the different nature of the conditional dependence structure, which yields different scaling functions, we will consider the cases $H = 1/2$ and $H > 1/2$ separately.

4.1. The case $H = 1/2$

In this case, it holds that

$$\mathbb{E}_M\big[S_{L,n}(X,q)\big] = c_q S_{L,n}(M,q/2),$$
$$\operatorname{var}_M\big(S_{L,n}(X,q)\big) = \sigma_q^2 S_{L,n}(M,q),$$

where $\sigma_q^2 = \operatorname{var}(|\mathbf{N}(0,1)|^q)$. By Proposition 3.1, if $q < q_\chi$, we get

$$L^{-1}2^{n\tau(q/2)}\mathbb{E}_M\big[S_{L,n}(X,q)\big] \to c_q m(q/2) \qquad \text{a.s.},$$
$$L^{-1}2^{n\tau(q)}\operatorname{var}_M\big(S_{L,n}(X,q)\big) \to \sigma_q^2 m(q) \qquad \text{a.s.}$$

This implies that $L^{-1}2^{n\tau(q/2)}S_{L,n}(X,q)$ converges in probability to $c_q m(q/2)$. Since $S_{L,n}(X,q)$ is the sum of $L2^n$ conditionally independent terms, by an application of Borel–Cantelli's lemma similar to the one used in the proof of Proposition 3.1, almost sure convergence also holds, that is,

$$L^{-1}2^{n\tau(q/2)}S_{L,n}(X,q) \to c_q m(q/2) \qquad \text{a.s.} \tag{4.4}$$

Using the notation (3.7) of the previous section, we have

$$\mathbb{E}_M\big[T_n(X,q)\big] = c_q 2^{\tau(q/2)}S_{L,n+1}(M,q/2) - c_q S_{L,n}(M,q/2) = -c_q \sum_{j=0}^{L-1}\sum_{k=0}^{2^n-1} D_{j,k,n,q}.$$

Thus, by Proposition 3.3, if $q < q_\chi$ then $L^{-1/2}2^{n\tau(q)/2}\mathbb{E}_M[T_n(X,q)]$ converges to a centered Gaussian random variable with variance $\Sigma(1/2,q)$, say. By the conditional independence of B and M, $T_n(X,q) - \mathbb{E}_M[T_n(X,q)]$ is a sum of centered and conditionally independent random variables with conditional variance

$$\operatorname{var}_M\big(T_n(X,q)\big) = \sigma_q^2 S_{L,n}(M,q) + \sigma_q^2\big(2^{2\tau(q/2)} - 2^{\tau(q/2)+1}\big)S_{L,n+1}(M,q).$$

By Proposition 3.1, $L^{-1}2^{n\tau(q)}\operatorname{var}_M(T_n(X,q))$ converges almost surely to the positive constant $\Gamma(1/2,q)$ defined by

$$\Gamma(1/2,q) = \sigma_q^2 m(q)\big\{1 + \big(2^{2\tau(q/2)} - 2^{\tau(q/2)+1}\big)2^{-\tau(q)}\big\}.$$

Thus,

$$L^{-1/2}2^{n\tau(q)/2}\big\{T_n(X,q) - \mathbb{E}_M\big[T_n(X,q)\big]\big\} \to_M \mathbf{N}\big(0,\Gamma(1/2,q)\big). \tag{4.5}$$

Since the variance is deterministic, this assures unconditional convergence to the stated Gaussian random variable. Moreover, the conditional independence of B and M also implies that the sequence of random vectors

$$L^{-1/2}2^{n\tau(q)/2}\big(T_n(X,q) - \mathbb{E}_M\big[T_n(X,q)\big], \mathbb{E}_M\big[T_n(X,q)\big]\big)$$

converges weakly to (Z_1, Z_2) where Z_1 and Z_2 are independent Gaussian random variables with zero mean and variance $\Gamma(1/2, q)$ and $\Sigma(1/2, q)$, respectively. The previous considerations yield the central limit theorem for $\tilde{\zeta}_X(q)$.

Theorem 4.1. *If $q < q_\chi$, then*

$$L^{1/2} 2^{n(\psi(q/2)-\psi(q)/2+1/2)} \{\tilde{\zeta}_X(q) - \zeta_{1/2}(q)\} \to_d \mathbf{N}\left(0, \frac{\Gamma(1/2, q) + \Sigma(1/2, q)}{c_q^2 m^2(q/2)}\right).$$

4.2. Case $H > 1/2$

We begin by studying $\mathbb{E}_M[T_n(X, q)]$. Define $a_{j,k,n,H} = \mathbb{E}_M^{1/2}[(\Delta X_{j,k,n})^2]$. Then

$$\mathbb{E}_M\big[T_n(X, q)\big] = c_q \sum_{j=0}^{L-1} \sum_{k=0}^{2^n-1} \big(2^{\tau_H(q)} \{a_{j,2k,n+1,H}^q + a_{j,2k+1,n+1,H}^q\} - a_{j,k,n,H}^q\big).$$

Denote $U_{j,k,n} = 2^{\tau_H(q)} \{a_{j,2k,n+1,H}^q + a_{j,2k+1,n+1,H}^q\} - a_{j,k,n,H}^q$ and define $U_{j,n} := \sum_{k=0}^{2^n-1} U_{j,k,n}$. Then the collection $\{U_{j,n}\}_{0 \le j \le L-1}$ is centered, 2-dependent and identically distributed. Remark that $\vartheta(q) = 2^{n\zeta_H(2q)} \operatorname{var}(U_{j,k,n})$ depends only on q. By stationarity, for $j = 0, \dots, L-1$,

$$\operatorname{var}(U_{j,n}) = 2^{-n\tau_H(2q)} v_q + 2 2^{-n\tau_H(q)} \sum_{k=1}^{2^n-1} \big(2^n - k\big) 2^{n\zeta_H(2q)} \operatorname{cov}(U_{0,0,n}, U_{0,k,n}).$$

By Lemma A.7, $2^{n\zeta_H(2q)} |\operatorname{cov}(U_{0,n,0}, U_{0,n,k})| \le C k^{-\{\psi(2q)-2\psi(q)+1\}}$. This series is convergent, thus the Cesaro mean above converges to its sum. Arguing as in the proof of Proposition 3.3, in order to prove the central limit theorem for $\mathbb{E}_M[T_n(X, q)]$, since the centered random variables $U_{j,n}$, $0 \le j \le L-1$, are 2-dependent, it suffices to show that

$$\lim_{n\to\infty} \frac{L^{1-p} \mathbb{E}[U_{0,k,n}^4]}{(\mathbb{E}[U_{0,k,n}^2])^2} = 0.$$

This is done as in Lemma A.5 using Lemma A.7. We then have the following result.

Proposition 4.2. *If $2q < q_\chi$, there exists a positive constant $\Sigma(H, q)$ such that*

$$L^{-1} 2^{-n\tau_H(2q)} \operatorname{var}\big(\mathbb{E}_M\big[T_n(X, q)\big]\big) \to \Sigma(H, q).$$

Moreover, if $4q < q_\chi$, then

$$L^{-1/2} 2^{-n\tau_H(2q)/2} \mathbb{E}_M\big[T_n(X, q)\big] \to_M \mathbf{N}\big(0, \Sigma(H, q)\big). \tag{4.6}$$

We next need a result which parallels (4.5). Its proof is more involved and is postponed to Section 5.

Proposition 4.3. *Let $H < 3/4$. If $2q < q_\chi$, then there exists a positive constant $\Gamma(H,q)$ such that*

$$L^{-1} 2^{n\tau_H(2q)} \operatorname{var}_M\big(T_{L,n}(X,q)\big) \to \Gamma(H,q) \qquad a.s. \tag{4.7}$$

and if $4q < q_\chi$, then

$$L^{-1/2} 2^{n\tau_H(2q)/2} \big\{T_n(X,q) - \mathbb{E}_M\big[T_n(X,q)\big]\big\} \to_M \mathbf{N}\big(0, \Gamma(H,q)\big). \tag{4.8}$$

As for the case $H = 1/2$, the fact that $\Gamma(H,q)$ is deterministic establishes unconditional convergence in distribution. The proof of (4.8) is based on the recent results of Nualart and Peccati [13] on the convergence of sequences of random variables in a Gaussian chaos. Altogether, (4.6) and (4.8) yield the asymptotic normality of the estimator.

Theorem 4.4. *If $4q < q_\chi$ and $H < 3/4$, then*

$$2^{n(1+\chi-\psi(2q)+2\psi(q))/2} \big\{\tilde{\zeta}_X(q) - \zeta_H(q)\big\} \to_d \mathbf{N}\left(0, \frac{\Gamma(H,q) + \Sigma(H,q)}{c_q^2 m_H^2(q)}\right).$$

5. Proofs

In all the proofs, without loss of generality, we set $T = 1$. We start by proving (3.6). The convexity of ψ and $\psi(1) = 0$ implies that $q_{\max} > 1$ if and only if $\psi'(1) < 1$, and $\psi'(q_{\max}) > 1$. This in turn implies that $1 < q_0 < q_{\max}$. The convexity of ψ also implies that the function $q \mapsto q\psi'(q) - \psi(q)$ is increasing, thus $q_\chi > q_0$ for all $\chi > 0$. Consider the positive and increasing function $p \mapsto \psi(pq) - p\psi(q)$. By convexity, for $p > 1$, $\psi(pq) - \psi(q) \le \psi'(pq)(pq - p)$. This yields, for $p > 1$ and $pq < q_\chi$,

$$\begin{aligned} 0 < \psi(pq) - p\psi(q) &= p\psi(pq) - p\psi(q) - (p-1)\psi(pq) \\ &\le (p-1)\big\{pq\psi'(pq) - \psi(pq)\big\} < (p-1)(1+\chi). \end{aligned}$$

This proves (3.6).

We will also repeatedly use an argument of m-dependence. If $\xi_1, \ldots, \xi_N$ are m-dependent random variables with zero mean and finite stationary pth moment, $1 \le p \le 2$, then there exists a constant C which depends only on p such that

$$\mathbb{E}\left[\left|\sum_{i=1}^N \xi_i\right|^p\right] \le C m^{p-1} N \mathbb{E}\big[|\xi_1|^p\big]. \tag{5.1}$$

5.1. Proof of Proposition 3.1

Let $n_0 \ge 2$ be an integer, $\alpha = 1/n_0$ and $l_n = 2^{-(1-\alpha)n}$. Fix $q < q_\chi$. We can choose $\alpha < \chi$ small enough so that $q < q_{\chi'}$ with $\chi' < \chi - \alpha$. Then, we can also choose $p > 1$, close enough to 1, such that $pq < q_{\chi'}$ and without loss of generality, we can also impose that $p - 1 < \alpha(q \vee 1)/2$. Define

$$\begin{aligned}\tilde{T}_{n,q} &= 2^{-n} L^{-1} \sum_{j=0}^{L-1} \sum_{k=0}^{2^n-1} \frac{\mathrm{e}^{q w_{l_n}(j+2^{-n}k)}}{\mathbb{E}[\mathrm{e}^{q w_{l_n}(0)}]} \\ &= 2^{-n} L^{-1} \mathrm{e}^{-\psi(q)} l_n^{\psi(q)} \sum_{j=0}^{L-1} \sum_{k=0}^{2^n-1} \mathrm{e}^{q w_{l_n}(j+2^{-n}k)}.\end{aligned} \tag{5.2}$$

We will prove that for α and $p > 1$ chosen as above, there exist constants $C, \eta > 0$ such that

$$\mathbb{E}\big[|\tilde{T}_{n,q} - 1|^p\big] \le C 2^{-n\eta}, \tag{5.3}$$

$$\mathbb{E}\left[\left|\tilde{T}_{n,q} - \frac{S_{L,n}(M,q)}{\mathbb{E}[S_{L,n}(M,q)]}\right|^p\right] \le C 2^{-n\eta}. \tag{5.4}$$

The above inequalities and an application of Borel–Cantelli's lemma yield that $\tilde{T}_{n,q} \to 1$, a.s. and

$$\frac{S_{L,n}(M,q)}{\mathbb{E}[S_{L,n}(M,q)]} - \tilde{T}_{n,q} \to 0 \qquad \text{a.s.}$$

For all j, k, n, we have $\mathbb{E}[M^q(\Delta_{k,n}^{(j)})] = 2^{-n\zeta(q)} m(q)$, so that $\mathbb{E}[S_{L,n}(M,q)] = L 2^{-n\tau(q)} m(q)$. Thus, Proposition 3.1 follows.

Proof of (5.3). Define $\varepsilon = p - 1$. The variables $\mathrm{e}^{q w_{l_n}(j+2^{-n}k)} - \mathbb{E}[\mathrm{e}^{q w_{l_n}(j+2^{-n}k)}]$ are 2-dependent (in j) and centered, so there exists a constant $C > 0$ such that

$$\mathbb{E}\big[|\tilde{T}_{n,q} - 1|^p\big] \le \frac{C}{L^\varepsilon} \mathbb{E}\left[\left|\frac{1}{2^n} \sum_{k=0}^{2^n-1} \frac{\mathrm{e}^{q w_{l_n}(2^{-n}k)}}{\mathrm{e}^{\psi(q)} l_n^{-\psi(q)}} - 1\right|^p\right].$$

By Lemma A.1, for any $\varepsilon' < \varepsilon$, there exists a constant C such that

$$\mathbb{E}\big[|\tilde{T}_{n,q} - 1|^p\big] \le C 2^{n\{(1-\alpha)\{\psi(pq) - p\psi(q) - \varepsilon'\} - \varepsilon\chi\}}.$$

By (3.6), since $pq < q_{\chi'} < q_\chi$, we have

$$\begin{aligned}(1-\alpha)\big\{\psi(pq) - p\psi(q) - \varepsilon'\big\} - \varepsilon\chi &< (1-\alpha)\big\{\varepsilon\big(1+\chi'\big) - \varepsilon'\big\} - \varepsilon\chi \\ &< (1-\alpha)\big\{\varepsilon(1+\chi-\alpha) - \varepsilon'\big\} - \varepsilon\chi \\ &= (1-\alpha)\big\{\varepsilon(1-\alpha) - \varepsilon'\big\} - \alpha\varepsilon\chi.\end{aligned}$$

This can be made negative by choosing $\varepsilon' > (1-\alpha)\varepsilon$. □

Proof of (5.4). We start by using again the argument of 2-dependence in j, to obtain, for some constant C,

$$\mathbb{E}\left[\left|\frac{S_{L,n}(M,q)}{\mathbb{E}[S_{L,n}(M,q)]} - \tilde{T}_{n,q}\right|^p\right] \le \frac{C}{L^\varepsilon}\mathbb{E}\left[\left|\frac{1}{2^n}\sum_{k=0}^{2^n-1}\frac{M^q(\Delta_{k,n}^{(0)})}{2^{-n\zeta(q)}m(q)} - \frac{\mathrm{e}^{qw_{l_n}(k2^{-n})}}{\mathrm{e}^{\psi(q)}l_n^{-\psi(q)}}\right|^p\right]. \tag{5.5}$$

For clarity, we now omit the superscript (0) in $\Delta_{k,n}^{(0)}$. Let M_n denote the random measure with density $\mathrm{e}^{-w_{l_n}}$ with respect to M. By construction, the measure M_n is independent of the process w_{l_n}. Indeed, for any Borel set A, $M_n(A) = \lim_{l\to 0}\int_A \mathrm{e}^{w_l(u)-w_{l_n}(u)}\,\mathrm{d}u$, and for $l < l_n$, $w_l - w_{l_n}$ is independent of w_{l_n}, by the independent increment property of the random measure P. Denote

$$\tilde{S}_n = 2^{n\tau(q)}\sum_{k=0}^{2^n-1}\mathrm{e}^{qw_{l_n}(k2^{-n})}M_n(\Delta_{k,n}).$$

Applying the bound (A.15) in Lemma A.2, we obtain

$$\mathbb{E}\left[\left|\tilde{S}_n - 2^{n\tau(q)}\sum_{k=0}^{2^n-1}M(\Delta_{k,n})\right|^p\right] \le C2^{-n\alpha(q\vee 1)/2}2^{n\{\psi(pq)-p\psi(q)\}}.$$

Since we have chosen $\varepsilon < \alpha(q\vee 1)/2$, by (3.6), we have

$$\psi(pq) - p\psi(q) - \alpha(q\vee 1)/2 - \varepsilon\chi < \varepsilon - \alpha(q\vee 1)/2 < 0.$$

Define $m_n(q) = \mathrm{e}^{\psi(q)}l_n^{-\psi(q)}2^{n\zeta(q)}\mathbb{E}[M_n^q(\Delta_{k,n})]$. By (A.14), we have $\lim_{n\to\infty} m_n(q) = m(q)$ and thus $\mathbb{E}[M_n^q(\Delta_{0,n})] \sim l_n^{\psi(q)}2^{-n\zeta(q)}\mathrm{e}^{-\psi(q)}m(q)$. Next, we note that the random variables $M_n(\Delta_{k,n})$ are $2^n l_n$-dependent and $\mathrm{e}^{w_{l_n}}$ is independent of M_n. Thus, applying (5.1) conditionally on w_{l_n} yields

$$\begin{aligned}
\mathbb{E}\left[\left|\frac{\tilde{S}_n}{m_n(q)} - 2^{-n}\sum_{k=0}^{2^n-1}\frac{\mathrm{e}^{qw_{l_n}(k2^{-n})}}{\mathrm{e}^{\psi(q)}l_n^{-\psi(q)}}\right|^p\right] &= \mathbb{E}\left[\left|2^{-n}\sum_{k=0}^{2^n-1}\frac{\mathrm{e}^{qw_{l_n}(k2^{-n})}}{\mathrm{e}^{\psi(q)}l_n^{-\psi(q)}}\left(\frac{M_n^q(\Delta_{k,n})}{\mathbb{E}[M_n^q(\Delta_{k,n})]} - 1\right)\right|^p\right] \\
&\le Cl_n^{-\psi(pq)+p\psi(q)-\varepsilon}\mathbb{E}\left[\left|\frac{M_n^q(\Delta_{0,n})}{\mathbb{E}[M_n^q(\Delta_{0,n})]}\right|^p\right] \\
&\le C2^{n\{\psi(pq)-p\psi(q)-\varepsilon(1-\alpha)\}}.
\end{aligned}$$

Using the fact that $pq < q_{\chi'}$, (3.6) and $\chi' < \chi - \alpha$, we obtain

$$\psi(pq) - p\psi(q) - (1-\alpha)\varepsilon - \varepsilon\chi \le \varepsilon\big(1+\chi'\big) - (1-\alpha)\varepsilon - \varepsilon\chi = \varepsilon\big(\chi' + \alpha - \chi\big) < 0.$$

This concludes the proof of (5.4). □

5.2. Proof of Proposition 4.3

Define $a_{j,k,n,H} = \mathbb{E}_M^{1/2}[(\Delta X_{j,k,n})^2]$ and the conditionally standard Gaussian random variables

$$Y_{j,k,n} = \Delta X_{j,k,n}/a_{j,k,n,H}.$$

Let $G_q(x) = |x|^q - c_q$. With this notation, we have

$$S_{L,n}(X,q) - \mathbb{E}_M\big[S_{L,n}(X,q)\big] = \sum_{j=0}^{L-1}\sum_{k=0}^{2^n-1} a^q_{j,k,n,H} G_q(Y_{j,k,n}).$$

Let $g_r(q)$, $r \geq 0$, be the coefficients of the expansion of G_q over the Hermite polynomials $\{H_r, r \geq 0\}$ (which are defined in such a way that $\mathbb{E}[H_k(X)H_l(X)] = k!$ if $k = l$ and 0 otherwise), that is, $g_r(q) = \mathbb{E}[H_r(V)G_q(V)]$ where V is a standard Gaussian random variable. Since G_q is a centered even function, $g_r(q) = 0$ for $r = 0, 1$. Since $\mathbb{E}[G_q^2(X)] < \infty$, the series $\sum_{r=2} g_r^2(q)/r!$ is summable and $G_q = \sum_{r=2}^{\infty} \frac{g_r(q)}{r!} H_r$. Then, by Mehler's formula (see, e.g., Arcones [2]), we have

$$L^{-1} 2^{n\tau_H(2q)} \operatorname{var}_M\big(S_{L,n}(X,q)\big) = \sum_{r=2}^{\infty} \frac{g_r(q)^2}{r!} \Gamma_n(r,q),$$

with

$$\begin{aligned}\Gamma_n(r,q) &= L^{-1} 2^{n\tau_H(q)} (r!)^{-1} \operatorname{var}_M\left(\sum_{j=0}^{L-1}\sum_{k=0}^{2^n-1} a^q_{j,k,n,H} H_r(Y_{j,k,n})\right) \\ &= L^{-1} 2^{n\tau_H(q)} \sum_{j_1,j_2=0}^{L-1} \sum_{k,k'=0}^{2^n-1} \rho^r_{H,n}\big(j_1,j_2,k,k'\big) a^q_{j_1,k,n,H} a^q_{j_2,k',n,H}\end{aligned}$$

for $r \in \mathbb{N}$, $r \geq 2$, and the conditional correlations (which are zero if $H = 1/2$) are

$$\rho_{H,n}\big(j_1,j_2,k,k'\big) = \operatorname{cov}_M(Y_{j_1,k,n}, Y_{j_2,k',n}) = \frac{\mathbb{E}_M[\Delta X_{j_1,k,n} \Delta X_{j_2,k',n}]}{a_{j_1,k,n,H} a_{j_2,k',n,H}}.$$

By Lemma 3.1 in Ludeña [10], for $j_1 < j_2$ and $k < k'$, we have the bound

$$\rho_{H,n}\big(j_1,j_2,k,k'\big) \leq \min\Big(1, C\big|(j_2 - j_1)2^n + \big(k' - k\big)\big|^{2H-2}\Big) \tag{5.6}$$

for some deterministic constant C. We start by proving that for $H < 3/4$ and $2q < q_\chi$, there exists a constant $\Gamma(r,q)$ such that

$$\lim_{n\to\infty} 2^{n(2\psi(q)-\psi(2q)+1+\chi)} \mathbb{E}\big[\Gamma_n(r,q)\big] = \Gamma(r,q). \tag{5.7}$$

By the scaling property,

$$\mathbb{E}\big[a^q_{j,k,n,H}\big] = 2^{-n\zeta_H(q)} m_H(q),$$

with $\zeta_H(q) = qH - \psi(q)$. Thus, denoting $v_\chi(q) = 2\psi(q) - \psi(2q) + 1 + \chi$, by stationarity, we have

$$\begin{aligned}
&2^{nv_\chi(q)}\mathbb{E}\big[\Gamma_n(r,q)\big] \\
&\quad = m_H(2q) + 2^{-n}2^{n\zeta_H(2q)}\sum_{k\neq k'}\mathbb{E}\big[\rho^r_{H,n}\big(0,0,k,k'\big)a^q_{0,k,n,H}a^q_{0,k',n,H}\big] \qquad (5.8)\\
&\qquad + 2^{-n(1+\chi)}2^{n\zeta_H(2q)}\sum_{j\neq j'}\sum_{k,k'}\mathbb{E}\big[\rho^r_{H,n}\big(j,j',k,k'\big)a^q_{j,k,n,H}a^q_{j',k',n,H}\big].
\end{aligned}$$

Consider the middle term. Recall that

$$\begin{aligned}
&\rho^r_{n,H}\big(0,0,k,k'\big)a^q_{0,k,n,H}a^q_{0,k',n,H}\\
&\quad = \left\{\int_{k2^{-n}}^{(k+1)2^{-n}}\int_{k'2^{-n}}^{(k'+1)2^{-n}}|u-v|^{2H-2}M(\mathrm{d}u)M(\mathrm{d}v)\right\}^r\\
&\qquad \times\left\{\int_{k2^{-n}}^{(k+1)2^{-n}}\int_{k2^{-n}}^{(k+1)2^{-n}}|u-v|^{2H-2}M(\mathrm{d}u)M(\mathrm{d}v)\right\}^{(q-r)/2}\\
&\qquad \times\left\{\int_{k'2^{-n}}^{(k'+1)2^{-n}}\int_{k'2^{-n}}^{(k'+1)2^{-n}}|u-v|^{2H-2}M(\mathrm{d}u)M(\mathrm{d}v)\right\}^{(q-r)/2}.
\end{aligned}$$

Assume that $k < k'$ and denote $\ell = k' - k + 1$. By the scaling property and the stationarity of the increments of M, we have

$$\begin{aligned}
&\rho^r_{n,H}\big(0,0,k,k'\big)a^q_{0,k,n,H}a^q_{0,k',n,H}\\
&\quad \overset{\text{(law)}}{=} \big(\ell 2^{-n}\big)^{r(2H-2)+2r}\mathrm{e}^{2r\Omega_{\ell 2^{-n}}}\left\{\int_0^{1/\ell}\int_{1-1/\ell}^{1}|u-v|^{2H-2}M(\mathrm{d}u)M(\mathrm{d}v)\right\}^r\\
&\qquad \times\big(\ell 2^{-n}\big)^{(q-r)(H-1)+q-r}\mathrm{e}^{(q-r)\Omega_{\ell 2^{-n}}}\left\{\int_0^{1/\ell}\int_0^{1/\ell}|u-v|^{2H-2}M(\mathrm{d}u)M(\mathrm{d}v)\right\}^{(q-r)/2}\\
&\qquad \times\big(\ell 2^{-n}\big)^{(q-r)(H-1)+q-r}\mathrm{e}^{(q-r)\Omega_{\ell 2^{-n}}}\left\{\int_{1-1/\ell}^{1}\int_{1-1/\ell}^{1}|u-v|^{2H-2}M(\mathrm{d}u)M(\mathrm{d}v)\right\}^{(q-r)/2}\\
&\quad = \big(\ell 2^{-n}\big)^{2qH}\mathrm{e}^{2q\Omega_{\ell 2^{-n}}}Q^r_\ell a^q_\ell b^q_\ell,
\end{aligned}$$

with

$$\begin{aligned}
a^2_\ell &= \int_0^{1/\ell}\int_0^{1/\ell}|u-v|^{2H-2}M(\mathrm{d}u)M(\mathrm{d}v),\\
b^2_\ell &= \int_{1-1/\ell}^{1}\int_{1-1/\ell}^{1}|u-v|^{2H-2}M(\mathrm{d}u)M(\mathrm{d}v),
\end{aligned}$$

$$Q_\ell = \frac{\int_0^{1/\ell}\int_{1-1/\ell}^1 |u-v|^{2H-2} M(\mathrm{d}u)M(\mathrm{d}v)}{a_\ell b_\ell}.$$

With this notation, the middle term in (5.8) can be expressed as

$$\begin{aligned}
&2\cdot 2^{n\zeta_H(2q)} \sum_{\ell=1}^{2^n-1} \bigl(1-\ell 2^{-n}\bigr)\bigl(\ell 2^{-n}\bigr)^{2qH} \mathbb{E}\bigl[\mathrm{e}^{2q\Omega_{\ell 2^{-n}}}\bigr]\mathbb{E}\bigl[Q_\ell^r a_\ell^q b_\ell^q\bigr] \\
&= 2\cdot 2^{n\zeta_H(2q)} 2^{-n\{2qH-\psi(2q)\}} \sum_{\ell=1}^{2^{n-1}} \bigl(1-\ell 2^{-n}\bigr)\ell^{2qH-\psi(2q)} \mathbb{E}\bigl[Q_\ell^r a_\ell^q b_\ell^q\bigr] \\
&= 2\sum_{\ell=1}^{2^{n-1}} \bigl(1-\ell 2^{-n}\bigr)\ell^{\zeta_H(2q)} \mathbb{E}\bigl[Q_\ell^r a_\ell^q b_\ell^q\bigr].
\end{aligned}$$

Moreover, $a_\ell \geq \ell^{2-2H} M([0,1/\ell])$, $b_\ell \geq \ell^{2-2H} M([1-1/\ell,1])$, and the numerator in Q_ℓ is bounded from above by $(1-2/\ell)^{2H-2} M([0,1/\ell])M([1-1/\ell,1])$. Thus,

$$Q_\ell \leq C\ell^{2H-2} \tag{5.9}$$

for some deterministic constant C. This and Hölder's inequality yields

$$\mathbb{E}\bigl[Q_\ell^r a_\ell^q b_\ell^q\bigr] \leq C\ell^{r(2H-2)} \mathbb{E}^{1/2}\bigl[a_\ell^{2q}\bigr]\mathbb{E}^{1/2}\bigl[b_\ell^{2q}\bigr].$$

Applying the scaling property of M yields $\mathbb{E}[a_\ell^{2q}] = \mathbb{E}[b_\ell^{2q}] = \ell^{-\zeta_H(2q)} m_H(q)$, hence

$$\ell^{\zeta_H(2q)} \mathbb{E}\bigl[Q_\ell^r a_\ell^q b_\ell^q\bigr] \leq C\ell^{r(2H-2)}.$$

Since $r \geq 2$ and $H < 3/4$, the series $\ell^{r(2H-2)}$ is summable, and thus

$$\lim_{n\to\infty} \sum_{\ell=1}^{2^{n-1}} \bigl(1-\ell 2^{-n}\bigr)\ell^{\zeta_H(2q)} \mathbb{E}\bigl[Q_\ell^r a_\ell^q b_\ell^q\bigr] = \sum_{\ell=1}^{\infty} \ell^{\zeta_H(2q)} \mathbb{E}\bigl[Q_\ell^r a_\ell^q b_\ell^q\bigr].$$

Consider now the last term in (5.8), say RR_n. Using the bound (5.6), the scaling property, the fact that the $a_{j,k,n,H}$ are 2-dependent, and $H < 3/4$, we have

$$RR_n \leq C2^{n\{\zeta_H(2q)-2\zeta_H(q)\}} \sum_{j=1}^{L}\sum_{k=1}^{2^n} \bigl(j2^n+k\bigr)^{2H-2} = \mathrm{O}\bigl(2^{n\{2\psi(q)-\psi(2q)\}}\bigr) = \mathrm{o}(1). \tag{5.10}$$

This proves (5.7). We now prove that if $H < 3/4$, for each $r \geq 2$,

$$\Gamma_n(r,q)/\mathbb{E}\bigl[\Gamma_n(r,q)\bigr] \to 1 \qquad \text{a.s.} \tag{5.11}$$

or equivalently

$$2^{n\{1+\chi-\psi(2q)+2\psi(q)\}}\Gamma_n(r,q)\to\Gamma(r,q)\qquad\text{a.s.}$$

Write $2^{n\{1+\chi-\psi(2q)+2\psi(q)\}}\Gamma_n(r,q)=S_{n,1}+S_{n,2}+S_{n,3}$ with

$$S_{n,1}=2^{n\tau_H(2q)}L^{-1}\sum_{j=0}^{L-1}\sum_{k=0}^{2^n-1}a_{j,k,n,H}^{2q},$$

$$S_{n,2}=2^{n\tau_H(2q)}L^{-1}\sum_{j=0}^{L-1}\sum_{0\le k\ne k'<2^n}\rho_{H,n}^r\big(j,j,k,k'\big)a_{j,k,n,H}^q a_{j,k',n,H}^q,$$

$$S_{n,3}=2^{n\tau_H(2q)}L^{-1}\sum_{0\le j\ne j'<L}\sum_{k,k'=0}^{2^n-1}\rho_{H,n}^r\big(j',j',k,k'\big)a_{j,k,n,H}^q a_{j',k',n,H}^q.$$

The bound (5.10) and Borel–Cantelli's lemma implies that $S_{n,3}\to 0$ a.s. Define $\tilde a_{j,k,n,H}=\mathrm{e}^{w_{l_n}(t_{j,k})}\tilde\delta_{j,k,n,H}$ with

$$\tilde\delta_{j,k,n,H}^2=\int_{\Delta_{k,n}^{(j)}}\int_{\Delta_{k,n}^{(j)}}|u-v|^{2H-2}M_n(\mathrm{d}u)M_n(\mathrm{d}v).$$

By Lemma A.6, we have, if $2q<q_\chi$,

$$\lim_{n\to\infty}2^{n\zeta_H(2q)}\mathrm{e}^{\psi(2q)}l_n^{-\psi(2q)}\mathbb{E}\big[\tilde\delta_{j,k,n,H}^{2q}\big]=m_H(2q).\tag{5.12}$$

By 2-dependence with respect to j, Jensen's inequality, (3.6) applied to $2q<q_\chi$ and the bound (A.28), we obtain, some $\eta>0$,

$$\begin{aligned}\mathbb{E}\Bigg[\Bigg|2^{n\tau_H(2q)}L^{-1}\sum_{j=0}^{L-1}\sum_{k=0}^{2^n-1}\big(a_{j,k,n,H}^{2q}-\tilde a_{j,k,n,H}^{2q}\big)\Bigg|^p\Bigg]&\le CL^{1-p}2^{np\zeta(2q)}\mathbb{E}\big[\big|a_{j,k,n,H}^{2q}-\tilde a_{j,k,n,H}^{2q}\big|^p\big]\\&\le CL^{1-p}2^{n\psi(2pq)-p\psi(2q)-\eta}\le C2^{n(p-1-\eta)}.\end{aligned}$$

Choosing $p-1<\eta$ and Borel–Cantelli's lemma yield that

$$2^{n\tau_H(2q)}L^{-1}\sum_{j=0}^{L-1}\sum_{k=0}^{2^n-1}\big(a_{j,k,n,H}^{2q}-\tilde a_{j,k,n,H}^{2q}\big)\to 0\qquad\text{a.s.}\tag{5.13}$$

Recall the definition of $\tilde T_{n,2q}$ in (5.2) and define further

$$\tilde S_{n,1}=2^{n\tau_H(2q)}L^{-1}\sum_{j=0}^{L-1}\sum_{k=0}^{2^n-1}\tilde a_{j,k,n,H}^{2q},$$

$$m_{n,H}(2q)=2^{n\zeta_H(q)}\mathbb{E}\big[\tilde a_{0,0,n,H}^{2q}\big]=2^{n\zeta_H(q)}\mathrm{e}^{\psi(q)}l_n^{-\psi(q)}\mathbb{E}\big[\tilde\delta_{0,0,n,H}^{2q}\big].$$

We have already shown in the proof of Proposition 3.1 that if $2q < q_\chi$, then $\tilde{T}_{n,2q} \to 1$ a.s. Moreover, by the argument of 2-dependence with respect to j, we have

$$\mathbb{E}\left[\left|\frac{\tilde{S}_{n,1}}{m_{n,H}(2q)} - \tilde{T}_{n,2q}\right|^p\right] \le CL^{1-p}\mathbb{E}\left[\left|2^{-n}\sum_{k=1}^{n}\frac{\mathrm{e}^{2qw_{l_n}(k2^{-n})}}{\mathrm{e}^{\psi(2q)}l_n^{-\psi(2q)}}\left(\frac{\tilde{\delta}_{0,k,n,H}^{2q}}{\mathbb{E}[\tilde{\delta}_{0,k,n,H}^{2q}]} - 1\right)\right|^p\right].$$

As in the proof of Proposition 3.1, we now use the fact that w_{l_n} is independent of the measure M_n, the $2^n l_n$-dependence of the variables $\tilde{\delta}_{0,k,n,H}$ and (5.12) to obtain

$$\begin{aligned}\mathbb{E}\left[\left|2^{-n}\sum_{k=1}^{n}\frac{\mathrm{e}^{2qw_{l_n}(k2^{-n})}}{\mathrm{e}^{\psi(2q)}l_n^{-\psi(2q)}}\left(\frac{\tilde{\delta}_{0,k,n,H}^{2q}}{\mathbb{E}[\tilde{\delta}_{0,k,n,H}^{2q}]} - 1\right)\right|^p\right] &\le Cl_n^{\varepsilon-\psi(2pq)+p\psi(2q)}\mathbb{E}\left[\frac{\tilde{\delta}_{0,0,n,H}^{2pq}}{(\mathbb{E}[\tilde{\delta}_{0,0,n,H}^{2q}])^p}\right] \\ &\le Cl_n^{\varepsilon}2^{n\{\psi(2pq)-p\psi(2q)\}}.\end{aligned}$$

Now, as in the proof of Proposition 3.1, we must choose α small enough so that $2q < q_{\chi'}$, for $\chi' < \chi - \alpha$, and ε such that $2pq < q_{\chi'}$ with $p = 1 + \varepsilon$. Such a choice and (3.6) applied with $2pq < q_{\chi'}$ yield

$$\mathbb{E}\left[\left|\frac{\tilde{S}_{n,1}}{m_{n,H}(2q)} - \tilde{T}_{n,2q}\right|^p\right] \le C2^{-\varepsilon\chi}l_n^{\varepsilon}2^{n\varepsilon(1+\chi')} = C2^{n\varepsilon(\chi'+\alpha-\chi)}.$$

This last bound and Borel–Cantelli's lemma yield that $m_{n,H}^{-1}(2q)\tilde{S}_{n,1} - \tilde{T}_{n,2q} \to 0$, a.s. This and (5.13) finally prove that $\tilde{S}_{n,1} \to m_H(2q)$ a.s.

In order to prove that $S_{n,2} \to 0$ a.s., by stationarity and 2-dependence in j, it is enough to prove that, for $p = 1 + \varepsilon$,

$$\mathbb{E}\left[\left|2^{n\tau_H(2q)}\sum_{0\le k\ne k'<2^n}\rho_{H,n}^r(0,0,k,k')a_{0,k,n,H}^q a_{0,k',n,H}^q\right|^p\right] = \mathrm{O}\big(2^{(\varepsilon\chi-\eta)n}\big) \tag{5.14}$$

for some $\eta > 0$ and apply Borel–Cantelli's lemma. Since all quantities involved are nonnegative, we can use the bound (5.6), and thus it suffices to obtain a bound for

$$\mathbb{E}\left[\left|2^{n\tau_H(2q)}\sum_{0\le k\ne k'<2^n}|k-k'|^{r(2H-2)}a_{0,k,n,H}^q a_{0,k',n,H}^q\right|^p\right].$$

Define

$$\tilde{\delta}_k^2 = \int_{\Delta_{n,k}}\int_{\Delta_{n,k}}|u-v|^{2H-2}M_n(\mathrm{d}u)M_n(\mathrm{d}v).$$

Then $\tilde{a}_{0,k,n,H} = \tilde{\delta}_k \mathrm{e}^{qw_{l_n}(k2^{-n})}$ and using the bound (A.29) and (3.6), we obtain

$$\mathbb{E}\left[\left|2^{n\tau_H(2q)}\sum_{0\le k\ne k'<2^n}|k-k'|^{r(2H-2)}\{a_{0,k,n,H}^q a_{0,k',n,H}^q - \tilde{a}_{0,k,n,H}^q\tilde{a}_{0,k',n,H}^q\}\right|^p\right] = \mathrm{O}\big(2^{(\varepsilon\chi-\eta)n}\big).$$

Thus, we need to obtain a bound for $\mathbb{E}[S_{n,4}^p]$ where

$$S_{n,4} = 2^{n\tau_H(2q)} \sum_{0 \le k \ne k' < 2^n} |k - k'|^{r(2H-2)} \tilde{a}_{0,k,n,H}^q \tilde{a}_{0,k',n,H}^q,$$

which we further decompose as $S_{n,4} = S_{n,5} + S_{n,6}$ with

$$S_{n,5} = 2^{n\tau_H(2q)} \sum_{0 \le k \ne k' < 2^n} |k - k'|^{r(2H-2)} \{\tilde{\delta}_k^q \tilde{\delta}_{k'}^q - \mathbb{E}[\tilde{\delta}_k^q \tilde{\delta}_{k'}^q]\} \mathrm{e}^{q w_{l_n}(k2^{-n}) + q w_{l_n}(k'2^{-n})},$$

$$S_{n,6} = 2^{n\tau_H(2q)} \sum_{0 \le k \ne k' < 2^n} |k - k'|^{r(2H-2)} \mathbb{E}[\tilde{\delta}_k^q \tilde{\delta}_{k'}^q] \mathrm{e}^{q w_{l_n}(k2^{-n}) + q w_{l_n}(k'2^{-n})}.$$

Since $H < 3/4$ and $r \ge 2$, we have that $r(2H-2) < -1$ and the series $\sum k^{r(2H-2)}$ is summable. Thus, applying Cauchy–Schwarz' inequality yields

$$\mathbb{E}\left[\left|2^{-n} \sum_{0 \le k \ne k' < 2^n} |k - k'|^{r(2H-2)} \mathrm{e}^{q w_{l_n}(k2^{-n}) + q w_{l_n}(k'2^{-n})}\right|^p\right] \le C\mathbb{E}\left[\left|2^{-n} \sum_{k=0}^{2^n-1} \mathrm{e}^{q w_{l_n}(k2^{-n})}\right|^{2p}\right].$$

Next, applying Lemma A.1 with p such that $2pq < q_\chi$ and $\varepsilon' < p - 1$ yields

$$\mathbb{E}\left[\left|2^{-n} \sum_{0 \le k \ne k' < 2^n} |k - k'|^{r(2H-2)} \mathrm{e}^{q w_{l_n}(k2^{-n}) + q w_{l_n}(k'2^{-n})}\right|^p\right] \le C l_n^{-\{\psi(2pq) - \varepsilon'\}}. \tag{5.15}$$

By (A.27), it holds that $\mathbb{E}[\tilde{\delta}_k^q \tilde{\delta}_{k'}^q] \sim C(k, k') l_n^{\psi(2q)} 2^{-n\zeta_H(q)}$ where $C(k, k')$ is uniformly bounded, thus

$$\mathbb{E}[S_{n,6}^p] \le C l_n^{-\{\psi(2pq) - p\psi(2q) - \varepsilon'\}}.$$

If $2pq < q_\chi$, applying (3.6), we have

$$(1 - \alpha)\{\psi(2pq) - p\psi(2q) - \varepsilon'\} - \varepsilon\chi \le (1 - \alpha)\varepsilon(1 + \chi) - \varepsilon' \le \varepsilon - \varepsilon' - \alpha\varepsilon(1 + \chi),$$

which can be made negative by choosing ε' close enough to ε. To deal with the last term, as in the proof of Proposition 3.1 we use the conditional $2^{\alpha n}$ dependence of the random variables δ_k. We obtain the bound

$$\mathbb{E}[S_{n,5}^p] \le C 2^{n\{\psi(2pq) - p\psi(2q) - \varepsilon\}} = \mathrm{O}(2^{n(\varepsilon\chi - \eta)})$$

for small some $\eta > 0$. We have proved (5.14), and thus (5.11) holds. We can now define

$$\Gamma_1(q) = \sum_{r=2}^{\infty} \frac{g_r(q)^2}{r!} \Gamma(r, q).$$

As $\sum_{r=2}^{\infty}(r!)^{-1}g_r(q)^2 < \infty$ and $\Gamma_n(r,q) \le \Gamma_n(2,q)$, then by the bounded convergence theorem, the previous series is convergent and thus we have obtained that

$$L^{-1}2^{n\tau_H(2q)}\operatorname{var}_M\bigl(S_{L,n}(X,q)\bigr) \to \Gamma_1(q) \qquad \text{a.s.}$$

This also yield that there exists a constant $\Gamma_2(q)$ such that

$$L^{-1}2^{n\tau_H(2q)}\operatorname{var}_M\bigl(2^{\tau_H(q)}S_{L,n+1}(X,q)\bigr) \to \Gamma_2(q) \qquad \text{a.s.}$$

By similar techniques, we also obtain that there exists a constant $\Gamma_3(q)$ such that

$$L^{-1}2^{n\tau_H(2q)}\operatorname{cov}_M\bigl(S_{L,n}(X,q), S_{L,n+1}(X,q)\bigr) \to \Gamma_3(q) \qquad \text{a.s.}$$

Finally, since $T_n(X,q) = S_{L,n}(X,q) - 2^{\tau_H(q)}S_{L,n+1}(X,q)$, the last three convergences yield (4.7).

Proof of (4.8). By Nualart and Peccati [13], Theorem 1, the proof will follow by checking that

$$L^{-2}2^{2n\tau_H(2q)}\mathbb{E}_M\bigl[\bigl\{T_n(X,q) - \mathbb{E}_M\bigl[T_n(X,q)\bigr]\bigr\}^4\bigr] \to 3\Gamma(H,q)^2 \qquad \text{a.s.} \tag{5.16}$$

Define

$$\begin{aligned} T_{n,r}(X,q) &= \sum_{j=0}^{L-1}\sum_{k=0}^{2^n-1} 2^{\tau_H(q)}\bigl\{a^q_{j,2k,n+1,H}H_r(Y_{j,2k,n+1}) + a^q_{j,2k+1,n+1,H}H_r(Y_{j,2k+1,n+1})\bigr\} \\ &\qquad - a^q_{j,k,n,H}H_r(Y_{j,k,n}). \end{aligned}$$

Then, from the definition of $T_n(X,q)$ and recalling the expansion $G_q = \sum_{r=2}^{\infty}\frac{g_r(q)}{r!}H_r$ in terms of the Hermite polynomials, to show (5.16) it is enough to check that

$$\mathbb{E}_M\bigl[\bigl(T_{n,r}(X,q)\bigr)^4\bigr] = \frac{3}{(r!)^2}\mathbb{E}_M^2\bigl[T_{n,r}^2(X,q)\bigr] + R_n(q,r), \tag{5.17}$$

with $L^{-2}2^{2n\tau_H(2q)}R_n(q,r) \to 0$ a.s. In order to calculate the fourth order moment in (5.17) we use a standard application of the Diagram formula, for which we use the notation in Surgailis [15]. Given a centered stationary Gaussian process $\{X_j\}_{j\ge 1}$ with positive covariance $c(t_i,t_j) = \operatorname{cov}(X_{t_i}, X_{t_j})$ and variance one, and a triangular array of positive elements $\{b_t\}_{t=1}^N$ define $S_N(b) := \sum_{t=1}^N b_t H_r(X_t)$. We introduce the following basic lattice notation. Let W be a 4 row table, whose rows correspond to the size r vectors $W_i = (i,\dots,i)$, $i = 1,\dots,4$. Consider the collection Γ of Gaussian flat connected diagrams γ, that is, of partitions of W defined by the disjoint subsets $\{V_\ell\}$ with $W = \bigcup_\ell V_\ell$, such that, respectively, $|V_\ell| = 2$, no $V_\ell \subset W_i$ and it is not possible to write $W = W_1 \cup W_2$, where W_1 and W_2 can be partitioned by the diagram separately.

Then, we have that (see, e.g., Surgailis [15])

$$\mathbb{E}\big[\big(S_N(b)\big)^4\big] = 3\Bigg(\sum_{t_1,t_2=1}^{N} b_{t_1} b_{t_2} c^r(t_1,t_2)\Bigg)^2 + \sum_{\gamma\in\Gamma}\sum_{t_1,\dots,t_4} b_{t_1}\cdots b_{t_4} \prod_{1\le i<j\le 4} c^{l_{i,j}}(t_i,t_j), \tag{5.18}$$

where $l_{i,j}$ is the number of elements V_ℓ in the diagram that pair row i with row j. Because the diagram is connected and each row must appear at least once, for each pair i, j we have $1 \le l_{i,j} < r$. Also, the fact that the diagrams in Γ are flat (i.e., that no $V_\ell \subset W_i$) assures that the second sum is over 4-tuples of pairwise distinct indices. On the other hand, since $0 \le c(i, j) \le 1$ and $r \ge 2$, for each $\gamma \in \Gamma$, by symmetry

$$\sum_{t_1,\dots,t_4} b_{t_1}\cdots b_{t_4} \prod_{1\le i<j\le 4} c^{l_{i,j}}(t_i,t_j) \le \sum_{t_1,\dots,t_4} b_{t_1}\cdots b_{t_4} c(t_1,t_2)c(t_2,t_3)c(t_3,t_4)c(t_4,t_1). \tag{5.19}$$

Applying (5.18) and (5.19) to $T_{n,r}(X,q)$, we obtain (5.17) if we show that

$$L^{-2}2^{2n\tau_H(2q)} \sum_{j_1,\dots,j_4=1}^{L-1} \sum_{k_1,\dots,k_4=1}^{2^n-1} \prod_{1\le i\le 4} a^q_{j_i,k_i,n,H}\,\rho_{H,n}(j_1,j_2,k_1,k_2)\rho_{H,n}(j_2,j_3,k_2,k_3) \times \rho_{H,n}(j_3,j_4,k_3,k_4) \times \rho_{H,n}(j_1,j_2,k_1,k_4) \to 0 \qquad \text{a.s.} \tag{5.20}$$

The fact that the sum is over pairwise distinct indices assures that $(j_i,k_i) \ne (j_\ell,k_\ell)$ for $i \ne \ell$, however it is necessary to distinguish several cases:

- Case $j_i \equiv j$ for all $i = 1,\dots,4$. We prove that

$$L^{-2}2^{2n\tau_H(2q)} \sum_{j=1}^{L-1} \sum_{k_1,\dots,k_4=1}^{2^n-1} \prod_{1\le i\le 4} a^q_{j,n,k_i,H}\,\rho_{H,n}(j_1,j_2,k_1,k_2) \times \rho_{H,n}(j,j,k_2,k_3)\rho_{H,n}(j,j,k_3,k_4) \times \rho_{H,n}(j,j,k_1,k_4) \to 0 \qquad \text{a.s.} \tag{5.21}$$

  This will be achieved by showing that the expectation of the l.h.s. of (5.21) tends to zero. By stationarity of increments and Hölder's inequality, we have

$$\mathbb{E}\Bigg[\prod_{1\le i\le 4} a^q_{0,n,k_i,H}\Bigg] \le \mathbb{E}^{1/2}\big[a^{2q}_{0,0,n,H}a^{2q}_{0,k_2-k_1+1,n,H}\big]\mathbb{E}^{1/2}\big[a^{2q}_{0,0,n,H}a^{2q}_{0,k_4-k_3+1,n,H}\big].$$

In addition, by the scaling property, we have that

$$\mathbb{E}\big[a_{0,0,n,H}^{2q}a_{0,k_2-k_1+1,n,H}^{2q}\big] = 2^{-n\zeta_H(4q)}(k_2-k_1+1)^{\zeta_H(4q)-2\zeta_H(2q)}C(k_1,k_2),$$

with $C(k_1,k_2)\le m_H(4q)$. This and the deterministic bounds on the covariance (5.6) yield that the expectation of the l.h.s. of (5.21) is bounded by

$$\begin{aligned}
&L^{-1}2^{-2n}2^{n\{\psi(4q)-2\psi(q)\}}\sum_{k_1,\dots,k_4=0}^{2^n-1}|k_1-k_2|^{2H-2-(\psi(4q)-2\psi(2q))/2}\\
&\qquad\times|k_3-k_4|^{2H-2-(\psi(4q)-2\psi(2q))/2}|k_2-k_3|^{2H-2}\\
&\qquad\times|k_1-k_4|^{2H-2}\\
&\le CL^{-1}2^{-2n}2^{n\{\psi(4q)-2\psi(q)\}}\sum_{k_1,k_2=0}^{2^n-1}|k_1-k_2|^{2(2H-2)}\left(\sum_{k=0}^{2^n-1}k^{2H-2-(\psi(4q)-2\psi(2q))/2}\right)^2\\
&\le CL^{-1}2^{-n}\sum_{k}^{2^n-1}k^{2(2H-2)}\left(2^{n\{\psi(4q)-2\psi(q)\}/2}\sum_{k=0}^{2^n-1}k^{2H-2-(\psi(4q)-2\psi(2q))/2}\right)^2.
\end{aligned}$$

Since $H>3/4$, the first series is summable, and since $\psi(4q)-2\psi(2q)>0$, the second one is of order $n2^{n(\{\psi(4q)-2\psi(2q)\}\vee(4H-2))/2}$ (where the factor n only arises if the two exponents are equal). Recalling that $\psi(4q)-2\psi(q)<1=\chi$ yields (5.21).

- Case $j_1=j_2=j_3=j$. In this case $|k_i-k_4|=\mathrm{O}(2^{-n(2H-2)})$, $i=1,2,3$ and by Hölder's inequality and independence of $a_{j',k_4,n,H}$ and $\prod_{1\le i\le 3}a_{j,k_i,n,H}$ we have

$$\mathbb{E}\left[a_{j',k_4,n,H}^{q}\prod_{1\le i\le 3}a_{j,k_i,n,H}^{q}\right]=\mathrm{O}\big(2^{-n\zeta(4q)/2}2^{-n\zeta(2q)/2}2^{-n\zeta(q)}\big)|k_2-k_3|^{(\psi(4q)-2\psi(2q))/2}.$$

Using again the bound (5.6), we obtain

$$\begin{aligned}
&L^{-2}2^{2n\tau_H(2q)}\sum_{j=0}^{L-1}\sum_{j'=0}^{L-1}\sum_{k_1,\dots,k_4=1}^{2^n-1}\mathbb{E}\Big[\prod_{1\le i\le 3}a_{j,k_i,n,H}^{q}a_{j',k_4,n,H}^{q}\rho_{H,n}^2\big(j,j',k_1,k_4\big)\\
&\qquad\times\rho_{H,n}(j,j,k_2,k_3)\rho_{H,n}(j,j,k_3,k_1)\Big] \qquad (5.22)\\
&\quad=\mathrm{O}\big(L^{-1}2^{n(4H-3)}2^{-n(\psi(2q)/2-\psi(q))}\big).
\end{aligned}$$

As before, $2^{n(4H-3)}\to 0$ under $H<3/4$ and $\psi(2q)/2-\psi(q)>0$ by convexity of function ψ.

- Case $j_1 = j_2$ and $j_3 = j_4$. The bound for the expectation of the l.h.s. of (5.20) is then

$$
\begin{aligned}
L^{-2} 2^{2n\tau_H(2q)} \sum_{j,j,j',j'}^{L-1} \sum_{k_1,\dots,k_4=1}^{2^n-1} \mathbb{E}\Big[& a^q_{j,k_1,n,H} a^q_{j,k_2,n,H} a^q_{j',k_3,n,H} a^q_{j',k_4,n,H} \\
& \times \rho^2_{H,n}\big(j,j',k_1,k_4\big) \\
& \times \rho_{H,n}(j,j,k_1,k_2)\rho_{H,n}\big(j',j',k_3,k_4\big)\Big] \\
\le C 2^{n(4H-3)}, &
\end{aligned}
\tag{5.23}
$$

by independence of $a^q_{j,n,k_1,H}$ and $a^q_{j',n,k_2,H}$ whenever $j \neq j'$.
- Case all j_i are different. The bound is then

$$
\begin{aligned}
L^{-2} 2^{2n\tau_H(2q)} \sum_{j_1,j_2,j_3,j_4}^{L-1} \sum_{k_1,\dots,k_4=1}^{2^n-1} \mathbb{E}\Big[& \prod_{1\le i\le 4} a^q_{j_i,n,k_i,H} \rho^2_{H,n}(j_1,j_2,k_1,k_4) \\
& \times \rho_{H,n}(j_2,j_3,k_2,k_3)\rho_{H,n}(j_3,j_4,k_3,k_4)\Big] \\
\le C 2^{n(-2\psi(2q)+4\psi(q))} 2^{n(2+\chi)(4H-3)}. &
\end{aligned}
\tag{5.24}
$$

As before, $2^{n(2+\chi)(4H-3)} \to 0$ under $H < 3/4$ and we use $\psi(2q) > 2\psi(q)$.

The proof follows by gathering (5.21), (5.22), (5.23) and (5.24). □

Appendix: Additional lemmas

Bounds for infinitely divisible random measures. We now state some results using the properties of infinitely divisible random measures. The infinitely divisible measure P introduced in Section 3 can be decomposed as $P = P_0 + P_1$ where P_0 and P_1 are independent and

$$
\mathbb{E}\big[e^{qP_i(A)}\big] = e^{\mu(A)\psi_i(q)},
$$

with

$$
\begin{aligned}
\psi_0(q) &= \frac{\sigma^2}{2} q^2 + mq + \int_{-1}^{\infty} \big\{e^{qx} - 1 - qx\mathbf{1}_{\{|x|\le 1\}}\big\} \nu(dx), \\
\psi_1(q) &= \int_{-\infty}^{-1} \big\{e^{qx} - 1\big\} \nu(dx).
\end{aligned}
$$

Note that by assumption, ψ_0 is infinitely differentiable on $[0,\infty)$, whereas ψ_1 is infinitely differentiable on $(0,\infty)$ only. Then, for A such that $\mu(A) \le 1$, $q > 0$ and $p \ge 1$ such that $pq < q^*$, it holds that

$$\mathbb{E}\big[\big|P_0(A)\big|^p\big] = \mathrm{O}\big(\big[\mu(A)\big]^{(p/2)\wedge 1}\big), \tag{A.1}$$

$$\mathbb{E}\big[\big|\mathrm{e}^{qP_0(A)} - 1 - qP_0(A)\big|^p\big] = \mathrm{O}\big(\mu(A)\big), \tag{A.2}$$

$$\mathbb{E}\big[\big|\mathrm{e}^{qP_1(A)} - 1\big|^p\big] = \mathrm{O}\big(\mu(A)\big). \tag{A.3}$$

Indeed, since $0 \le \mathrm{e}^x - 1 - x \le x^2\mathrm{e}^{x_+} \le x^2(\mathrm{e}^x + 1)$, with $x_+ = \max(x, 0)$, we have

$$\begin{aligned}&\mathbb{E}\big[\big|\mathrm{e}^{qP_0(A)} - 1 - qP_0(A)\big|^p\big]\\ &\quad\le C\mathbb{E}\big[P_0^{2p}(A)\mathrm{e}^{pqP_0(A)}\big] + C\mathbb{E}\big[P_0^{2p}(A)\big].\end{aligned}$$

Denote $L(s) = \mathbb{E}[\mathrm{e}^{sP(A)}] = \mathrm{e}^{\psi_0(s)}\mu(A)$. The function L is infinitely differentiable on $[0, q^*)$ and $L^{(n)}(q) = \mathrm{O}(\mu(A))$ for all $q \ge 0$ and $n \ge 1$. This yields (A.1) by the Cauchy–Schwarz inequality. Let n be an integer greater than p. Then, for $0 \le q < q^*$, (A.3) follows from the following bound:

$$\begin{aligned}\mathbb{E}\big[P_0^{2p}(A)\mathrm{e}^{pqP_0(A)}\big] &\le \mathbb{E}\big[P_0^{2}(A)\mathrm{e}^{pqP_0(A)}\big] + \mathbb{E}\big[P_0^{2n}(A)\mathrm{e}^{pqP_0(A)}\big]\\ &= L''(pq) + L^{(2n)}(pq).\end{aligned}$$

To prove (A.2), note that $P_1(A)$ is a coumpond Poisson distribution with negative jumps, thus $P_1(A) < 0$ for all A, and for all $p \ge 1$,

$$\mathbb{E}\big[\big|\mathrm{e}^{qP_1(A)} - 1\big|^p\big] \le 1 - \mathrm{e}^{\psi_1(q)\mu(A)} = \mathrm{O}\big(\mu(A)\big).$$

Further, write

$$\begin{aligned}&\mathrm{e}^{qP(A)} - 1 - qP_0(A)\\ &\quad = \big\{\mathrm{e}^{qP_1(A)} - 1\big\}\mathrm{e}^{qP_0(A)} + \mathrm{e}^{qP_0(A)} - 1 - qP_0(A).\end{aligned} \tag{A.4}$$

This decomposition, (A.2), (A.3) and the independence of P_0 and P_1 yield, for $q > 0$ and $p \ge 1$,

$$\mathbb{E}\big[\big|\mathrm{e}^{qP(A)} - 1 - qP_0(A)\big|^p\big] = \mathrm{O}\big(\mu(A)\big). \tag{A.5}$$

Since P, P_0 and P_1 are independently scattered, these inequalities yield martingale maximal inequalities. For A such that $\mu(A) \le 1$, and for C_u an increasing sequence of measurable subsets of A, it holds that

$$\mathbb{E}\Big[\sup_u \big|P_0(C_u)\big|^p\Big] = \mathrm{O}\big(\mu^{(p/2)\vee 1}(A)\big), \qquad p \ge 1, \tag{A.6}$$

$$\mathbb{E}\Big[\sup_u \big|\mathrm{e}^{qP(C_u)} - 1\big|^p\Big] = \mathrm{O}\big(\mu(A)^{(p/2)\vee 1}\big), \qquad p \ge 1, \tag{A.7}$$

$$\mathbb{E}\Big[\sup_u \big|\mathrm{e}^{qP(C_u)} - 1 - qP_0(C_u)\big|^p\Big] = \mathrm{O}\big(\mu(A)\big), \qquad p \ge 1. \tag{A.8}$$

Approximation and covariance bounds for the MRM.

Lemma A.1. *Let $\alpha = 1/n_0$ for some arbitrary integer $n_0 \geq 2$. For all $p > 1$ such that $\mathbb{E}[\mathrm{e}^{pqw_l(0)}] < \infty$, for any $\varepsilon' \in (0, p-1)$, there exists a constant C such that*

$$\mathbb{E}\left[\left(\int_0^1 \frac{\mathrm{e}^{qw_{l_n}(u)}}{\mathbb{E}[\mathrm{e}^{qw_{l_n}(0)}]}\,\mathrm{d}u\right)^p\right] \leq C l_n^{-\{\psi(pq)-p\psi(q)-\varepsilon'\}}, \tag{A.9}$$

$$\mathbb{E}\left[\left(2^{-n}\sum_{k=0}^{2^n-1} \frac{\mathrm{e}^{qw_{l_n}(k2^{-n})}}{\mathbb{E}[\mathrm{e}^{qw_{l_n}(0)}]}\right)^p\right] \leq C l_n^{-\{\psi(pq)-p\psi(q)-\varepsilon'\}}. \tag{A.10}$$

Proof. The choice of α implies that $(1-\alpha)n_0 = n_0 - 1$ is an integer. Denote $g_n(u) = \mathrm{e}^{qw_{l_n}(u)}/\mathbb{E}[\mathrm{e}^{qw_{l_n}(0)}]$. Fix some integer k_0, and define $n_1 = k_0 n_0$. If $n_1 < n$, then

$$\begin{aligned}\int_0^1 g_n(u)\,\mathrm{d}u &= \int_0^1 g_{n_1}(u)\,\mathrm{d}u + \int_0^1 \{g_n(u) - g_{n_1}(u)\}\,\mathrm{d}u \\ &= \int_0^1 g_{n_1}(u)\,\mathrm{d}u + \sum_{k=0}^{2^{(1-\alpha)n_1}-1} \int_{\Delta_{k,(1-\alpha)n_1}} \{g_n(u) - g_{n_1}(u)\}\,\mathrm{d}u.\end{aligned} \tag{A.11}$$

We bound the first integral by applying Jensen's inequality:

$$\mathbb{E}\left[\left(\int_0^1 g_{n_1}(u)\,\mathrm{d}u\right)^p\right] \leq \mathbb{E}[g_{n_1}^p(0)] = 2^{(1-\alpha)n_1\{\psi(pq)-p\psi(q)\}}. \tag{A.12}$$

Since $w_{l_{n_1}}$ is independent of $w_{l_n} - w_{l_{n_1}}$, we can write

$$g_n(u) - g_{n_1}(u) = g_{n_1}(u)\left\{\frac{\mathrm{e}^{qw_{l_n}(u)-qw_{l_{n_1}}(u)}}{\mathbb{E}[\mathrm{e}^{qw_{l_n}(0)-qw_{l_{n_1}}(0)}]} - 1\right\}.$$

Thus we see that the integrals $\int_{\Delta_{j,n_1}} \{g_n(u) - g_{n_1}(u)\}\,\mathrm{d}u$ are centered and 2-dependent conditionally on $\mathcal{F}_{n_1}$ the sigma-field generated by $\{w_{l_{n_1}}(u), u \in [0,1]\}$. Thus by von Bahr and Esseen [16], Theorem 2, there is a constant C such that

$$\begin{aligned}&\mathbb{E}\left[\left|\sum_{k=0}^{2^{(1-\alpha)n_1}-1} \int_{\Delta_{k,(1-\alpha)n_1}} \{g_n(u) - g_{n_1}(u)\}\,\mathrm{d}u\right|^p\right] \\ &\quad\leq C 2^{(1-\alpha)n_1}\mathbb{E}\left[\left|\int_{\Delta_{0,(1-\alpha)n_1}} \{g_n(u) - g_{n_1}(u)\}\,\mathrm{d}u\right|^p\right] \\ &\quad\leq C 2^{p-1} 2^{(1-\alpha)n_1}\mathbb{E}\left[\left|\int_{\Delta_{0,(1-\alpha)n_1}} g_n(u)\,\mathrm{d}u\right|^p\right] + C 2^{p-1} 2^{(1-\alpha)n_1}\mathbb{E}\left[\left|\int_{\Delta_{0,(1-\alpha)n_1}} g_{n_1}(u)\,\mathrm{d}u\right|^p\right] \\ &\quad\leq C 2^{p-1} 2^{(1-\alpha)n_1}\mathbb{E}\left[\left(\int_{\Delta_{0,(1-\alpha)n_1}} g_n(u)\,\mathrm{d}u\right)^p\right] + C 2^{p-1} 2^{\{1-p+\psi(pq)-p\psi(q)\}(1-\alpha)n_1}.\end{aligned}$$

Since $l_n/l_{n_1} = l_{n-n_1}$, by the scaling property (3.3), we have

$$\int_{\Delta_{0,(1-\alpha)n_1}} \mathrm{e}^{q w_{l_n}(u)}\,\mathrm{d}u = l_{n_1} \int_0^1 \mathrm{e}^{q w_{l_{n-n_1} l_{n_1}}(l_{n_1} u)}\,\mathrm{d}u \stackrel{\text{law}}{=} l_{n_1} \mathrm{e}^{q\Omega_{l_{n_1}}} \int_0^1 \mathrm{e}^{q w_{l_{n-n_1}}(u)}\,\mathrm{d}u.$$

Thus,

$$\begin{aligned}\mathbb{E}\biggl[\biggl(\int_{\Delta_{0,(1-\alpha)n_1}} g_n(u)\,\mathrm{d}u\biggr)^p\biggr] &= 2^{(1-\alpha)n_1(\psi(pq)-p)} \frac{(\mathbb{E}[\mathrm{e}^{q w_{l_{n-n_1}}(0)}])^p}{(\mathbb{E}[\mathrm{e}^{q w_{l_n}(0)}])^p} \mathbb{E}\biggl[\biggl(\int_0^1 g_{n-n_1}(u)\,\mathrm{d}u\biggr)^p\biggr] \\ &= 2^{(1-\alpha)n_1(\psi(pq)-p\psi(q)-p)} \mathbb{E}\biggl[\biggl(\int_0^1 g_{n-n_1}(u)\,\mathrm{d}u\biggr)^p\biggr].\end{aligned}$$

Thus we have obtained

$$\begin{aligned}&\mathbb{E}\Biggl[\Biggl|\sum_{k=0}^{2^{(1-\alpha)n_1}-1} \int_{\Delta_{k,(1-\alpha)n_1}} \bigl\{g_n(u) - g_{n_1}(u)\bigr\}\,\mathrm{d}u\Biggr|^p\Biggr] \\ &\quad\le C 2^{(1-\alpha)n_1(\psi(pq)-p\psi(q)-p)} \mathbb{E}\biggl[\biggl(\int_0^1 g_{n-n_1}(u)\,\mathrm{d}u\biggr)^p\biggr].\end{aligned} \tag{A.13}$$

Denote $u_n = \mathbb{E}[(\int_0^1 g_n(u)\,\mathrm{d}u)^p]$. Gathering (A.11), (A.12) and (A.13), we obtain the following recurrence:

$$u_n \le B + C 2^{(1-\alpha)n_1(1-p+\psi(pq)-p\psi(q))} u_{n-n_1}.$$

By choosing k_0 large enough, this yields that for any $\varepsilon' \in (0, \varepsilon)$,

$$u_n \le B + 2^{(1-\alpha)n_1(\psi(pq)-p\psi(q)-\varepsilon')} u_{n-n_1}.$$

Thus, there exists a constant D such that

$$u_n \le D 2^{(1-\alpha)n(\psi(pq)-p\psi(q)-\varepsilon')}.$$

This proves (A.9). The bound (A.10) follows by replacing the measure du with a discrete measure. □

Lemma A.2. *Let* $0 < \alpha < 1$ *and* $l_n = 2^{-(1-\alpha)n}$. *For* $p \ge 1$ *and* $q > 0$ *such that* $pq < q_\chi$, *there exists a positive constant C such that*

$$\lim_{n\to\infty} 2^{n\zeta(q)} \mathrm{e}^{\psi(q)} l_n^{-\psi(q)} \mathbb{E}\bigl[M_n^q(\Delta_{0,n})\bigr] = m(q), \tag{A.14}$$

$$\mathbb{E}\bigl[\bigl|\mathrm{e}^{q w_{l_n}(0)} M_n^q(\Delta_{0,n}) - M^q(\Delta_{0,n})\bigr|^p\bigr] \le C 2^{-\alpha(q\vee 1)n/2} 2^{-n\zeta(pq)}. \tag{A.15}$$

Proof. Note that (A.15) implies (A.14). So we only need to prove (A.15). Define the sets I_n, $B_n(u)$, $u \in [0, 2^{-n}]$ by

$$I_n = \bigcap_{0 \le u \le 2^{-n}} A_{l_n}(u) = A_{l_n(0)} \cap A_{l_n(2^{-n})}, \qquad B_n(u) = A_{l_n}(u) \setminus I_n.$$

See Figure 2 for an illustration. By definition of the function ψ and the measure μ, we have, $\mathbb{E}[\mathrm{e}^{qP(I_n)}] = \mathrm{e}^{\psi(q)\mu(I_n)}$ and

$$\begin{aligned}\mu(I_n) &= \int_{I_n} \frac{\mathrm{d}s\,\mathrm{d}t}{t^2} = \int_{l_n}^{1} \frac{t - 2^{-n}}{t^2}\,\mathrm{d}t + \int_1^\infty \frac{1 - 2^{-n}}{t^2}\,\mathrm{d}t \\ &= -\log(l_n) - 2^{-n}\bigl(l_n^{-1} - 1\bigr) + 1 - 2^{-n} = 1 - \log(l_n) - 2^{-\alpha n} = \mu\bigl(A_{l_n}(0)\bigr) - 2^{-\alpha n}.\end{aligned}$$

This yields $w_{l_n}(u) = P(I_n) + P(B_n(u))$ where the two summands are independent and

$$\mathbb{E}\bigl[\mathrm{e}^{qP(I_n)}\bigr] = \mathbb{E}\bigl[\mathrm{e}^{qw_{l_n}(0)}\bigr]\bigl\{1 + \mathrm{O}\bigl(2^{-\alpha n}\bigr)\bigr\}. \tag{A.16}$$

Write further

$$\begin{aligned}M(\Delta_{0,n}) &= \int_0^{2^{-n}} \mathrm{e}^{w_{l_n}(u)} M_n(\mathrm{d}u) = \mathrm{e}^{P(I_n)} \int_0^{2^{-n}} \mathrm{e}^{P(B_n(u))} M_n(\mathrm{d}u) \\ &= \xi_n \int_0^{2^{-n}} \mathrm{e}^{P(B_n(u))} \bar{M}_n(\mathrm{d}u),\end{aligned}$$

with $\xi_n = \mathrm{e}^{P(I_n)} M_n(\Delta_{0,n})$ and $\bar{M}_n(\mathrm{d}u) = M_n(\mathrm{d}u)/M_n(\Delta_{0,n})$ is a random probability measure on $\Delta_{0,n}$. We thus obtain

$$M^q(\Delta_{0,n}) - \mathrm{e}^{qw_{l_n}(0)} M_n^q(\Delta_{0,n}) = \xi_n^q \left\{ \left(\int_0^{2^{-n}} \mathrm{e}^{P(B_n(u))} \bar{M}_n(\mathrm{d}u) \right)^q - \mathrm{e}^{qP(B_n(0))} \right\}.$$

Noting that for $x > -1$ and $q > 0$, it holds that $0 \le |1 - (1+x)^q| \le C_q(|x| + |x|^q)$ and since $P(I_n)$, $M_n(\Delta_{0,n})$ and $P(B_n(u))$, $0 \le u \le 2^{-n}$, are mutually independent, we have

$$\begin{aligned}&\mathbb{E}\left[\xi_n^{pq} \left| \left(\int_0^{2^{-n}} \mathrm{e}^{P(B_n(u))} \bar{M}_n(\mathrm{d}u) \right)^q - \mathrm{e}^{qP(B_n(0))} \right|^p \right] \\ &\quad \le C\mathbb{E}\bigl[\xi_n^{pq}\bigr] \left\{ \left(\mathbb{E}\Bigl[\sup_{0 \le u \le 2^{-n}} \bigl| \mathrm{e}^{P(B_n(u))} - 1 \bigr|^{p(q\vee 1)} \Bigr] \right)^{q\wedge 1} + \mathbb{E}\Bigl[\sup_{0 \le u \le 2^{-n}} \bigl| \mathrm{e}^{P(B_n(u))} - 1 \bigr|^p \Bigr] \right\}.\end{aligned}$$

Thus, applying (A.7) yields

$$\mathbb{E}\bigl[\bigl|M^q(\Delta_{0,n}) - \mathrm{e}^{qw_{l_n}(0)} M_n^q(\Delta_{0,n})\bigr|^p\bigr] = \mathrm{O}\bigl(2^{-\alpha n(q\wedge 1)/2}\bigr)\mathbb{E}\bigl[\xi_n^{pq}\bigr]. \qquad \square$$

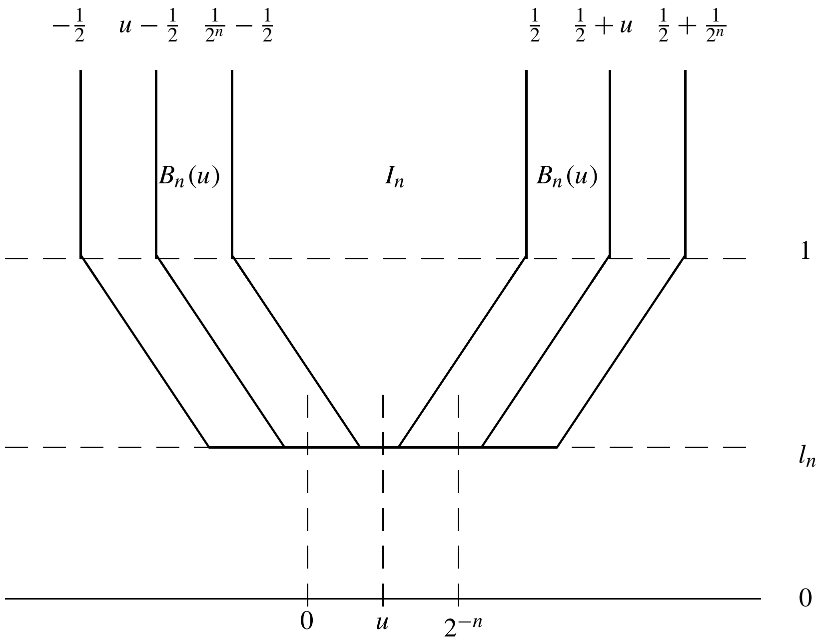

Figure 2. The sets I_n and $B_n(u)$.

Lemma A.3. *If $q+q' < q_{\max}$, then for $s, t \in (0,1)$ such that $s+t < 1/2$,*

$$\operatorname{cov}\big(M^q([0,s]), M^{q'}([1-t,1])\big) = \mathrm{O}\big((s+t)^{\{\zeta(q)+\zeta(q')+1\}}\big). \tag{A.17}$$

Proof. Define $l = 1-s-t$ and $M_l(\mathrm{d}u) = \mathrm{e}^{-w_l(u)} M(\mathrm{d}u)$. By construction, the measure M_l is independent of $\{w_l(u)\}$ and $M_l([0,s])$ is independent of $M_l([1-t,1])$. Define the sets $A_{s,t}$ and $B_{s,t}$ by

$$A_{s,t} = A_l(s) \setminus A_l(1-t), \qquad B_{s,t} = A_l(1-t) \setminus A_l(s).$$

For $u \le s$ and $v \ge 1-t$, define

$$\begin{aligned} C_{u,v} &= A_l(u) \cap A_l(v), \\ D_{s,u} &= C_{s,v} \setminus C_{u,v}, \qquad D'_{s,u} = A_l(u) \setminus A_l(s), \\ E_{t,u} &= C_{u,1-t} \setminus C_{u,v}, \qquad E'_{t,v} = A_l(v) \setminus A_l(1-t). \end{aligned}$$

See Figure 3 for an illustration. Note that all these sets are above the horizontal line at level $l = 1-s-t$, hence $P(A)$ is independent of M_l and $P(A)$ is independent of $P(B)$, where A, B are any two of these sets. Note also that $\bigcup_{u \le s, v \ge 1-t} C_{u,v} = C_{s,1-t}$, $D_{s,u} \subset C_{s,1-t}$, $E_{t,v} \subset C_{s,1-t}$,

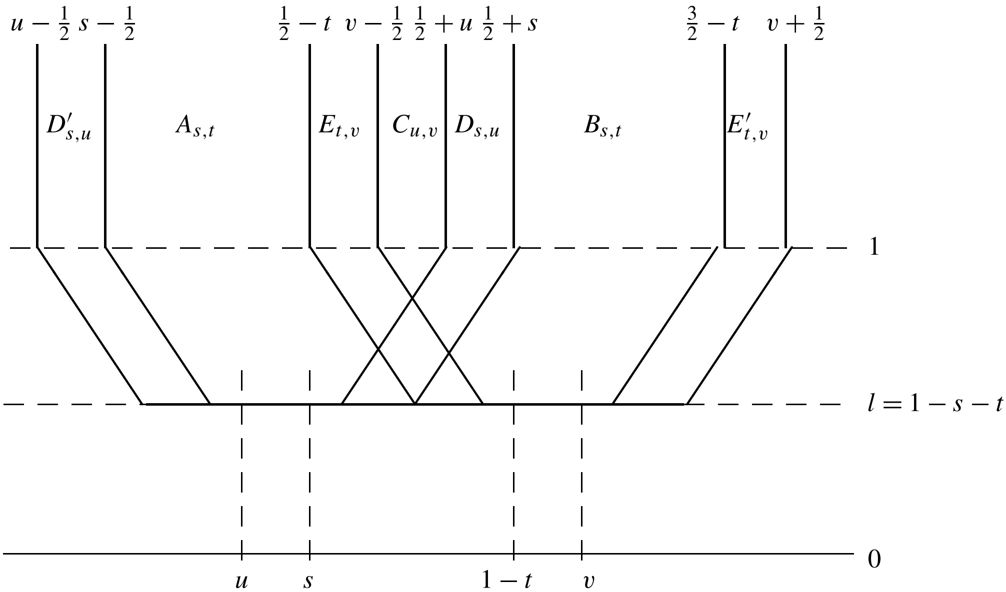

Figure 3. The sets A, B, C, D, D', E, E'.

$D'_{s,u} \subset D'_{s,0}$ and $E'_{t,v} \subset E'_{t,1}$. We moreover have

$$\mu(A_{s,t}) = \mu(B_{s,t}) = 1,$$
$$\mu(C_{s,1-t}) = -\log(1-s-t),$$
$$\mu\big(D'_{s,0}\big) = \frac{s}{1-s-t}, \qquad \mu\big(E'_{t,1}\big) = \frac{t}{1-s-t}.$$

Moreover, for $u \le s$ and $v \ge 1-t$, we have the following decompositions:

$$w_l(u) = P(A_{s,t}) + P(C_{u,v}) + P\big(D'_{s,u}\big) + P(E_{t,v}),$$
$$w_l(v) = P(B_{s,t}) + P(C_{u,v}) + P(D_{s,u}) + P\big(E'_{t,v}\big).$$

Recall that the random measure P can be split into two independent random measures P_0 and P_1 such that $P = P_0 + P_1$. For $i = 0, 1$ and $u \in [0, s]$, define $\pi_{i,l}(u) = P_i(D'_{s,u}) + P_i(C_{u,1-t})$ and

$$\pi_l(u) = \pi_{0,l}(u) + \pi_{1,l}(u).$$

Similarly, for $i = 0, 1$ and $v \in [1-t, 1]$, define $\pi'_{i,l}(v) = P_i(E'_{t,v}) + P_i(C_{s,v})$ and

$$\pi'_l(v) = \pi'_{0,l}(v) + \pi'_{1,l}(v).$$

Let $\bar{M}_l$ and $\bar{M}_l'$ denote the normalized measures $M_l/M_l([0,s])$ and $M_l/M_l([1-t,1])$ and

$$\begin{aligned}
&\zeta_l = M_l\big([0,s]\big), \qquad \xi_l = M_l\big([1-t,1]\big),\\
&\gamma_l = \int_0^s \big\{e^{\pi_l(u)} - 1\big\}\bar{M}_l(\mathrm{d}u), \qquad \gamma_l' = \int_{1-t}^1 \big\{e^{\pi_l'(v)} - 1\big\}\bar{M}_l'(\mathrm{d}v),\\
&R_l = (1+\gamma_l)^q - 1 - q\gamma_l, \qquad R_l'\big(1+\gamma_l'\big)^{q'} - 1 - q'\gamma_l'.
\end{aligned}$$

This yields

$$\begin{aligned}
M^q\big([0,s]\big) &= e^{qP(A_{s,t})}\zeta_l^q \times \{1 + q\gamma_l + R_l\},\\
M^{q'}\big([1-t,1]\big) &= e^{q'P(B_{s,t})}\xi_l^{q'} \times \big\{1 + q'\gamma_l + R_l'\big\}.
\end{aligned}$$

Note that ζ_l and ξ_l are independent and independent of π_l and π_l' which are independent of M_l. Thus, ξ_l is also independent of γ_l and R_l, and ζ_l is independent of γ_l' and R_l'. Also, $P(A_{s,t})$ and $P(B_{s,t})$ are independent of all the other quantities, and $\mathbb{E}[e^{qP(A_{s,t})}] = \mathbb{E}[e^{qP(B_{s,t})}] = e^{\psi(q)}$. Thus,

$$\begin{aligned}
&e^{-\psi(q)-\psi(q')}\operatorname{cov}\big(M^q\big([0,s]\big), M^{q'}\big([1-t,1]\big)\big)\\
&\quad = qq'\operatorname{cov}\big(\zeta_l^q\gamma_l, \xi_l^{q'}\gamma_l'\big) + q\mathbb{E}\big[\xi_l^q\zeta_l^{q'}\gamma_l R_l'\big] - q\mathbb{E}\big[\zeta_l^q\gamma_l\big]\mathbb{E}\big[\xi_l^{q'}\gamma_l'\big] &\text{(A.18)}\\
&\qquad + q'\mathbb{E}\big[\zeta_l^q\xi_l^{q'}R_l\gamma_l'\big] - q'\mathbb{E}\big[\zeta_l^q R_l\big]\mathbb{E}\big[\xi_l^{q'}\gamma_l'\big] + \mathbb{E}\big[\zeta_l^q\xi_l^{q'}R_l R_l'\big] - \mathbb{E}\big[\zeta_l^q R_l\big]\mathbb{E}\big[\xi_l^{q'}R_l'\big]. &\text{(A.19)}
\end{aligned}$$

We will show that all the terms on the right-hand side are of order $(s+t)^{-1}\mathbb{E}[\xi_k^q]\mathbb{E}[\zeta_k^{q'}]$. Since π_l and π_l' are independent of the measure M_l, using the definition of π_l and π_l' and the fact that the random measure P' has independent increments, and $\mathbb{E}[e^{P(A)}] = 1$ for all measurable set A with finite μ measure, we have

$$\begin{aligned}
\operatorname{cov}\big(\zeta_l^q\gamma_l, \xi_l^{q'}\gamma_l\big) &= \mathbb{E}\left[\zeta_l^q\xi_l^{q'}\int_0^s\int_{1-t}^1 \operatorname{cov}\big(e^{\pi_l(u)}, e^{\pi_l'(v)}\big)\bar{M}_l(\mathrm{d}u)\bar{M}_l'(\mathrm{d}v)\right]\\
&= \mathbb{E}\left[\zeta_l^q\xi_l^{q'}\int_0^s\int_t^1 \operatorname{var}\big(e^{P(C_{u,v})}\big)\bar{M}_l(\mathrm{d}u)\bar{M}_l'(\mathrm{d}v)\right]\\
&= \mathbb{E}\left[\zeta_l^q\xi_l^{q'}\int_0^s\int_{1-t}^1 \big\{e^{\psi(2)\mu(C_{u,v})} - 1\big\}\bar{M}_l(\mathrm{d}u)\bar{M}_l'(\mathrm{d}v)\right]\\
&\le \mathbb{E}\big[\zeta_l^q\big]\mathbb{E}\big[\xi_l^{q'}\big]\big\{e^{\psi(2)\mu(C_{s,1-t})} - 1\big\} \le C\mathbb{E}\big[\zeta_l^q\big]\mathbb{E}\big[\xi_l^{q'}\big](s+t).
\end{aligned}$$

If $q > 0$, a second order Taylor expansion yields that there exists a constant $C_q \ge 1$ such that for all $x \ge -1$,

$$\big|(1+x)^q - 1 - qx\big| \le C_q\big(x^2 + |x|^{q\vee 2}\big). \tag{A.20}$$

Applying (A.20) and Jensen's inequality (since by definition $\bar{M}_l$ is a probability measure on $[0, s]$), we obtain, with $r_l = \sup_{u\in[0,s]} |\mathrm{e}^{\pi_l(u)} - 1|$, which is independent of M_l,

$$|R_l| \le C \int_0^s \{|\mathrm{e}^{\pi_l(u)} - 1|^{q\vee 2} + |\mathrm{e}^{\pi_l(u)} - 1|^2\} \bar{M}_l(\mathrm{d}u) \le C(r_l^{q\vee 2} + r_l^2).$$

Define $r_l' = \sup_{u\in[0,s]} |\mathrm{e}^{\pi_l(u)} - 1|$ and note that $|\gamma_l| \le r_l$ and $|\gamma_l'| \le r_l'$. We thus get

$$\mathbb{E}[\zeta_l^q \xi_l^{q'} R_l \gamma_l'] \le \mathbb{E}[\zeta_l^q \xi_l^{q'}]\mathbb{E}[(r_l^2 + r_i^{q\vee 2})r_l'] \le \mathbb{E}[\zeta_l^q \xi_l^{q'}]\mathbb{E}^{1/2}[(r_l^2 + r_i^{q\vee 2})^2]\mathbb{E}^{1/2}[r_l'^2].$$

Applying (A.7), we obtain, for any $h \ge 2$,

$$\mathbb{E}[r_l^h] = \mathrm{O}(\mu(C_{s,1-t}) + \mu(D_{0,s}')) = \mathrm{O}(s + t),$$
$$\mathbb{E}[r_l'^h] = \mathrm{O}(\mu(C_{s,1-t}) + \mu(E_{t,1}')) = \mathrm{O}(s + t).$$

Thus finally

$$\mathbb{E}[\zeta_l^q \xi_l^{q'} R_l \gamma_l'] \le C(s + t)\mathbb{E}[\zeta_l^q]\mathbb{E}[\xi_l^{q'}].$$

The remaining terms in (A.18) and (A.19) are dealt with similarly and we obtain

$$|\mathrm{cov}(M^q([0, s]), M^{q'}([0, t]))| \le C(s + t)\mathbb{E}[\zeta_l^q]\mathbb{E}[\xi_l^{q'}].$$

The previous considerations also yield that

$$s^{\zeta(q)} = \mathbb{E}[M^q([0, s])] = \mathrm{e}^{\psi(q)}\mathbb{E}[\zeta_l^q]\{1 + \mathrm{O}(s + t)\},$$
$$t^{\zeta(q)} = \mathbb{E}[M^{q'}([1 - t, 1])] = \mathrm{e}^{\psi(q')}\mathbb{E}[\xi_l^{q'}]\{1 + \mathrm{O}(s + t)\}$$

and all the previous bounds finally yield (A.17). □

Lemma A.4. *If* $2q < q_{\max}$, *then for* $k = 1, \ldots, 2^n - 1$,

$$2^{n\zeta(2q)}\mathbb{E}[D_{0,0,n,q} D_{0,k,n,q}] = \mathrm{O}(k^{-\{\psi(2q)-2\psi(q)+1\}}). \tag{A.21}$$

Proof. By the scaling property, and since $\mathbb{E}[D_{0,k,n,q}] = 0$, we have

$$\begin{aligned}
&2^{n\zeta(2q)}\mathbb{E}[D_{0,0,n,q} D_{0,k,n,q}] \\
&\quad = k^{\zeta(2q)} \mathrm{cov}\left(M^q\left(\left[0, \frac{1}{k}\right]\right), M^q\left(\left[1 - \frac{1}{k}, 1\right]\right)\right) \\
&\qquad - 2^{\tau(q)}\left(k - \frac{1}{2}\right)^{\zeta(2q)} \mathrm{cov}\left(M^q\left(\left[0, \frac{1}{k - 1/2}\right]\right), M^q\left(\left[1 - \frac{1}{2k - 1}, 1\right]\right)\right) \\
&\qquad - 2^{\tau(q)}k^{\zeta(2q)} \mathrm{cov}\left(M^q\left(\left[0, \frac{1}{k}\right]\right), M^q\left(\left[1 - \frac{1}{2k}, 1\right]\right)\right)
\end{aligned}$$

$$
\begin{aligned}
&- 2^{\tau(q)} k^{\zeta(2q)} \operatorname{cov}\Big(M^q\Big(\Big[0, \frac{1}{2k}\Big]\Big), M^q\Big(\Big[1-\frac{1}{k}, 1\Big]\Big)\Big) \\
&+ 2^{2\tau(q)} (k-1/2)^{\zeta(2q)} \operatorname{cov}\Big(M^q\Big(\Big[0, \frac{1}{2k-1}\Big]\Big), M^q\Big(\Big[1-\frac{1}{2k-1}, 1\Big]\Big)\Big) \\
&+ 2^{2\tau(q)} k^{\zeta(2q)} \operatorname{cov}\Big(M^q\Big(\Big[0, \frac{1}{2k}\Big]\Big), M^q\Big(\Big[1-\frac{1}{2k}, 1\Big]\Big)\Big) \\
&- 2^{\tau(q)} k^{\zeta(2q)} \operatorname{cov}\Big(M^q\Big(\Big[\frac{1}{2k}, \frac{1}{k}\Big]\Big), M^q\Big(\Big[1-\frac{1}{k}, 1\Big]\Big)\Big) \\
&+ 2^{2\tau(q)} (k-1/2)^{\zeta(2q)} \operatorname{cov}\Big(M^q\Big(\Big[\frac{1}{2k-1}, \frac{2}{2k-1}\Big]\Big), M^q\Big(\Big[1-\frac{1}{2k-1}, 1\Big]\Big)\Big) \\
&+ 2^{2\tau(q)} k^{\zeta(2q)} \operatorname{cov}\Big(M^q\Big(\Big[\frac{1}{2k}, \frac{1}{k}\Big]\Big), M^q\Big(\Big[1-\frac{1}{2k}, 1\Big]\Big)\Big).
\end{aligned}
$$

Applying Lemma A.3, with s and t replaced by k and $2k$ and $q = q'$, we obtain that each covariance term that appears above is of order $k^{-2\zeta(q)-1}$, which yields $2^{n\zeta(2q)}\mathbb{E}[D_{0,0,n,q} D_{0,k,n,q}] = O(k^{\zeta(2q)-2\zeta(q)-1})$, and since $\zeta(2q) - 2\zeta(q) = 2\psi(q) - \psi(2q)$, the bound (A.21) is proved. □

Lemma A.5. *If* $4q < q_\chi$, *then*

$$
\mathbb{E}\big[D_{0,n,q}^4\big] = O\big(n2^{-n\tau(4q)} + 2^{-2n\tau(2q)}\big).
$$

Proof. Let us compute the fourth moment of $D_{0,n,q}$. For brevity, let the centered random variables $D_{0,k,n,q}$ be simply denoted by x_k. We have

$$
\begin{aligned}
\mathbb{E}\big[D_{0,n,q}^4\big] = 2^n \mathbb{E}\big[x_0^4\big] &+ \sum_{0 \le i \ne j \le 2^n - 1} \mathbb{E}\big[x_i^2 x_j^2\big] + \sum_{0 \le i \ne j \le 2^n - 1} \mathbb{E}\big[x_i^3 x_j\big] \\
&+ \sum_{\substack{1 \le i,j,k \le 2^n \\ \#\{i,j,k\} = 3}} \mathbb{E}\big[x_i^2 x_j x_k\big] + \sum_{\substack{1 \le i,j,k,l \le 2^n \\ \#\{i,j,k,l\} = 4}} \mathbb{E}[x_i x_j x_k x_l].
\end{aligned}
\tag{A.22}
$$

By the scaling property and Lemma A.3, obtain that

$$
2^{n\zeta(4q)} k^{-\zeta(4q)} \mathbb{E}\big[x_1^2 x_k^2\big] = O\big(k^{-2\zeta(2q)}\big).
$$

Since $\zeta(4q) < 2\zeta(2q)$, this yields

$$
\sum_{0 \le i \ne j \le 2^n - 1} \mathbb{E}\big[x_i^2 x_j^2\big] = O\left(2^{-n\tau(4q)} \sum_{k=0}^{2^n - 1} k^{\zeta(4q) - 2\zeta(2q)}\right) = O\big(n2^{-n\tau(4q)} + 2^{-2n\tau(2q)}\big).
$$

Again, by Lemma A.3, we have

$$
2^{n\zeta(4q)} k^{-\zeta(4q)} \mathbb{E}\big[x_1^3 x_k\big] = 2^{n\zeta(4q)} k^{-\zeta(4q)} \operatorname{cov}\big(x_1^3, x_k\big) = O\big(k^{-\zeta(3q) - \zeta(q) - 1}\big).
$$

By (3.6), if $4q < q_\chi$, then $\psi(4q) > 4\psi(3q)/3$ and $\psi(3q)/3 > \psi(q)$, so $\zeta(4q) - \zeta(3q) - \zeta(q) < 0$, thus

$$\sum_{0 \le i \ne j \le 2^n - 1} \mathbb{E}\big[x_i^3 x_j\big] = \mathrm{O}\left(2^{-n\tau(4q)} \sum_{k=0}^{2^n-1} k^{\zeta(4q)-\zeta(3q)-\zeta(q)-1}\right) = \mathrm{O}\big(2^{-n\tau(4q)}\big).$$

We now calculate the fourth term in the expansion (A.22) of $\mathbb{E}[D^4_{0,n,q}]$. By stationarity we may assume $i = 0$ and without loss of generality assume $j < k/2$. Set $y_\ell = D_{0,\ell,\log_2(k),q}$ for $\ell = 1, \dots, k$. Then by the scaling property

$$\mathbb{E}\big[x_i^2 x_j x_k\big] = \big(k/2^n\big)^{\zeta(4q)} \mathbb{E}\big[y_1^2 y_j y_k\big].$$

Since $\mathbb{E}[y_k] = 0$, from the definition of $D_{\ell,\log_2(k),q}$ we may write

$$\begin{aligned} \mathbb{E}\big[y_1^2 y_j y_k\big] &= \mathrm{cov}\big(y_1^2 y_j, y_k\big) \\ &= \sum_{l,s,t} \beta_l \alpha_s \eta_t \,\mathrm{cov}\big(M^{r_l q}(\Delta_{1,\log_2(b_l k)}) M^{(2-r_l)q}(\Delta_{1,\log_2(b_l k)}) \\ &\qquad \times M^q(\Delta_{j,\log_2(b_s k)}), M^q(\Delta_{k,\log_2(b_t k)})\big), \end{aligned} \tag{A.23}$$

where $r_l \in \{1,2\}$ and $b_l, b_s, b_t \in \{1,2\}$ indicate whether the scale is k or $2k$. Set $\ell = 1 - j/k$. In the notation of Lemmas A.2 and A.3 set $C = A_\ell((j-1)/k, j/k) \cap A_\ell((k-1)/k, 1)$, $A_1 = A_\ell(1/k, (j-1)/k)$, $A_2 = A_\ell((j-1)/k, j/k) \cap A_\ell(0, 1/k)$ and $A_3 = B_\ell(j/k, (k-1)/k)$. So that $A_i \cap A_3 = \varnothing$ for $i = 1, 2$ and $A_i \cap C = \varnothing$ for $i = 1, 2, 3$. Also define

$$\zeta_{l,1} = M_\ell^{r_l q}(\Delta_{i,\log_2(b_l k)}) M_\ell^q(\Delta_{j,\log_2(b_s k)}), \qquad \zeta_{l,2} = M_\ell^q(\Delta_{k,\log_2(b_t k)}),$$

which by construction are independent of $\mathrm{e}^{P(A_i)}$ and $\mathrm{e}^{P(C)}$. Then

$$\begin{aligned} M^{r_l q}(\Delta_{i,\log_2(b_l k)}) M^q(\Delta_{j,\log_2(b_s k)}) &= \mathrm{e}^{q r_l (P(A_1)+P(A_2))} \mathrm{e}^{q(P(A_1)+P(C))} \zeta_{l,1} \times \{1 + q\gamma_{l,1} + R_{l,1}\}, \\ M^q(\Delta_{k,\log_2(b_t k)}) &= \mathrm{e}^{q(P(A_3)+P(C))} \zeta_{l,2} \times \{1 + q\gamma_{l,2} + R_{l,2}\}, \end{aligned}$$

where $\gamma_{l,i}$ and $R_{l,i}$ are independent of $\zeta_{l,i}$, $\mathrm{e}^{P(A_i)}$ and $\mathrm{e}^{P(C)}$ and satisfy $\mathbb{E}[\gamma_{l,1}\gamma_{l,2}] = \mathrm{O}(1/k)$, $\mathbb{E}[\gamma_{l,i} R_{l,i}] = \mathrm{O}(1/k)$ and $\mathbb{E}[R_{l,1} R_{l,2}] = \mathrm{O}(1/k)$. Finally set $K_{l,1} = \mathbb{E}[\zeta_{l,1} \mathrm{e}^{q r_l (P(A_1)+P(A_2))} \times \mathrm{e}^{q P(A_1)}]$ and $K_{l,2} = \mathbb{E}[\zeta_{l,2} \mathrm{e}^{q P(A_3)}]$. Then, for each of the terms in (A.23)

$$\begin{aligned} &\mathrm{cov}\big(M^{r_l q}(\Delta_{i,\log_2(b_l k)}) M^q(\Delta_{j,\log_2(b_s k)}), M^q(\Delta_{k,\log_2(b_t k)})\big) \\ &\quad = K_{l,1} K_{l,2} \,\mathrm{var}(C)\big(1 + \mathrm{O}(1/k)\big) \\ &\quad = \frac{1}{\mathbb{E}^2[\mathrm{e}^{q P(C)}]} \mathbb{E}\big[M^{r_l q}(\Delta_{i,\log_2(b_l k)}) M^q(\Delta_{j,\log_2(b_s k)})\big] \\ &\qquad \times \mathbb{E}\big[M_\ell^q(\Delta_{k,\log_2(b_t k)})\big] \log(1 - j/k)\big(1 + \mathrm{O}(1/k)\big)^2. \end{aligned} \tag{A.24}$$

Adding up all the terms in (A.23) and using $\mathbb{E}[M_\ell^q(\Delta_{k,\log_2(b_t k)})] = \mathrm{O}(k^{-\zeta(q)})$ for all ℓ yields

$$\mathbb{E}\big[y_1^2 y_j y_k\big] = \mathrm{O}(j/k)\mathbb{E}\big[y_1^2 y_j\big]k^{\zeta(q)}.$$

On the other hand, again by the scaling property, and because $\mathbb{E}[y_j] = 0$, $\mathbb{E}[y_1^2 y_j] = \mathrm{cov}(y_1^2, y_j)$ and applying Lemma A.3 we have

$$\mathbb{E}\big[y_1^2 y_j\big] = \mathrm{O}\big(k^{-\zeta(3q)} j^{\zeta(3q)-\zeta(2q)-\zeta(q)-1}\big). \tag{A.25}$$

By (A.24) and (A.25) we obtain the bound

$$\mathbb{E}\big[x_0^2 x_j x_k\big] = \mathrm{O}\big(2^{-n\zeta(4q)} j^{\zeta(3q)-\zeta(2q)-\zeta(q)} k^{\zeta(4q)-\zeta(3q)-\zeta(q)-1}\big).$$

Noting that by convexity of ψ, it holds that $2\psi(q) < \psi(2q)$, this yields

$$\sum_{\substack{1\le i,j,k\le 2^n \\ \#\{i,j,k\}=3}} \mathbb{E}\big[x_i^2 x_j x_k\big] = \mathrm{O}\big(2^{-2n\tau(2q)} + n2^{-n\tau(4q)}\big).$$

For the last term in (A.22) by stationarity set $i = 0$, and assume $j < \ell < k$ and moreover that $\ell - j < k/2$. Write

$$\mathbb{E}[x_i x_j x_\ell x_k] = \mathrm{cov}(x_i x_j, x_\ell x_k) + \mathbb{E}[y_i y_j]\mathbb{E}[y_\ell y_k].$$

The term $\mathrm{cov}(y_1 y_j, y_\ell y_k)$ can be shown to be of smaller order than the product of expectations. Thus, applying Lemma A.4, we finally obtain

$$\sum_{\substack{1\le i,j,k,l\le 2^n \\ \#\{i,j,k,l\}=4}} \mathbb{E}[x_i x_j x_k x_\ell] = \mathrm{O}\big(2^{-2n\tau(q)}\big).$$

□

Bounds for the MRW, case $H > 1/2$. Define $\tilde{a}_{j,k,n,H} = \mathrm{e}^{w_{l_n}(t_{j,k})}\tilde{\delta}_{j,k,n,H}$ with

$$\tilde{\delta}^2_{j,k,n,H} = \int_{\Delta^{(j)}_{k,n}} \int_{\Delta^{(j)}_{k,n}} |u-v|^{2H-2} M_n(\mathrm{d}u) M_n(\mathrm{d}v)$$

and for $j_1 \neq j_2$,

$$\tilde{\rho}_H\big(j_1, j_2, k, k'\big) = \frac{\int_{\Delta^{(j_1)}_{k,n}} \int_{\Delta^{(j_2)}_{k',n}} |u-v|^{2H-2} M_n(\mathrm{d}u) M_n(\mathrm{d}v)}{\delta_{j_1,k,n,H}\,\delta_{j_2,k,n,H}}.$$

Lemma A.6. *For $p \ge 1$ such that $2pq < q_\chi$ and for $r \ge 2$, there exist $\eta, C > 0$ and uniformly bounded constants $c_{q,H}(k,k')$ such that*

$$\big|2^{n\zeta_H(2q)}\mathrm{e}^{\psi(2q)} l_n^{-\psi(2q)}\mathbb{E}\big[\tilde{\delta}^{2q}_{j,k,n,H}\big] - m_H(2q)\big| = \mathrm{O}\big(2^{-n\eta}\big), \tag{A.26}$$

$$\left|2^{n\zeta_H(2q)} \mathrm{e}^{\psi(2q)} l_n^{-\psi(2q)} \mathbb{E}\big[\tilde{\delta}_{0,k,n,H}^q \tilde{\delta}_{0,k',n,H}^q\big] - c_{q,H}\big(k,k'\big)\right| = \mathrm{O}\big(2^{-n\eta}\big), \tag{A.27}$$

$$2^{n\zeta_H(2pq)} \mathbb{E}\big[\big|a_{0,k,n,H}^{2q} - \tilde{a}_{0,k,n,H}^{2q}\big|^p\big] = \mathrm{O}\big(2^{-n\eta}\big), \tag{A.28}$$

$$2^{n\zeta_H(2pq)} \mathbb{E}\big[\big|a_{0,k,n,H}^q a_{0,k',n,H}^q - \tilde{a}_{0,k,n,H}^q \tilde{a}_{0,0,k',n,H}^q\big|^p\big] = \mathrm{O}\big(2^{-n\eta}\big). \tag{A.29}$$

Proof. Note that (A.29) implies (A.27) and (A.28) implies (A.26). By stationarity of increments, we can assume without loss of generality that $k' = 0$. For brevity, denote $a_k = a_{0,k,n,H}$, $\tilde{a}_k = \tilde{a}_{0,k,n,H}$ and $\tilde{\delta}_k = \tilde{\delta}_{0,k,n,H}$. Generalizing the notation of the proof of Lemma A.2, we can write $a_k^2 = \xi_k^2(R_k + 1)$ with $\xi_k = \mathrm{e}^{P(I_n(k))}\tilde{\delta}_k$, $I_n(k) = A_{l_n}(k2^{-n}) \setminus A_{l_n}(2^{-n})$, $B_k(u) = A_{l_n}(u) \setminus I_n(k)$ and

$$R_k = \int_{\Delta_{k,n}} \int_{\Delta_{k,n}} \big\{\mathrm{e}^{P(B_k(u))+P(B_k(v))} - 1\big\} |u-v|^{2H-2} \tilde{M}_k(\mathrm{d}u) \tilde{M}_k(\mathrm{d}v).$$

Denote $r_k = \sup_{u\in\Delta_{k,n}} |\mathrm{e}^{P(B_k(u))} - 1|$. Then $|R_k| \le (1 + r_k)^2 - 1$, the sequence $\{r_k, k = 0, \dots, 2^n - 1\}$ is independent of the measures $\tilde{M}_k$, $0 \le k \le 2^n - 1$ and by (A.7) and Hölder's inequality, we have, for $p \ge 1$, $\mathbb{E}[|r_0|^p] = \mathrm{O}(\sqrt{\mu(B_0(2^{-n})}) = \mathrm{O}(2^{-\alpha n/2})$. Thus

$$\mathbb{E}\big[\big|a_k^q - \mathrm{e}^{qP(I_n(k))}\tilde{\delta}_k^q\big|^p\big] \le \mathbb{E}\big[\mathrm{e}^{pqP(I_n(k))}\big]\mathbb{E}\big[\tilde{\delta}_0^{pq}\big]\mathrm{O}\big(2^{-\alpha n}\big),$$

which proves (A.28). Since $\mathbb{E}[\mathrm{e}^{qP(I_n(k))}] \sim \mathbb{E}[\mathrm{e}^{qw_{l_n}(0)}] = \mathrm{e}^{\psi(q)} l_n^{-\psi(q)}$, this implies that $\mathbb{E}[\tilde{\delta}_0^q] \sim c l_n^{\psi(q)} 2^{-n\zeta_H(q)}$. Next, using the bound $|(1+x)^q - 1| \le C(|x| + |x|^{q\wedge 1})$ valid for $x \ge 0$, we obtain

$$\begin{aligned}
\mathbb{E}\big[\big|a_0^q a_k^q - \xi_0^q \xi_k^q\big|^p\big] &\le \mathbb{E}\big[\xi_0^{pq}\xi_k^{pq}\big|(R_0+1)^{q/2}(R_k+1)^{q/2} - 1\big|^p\big] \\
&\le \mathbb{E}\big[\xi_0^{pq}\xi_k^{pq}\big]\mathbb{E}\big[\big|(r_0+1)^q(r_k+1)^q - 1\big|^p\big] \le C2^{-\eta n}\mathbb{E}\big[\xi_0^{2pq}\big]
\end{aligned}$$

for some $\eta > 0$. This proves (A.29). □

Lemma A.7. *If $2q < q_\chi$, then*

$$2^{n\zeta_H(2q)} \big|\mathbb{E}[U_{0,n,0}, U_{0,n,k}]\big| \le Ck^{-\{\psi(2q)-2\psi(q)+1\}}. \tag{A.30}$$

Proof. For $k \ge 1$, denote

$$\begin{aligned}
U_k &= \int_0^{1/k} \int_0^{1/k} |u-v|^{2H-2} M(\mathrm{d}u) M(\mathrm{d}v), \\
U_k' &= \int_{1/2k}^{1/k} \int_{1/2k}^{1/k} |u-v|^{2H-2} M(\mathrm{d}u) M(\mathrm{d}v), \\
V_k &= \int_{1-1/k}^{1} |u-v|^{2H-2} M(\mathrm{d}u) M(\mathrm{d}v), \\
V_k' &= \int_{1-1/k}^{1-1/2k} \int_{1-1/k}^{1-1/2k} |u-v|^{2H-2} M(\mathrm{d}u) M(\mathrm{d}v).
\end{aligned}$$

Then, by the scaling property, we have

$$
\begin{aligned}
&2^{n\zeta_H(2q)}\mathbb{E}[U_{0,n,0}U_{0,n,k}] \\
&\quad = k^{\zeta_H(2q)}\operatorname{cov}\big(U_k^q, V_k^q\big) - 2^{\tau_H(q)}(k-1/2)^{\zeta_H(2q)}\operatorname{cov}\big(U_k^q, V_{2k}^q\big) \\
&\qquad - 2^{\tau_H(q)}k^{\zeta_H(2q)}\big\{\operatorname{cov}\big(U_k^q, V_k'^q\big) - \operatorname{cov}\big(U_{2k}^q, V_k^q\big) + \operatorname{cov}\big(U_k'^q, V_k^q\big)\big\} \\
&\qquad + 2^{2\tau_H(q)}(k-1/2)^{\zeta_H(2q)}\big\{\operatorname{cov}\big(U_{2k}^q, V_{2k}^q\big) + \operatorname{cov}\big(U_k'^q, V_{2k}^q\big)\big\} \\
&\qquad + 2^{2\tau_H(q)}k^{\zeta_H(2q)}\big\{\operatorname{cov}\big(U_{2k}^q, V_k'^q\big) + \operatorname{cov}\big(U_k'^q, V_k'^q\big)\big\}.
\end{aligned}
$$

All the covariance terms are of the same order, and we only consider the first one, $\operatorname{cov}(U_k^q, V_k^q)$. Denote $l = 1 - 2/k$, define the measure $M_l(\mathrm{d}u) = \mathrm{e}^{-w_l(u)}M(\mathrm{d}u)$ and

$$
\begin{aligned}
\zeta_{k,H} &= \int_0^{1/k}\int_0^{1/k} |u-v|^{2H-2}M_l(\mathrm{d}u)M_l(\mathrm{d}v), \\
\xi_{k,H} &= \int_{1-1/k}^1\int_{1-1/k}^1 |u-v|^{2H-2}M_l(\mathrm{d}u)M_l(\mathrm{d}v), \\
A_k &= A_l(1/k)\setminus A_l(1-1/k), \qquad B_k = A_l(1-1/k)\setminus A_l(1/k), \\
\bar{A}_k(u) &= A_l(u)\setminus A_k, \qquad \bar{B}_k(u) = A_l(u)\setminus B_k, \\
\pi_k(u,v) &= P_0\big(\bar{A}_k(u)\big) + P_0\big(\bar{A}_k(v)\big), \\
\pi_k'(u,v) &= P_0\big(\bar{B}_k(u)\big) + P_0\big(\bar{B}_k(v)\big), \\
\tilde{\alpha}_k &= \zeta_{k,H}^{-1}\int_0^{1/k}\int_0^{1/k} |u-v|^{2H-2}\pi_k(u,v)M_l(\mathrm{d}u)M_l(\mathrm{d}v), \\
\tilde{\beta}_k &= \xi_{k,H}^{-1}\int_{1-1/k}^1\int_{1-1/k}^1 |u-v|^{2H-2}\pi_k'(u,v)M_l(\mathrm{d}u)M_l(\mathrm{d}v).
\end{aligned}
$$

Then we can write

$$
\begin{aligned}
U_k^q &= \mathrm{e}^{2qP(A_k)}\zeta_{k,H}^q + \mathrm{e}^{2qP(A_k)}\zeta_{k,H}^q\tilde{\alpha}_k + \mathrm{e}^{2qP(A_k)}\zeta_{k,H}^q R_k, \\
V_k^q &= \mathrm{e}^{2qP(B_k)}\xi_{k,H}^q + \mathrm{e}^{2qP(B_k)}\xi_{k,H}^q\tilde{\beta}_k + \mathrm{e}^{2qP(B_k)}\xi_{k,H}^q R_k',
\end{aligned}
$$

with

$$
\begin{aligned}
R_k &= \left(\int_0^{1/k}\int_0^{1/k} |u-v|^{2H-2}\mathrm{e}^{P(\bar{A}_k(u))+P(\bar{A}_k(v))}\tilde{M}_l(\mathrm{d}u)\tilde{M}_l(\mathrm{d}v)\right)^q - 1 - q\tilde{\alpha}_k, \\
R_k' &= \left(\int_{1-1/k}^1\int_{1-1/k}^1 |u-v|^{2H-2}\mathrm{e}^{P(\bar{B}_k(u))+P(\bar{B}_k(v))}\tilde{M}_l'(\mathrm{d}u)\tilde{M}_l'(\mathrm{d}v)\right)^q - 1 - q\tilde{\beta}_k.
\end{aligned}
$$

Note that $P(\bar{A}_k(u))$, $P(\bar{B}_k(u))$, ζ_k and ξ_k are mutually independent and by (A.20),

$$|R_k| \le C \sup_{u,v\in[0,1/k]} \left|\mathrm{e}^{P(\bar{A}_k(u))+P(\bar{A}_k(v))} - 1\right|^{q\vee 2} + C \sup_{u,v\in[0,1/k]} \left|\mathrm{e}^{P(\bar{A}_k(u))+P(\bar{A}_k(v))} - 1\right|^2$$
$$+ C \sup_{u,v\in[0,1/k]} \left|\mathrm{e}^{P(\bar{A}_k(u))+P(\bar{A}_k(v))} - 1 - \pi_k(u,v)\right|.$$

Applying now the bounds (A.7) and (A.8) we obtain that

$$\mathbb{E}\bigl[U_k^q\bigr] = \mathrm{e}^{2\psi(q)}\mathbb{E}\bigl[\zeta_{k,H}^q\bigr]\bigl\{1+\mathrm{O}\bigl(k^{-1}\bigr)\bigr\} + q\mathrm{e}^{2\psi(q)}\mathbb{E}\bigl[\zeta_{k,H}^q\tilde{\alpha}_k\bigr],$$
$$\mathbb{E}\bigl[V_k^q\bigr] = \mathrm{e}^{2\psi(q)}\mathbb{E}\bigl[\xi_{k,H}^q\bigr]\bigl\{1+\mathrm{O}\bigl(k^{-1}\bigr)\bigr\} + q\mathrm{e}^{2\psi(q)}\mathbb{E}\bigl[\xi_{k,H}^q\tilde{\beta}_k\bigr],$$
$$\mathbb{E}\bigl[U_k^q V_k^q\bigr] = \mathrm{e}^{4\psi(q)}\mathbb{E}\bigl[\zeta_{k,H}^q\bigr]\mathbb{E}\bigl[\xi_{k,H}^q\bigr]\bigl\{1+\mathrm{O}\bigl(k^{-1}\bigr)\bigr\} + q\mathrm{e}^{2\psi(q)}\bigl\{\mathbb{E}\bigl[\xi_{k,H}^q\bigr]\mathbb{E}\bigl[\zeta_{k,H}^q\tilde{\alpha}_k\bigr]$$
$$+ \mathbb{E}\bigl[\zeta_{k,h}^q\bigr]\mathbb{E}\bigl[\xi_{k,H}^q\tilde{\beta}_k\bigr]\bigr\}.$$

Combining these bounds yields the requested bound for $\operatorname{cov}(U_k^q, V_k^q)$ and (A.30). □

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