Estimating the support of a scaling vector

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Abstract

An estimate is given for the support of each component function of a compactly supported scaling vector satisfying a matrix refinement equation with finite number of terms. The estimate is based on the highest and lowest degree of each polynomial in the corresponding matrix symbol. Only basic techniques from matrix theory are involved in the derivation.

1 Introduction

In this paper we are interested in measurable functions from the reals $\mathbf{R}$ to the complex $\mathbf{C}$; two functions are equal if they are identical almost everywhere. Let r be a positive integer and $F = [f_1 \ \ldots \ f_r]^T$ be a complex vector-valued function on $\mathbf{R}$, where ^{T} denotes the transpose of a matrix. A point $t \in \mathbf{R}$ is called a *support point* of F if the measure of the intersection $\{x : F(x) \neq 0\} \cap (t-\epsilon, t+\epsilon)$ is not zero for any $\epsilon > 0$. The support of F, denoted by $supp(F)$, is defined as the convex hull of the set of support points of F. Hence equal functions have same supports; and the support of a nonzero function is always a close interval with positve length. Note that, in the literature of wavelet theory with $r = 1$, the support of a scaling function is always taken to be a closed interval because of the result in [5] (also see [1, pg 252]).

Recent interests in multiwavelets lead to the study of scaling vector $\Phi = [\phi_1 \ \ldots \ \phi_r]^T$ which is a vector-valued function satisfying a matrix refinement

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equation (MRE) with finite number of terms

$$\Phi(x) = \sum_{k=0}^{N} C_k \Phi(2x-k) \tag{1}$$

where C_k's are $r \times r$ matrices. In applications, shortly supported multiwavelets are always desired. Support of multiwavelets can be obtained easily from the support of the corresponding scaling vectors. Hence it is useful to estimate the support of scaling vectors from the defining MRE. However the determination of the support of a scaling vector is not straightforward. In [3], Heil and Colella observed that $supp(\Phi) \subset [0, N]$ if Φ is compactly supported. But this estimate is too crude as the following example, due to Geronimo, Hardin, and Massopust [2], shows.

Example. Let $\Phi = [\phi_1 \ \phi_2]^T$ be a scaling vector satisfying the MRE (1) with matrix coefficients:

$$C_0 = \frac{1}{20}\begin{bmatrix} 12 & 16\sqrt{2} \\ -\sqrt{2} & -6 \end{bmatrix}, \quad C_1 = \frac{1}{20}\begin{bmatrix} 12 & 0 \\ 9\sqrt{2} & 20 \end{bmatrix},$$

$$C_2 = \frac{1}{20}\begin{bmatrix} 0 & 0 \\ 9\sqrt{2} & -6 \end{bmatrix}, \quad C_3 = \frac{1}{20}\begin{bmatrix} 0 & 0 \\ -\sqrt{2} & 0 \end{bmatrix}.$$

Note that $supp(\phi_1) = [0,1]$, $supp(\phi_2) = [0,2]$ and so $supp(\Phi) = [0,2] \neq [0,3]$.

An explanation is the existence of nilpotent matrices. Note that C_3 in the above example is nilpotent. In [6], Massopust, Ruch and Van Fleet showed that $supp(\Phi) \subset [0, N - \frac{1}{2^r-1}]$ if C_N is nilpotent, and $supp(\Phi) \subset [\frac{1}{2^r-1}, N]$ if C_0 is nilpotent. However such improved estimates are still not good enough to explain the above example.

In this paper, we give an estimate for each componentwise support $supp(\phi_i)$ and hence the global support $supp(\Phi)$. Sufficient conditions are given for these estimates to be acheived. The rest of the paper is organized as follows. Our main results are stated in §2 with an illustration. Proofs are given in §3. §4 is devoted to the study of the global support of a scaling vector.

2 Componentwise support of a scaling vector

For the rest of the paper, let $\Phi = [\phi_1 \ \ldots \ \phi_r]^T$ be a compactly supported scaling vector satisfying the MRE (1). In this section we are interested in estimating the support $supp(\phi_i)$ for $1 \le i \le r$. To this end, we define the associated matrix symbol by

$$P(z) = \sum_{k=0}^{N} C_k z^k,$$

which is a $r \times r$ matrix with polynomial entries. Let $h(i,j)$ (resp. $l(i,j)$) be the highest (resp. lowest) degree of the (i,j)-entry of $P(z)$. We adopt the convention that the highest (resp. lowest) degree of the zero polynomial is $-\infty$ (resp. ∞).

I_k denotes the $k \times k$ identity matrix and e_k denotes the k-th column of the identity matrix whose dimension is determined from the context. For positive integers a, b, E_{ab} denotes the matrix $e_a e_b^T$.

Let $\mathcal{J}$ be the set of all integer sequences $J = (j_1, \ldots, j_r)$ where $1 \le j_1, \ldots, j_r \le r$. For each $J = (j_1, \ldots, j_r) \in \mathcal{J}$, define

$$E_J = 2I_r - E_{1j_1} - \cdots - E_{rj_r},$$

$$h_J = [h(1,j_1) \ \cdots \ h(r,j_r)]^T \qquad \text{and} \qquad l_J = [l(1,j_1) \ \cdots \ l(r,j_r)]^T.$$

Note that E_J is always invertible (see Lemma 3.2).

Theorem 2.1. For $1 \le i \le r$, the support of ϕ_i is a finite closed interval $[L_i, R_i]$ where

$$R_i \le \max\left\{ e_i^T E_J^{-1} h_J : J \in \mathcal{J} \right\}$$

and

$$L_i \ge \min\left\{ e_i^T E_J^{-1} l_J : J \in \mathcal{J} \right\}.$$

In Theorem 2.1, both maximization and minimization are with respect to the set $\mathcal{J}$ which has r^r elements. In order to reduce the complexity we introduce the following concepts. For each $J = (j_1, \ldots, j_r) \in \mathcal{J}$ and $1 \le i \le r$, define a new integer sequence $\gamma = (\gamma_0, \gamma_1, \ldots, \gamma_t)$ satisfying the following conditions:

1. $1 \le t \le r$,
2. $\gamma_0 = i$,

3. $\gamma_k = j_{\gamma_{(k-1)}}$ for $k = 1, \ldots, t$,

4. $\gamma_0, \ldots, \gamma_{t-1}$ are distinct,

5. $\gamma_t = \gamma_{s-1}$ for some $1 \leq s \leq t$.

The existence of γ, s and t is clear and they are uniquely determined by the sequence $J = (j_1, \ldots, j_r)$ and the integer i. As examples, take $r = 4$. If $J = (3, 2, 4, 3)$ and $i = 1$, then $\gamma = (1, 3, 4, 3)$ $t = 3$, and $s = 2$. If $J = (3, 2, 4, 3)$ and $i = 2$, then $\gamma = (2, 2)$, $t = 1$, and $s = 1$.

For fixed i, let Γ_i be the collection of all such γ's. Let s and t be the numbers corresponding to a given $\gamma \in \Gamma_i$. Define a $t \times t$ matrix by

$$A_\gamma = 2I_t - E_{12} - E_{23} - \cdots - E_{(t-1)t} - E_{ts}.$$

Note that $A_\gamma = E_J$ for $J = (2, 3, \ldots, t-1, s)$ and so A_γ is invertible (see Lemma 3.2). Define

$$h_\gamma = [h(\gamma_0, \gamma_1)\ h(\gamma_1, \gamma_2)\ \ldots\ h(\gamma_{t-1}, \gamma_t)]^T$$

and

$$l_\gamma = [l(\gamma_0, \gamma_1)\ l(\gamma_1, \gamma_2)\ \ldots\ l(\gamma_{t-1}, \gamma_t)]^T.$$

Theorem 2.2. For $1 \leq i \leq r$, the support of ϕ_i is a finite closed interval $[L_i, R_i]$ where

$$R_i \leq \max\left\{e_1^T A_\gamma^{-1} h_\gamma : \gamma \in \Gamma_i\right\}$$

and

$$L_i \geq \min\left\{e_1^T A_\gamma^{-1} l_\gamma : \gamma \in \Gamma_i\right\}.$$

In Theorem 2.2, both maximization and minimization are with respect to the set Γ_i. The number of elements in Γ_i is $\sum_{k=0}^{r-1} \left(\begin{array}{c} r-1 \\ k \end{array}\right) (k+1)!$ which can be proved to be equal to the integral part of the positive number $(r-1)!(r-1)e + 1$, where e is the base of natural logarithm. Hence the complexity of the optimization is reduced to $(r-1)!(r-1)e + 1$ from r^r in Theorem 2.1.

Using the classical adjoint formula for matrix inverse [4, pg 20], it is not hard to see that the first row of A_γ^{-1} is

$$e_1^T A_\gamma^{-1} = \left[\frac{1}{2} \quad \cdots \quad \frac{1}{2^{s-1}} \quad (\frac{2^t}{2^t - 2^{s-1}})\frac{1}{2^s} \quad \cdots \quad (\frac{2^t}{2^t - 2^{s-1}})\frac{1}{2^t}\right].$$

Therefore Theorem 2.2 can be restated explicitly as follows.

Theorem 2.3. For $1 \le i \le r$, the support of ϕ_i is a finite closed interval $[L_i, R_i]$ where

$$R_i \le \max_{\gamma \in \Gamma_i} \left\{ \sum_{k=1}^{s-1} \frac{1}{2^k} h(\gamma_{(k-1)}, \gamma_k) + \frac{2^t}{2^t - 2^{s-1}} \sum_{k=s}^{t} \frac{1}{2^k} h(\gamma_{(k-1)}, \gamma_k) \right\},$$

and

$$L_i \ge \min_{\gamma \in \Gamma_i} \left\{ \sum_{k=1}^{s-1} \frac{1}{2^k} l(\gamma_{(k-1)}, \gamma_k) + \frac{2^t}{2^t - 2^{s-1}} \sum_{k=s}^{t} \frac{1}{2^k} l(\gamma_{(k-1)}, \gamma_k) \right\}.$$

A family $\{f_i\}$ of functions on $\mathbf{R}$ is *locally linearly independent* if $\sum_i c_i f_i(x) = 0$ on any nontrivial interval (a, b) then $c_i = 0$ for those $supp(f_i) \cap (a, b) \neq \emptyset$. $\Phi = [\phi_1, \ldots, \phi_r]^T$ is called a *locally linearly independent* scaling vector if the family $\{\phi_j(x - k) : 1 \le j \le r, k \in \mathbf{Z}\}$ is locally linearly independent. In this case, the family $\{\phi_j(2x - k) : 1 \le j \le r, k \in \mathbf{Z}\}$ is also locally linearly independent. This fact will be used in Lemma 3.4.

Theorem 2.4. If Φ is a locally linearly independent scaling vector then all inequalities become equalities in Theorems 2.1, 2.2, and 2.3.

Choosing $r = 2$ in Theorem 2.3, it yields

$$R_1 \le \max \left\{ h(1,1), \;\; \frac{2}{3}h(1,2) + \frac{1}{3}h(2,1), \;\; \frac{1}{2}h(1,2) + \frac{1}{2}h(2,2) \right\},$$

$$R_2 \le \max \left\{ h(2,2), \;\; \frac{2}{3}h(2,1) + \frac{1}{3}h(1,2), \;\; \frac{1}{2}h(2,1) + \frac{1}{2}h(1,1) \right\},$$

$$L_1 \ge \min \left\{ l(1,1), \;\; \frac{2}{3}l(1,2) + \frac{1}{3}l(2,1), \;\; \frac{1}{2}l(1,2) + \frac{1}{2}l(2,2) \right\},$$

and

$$L_2 \ge \min \left\{ l(2,2), \;\; \frac{2}{3}l(2,1) + \frac{1}{3}l(1,2), \;\; \frac{1}{2}l(2,1) + \frac{1}{2}l(1,1) \right\}.$$

As an illustration, we use these formulas to estimate the support of the scaling vector mentioned in the example of §1. The highest and lowest degree matrices are respectively $h = \begin{bmatrix} 1 & 0 \\ 3 & 2 \end{bmatrix}$ and $l = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$. Hence $0 \le L_1 \le R_1 \le 1$ and $0 \le L_2 \le R_2 \le 2$. Furthermore, Φ is known to be locally linearly independent [2] and so we have $supp(\phi_1) = [L_1, R_1] = [0, 1]$ and $supp(\phi_2) = [L_2, R_2] = [0, 2]$.

3 Proofs

We need two lemmas for the proof of Theorem 2.1.

Lemma 3.1. Let $\{f_i\}$ be a family of functions on $\mathbf{R}$. Then

$$supp\left(\sum_i c_i f_i\right) \subset \mathrm{conv}\left(\cup_i supp(f_i)\right)$$

where 'conv' denotes the convex hull of a set.

Lemma 3.2. For $J \in \mathcal{J}$, the matrix E_J is invertible and its inverse has nonnegative entries.

Proof. Let $E = E_{1j_r} + \cdots + E_{rj_r}$. Note that $\|E\| = 1$ where $\|\cdot\|$ is the maximum row sum norm. Then $E_J = 2I_r - E$ is invertible and actually

$$E_J^{-1} = \sum_{k=0}^{\infty} \frac{1}{2^{k+1}} E^k$$

which has nonnegative entries because E has nonnegative entries. □

We are ready to prove Theorem 2.1.

Proof of Theorem 2.1. For each $1 \le i \le r$, using the MRE (1), we have

$$\begin{aligned} \phi_i(x) &= \sum_{k=0}^{N}\sum_{j=1}^{r} C_k(i,j)\phi_j(2x-k) \\ &= \sum_{j=1}^{r}\sum_{k=0}^{N} C_k(i,j)\phi_j(2x-k) \\ &= \sum_{j=1}^{r}\sum_{k=l(i,j)}^{h(i,j)} C_k(i,j)\phi_j(2x-k) \end{aligned}$$

where $C_k(i,j)$ is the (i,j)-entry of the matrix C_k. Since ϕ_i has compact support, we let $supp(\phi_i) = [L_i, R_i]$. By Lemma 3.1, we have

$$\begin{aligned} [L_i, R_i] &\subset \mathrm{conv}\left(\cup_{j=1}^{r} supp\left(\sum_{k=l(i,j)}^{h(i,j)} C_k(i,j)\phi_j(2x-k)\right)\right) \\ &= \mathrm{conv}\left(\cup_{j=1}^{r}\left[\frac{1}{2}(L_j + l(i,j)), \frac{1}{2}(R_j + h(i,j))\right]\right). \end{aligned}$$

Hence we have

$$2R_i \leq \max\{R_j + h(i,j) : 1 \leq j \leq r\},$$

and

$$2L_i \geq \min\{L_j + l(i,j) : 1 \leq j \leq r\}.$$

For each $1 \leq i \leq r$, there exist integers $1 \leq j_1, \ldots, j_r \leq r$ such that

$$2R_i \leq R_{j_i} + h(i, j_i).$$

In matrix form,

$$E_J \begin{bmatrix} R_1 \\ \vdots \\ R_r \end{bmatrix} = (2I - \sum_{t=1}^{r} E_{tj_t}) \begin{bmatrix} R_1 \\ \vdots \\ R_r \end{bmatrix} \leq \begin{bmatrix} h(1,j_1) \\ \vdots \\ h(r,j_r) \end{bmatrix} = h_J$$

where $J = (j_1, \ldots, j_r)$. By Lemma 3.2, E_J^{-1} is nonnegative matrix and so

$$\begin{bmatrix} R_1 \\ \vdots \\ R_r \end{bmatrix} \leq E_J^{-1} h_J.$$

Hence

$$R_i \leq e_i^T E_J^{-1} h_J \leq \max_{J \in \mathcal{J}} e_i^T E_J^{-1} h_J.$$

Similarly, the lower bound for L_i is obtained. □

Lemma 3.3. Let p be a permutation on $\{1, \ldots, r\}$ and P be the $r \times r$ matrix associated with p. Then $P^{-1} = P^T, e_k^T P = e_{p^{-1}(k)}^T, P^T E_{ab} P = E_{p^{-1}(a)p^{-1}(b)}$, and $[v_1\ v_2\ \ldots\ v_r]P = [v_{p(1)}\ v_{p(2)}\ \ldots\ v_{p(r)}]$.

Finally we give the proof of Theorem 2.2.

Proof of Theorem 2.2. Given $J \in \mathcal{J}$ and $1 \leq i \leq r$, let γ, s, t be the corresponding sequence and numbers defined in §2. It suffices to prove that

$$e_i^T E_J^{-1} h_J = e_1^T A_\gamma^{-1} h_\gamma.$$

Take a permutation p on $\{1, \ldots, n\}$ such that $p(k) = \gamma_{(k-1)}$ for $k = 1, \ldots, t$. Such permutation exists because the integers $\gamma_0, \ldots, \gamma_{t-1}$ are distinct. Using Lemma 3.3, we have $P^T e_{\gamma_{(k-1)}} = e_{p^{-1}(\gamma_{(k-1)})} = e_k$ for

$k = 1, \ldots, t$. It follows that $e_i^T P = e_{p^{-1}(i)}^T = e_1^T$ because $p(1) = \gamma_0 = i$, $P^T h_J = \begin{bmatrix} h_\gamma \\ * \end{bmatrix}$, and

$$\begin{aligned} P^T E_J P &= P^T \left(2I_r - \sum_{k=1}^{r} E_{kj_k} \right) P \\ &= 2I_r - P^T \left(\sum_{k=1}^{t} E_{\gamma_{(k-1)}\gamma_k} + \sum_{k \notin \gamma} E_{kj_k} \right) P \\ &= 2I_r - \sum_{k=1}^{t} E_{p^{-1}(\gamma_{k-1})p^{-1}(\gamma_k)} - \sum_{k \notin \gamma} E_{p^{-1}(k)p^{-1}(j_k)} \\ &= 2I_r - \sum_{k=1}^{t} E_{k\ (k+1)} - \sum_{k>t} E_{kj_k} \\ &= \begin{bmatrix} A_\gamma & 0 \\ * & * \end{bmatrix}. \end{aligned}$$

Finally,

$$\begin{aligned} e_i^T E_J^{-1} h_J &= (e_i^T P)\ (P^T E_J^{-1} P)\ (P^T h_J) \\ &= (e_i^T P)\ (P^T E_J P)^{-1}\ (P^T h_J) \\ &= e_1^T \begin{bmatrix} A_\gamma & 0 \\ * & * \end{bmatrix}^{-1} \begin{bmatrix} h_\gamma \\ * \end{bmatrix} \\ &= e_1^T A_\gamma^{-1} h_\gamma. \ \Box \end{aligned}$$

Lemma 3.4. Let $\{f_1, \ldots f_n\}$ be a family of locally linearly independent functions on $\mathbf{R}$ such that $supp(f_i) = [a_i, b_i]$ where $a_i < b_i$. Then

$$supp \left(\sum_{i=1}^{n} c_i f_i \right) = [a, b]$$

where $a = min\,\{a_i : c_i \neq 0\}$ and $b = max\,\{b_i : c_i \neq 0\}$.

Proof. Let $a_l = min\,\{a_i : c_i \neq 0\}$ and $b_h = max\,\{b_i : c_i \neq 0\}$. By Lemma 3.1, $\sum_{i=1}^{n} c_i f_i$ is compactly supported and

$$supp \left(\sum_{i=1}^{n} c_i f_i \right) = [a, b] \subset [a_l, b_h] = \operatorname{conv}\left(\cup_i supp(f_i)\right).$$

It remains to show that $a = a_l$ and $b = b_h$. Assume the contrary that $b < b_h$. Then $\sum_{i=1}^{n} c_i f_i(x) = 0$ on $(b_h - \epsilon, b_h)$ for $0 < \epsilon < \min_i \left\{ \frac{b_i - a_i}{2} \right\}$. Note that $[a_h, b_h] \cap (b_h - \epsilon, b_h) \neq \emptyset$. By the local linear independence of $\{f_i\}$, $c_h = 0$ which is impossible by the definition of b_h. The argument for $a = a_l$ is similar. $\square$.

Proof of Theorem 2.4. It suffices to give the proof involving Theorem 2.1. For each $1 \leq i \leq r$, using the MRE (1), we have

$$\begin{aligned} \phi_i(x) &= \sum_{k=0}^{N} \sum_{j=1}^{r} C_k(i,j) \phi_j(2x-k) \\ &= \sum_{j=1}^{r} \sum_{k=0}^{N} C_k(i,j) \phi_j(2x-k) \\ &= \sum_{j=1}^{r} \sum_{k=l(i,j)}^{h(i,j)} C_k(i,j) \phi_j(2x-k) \end{aligned}$$

where $C_k(i,j)$ is the (i,j)-entry of the matrix C_k. Since ϕ_i has compact support, we let $supp(\phi_i) = [L_i, R_i]$. By Lemma 3.4, we have

$$\begin{aligned} [L_i, R_i] &= \mathrm{conv}\left(\cup_{j=1}^{r} supp \left(\sum_{k=l(i,j)}^{h(i,j)} C_k(i,j) \phi_j(2x-k) \right) \right) \\ &= \mathrm{conv}\left(\cup_{j=1}^{r} \left[\frac{1}{2}(L_j + l(i,j)), \frac{1}{2}(R_j + h(i,j)) \right] \right). \end{aligned}$$

The rest of the proof is exactly the same as the proof of Theorem 2.1 with the modification that all inequalities are changed to equalities. $\square$

4 Global support of a scaling vector

In this section we are interested in the global support $supp(\Phi)$ of Φ satisfying the MRE (1). From the last section, we know that $supp(\phi_i) = [L_i, R_i]$ for $1 \leq i \leq r$. Hence $supp(\Phi) = [L, R]$ where $R = \max\{R_i : 1 \leq i \leq r\}$ and $L = \min\{L_i : 1 \leq i \leq r\}$. Theorem 2.3 gives the estimates as

$$R \leq \max_{1 \leq i \leq r} \max_{\gamma \in \Gamma_i} \left\{ \sum_{k=1}^{s-1} \frac{1}{2^k} h(\gamma_{k-1}, \gamma_k) + \frac{2^t}{2^t - 2^{s-1}} \sum_{k=s}^{t} \frac{1}{2^k} h(\gamma_{k-1}, \gamma_k) \right\},$$

and

$$L \geq \min_{1\leq i\leq r} \min_{\gamma\in\Gamma_i} \left\{ \sum_{k=1}^{s-1} \frac{1}{2^k} l(\gamma_{k-1},\gamma_k) + \frac{2^t}{2^t - 2^{s-1}} \sum_{k=s}^{t} \frac{1}{2^k} l(\gamma_{k-1},\gamma_k) \right\}.$$

Theorem 4.1. (i) If C_N is a nilpotent matrix of index m i.e. $(C_N)^m = 0$, then $R \leq N - \frac{1}{2^m-1}$.

(ii) If C_0 is a nilpotent matrix of index m i.e. $(C_0)^m = 0$, then $L \geq \frac{1}{2^m-1}$.

Proof. Using Lemma 3.1, it is not hard to see that $supp(\Phi) = supp(A\Phi)$ for any invertible matrix A.

(i) Without loss of generality, we can assume that C_N is reduced to the Jordan form $J_m(0) \oplus \cdots$ where $J_m(0)$ is a lower triangular Jordan block with largest size. Hence the highest degree matrix satisfies

$$h \leq (N-1)One(r) + C_N$$

where $One(r)$ is a $r \times r$ matrix with all entries equal to 1. Now it is not hard to see that the maximum is attained at $i = m$ and $\gamma = (m, m-1, \ldots, 1, m)$. Actually, the maximum is equal to

$$\frac{2^m}{2^m - 1} \sum_{k=1}^{m} \frac{1}{2^k} h(\gamma_{k-1},\gamma_k) \leq N - \frac{1}{2^m - 1}.$$

(ii) Without loss of generality, we can C_0 is reduced to the Jordan form $J_m(0) \oplus \cdots$ where $J_m(0)$ is a lower triangular Jordan block with largest size. Hence the lowest degree matrix satisfies

$$l \geq One(r) - C_0.$$

Now it is not hard to see that the minimum is attained at $i = m$ and $\gamma = (m, m-1, \ldots, 1, m)$. Actually, the minimum is equal to

$$\frac{2^m}{2^m - 1} \sum_{k=1}^{m} \frac{1}{2^k} l(\gamma_{k-1},\gamma_k) \geq \frac{1}{2^m - 1}. \quad \square$$

Setting $m = r$, we obtain the result of Massopust, Ruch and Van Fleet mentioned in the introduction.

Corollary 4.2. (i) If C_N is nilpotent, then $supp(\Phi) \subset [0, N - \frac{1}{2^r-1}]$.

(ii) If C_0 is nilpotent, then $supp(\Phi) \subset [\frac{1}{2^r-1}, N]$.

References

[1] I. Daubechies, *Ten Lectures on Wavelets*, SIAM Press, Philadelphia, 1992.

[2] J.S. Geronimo, D.P. Hardin, and P.R. Massopust, Fractal functions and wavelet expansions based on several scaling functions, *J. Approximation Theory*, 78 (1994) 373–401.

[3] C. Heil and D. Colella, Matrix refinement equations: existence and uniqueness, *Preprint*, 1994.

[4] R. Horn and C. Johnson, *Matrix Analysis*, Cambridge, New York, 1989.

[5] P.G. Lemarie and G. Malgouyres, Support des fonctions de base daus une analyse multiresolution. *Comptes Rendus del l'Acad. Sci. Paris*, 313 (1991)377–380.

[6] P. Massopust, D. Ruch, and P. Van Fleet, On the support properties of scaling vectors, *Preprint*, 1994.