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Estimation and decomposition of downside risk for portfolios with non-normal returns

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KBI 0730

Estimation and Decomposition of Downside Risk for Portfolios with Non-Normal Returns

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First version: 17 August, 2007

This version: 29 September, 2007

Abstract

Modified Value at Risk (VaR) is an estimator of VaR based on the Cornish-Fisher expansion. It is fast to compute and reliable for non-normal returns. In this paper, we introduce modified Expected Shortfall as a new analytical estimator for Expected Shortfall (ES), another popular measure of downside risk. We give all the necessary formulas for computing portfolio modified VaR and ES and for decomposing these risk measures into the contributions made by each of the portfolio holdings. This new methodology is shown to be very useful for analyzing the risk properties of portfolios of alternative investments.

Keywords: Alternative investments; Component Value at Risk; Cornish-Fisher expansion; downside risk; expected shortfall; portfolio; risk contribution; Value at Risk.

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The authors have benefited from helpful comments by Peter Carl. Financial support from the FWO and Research Fund K.U.Leuven (contract number: GOA/2007/04) is gratefully acknowledged. The usual disclaimer applies.

1 Introduction

Value at Risk (VaR) and Expected Shortfall (ES) have emerged as industry standards for measuring downside risk. VaR was first published as a standard measure in 1994 by J.P.Morgan's RiskMetrics Group. VaR is the maximum potential loss incurred by an investment at a given time horizon such that higher losses will only occur with at most a preset probability level, denoted α, which in general is between 1 and 5 per cent. Expected shortfall is the expected value of losses that exceed the α quantile (Acerbi and Tache, 2002). From these definitions follows that the computation of VaR and ES requires the estimation of the probabilities of future losses.

Despite the variety of complex estimation methods based on Monte Carlo simulation, extreme value theory and quantile regression proposed in the literature (see Kuester *et al.*, 2007, for a recent review), many practitioners either use the empirical or the Gaussian distribution function to predict portfolio downside risk. The potential advantage of using the empirical distribution function over the hypothetical Gaussian distribution function is that only the information in the return series is used to estimate downside risk, without any distributional assumptions. The disadvantage is that the resulting estimates of VaR and ES, called historical VaR and ES, are less accurate. Because of this, it is common to use an estimate based on a parametric class of distribution function. J.P.Morgan/Reuters' RiskMetrics (1996) parametric VaR methodology assumes the Gaussian distribution function. Gaussian VaR and ES neglect the well established fact that many financial time series are skewed and fat tailed. It is intuitively clear that incorporating the asymmetry and the thickness of tails of the density function into the downside risk estimates will lead to more accurate risk forecasts. This statement has been empirically verified by Giot and Laurent (2003).

Ideally, estimators of portfolio downside risk provide not only accurate estimates of the downside risk of the whole portfolio, but also allow decomposition into the component risk contribution of individual portfolio assets. Estimates of how much risk each asset in the portfolio contributes to the total portfolio risk are extremely important for portfolio risk allocation and for portfolio risk monitoring. They constitute a central tool to help financial institutions enforce a risk budget stating the bounds within which the risky asset positions have to remain (Sharpe, 2002).

Garman (1997) and Gouriéroux *et al.* (2000) show that the derivative of VaR with respect to the portfolio weight of an asset multiplied by the portfolio weight of that asset qualifies as a good estimate for the risk contribution of that asset, which is called "Component VaR" by Garman (1997). Under this approach, risk contributions can be easily computed for Gaussian VaR, but not for historical VaR, since this risk measure cannot be expressed as an analytical function of the portfolio weights.

Modified VaR as proposed by Zangari (1996) is an estimator for VaR that corrects Gaussian VaR for skewness and excess kurtosis in the return series while preserving the faculty to decompose portfolio risk into the components due to the different assets in the portfolio. It relies on adjusting the Gaussian quantile function for skewness and kurtosis, using the Cornish-Fisher expansion (Cornish and Fisher, 1937). Modified VaR has been used by Favre and Galeano (2002a,b) and by Amenc *et al.* (2003) to construct mean-VaR efficient portfolios and by Qian (2006) to do risk attribution for a balanced portfolio.

In this paper, we derive a definition for modified ES that, like modified VaR, uses asymptotic expansions to adjust the Gaussian distribution function for the non-normality in the observed return series. We are the first to give all the formulas needed for decomposing modified VaR and ES into the risk contributions of the assets in the portfolio and to illustrate their practical use for a portfolio of alternative investments. The practical application of risk decomposition to a portfolio containing multiple assets with non-normal distributions should allow for portfolio construction that more closely resembles investor preferences, and allow risk managers to better monitor and control risk in the portfolio.

The remainder of this paper is organized as follows. In Section 2, we review important results from the literature on computation and attribution of portfolio risk. Section 3 first introduces the Edgeworth and Cornish-Fisher expansion and then uses them to define modified ES. Section 4 investigates how well modified VaR and ES approximate VaR and ES for the skewed Student t distribution. For computing modified VaR and ES we need estimates of the multivariate moments of the return series. Since higher order moments such as skewness and kurtosis are extremely sensitive to outliers, we propose a robust estimation scheme in Section 5. The usefulness of this new methodology is illustrated in Section 6 where we analyze downside risk of the maximum Sharpe ratio portfolio of different hedge fund investment style indices. Section 7 summarizes our conclusions and outlines the implications for further research.

2 Computation and decomposition of portfolio risk

This section reviews useful results from the literature on computation and decomposition of portfolio risk. We focus on the parametric approach assuming the returns to be Gaussian distributed and the Cornish-Fisher approach under which the returns can be non-normal. We stress Sharpe (2002)'s intuition that a mere mathematical decomposition of total portfolio risk does not necessarily qualify as risk contribution and investigate the financial interpretation of the proposed definition of risk contribution. We recall Garman (1997)'s result that portfolio risk and the contribution to the total portfolio risk by each component in a portfolio are readily computed under the assumption of normality. We conclude this review with Zangari (1996)'s proposal to use the Cornish-Fisher expansion to obtain reliable estimates of VaR of portfolios with non-normal returns and provide explicit formulae for the calculation of risk contributions under this approach.

We consider an investor who allocates his portfolio between n assets, with weights $w = (w_1, ..., w_n)'$. We stack the n asset returns into the random vector $r = (r_1, ..., r_n)'$, which we assume to be strictly stationary with mean μ and covariance matrix Σ. It follows that the portfolio return r_p has mean $w'\mu$ and variance $w'\Sigma w$. Under the additional assumption that the portfolio return distribution $F(\cdot)$ is continuous[1], the VaR and ES of r_p as a function of α, are defined as follows

$$\begin{aligned} \text{VaR}(\alpha) &= -F^{-1}(\alpha) \\ \text{ES}(\alpha) &= -E_F\left[r_p | r_p \leq F^{-1}(\alpha)\right], \end{aligned} \tag{1}$$

with $F^{-1}(\cdot)$ the quantile function associated to $F(\cdot)$ and $E_F[\cdot|\cdot]$ the operator that takes the conditional expectation under $F(\cdot)$.

Portfolio risk decomposition. For the purpose of portfolio risk decomposition, we follow Martin *et al.* (2001) in requiring the risk measures to be 1-homogeneous, meaning that if the weight vector is multiplied by some scalar b, then also these risk measures are multiplied by b. From a mathematical perspective, risk decomposition is straightforward for such risk measures, thanks to Euler's homogeneous function theorem stating that for 1-homogenous $f(w)$, we have

$$f(w) = \sum_{i=1}^{n} w_i \partial_i f(w),$$

[1] Definitions of VaR and ES that apply also to discontinuous distribution functions can be found in Acerbi and Tasche (2002).

where $\partial_i f(w) = \partial f(w)/\partial w_i$. Under this decomposition, the Contribution of asset i to the risk measure $f(w)$, $C_i f(w)$, and its percentage Contribution, $\%C_i f(w)$, which are also called the (percentage) Component of asset i in the portfolio risk measure $f(w)$, equal

$$C_i f(w) = w_i \partial_i f(w) \quad \text{and} \quad \%C_i f(w) = C_i f(w)/f(w). \tag{2}$$

Sharpe (2002) warns that a mere mathematical decomposition of portfolio risk does not necessarily qualify as risk contribution. Gouriéroux *et al.* (2000) and Qian (2006), however, show that for VaR, this mathematical decomposition of portfolio risk has a financial meaning. It equals the negative value of the asset's expected contribution to the portfolio return when the portfolio return equals the negative portfolio VaR:

$$C_i \text{VaR}(\alpha) = w_i \partial_i \text{VaR}(\alpha) = -E\left[w_i r_i | r_p = -\text{VaR}(\alpha)\right]. \tag{3}$$

In Appendix A we establish that contribution to $\text{ES}(\alpha)$ can be interpreted as the expected contribution to portfolio return when the portfolio return is at least the negative value of $\text{VaR}(\alpha)$:

$$C_i \text{ES}(\alpha) = w_i \partial_i \text{ES}(\alpha) = -E\left[w_i r_i | r_p \leq -\text{VaR}(\alpha)\right]. \tag{4}$$

Derivative of portfolio moments. The definition of risk contributions in (2) will only be useful in practice for risk measures for which the derivative with respect to the portfolio weights can be easily computed. This paper considers downside risk measures that depend on the portfolio moments. It is computationally convenient to express the portfolio moments as a function of the multivariate moments of the returns on the underlying assets, using the $N \times N^2$ co-skewness matrix

$$M_3 = E\left[(r-\mu)(r-\mu)' \otimes (r-\mu)'\right]$$

and $N \times N^3$ co-kurtosis matrix

$$M_4 = E\left[(r-\mu)(r-\mu)' \otimes (r-\mu)' \otimes (r-\mu)'\right],$$

where $\otimes$ stands for the Kronecker product (see e.g. Jondeau and Rockinger, 2006). Under this representation, the derivatives of the portfolio moments are easy to compute. Denote the q-th centered portfolio moment $m_q = E\left[(r_p - w'\mu)^q\right]$ and let $\partial_i m_q$ be its partial derivative with respect to w_i. We have that

$$\begin{array}{ll} m_2 = w'\Sigma w & \partial_i m_2 = 2(\Sigma w)_i \\ m_3 = w' M_3 (w \otimes w) & \partial_i m_3 = 3(M_3(w \otimes w))_i \\ m_4 = w' M_4 (w \otimes w \otimes w) & \partial_i m_4 = 4(M_4(w \otimes w \otimes w))_i. \end{array} \tag{5}$$

The portfolio skewness s_p and excess kurtosis k_p and their partial derivative are then given by

$$
\begin{array}{lll}
(6) \quad & s_p = m_3/m_2^{3/2} & \partial_i s_p = \left(2m_2^{3/2}\partial_i m_3 - 3m_3 m_2^{1/2}\partial_i m_2\right)/2m_2^3 \\
 & k_p = m_4/m_2^2 - 3 & \partial_i k_p = \left(m_2\partial_i m_4 - 2m_4\partial_i m_2\right)/m_2^3.
\end{array}
$$

Gaussian VaR. The portfolio standard deviation, VaR and ES are all 1-homogenous functions. For portfolio standard error, for example, the risk contribution of asset i is given by $0.5w_i\partial_i m_2/\sqrt{m_2}$. For many estimation methods, the computation of the derivative of the estimated VaR and ES is challenging because the estimator cannot be expressed as an explicit function of the portfolio weights. A notable exception is when VaR and ES are computed under the assumption of normality (Garman, 1997). Replacing $F(\cdot)$ by the Gaussian distribution function in (1), we obtain the following expressions for Gaussian VaR (GVaR) and ES (GES)

$$
\begin{aligned}
\text{GVaR}(\alpha) &= -w'\mu - \sqrt{m_2}\Phi^{-1}(\alpha) \\
\text{GES}(\alpha) &= -w'\mu + \sqrt{m_2}\tfrac{1}{\alpha}\phi[\Phi^{-1}(\alpha)],
\end{aligned} \tag{7}
$$

where $\phi(\cdot)$, $\Phi(\cdot)$ and $\Phi^{-1}(\cdot)$ are the standard Gaussian density, distribution and quantile functions, respectively. Note that Gaussian VaR and ES depend only on the portfolio mean and variance. Hence, risk contributions (2) can be easily computed, using the partial derivatives

$$
\begin{aligned}
\partial_i\text{GVaR}(\alpha) &= -\mu_i - \tfrac{\partial_i m_2}{2\sqrt{m_2}}\Phi^{-1}(\alpha) \\
\partial_i\text{GES}(\alpha) &= -\mu_i + \tfrac{\partial_i m_2}{2\sqrt{m_2}}\tfrac{1}{\alpha}\phi[\Phi^{-1}(\alpha)].
\end{aligned} \tag{8}
$$

The partial derivative of the portfolio variance, $\partial_i m_2$, is given in (5).

Modified VaR. Gaussian VaR estimates VaR utilizing only the first two portfolio moments. This approach is no longer optimal for portfolios with non-normal returns. For this reason, Zangari (1996) proposed to generalize Gaussian VaR by correcting the Gaussian quantile for the portfolio skewness and excess kurtosis. As can be seen in (6), the portfolio skewness s_p is a measure for the amount of asymmetry in the portfolio return distribution. The larger the absolute size of the skewness statistic, the more asymmetric is the distribution. A large positive (negative) value indicates a long right (left) tail. The portfolio excess kurtosis k_p measures the thickness of the tails of the portfolio return distribution relatively to those of the normal distribution. A positive (negative) excess kurtosis means that the distribution has more (less) probability mass in the tails than the normal distribution. Zangari (1996)'s new estimator for VaR, called modified VaR (mVaR), is defined by

$$\begin{aligned} \text{mVaR}(\alpha) &= \text{GVaR}(\alpha) \\ &+\sqrt{m_2}\left[-\frac{1}{6}(z_\alpha^2-1)s_p-\frac{1}{24}(z_\alpha^3-3z_\alpha)k_p+\frac{1}{36}(2z_\alpha^3-5z_\alpha)s_p^2\right]. \end{aligned} \tag{9}$$

Note that when skewness and excess kurtosis are zero, which is the case under normality, modified VaR equals Gaussian VaR. As we will see in the next section, modified VaR is an estimator for VaR that estimates the true, unknown quantile function $F^{-1}(\cdot)$ in (1) by its second order Cornish-Fisher expansion around the Gaussian quantile function. The component of asset i in the portfolio's mVaR is fairly easy to compute, using the following partial derivative:

$$\begin{aligned} \partial_i \text{mVaR}(\alpha) &= \partial_i \text{GVaR}(\alpha) \\ &+\frac{\partial_i m_2}{\sqrt{m_2}}\left[-\frac{1}{12}(z_\alpha^2-1)s_p-\frac{1}{48}(z_\alpha^3-3z_\alpha)k_p+\frac{1}{72}(2z_\alpha^3-5z_\alpha)s_p^2\right] \\ &+\sqrt{m_2}\left[-\frac{1}{6}(z_\alpha^2-1)\partial_i s_p-\frac{1}{24}(z_\alpha^3-3z_\alpha)\partial_i k_p+\frac{1}{18}(2z_\alpha^3-5z_\alpha)s_p\partial_i s_p\right]. \end{aligned} \tag{10}$$

It is thus possible both to compute mVaR numerically, and to decompose this in the risk contributions of the different assets in the portfolio using the equations presented above. This result should be very useful to a portfolio or risk manager wishing to understand the contribution to total risk from each element of the portfolio. This result may also be applied to a variety of portfolio construction and optimization approaches.

3 Modified expected shortfall

Downside risk measures describe the left tail of the return distribution. For a loss probability α, it is interesting not only to estimate the location of this tail quantile, which is VaR, but also to have a central estimate of this tail, which is the Expected Shortfall (ES) as defined in Section 1. Given that ES is a useful downside risk measure, it is natural to consider an estimator of portfolio ES which, like modified VaR, uses asymptotic expansions to take into account the skewness and excess kurtosis in the asset returns. At first sight, one could use the result that for continuous distribution functions

$$\text{ES}(\alpha) = \frac{1}{\alpha}\int_{\beta=0}^{\alpha} \text{VaR}(\beta)d\beta \tag{11}$$

to extend modified VaR to modified ES in a straightforward manner as the right hand side of the equation (11) in which VaR(β) is replaced with mVaR(β). In practice, this is not a good idea because as we will see in the next Section, mVaR(β) is less reliable when β is close to zero. However, combining the properties of the Cornish-Fisher and Edgeworth expansions, we will show that it is possible to derive a definition of modified ES that, for a loss probability α, only depends on mVaR(α) and not on mVaR(β) with $\beta < \alpha$. Before giving a formal definition of modified ES, we first recall the main ideas behind these asymptotic expansions.

3.1 Cornish-Fisher and Edgeworth expansions

It is convenient to first express the portfolio return under its location-scale representation

$$r_p = w'\mu + \sqrt{m_2}u, \tag{12}$$

where u is a zero mean, unit variance random variable with distribution function $G(\cdot)$. In an empirical setting, $G(\cdot)$ is generally assumed to be approximately normal. The approximation can be improved by adjusting it for higher moments in the data. This can be done, using the r^{th} order Edgeworth expansion of $G(\cdot)$ around the standard Gaussian distribution function $\Phi(\cdot)$:

$$G_r(z) = \Phi(z) - \phi(z)\sum_{i=1}^{r} P_i(z), \tag{13}$$

where $P_i(z)$ is a polynomial in z. The corresponding r^{th} order Cornish-Fisher expansion of the quantile function $G^{-1}(\cdot)$ around the Gaussian quantile function $\Phi(\cdot)$, equals

$$G_r^{-1}(\alpha) = z_\alpha + \sum_{i=1}^{r} P_i^*(z_\alpha), \tag{14}$$

with $z_\alpha = \Phi^{-1}(\alpha)$. Exact formulas for the first eight terms in the Edgeworth and Cornish-Fisher expansions can be found in Draper and Tierney (1973). Since modified VaR and ES are defined using the second order Cornish-Fisher and Edgeworth expansion, we only need the following terms:

$$\begin{aligned} P_1(z) &= P_1^*(z) = \frac{1}{6}(z^2 - 1)s_p \\ P_2(z) &= \frac{1}{24}(z^3 - 3z)k_p + \frac{1}{72}(z^5 - 10z^3 + 15z)s_p^2 \\ P_2^*(z) &= \frac{1}{24}\left(z^3 - 3z\right)k_p - \frac{1}{36}\left(2z^3 - 5z\right)s_p^2, \end{aligned}$$

where s_p and k_p are the skewness and excess kurtosis of the portfolio return.

3.2 Definition

Under the location-scale representation (12), the VaR and ES of the portfolio return are given by

$$\begin{aligned} \text{VaR}(\alpha) &= -w'\mu - \sqrt{m_2}\; G^{-1}(\alpha) \\ \text{ES}(\alpha) &= -w'\mu - \sqrt{m_2} E_G\left[z|z \leq G^{-1}(\alpha)\right], \end{aligned} \tag{15}$$

with $G^{-1}(\cdot)$ the quantile function associated to $G(\cdot)$. Comparing (15) with (14), it is not difficult to verify that modified VaR is an estimator for VaR that approximates the true quantile $G^{-1}(\alpha)$ with its second[2] order Cornish-Fisher expansion.

For a loss probability α, we define modified ES (mES) as the expected value of all returns below the α Cornish-Fisher quantile and where the expectation is computed under the second order Edgeworth expansion of the true distribution function $G(\cdot)$:

$$\text{mES}(\alpha) = -w'\mu - \sqrt{m_2}\; E_{G_2}\left[z|z \leq g_\alpha\right], \tag{16}$$

[2]Other authors have considered higher order Cornish-Fisher expansions but find that increasing the order r in (14) does not necessarily improve the approximation (see e.g. Baillie and Bollerslev (1992), p.105, and Jaschke (2001), p.6).

with $g_\alpha = G_2^{-1}(\alpha)$. After tedious computations (see Appendix B for details), we obtain:

$$\begin{aligned} E_{G_2}\left[z|z \leq g_\alpha\right] &= -\frac{1}{\alpha}\Big\{\phi(g_\alpha) + \frac{1}{24}\left[I^4 - 6I^2 + 3\phi(g_\alpha)\right]k_p + \frac{1}{6}\left[I^3 - 3I^1\right]s_p \\ &\quad + \frac{1}{72}\left[I^6 - 15I^4 + 45I^2 - 15\phi(g_\alpha)\right]s_p^2\Big\} \end{aligned}$$

where

$$I^q = \begin{cases} \sum_{i=1}^{q/2}\left(\frac{\prod_{j=1}^{q/2} 2j}{\prod_{j=1}^{i} 2j}\right) g_\alpha^{2i}\phi(g_\alpha) + \left(\prod_{j=1}^{q/2} 2j\right)\phi(g_\alpha) & \text{for } q \text{ even} \\ \sum_{i=0}^{q^*}\left(\frac{\prod_{j=0}^{q^*}(2j+1)}{\prod_{j=0}^{i}(2j+1)}\right) g_\alpha^{2i+1}\phi(g_\alpha) - \left(\prod_{j=0}^{q^*}(2j+1)\right)\Phi(g_\alpha) & \text{for } q \text{ odd} \end{cases}$$

and $q^* = (q-1)/2$. In Appendix C, we provide a long but explicit formula for computing the derivative of mES. Although the resulting formulae are rather complex, they lend themselves to efficient translation into a simple algorithm that computes in less than a second mES and component mES, even for portfolios with a very large number of assets.[3]

[3]The data together with the programs used in the paper can be downloaded from http://econ.kuleuven.be/kris.boudt/public.

4 Approximation quality

In this Section we assess, in the absence of estimation error in the portfolio moments, how well modified VaR and ES approximate VaR and ES, when the portfolio return is distributed as a random variable, with zero mean and unit variance and various degrees of skewness and excess kurtosis. More specifically, we consider the skewed Student t density function proposed by Fernández and Steel (1998) and standardized by Lambert and Laurent (2001). It has been used by Giot and Laurent (2004), among others, for forecasting one-day-ahead VaR for long and short trading positions on daily stocks and stock indices.

The standardized Student t density function has two parameters: ξ and ν. The skewness parameter $\xi > 0$ is defined such that the ratio of probability masses above and below the mean is

$$\frac{\Pr\left(r_p \geq 0|\xi\right)}{\Pr\left(r_p < 0|\xi\right)} = \xi^2.$$

Note that the density function is skewed to the left for $\xi < 1$, symmetric for $\xi = 1$ and skewed to the right for $\xi > 1$. The parameter $\nu > 0$ models the tail thickness. Holding ξ fixed, we have that the smaller ν is, the thicker the tails are. When $\xi = 1$, the standardized skewed Student t distribution coincides with the standardized Student t distribution, and for $\nu \to \infty$, the standard Gaussian distribution is the limiting case.

In Appendix D we recall Lambert and Laurent (2001)'s expression for the skewness, kurtosis and quantile function of this skewed Student t distribution. Using these expressions, VaR and ES, and their estimators, modified VaR and ES can be computed at the theoretical values of the moments of the distribution. Let us now study how sensitive these statistics are to the parameters ξ and ν of the skewed Student t distribution function and the loss probability α for which they are computed.

In Table 1, we report the true, Gaussian and modified VaR and ES computed for $\alpha = 0.05$, for various values of ξ and ν. The benchmark values are $\xi = 1$ and $\nu = \infty$, for which the return distribution is the standard Gaussian one and for which, by definition, the Gaussian and modified estimators for VaR and ES coincide with the true VaR and ES. The more ξ deviates from 1 and the smaller ν is, the higher skewness and excess kurtosis are in absolute values and the more distant the skewed Student t will be from the Gaussian distribution. We find that for moderate values of skewness and kurtosis, modified VaR and ES are good approximations of the true VAR and ES, and they certainly do better than Gaussian VaR and ES which are independent of ξ and ν. The more extreme the skewness and excess kurtosis, the less reliable Gaussian and modified VaR and ES are. Modified ES is more sensitive to extreme deviations from normality than modified VaR. Note also that when the data is negatively skewed, modified VaR and ES tend to be too pessimistic and Gaussian VaR and ES too optimistic. The opposite result is observed when the data is positively skewed.

ξ	0.5			1			1.5		
ν	5	8	∞	5	8	∞	5	8	∞
Skewness	-2.06	-1.32	-0.79	0	0	0	1.52	0.96	0.56
Excess kurtosis	14.54	3.53	0.51	6	1.5	0	10.42	2.53	0.24
VaR	1.82	1.87	1.88	1.56	1.61	1.64	1.27	1.34	1.43
GVaR-VaR	-0.18	-0.23	-0.24	0.08	0.03	0	0.37	0.30	0.21
mVaR-VaR	0.04	0.05	-0.03	-0.04	0	0	-0.31	-0.04	0.05
ES	2.82	2.69	2.46	2.24	2.18	2.06	1.65	1.68	1.70
GES-ES	-0.76	-0.63	-0.39	-0.18	-0.11	0	0.42	0.38	0.36
mES-ES	2.49	0.41	-0.08	0.10	0.07	0	-1.38	-0.14	0.05

Table 1: Sensitivity of skewness, excess kurtosis, VaR, ES and the estimation errors of Gaussian and modified VaR and ES (for $\alpha = 0.05$) to the parameters ξ and ν of the skewed Student t distribution function.

Figure 1 shows the sensitivity of the true and the approximated values of VaR and ES to the loss probability α for the skewed Student t distribution with $\xi = 0.5$ and $\nu = 8$. Consistent with Jaschke (2001, p.6), we find that all approximations have the "wrong tail behavior" in the sense that the approximation becomes less and less reliable for $\alpha \to 0$. For $\alpha \to 0$, mES drops to zero. This result occurs because the Edgeworth approximation to the density function tends to zero when evaluated at very large losses. As a method of avoiding the unwanted result that mES is smaller than mVaR, we will henceforth use the following *operational* definition of modified ES:

$$\text{(17)} \qquad \text{mES}^*(\alpha) = -w'\mu - \sqrt{m_2}\ \min\{\ E_{G_2}\left[z|z \leq g_\alpha\right]\ ,\ g_\alpha\ \}$$

where G_2 and g_α are as defined in (16). In Figure 1 we see that by construction mES* coincides with mES as long as mES is greater than mVaR. For all values of α, mES* is a better estimator of ES than GES.

If your data is fat-tailed and/or skewed enough to cause the breakdown in mVaR and mES, or if you want to estimate the very extreme downside risk ($\alpha = 0.01$ or smaller), a copula-based approach as proposed by Embrechts *et al.* (2002) may be a more appropriate estimator of VaR and ES. Under this approach there is, however, no simple procedure available for estimating

Component VaR and Component ES. In cases where mVaR and mES are still providing a reliable estimation of downside risk, calculation of (Component) VaR and ES under the Cornish-Fisher and Edgeworth approach is certainly more computationally tractable and does not suffer the fitting subtleties of a copula approach.

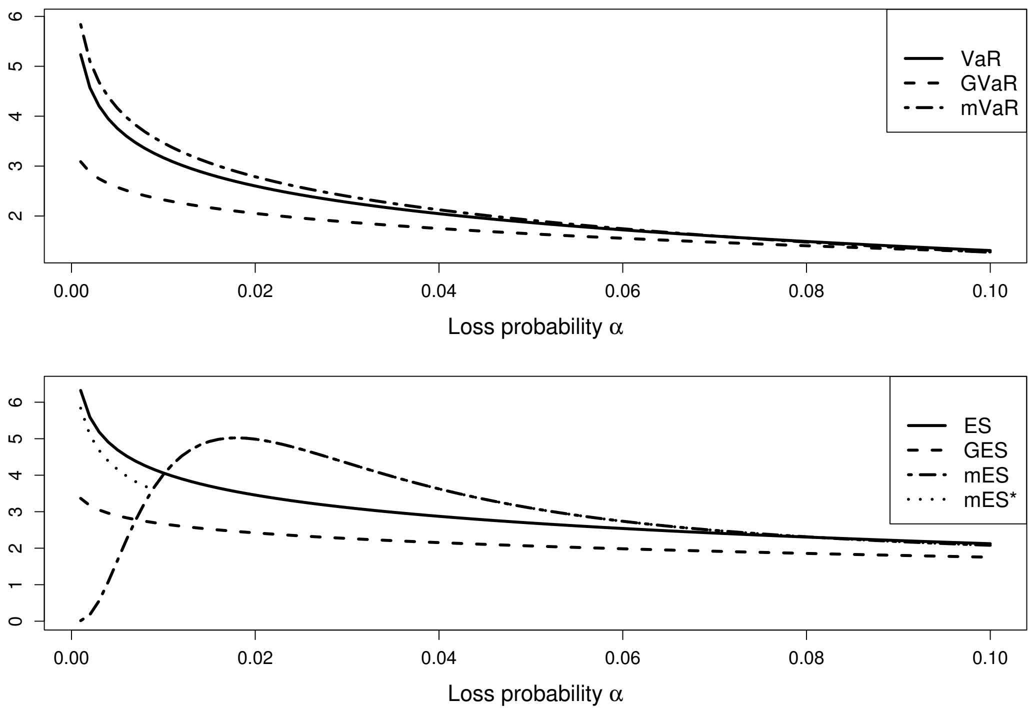

Figure 1: Sensitivity of true, Gaussian and modified VaR and ES to the loss probability α for the skewed Student t distribution with $\nu = 8$ and $\xi = -0.5$.

5 Robust estimation

Gaussian (modified) VaR and ES estimate VaR and ES using the first two (four) moments of the portfolio return distribution. Formulas (5)-(6) express the portfolio moments as a computationally convenient function of the multivariate moments of the return series of the assets in the portfolio. From the theoretical study in the previous section, it follows that modified VaR and ES are good approximations of VaR and ES, provided the true distribution function does not deviate too much from normality. For this reason, it seems appropriate to consider estimates of the multivariate moments that are robust to return observations that deviate extremely from the Gaussian distribution. There are two main approaches in defining robust alternatives to estimate the multivariate moments by their sample means (see e.g. Maronna *et al.*, 2007). One approach is to consider a more robust estimator than the sample means. Another one is to first clean (in a robust way) the data and then take the sample means of the cleaned data.

Our cleaning method follows the second approach. It is designed in such a way that, if we want to estimate downside risk with loss probability α, it will never clean observations that belong to the $1-\alpha$ least extreme observations. Suppose we have an n-dimensional vector time series of length T: $r_1, ..., r_T$. We clean this time series in three steps.

1. *Ranking the observations in function of their extremeness.* Denote μ and Σ the mean and covariance matrix of the bulk of the data and let $\lfloor \cdot \rfloor$ be the operator that takes the integer part of its argument. As a measure of the extremeness of the return observation r_t, we use its squared Mahalanobis distance $d_t^2 = (r_t - \mu)'\Sigma^{-1}(r_t - \mu)$.We follow Rousseeuw (1985) by estimating μ and Σ as the mean vector and covariance matrix of the subset of size $\lfloor (1-\alpha)T \rfloor$ for which the sum of d_t^2 computed over the elements in that subset is the smallest. These estimates will be robust against the α most extreme returns. Let $d_{(1)}^2, ..., d_{(T)}^2$ be the ordered sequence of the estimated squared Mahalanobis distances such that $d_{(i)}^2 \leq d_{(i+1)}^2$.

2. *Outlier identification.* Return observations are qualified as outliers if their estimated squared Mahalanobis distance d_t^2 is greater than the empirical $1-\alpha$ quantile $d^2_{(\lfloor (1-\alpha)T \rfloor)}$ and exceeds a very extreme quantile of the Chi squared distribution function with n degrees of freedom, which is the distribution function of d_t^2 when the returns are normally distributed. In the application we take the 99.9% quantile, denoted $\chi^2_{n,0.999}$.

3. *Data cleaning.* We follow Khan *et al.* (2007) in cleaning the outlying

returns r_t by replacing them with $r_t\sqrt{\chi^2_{n,0.999}/d_t^2}$. The cleaned return vector has the same orientation as the original return vector, but its magnitude is smaller.[4]

Note that the primary value of data cleaning lies in creating a more robust and stable estimation of the distribution generating the large majority of the return data. The increased robustness and stability of the estimated moments utilizing cleaned data should be used for portfolio construction. If a portfolio manager wishes to have a more conservative risk estimate, cleaning may not be indicated for risk monitoring. It is also important to note that the robust method proposed here does not remove data from the series, but only decreases the magnitude of the extreme events. It may also be appropriate in practice to use a cleaning threshold somewhat outside the VaR threshold that the manager wishes to consider. In actual practice, it is probably best to back-test the results of both cleaned and uncleaned series to see what works best with the particular combination of assets under consideration.

[4] Khan *et al.* (2007) call this procedure of limiting the value of d_t^2 to a quantile of the χ^2_n distribution, "multivariate Winsorization".

6 Application: component risk analysis of hedge fund portfolios

The correct measurement of financial risk of alternative investments is a concern for many portfolio managers. Because of the non-normality of these returns, they constitute an excellent case for illustrating the use of modified VaR and ES in a context of portfolio construction. Our data consists of monthly returns on 6 Credit Suisse/Tremont hedge fund investment style indices for the period January 1995 - August 2007. The advantages of the Credit Suisse/Tremont indices over their competitors and a description of the investment strategies can be found in Amenc and Martellini (2002). Before studying the estimates of modified VaR and ES for portfolios of these indices and the component risk allocation in the sample portfolios, we investigate the appropriateness of cleaning the data prior to estimating downside risk.

Out-of-sample evidence in favor of data cleaning. The original and cleaned series are plotted in Figure 2. Note that cleaning the data leads to an important reduction in the magnitude of the extreme returns corresponding to the Russian financial crisis in August-October 1998. Because there is no similar succession of extreme returns in the time series, it is reasonable to impose that the effect of these returns on the multivariate moment estimates, and thus on the values of mVaR and mES, should be bounded. This reasoning is also verified empirically. Table 2 compares the failure rate of the out-of-sample Gaussian and modified VaR estimators ($\alpha = 0.05$) using the raw and cleaned data sets for the period January 1999 - August 2007. The failure rate is defined as the percentage of negative returns smaller than the negative value of the out-of-sample one-step ahead VaR forecast. If the VaR calculation method is accurate, then the failure rate should be close to α. Because extreme returns blow up the sample standard deviation and kurtosis, we see in Table 2 that cleaning the data leads to a lower estimate of the Gaussian and modified VaR and thus a higher failure rate. Modified VaR is more affected than Gaussian VaR because it depends on the sample estimates of skewness and kurtosis which raise outliers to the third and fourth power. An important argument in favor of cleaning the data is that the failure rates obtained using the cleaned data to estimate the one-step ahead VaR forecast, are closer to $\alpha = 0.05$. For this reason we will use the cleaned data set in the remainder of the application. From Table 2 we cannot conclude whether the Gaussian or modified VaR estimator has a better out-of-sample performance.

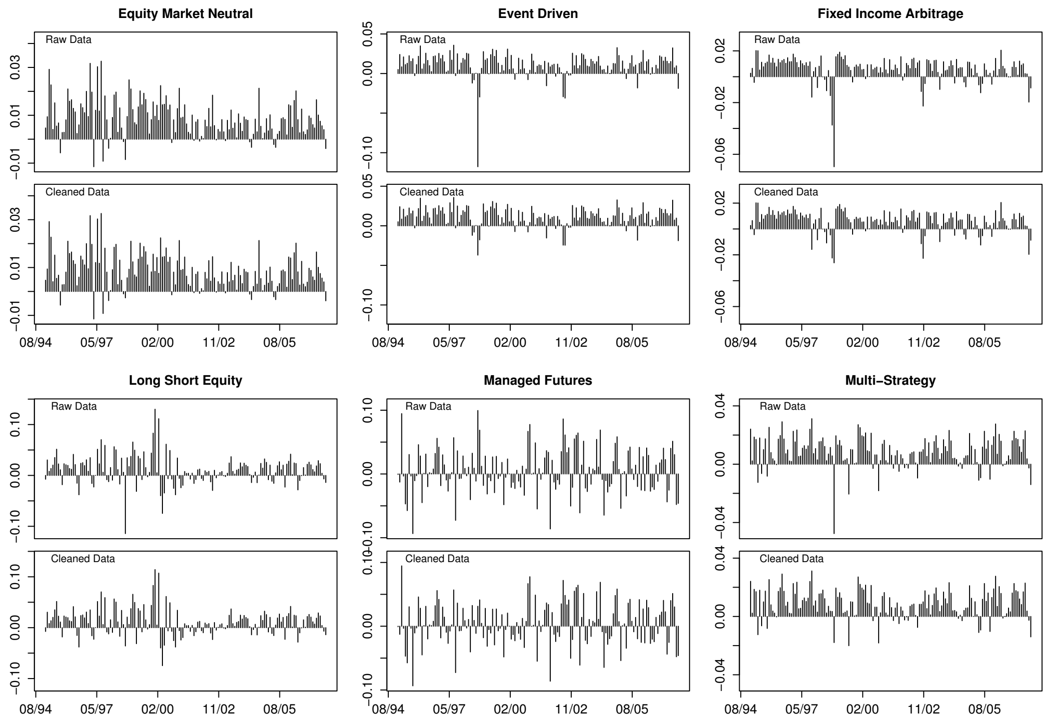

Figure 2: Original and cleaned monthly returns on January 1995-August 2007 Credit Suisse/Tremont hedge fund investment style indices.

Hedge Fund Style Index	Raw data		Cleaned data	
	GVaR	mVaR	GVaR	mVaR
Equity Market Neutral (EMN)	0	0.029	0.010	0.029
Event Driven (ED)	0.039	0.019	0.049	0.049
Fixed Income Arbitrage (FIA)	0.029	0.019	0.068	0.039
Long Short Equity (LSE)	0.019	0.019	0.029	0.058
Managed Futures (MF)	0.039	0.049	0.049	0.049
Multi-Strategy (MS)	0.058	0.029	0.068	0.068

Table 2: Out-of-sample failure rate for one-step ahead VaR predictions ($\alpha = 0.05$) made by the Gaussian and modified VaR estimators for the period January 1999 - August 2007 for six Credit Suisse/Tremont hedge fund investment style indices.

Index	Mean	Sd	Skew	Exc Kur	JB	GVaR	mVaR	GES	mES
EMN	0.009	0.008	0.534	0.724	0.005	0.004	0.003	0.007	0.005
ED	0.011	0.012	-0.807	1.477	0.000	0.009	0.011	0.014	0.021
FIA	0.006	0.008	-1.294	2.237	0.000	0.008	0.010	0.011	0.017
LSE	0.012	0.026	0.597	2.233	0.000	0.032	0.026	0.043	0.033
MF	0.004	0.034	-0.106	-0.047	0.861	0.052	0.053	0.066	0.068
MS	0.009	0.010	-0.320	0.150	0.255	0.007	0.008	0.012	0.013

Table 3: Sample mean, standard deviation, skewness, excess kurtosis, P-value of Jarque-Bera test for normality and 5% Gaussian and modified VaR and ES estimates for the January 1995-August 2007 cleaned monthly return series of Credit Suisse/Tremont hedge fund investment style indices.

Downside risk analysis of individual hedge fund return series. Table 3 reports a sample of descriptive statistics for the monthly return series of each of the six style indices. Note that the monthly returns of the equity market neutral, event driven, fixed income arbitrage and long short equity investment style indices are non-normal. Their skewness and excess kurtosis are high enough for the P-value of their Jarque-Bera test statistic for normality to be less than 1%. Their skewness and excess kurtosis are in the range of values for which we found in Section 4 that for $\alpha = 0.05$, modified VaR and ES are reliable estimators of VaR and ES and more accurate than Gaussian VaR and ES.

Regarding the sources of downside risk, we find that the mean and standard deviation of the return series are the main drivers of downside risk and that skewness and excess kurtosis are important fine-tuning parameters for adjusting the Gaussian downside risk estimate for the non-normality in the return series. Indeed, the estimates of downside risk are the highest for the return series of the managed futures index, which has negligible skewness and excess kurtosis, but the lowest mean return and highest standard deviation. For $\alpha = 0.05$, its mVaR and mES[5] equal 5.3% and 6.8%, respectively. If the estimates are correct, this means that there is a 5% probability of observing monthly losses that exceed -5.3% and the expected value of these extreme losses equals -6.8%. When the return series exhibits significant skewness and excess kurtosis, modified VaR and ES can give very different estimates for VaR and ES than Gaussian VaR and ES. Since the returns on the equity market neutral index are positively skewed, modified VaR and ES are more optimistic about downside risk than Gaussian VaR and ES. The opposite result is found

[5] In this Section, all values of mES as defined in (16) equal the values of mES* as defined in (17).

for the return series for the event driven and fixed income arbitrage indices. Since these returns are very much skewed to the left and heavy tailed, downside risk as estimated by modified VaR and ES is much greater than when the Gaussian estimators are used.

Portfolio downside risk analysis. We will first discuss the portfolio construction for our application examples, and then discuss in more detail the risk decomposition of each portfolio. We will see that the contribution to risk of each portfolio component varies widely from the portfolio weights and standard deviation of the individual components, while following a pattern that can be intuited from the construction of the component risk model and the individual properties of the portfolio holdings.

For illustrating the computation and interpretation of portfolio modified VaR and ES, we consider the balanced Equal-Weighted (EW) portfolio, and the Maximum Sharpe Ratio (MSR) portfolios. We consider these portfolio constructions to be symbolically representative of actual practice at many hedge fund investors: investors often pursue either style diversification (thus our example of the equal-weighted balanced view), or they pursue the perceived best risk-reward ratio (represented by the MSR view). The Sharpe Ratio is the most popular risk-adjusted return measure reported by distributors of hedge funds (Amenc *et al.*, 2003). Construction of portfolios that lie on the Markowitz efficient frontier is probably the most common portfolio optimization method employed in finance. For the purposes of this example, we have constrained the portfolio optimization to a long-only portfolio with minimum weights of 5% in each hedge fund style to avoid the creation of overly concentrated positions. Figure 3 compares the mean-variance characteristics of the hedge fund indices and the EW and MSR portfolios with the mean-variance efficient frontier of these portfolios. The long-only and 5% minimum constraint, as could be expected, creates a smaller efficient frontier portfolio space than would be created in an unconstrained portfolio. As designed, these constraints eliminate "corner" portfolios where any single instrument can dominate the entire portfolio. Figure 4 plots the monthly returns of these portfolios.

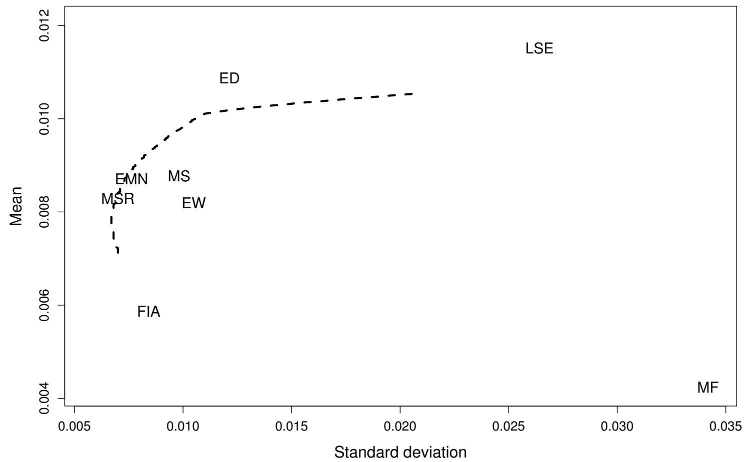

Figure 3: Mean and standard deviation of Credit Suisse/Tremont indices and of Equal-Weighted and Maximum Sharpe Ratio portfolios as well as the mean-variance efficient frontier of long-only portfolios with minimum weights of 5% in each hedge fund style (dashed).

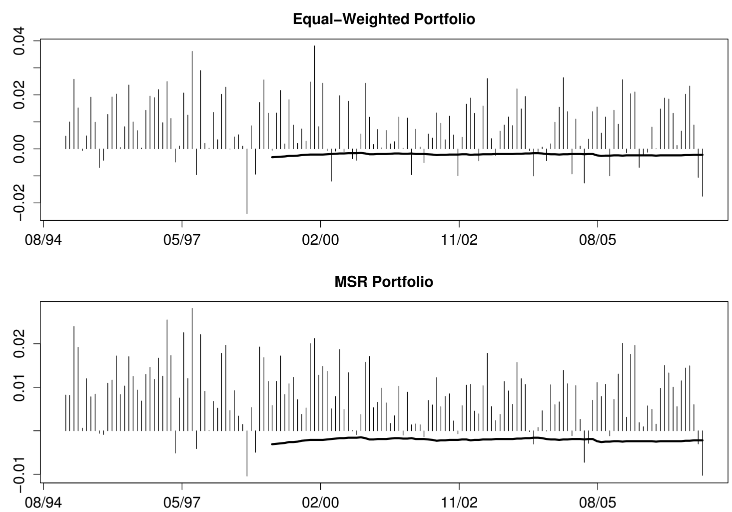

Figure 4: Bar plot of January 1995-August 2007 monthly return series for Equal-Weighted and Maximum Sharpe Ratio portfolios of Credit Suisse/Tremont indices and line graph of negative value of out-of-sample one step ahead modified VaR forecast ($\alpha = 0.05$) for the monthly portfolio return in January 1999-August 2007.

Table 4 reports the risk measures for the two portfolios and compares the portfolio holdings and percentage risk contributions calculated per (2). A direct comparison of all the portfolio risk measures (standard deviation, Gaussian VaR and ES, and modified VaR and ES) shows improvement for the MSR portfolio over the balanced portfolio, even though the mean return of the two portfolios is the same. We also find on examining the component decomposition of risk that for VaR and ES the capital allocation given by the portfolio weights can be very different from the risk exposures (Qian, 2006), and also that the risk allocation depends on the risk measure used.[6] This can be explained in two ways:

1. A mechanical explanation follows from the definition in (2) of asset i's percentage risk contribution as the derivative of the risk measure with respect to the weight of that component multiplied by the component's weight in the portfolio and divided by the value of that risk measure. For Gaussian VaR and ES, the risk contributions are dominated by the portfolio weights. On the contrary, for Modified VaR and ES, the derivative of the risk measure with respect to the component's weight also plays a very important role.

2. A perhaps more intuitive understanding for the difference in the percentage risk contributions can be obtained using the financial interpretation given to VaR and ES in Section 2. Asset i's contribution to VaR (ES) equals the negative value of the expected contribution to portfolio return when the portfolio return equals (is less than or equals) the negative value of VaR. For an equal-weighted portfolio, the expected contribution to the portfolio downside risk will be higher for assets with negatively skewed and thick-tailed returns than for assets with normal return or positively skewed returns.

One of the first major observations on the sample portfolios is the impact the standard deviation has on both the risk decomposition and on the MSR portfolio construction. The managed futures style, with standard deviation (0.034) the largest of the styles, but minimal skewness (-0.106) and excess kurtosis (-0.047), accounts for the largest portion of both the Gaussian and the modified risk measures in the balanced portfolio. In the MSR portfolio, despite being penalized to the minimum 5% weight for its riskiness, the managed futures component of the portfolio still has the highest component risk contribution of any asset in the portfolio. The long short equity style, with the

[6]In Appendix E we show that for the unconstrained MSR portfolio the percentage contribution to portfolio standard deviation, Gaussian VaR and Gaussian ES coincide. This result may no longer hold under a long-only constraint.

Equal-Weighted Portfolio						
	Mean	Sd	GVaR	GES	mVaR	mES
Total	0.0083	0.0105	0.0091	0.0135	0.0089	0.0121
	w_i	% contribution:				
EMN	0.17	0.06	-0.04	-0.01	-0.05	-0.02
ED	0.17	0.13	0.05	0.08	0.06	0.08
FIA	0.17	0.06	0.00	0.02	0.01	0.02
LSE	0.17	0.31	0.38	0.36	0.30	0.28
MF	0.17	0.36	0.61	0.53	0.69	0.62
MS	0.17	0.08	-0.01	0.02	0.00	0.03

MSR Portfolio with 5% minimum weights						
	Mean	Sd	GVaR	GES	mVaR	mES
Total	0.0083	0.0070	0.0031	0.0060	0.0029	0.0053
	w_i	% contribution:				
EMN	0.49	0.44	0.25	0.34	0.04	0.22
ED	0.06	0.07	0.04	0.05	0.06	0.07
FIA	0.11	0.06	0.02	0.04	0.06	0.06
LSE	0.05	0.12	0.24	0.18	0.21	0.12
MF	0.05	0.11	0.32	0.21	0.37	0.29
MS	0.24	0.21	0.12	0.16	0.26	0.24

Table 4: Portfolio totals, weights and percentage risk contributions for the equal-weighted and long-only 5% minimum weight constrained Maximum Sharpe Ratio portfolio of Credit Suisse/Tremont hedge fund investment style indices. Gaussian and modified VaR and ES are computed for $\alpha = 5\%$.

highest mean (0.012) and second highest standard deviation (0.026), and fat-tailed excess kurtosis (2.233), shows similar effects, being the second largest contributor on all component risk measures in both portfolios, and being similarly penalized in the MSR portfolio by a minimum weight.

The risk added by an asset to the portfolio depends on the asset's risk properties, relatively to the risk properties of the other assets in the portfolio. This is very well illustrated by the fixed income arbitrage index. It has at the same time a lower portfolio weight and a higher risk contribution in the MSR portfolio than in the balanced portfolio. This is because assets such as the long short equity and managed futures indices with a relatively higher downside risk than the fixed income arbitrage index, also have a lower weight in the MSR portfolio. A similar effect may be noted with the multi-strategy style index, which has a negligible impact on all component VaR and ES risk measures in the balanced portfolio (-1% to 3%), but a much larger impact on all these measures in the MSR portfolio. The equity market neutral index shows the risk benefits of positive skew (0.534) and moderate excess kurtosis (0.724). In the balanced portfolio, it has a negative risk contribution to Gaussian and modified VaR. This means that it serves as a hedge to the VaR of the rest of the portfolio (Garman, 1997). The equity market neutral component risk measures increase in the MSR portfolio, but are still below the component weight of 49%.

Figure 4 compares the monthly returns on the equal-weighted and MSR portfolios with the negative value of their out-of-sample one step ahead mVaR forecast ($\alpha = 0.05$) for the period January 1999-August 2007.[7] In Table 5 we report all the months in this period for which the return on the equal-weighted and MSR portfolios was below the negative value of the out-of-sample one step ahead mVaR forecast. Any such examination of realized contribution to extreme portfolio losses is by nature imprecise, but this comparison should still be informative. ES attempts to predict the *average* contribution to loss, not the specific contribution to loss on each period in which the loss exceeds the VaR estimate.

The complicated interaction between the stochastic properties of the return series of the portfolio assets makes the point prediction of future realizations of percentage risk contributions a very difficult task. For the six hedge fund style indices, we compare the realized percentage contribution to the portfolio return ($w_i r_i / r_p$) with its predicted value by the mES estimate ($\%C_i$mES). We find that the realized loss contributions deviate a lot from the capital allocation and from the percentage contributions to the standard deviation of each style. We find that in almost all cases, the mES correctly predicts the largest contributors to potential losses.

We also compare the average of the realized losses to the average one-step-ahead prediction of mES. Even with a relatively limited historical series, we obtain that on average the realized contributions to extreme losses are well predicted by the percentage contribution to modified ES. We find that the average of the realized extreme losses and the average of the predicted value of mES compare closely to each other. This result is to be expected if the mES predicting process truly takes into account the shape of past losses. Examination of the past performance versus the future prediction offers another confirmation that the mES method is likely to be a reasonable predictor of future loss distributions. Overall, Table 5 indicates that combining the capital allocation and the estimated risk contributions will help the risk manager in forming a better opinion of the sources and magnitude of future portfolio risk.

A practical conclusion from examining the component risk contributions

[7] The out-of-sample one step ahead estimate for modified VaR and ES of month t is obtained using the cleaned returns from January 1995 up to month $t-1$.

to the two sample portfolios would be that the long-term performance of the portfolios could be improved by adjusting the component weights to better match a deliberate risk profile that was complimentary to the investor's goals. In any portfolio holding a sufficient numbers of assets, there will be many possible portfolios with similar mean return and standard deviation, so additional information provided by the modified VaR and ES and portfolio risk decomposition techniques presented here adds significant information to the portfolio selection process. Further work should examine these techniques in relation to various risk budgeting and portfolio optimization methodologies.

EW Portfolio		EMN	ED	FIA	LSE	MF	MS
	w_i	0.17	0.17	0.17	0.17	0.17	0.17
April 2000	$w_i r_i / r_p$	-0.20	0.10	-0.13	1.04	0.32	-0.13
($r_p = -0.012$)	$\%C_i$mES	0.03	0.10	-0.03	0.36	0.64	-0.11
November 2001	$w_i r_i / r_p$	-0.14	-0.15	0.05	-0.19	1.50	-0.07
($r_p = -0.010$)	$\%C_i$mES	-0.02	0.11	-0.05	0.52	0.50	-0.07
October 2002	$w_i r_i / r_p$	-0.07	0.03	0.38	-0.01	0.84	-0.17
($r_p = -0.010$)	$\%C_i$mES	-0.02	0.09	-0.04	0.44	0.59	-0.05
April 2004	$w_i r_i / r_p$	0.06	-0.08	-0.22	0.23	1.07	-0.05
($r_p = -0.010$)	$\%C_i$mES	-0.04	0.06	0.01	0.36	0.67	-0.06
January 2005	$w_i r_i / r_p$	-0.06	-0.04	-0.02	0.15	0.97	0.00
($r_p = -0.009$)	$\%C_i$mES	-0.03	0.06	-0.01	0.34	0.69	-0.05
April 2005	$w_i r_i / r_p$	0.03	0.08	0.08	0.20	0.45	0.14
($r_p = -0.013$)	$\%C_i$mES	-0.03	0.05	-0.01	0.33	0.70	-0.04
October 2005	$w_i r_i / r_p$	-0.14	0.30	-0.04	0.38	0.33	0.17
($r_p = -0.010$)	$\%C_i$mES	-0.02	0.05	0.00	0.31	0.68	-0.02
July 2007	$w_i r_i / r_p$	-0.06	-0.15	0.31	0.11	0.76	0.04
($r_p = -0.010$)	$\%C_i$mES	-0.03	0.06	-0.01	0.33	0.66	0.00
August 2007	$w_i r_i / r_p$	0.04	0.18	0.08	0.13	0.44	0.13
($r_p = -0.018$)	$\%C_i$mES	-0.03	0.05	0.01	0.32	0.65	0.00
Average	$w_i r_i / r_p$	-0.06	0.03	0.05	0.23	0.74	0.01
	$\%C_i$mES	-0.02	0.07	-0.02	0.37	0.64	-0.04

MSR Portfolio		EMN	ED	FIA	LSE	MF	MS
	w_i	0.49	0.06	0.11	0.05	0.05	0.24
April 2004	$w_i r_i / r_p$	0.55	-0.09	-0.49	0.23	1.05	-0.24
($r_p = -0.003$)	$\%C_i$mES	0.40	0.07	0.03	0.18	0.26	0.06
April 2005	$w_i r_i / r_p$	0.15	0.05	0.10	0.11	0.24	0.36
($r_p = -0.007$)	$\%C_i$mES	0.36	0.06	-0.01	0.18	0.32	0.09
May 2005	$w_i r_i / r_p$	0.58	-0.11	0.48	-0.09	-0.62	0.76
($r_p = -0.003$)	$\%C_i$mES	0.32	0.06	0.02	0.15	0.30	0.14
July 2007	$w_i r_i / r_p$	-0.65	-0.18	0.72	0.11	0.79	0.20
($r_p = -0.003$)	$\%C_i$mES	0.27	0.06	0.02	0.16	0.29	0.20
August 2007	$w_i r_i / r_p$	0.19	0.10	0.09	0.07	0.22	0.32
($r_p = -0.010$)	$\%C_i$mES	0.24	0.05	0.05	0.16	0.30	0.20
Average	$w_i r_i / r_p$	0.16	-0.05	0.18	0.09	0.34	0.28
	$\%C_i$mES	0.32	0.06	0.02	0.16	0.30	0.14

Table 5: Realized percentage contribution to portfolio return ($w_i r_i / r_p$) and percentage contributions to out-of-sample mES estimates ($\%C_i$mES) for the months in the period January 1999-August 2007, in which the portfolio loss of the equal weighted portfolio exceeds the one-step ahead modified VaR forecast ($\alpha = 0.05$). The portfolio return r_p is in parenthesis.

7 Concluding remarks

This paper contributes to the literature on downside risk measurement in multiple ways. First of all, we introduce a new estimator for Expected Shortfall (ES), called modified ES which is based on the Cornish-Fisher and Edgeworth approximations of the portfolio return quantile and distribution functions. The definition of this new estimator is consistent with Zangari (1996)'s definition of modified Value at Risk (VaR). Modified VaR and ES can be considered as parametric Gaussian VaR and ES corrected for skewness and excess kurtosis in the data. We investigate how well modified VaR and ES proxy the true VaR and ES for the skewed Student t distribution function and find that for moderate values of skewness and excess kurtosis, modified VaR and ES are better estimators of VaR and ES than Gaussian VaR and ES, respectively. Some caution is necessary when modified VaR and ES are computed for very small loss probabilities α and for return distributions that deviate a lot from normality. We provide computationally convenient formulas for calculating these risk measures for portfolios and for decomposing them into the risk added to the portfolio by each of the assets in the portfolio.

We illustrate the usefulness of this new methodology for a set of hedge fund investment style indices. We investigate how the non-normality of the returns on these indices affects their downside risk as estimated by Gaussian and modified VaR and ES. For the equal-weighted and maximum Sharpe ratio portfolios, we find that capital allocation can be very different from risk allocation and that the estimated risk allocation depends on the risk measure used. We conclude that estimating the risk contributions of the portfolio holdings will help the investor in adjusting the portfolio composition to better match the desired portfolio risk profile.

Throughout the paper, we assume the portfolio return distribution to be continuous and the conditional portfolio moments to be constant. In future work we will investigate the relaxation of these two assumptions. Further work should also test implementation of risk monitoring and portfolio construction systems that use the formulae given in this paper to assure that the actual risk positions remain within the bounds stated in the risk budget.

A Financial interpretation of contribution to expected shortfall

Here we establish the following financial interpretation of contribution to expected shortfall

$$C_i\text{ES}(\alpha) = w_i\partial_i\text{ES}(\alpha) = -E\left[w_ir_i|r_p \leq -\text{VaR}(\alpha)\right].$$

Without loss of generality, we assume a portfolio of two assets, whose returns have the joint probability density function $f(r_1, r_2)$. For a loss probability α, the expected shortfall of the portfolio return $r_p = w_1r_1 + w_2r_2$ is defined as

$$\begin{aligned}\text{ES}(\alpha) &= -E[w_1r_1 + w_2r_2|w_1r_1 + w_2r_2 \leq -\text{VaR}(\alpha)] \\ &= -\frac{1}{\alpha}\int_{-\infty}^{\infty}\int_{-\infty}^{(-\text{VaR}(\alpha)-w_1r_1)/w_2}(w_1r_1 + w_2r_2)f(r_1, r_2)dr_2dr_1.\end{aligned}$$

By Leibniz's rule for differentiation, we have that the partial derivative of $\text{ES}(\alpha)$ with respect to w_1 equals

$$\begin{aligned}\partial_1\text{ES}(\alpha) = &-\frac{1}{\alpha}\int_{-\infty}^{\infty}\int_{-\infty}^{(-\text{VaR}(\alpha)-w_1r_1)/w_2} r_1f(r_1, r_2)dr_2dr_1 \\ &+\frac{\text{VaR}(\alpha)}{\alpha}\int_{-\infty}^{\infty} f\left(r_1, \frac{-\text{VaR}(\alpha) - w_1r_1}{w_2}\right)\partial_1\left(\frac{-\text{VaR}(\alpha) - w_1r_1}{w_2}\right)dr_1.\end{aligned}$$

The first term equals $-E\left[r_1|r_p \leq -\text{VaR}(\alpha)\right]$. The second term is zero since the integral in the second term is the derivative of the loss probability α. To see this, it suffices to compare this integral with the partial derivative of the left-hand-side of

$$\text{Prob}(w_1r_1 + w_2r_2 \leq -\text{VaR}(\alpha)) = \int_{-\infty}^{\infty}\int_{-\infty}^{(-\text{VaR}(\alpha)-w_1r_1)/w_2} f(r_1, r_2)dr_2dr_1 = \alpha.$$

The financial interpretation of contribution to expected shortfall (4) follows directly. This proof is similar to Gouriéroux *et al.* (2000)'s and Qian (2006)'s proof for the financial interpretation of contribution to value-at-risk.

B Modified expected shortfall

Here we show how

$$\begin{aligned}E_{G_2}[z|z \leq g_\alpha] &= \frac{1}{\alpha}\int_{-\infty}^{g_\alpha} zdG_2(z) = \frac{1}{\alpha}\int_{-\infty}^{g_\alpha} zd\left[\Phi(z) - \phi(z)\sum_{i=1}^{2}P_i(z)\right] \\ &= -\frac{1}{\alpha}\int_{-\infty}^{g_\alpha}\phi'(z)dz + zd\left[\phi(z)\sum_{i=1}^{2}P_i(z)\right]\end{aligned}$$

can be expressed as a polynomial in g_α with coefficients that depend on the portfolio skewness s_p and excess kurtosis k_p and on the standard Gaussian density function $\phi(\cdot)$. Using the property $\phi'(z) = -z\phi(z)$, we first rewrite the differentials $zd\phi(z)P_i(z)$ $(i = 1, 2)$ as a function of $z^q\phi'(z)dz$, with q a positive integer:

$$\begin{aligned} zd\phi(z)P_1(z) &= \frac{s_p}{6}\left(z^3 - 3z\right)\phi'(z)dz \\ zd\phi(z)P_2(z) &= \frac{k_p}{24}\left(z^4 - 6z^2 + 3\right)\phi'(z)dz \\ &\quad + \frac{s_p^2}{72}\left(z^6 - 15z^4 + 45z^2 - 15\right)\phi'(z)dz. \end{aligned}$$

Through integration by parts, we find that for $q = 1$,

$$I^1 = \int_{-\infty}^{g_\alpha} z\phi'(z)dz = g_\alpha\phi(g_\alpha) - \Phi(g_\alpha)$$

and for $q > 1$:

$$I^q = \int_{-\infty}^{g_\alpha} z^q\phi'(z)dz = g_\alpha^q\phi(g_\alpha) + q\int_{-\infty}^{g_\alpha} z^{q-2}\phi'(z)dz.$$

From the development of this recursive formula and using the result that by l'Hopital's theorem, $z^q\phi(z)$ is zero for $z = -\infty$, expression (16) follows straightforwardly.

C Derivative of modified expected shortfall

As with modified VaR, the derivative of modified ES can be computed analytically. Using the property $\phi'(z) = -z\phi(z)$, we obtain

$$\partial_i \mathrm{mES}(\alpha) = -\mu_i - \frac{\partial_i m_2}{2\sqrt{m_2}} E_{G_2}\left[z | z \leq g_\alpha\right] + \sqrt{m_2}\,\frac{1}{\alpha}\Bigg\{ \frac{1}{24}\left[I^4 - 6I^2 + 3\phi(g_\alpha)\right]\partial_i k_p$$
$$+ \frac{1}{6}\left[I^3 - 3I^1\right]\partial_i s_p + \frac{1}{36}\left[I^6 - 15I^4 + 45I^2 - 15\phi(g_\alpha)\right] s_p \partial_i s_p$$
$$+ \partial_i g_\alpha \Bigg[- g_\alpha \phi(g_\alpha) + \frac{1}{24}\left[\partial_i I^4 - 6\partial_i I^2 - 3 g_\alpha \phi(g_\alpha)\right] k_p + \frac{1}{6}\left[\partial_i I^3 - 3\partial_i I^1\right] s_p$$
$$+ \frac{1}{72}\left[\partial_i I^6 - 15\partial_i I^4 + 45\partial_i I^2 + 15 g_\alpha \phi(g_\alpha)\right] s_p^2 \Bigg]\Bigg\}.$$

Let $z_\alpha = \Phi^{-1}(\alpha)$, then

$$\partial_i g_\alpha = \frac{1}{6}(z_\alpha^2 - 1)\partial_i s_p + \frac{1}{24}(z_\alpha^3 - 3z_\alpha)\partial_i k_p - \frac{1}{18}(2z_\alpha^3 - 5z) s_p \partial_i s_p.$$

For q even, we have

$$\partial_i I^q = \sum_{i=1}^{q/2} \left(\frac{\prod_{j=1}^{q/2} 2j}{\prod_{j=1}^{i} 2j} \right) g_\alpha^{2i-1}(2i - g_\alpha^2)\phi(g_\alpha) - \left(\prod_{j=1}^{q/2} 2j \right) g_\alpha \phi(g_\alpha)$$

and for q odd

$$\partial_i I^q = \sum_{i=0}^{q^*} \left(\frac{\prod_{j=0}^{q^*}(2j+1)}{\prod_{j=0}^{i}(2j+1)} \right) g_\alpha^{2i}(2i + 1 - g_\alpha^2)\phi(g_\alpha) - \left(\prod_{j=0}^{q^*}(2j+1) \right) \phi(g_\alpha),$$

with $q^* = (q-1)/2$. At first sight, this expression may seem daunting, but it is fairly easy to implement using the computationally convenient formulas for portfolio skewness and excess kurtosis and their derivative in (6).

D Skewed Student t

Let $t_\nu(\cdot)$ and $T_\nu^{-1}(\cdot)$ be the density and quantile functions of the classical, non standardized Student t density functions with ν degrees of freedom, mean zero and standard deviation $\sigma = \sqrt{\nu/(\nu-2)}$. The random variable z is said to be (standardized) skewed Student t distributed with ν degrees of freedom and skewness parameter ξ, if its density function equals

$$t_{\xi,\nu}(z) = \begin{cases} \frac{2s\sigma}{\xi+\xi^{-1}} t_\nu\left[\sigma\xi(sz+m)\right] & \text{if} \quad z < -m/s \\ \\ \frac{2s\sigma}{\xi+\xi^{-1}} t_\nu\left[\sigma(sz+m)/\xi\right] & \text{if} \quad z \geq -m/s, \end{cases}$$

where m and s are the mean and standard deviation of the non-standardized skewed Student, respectively:

$$\begin{aligned} m &= \left[\sqrt{\pi}\Gamma\left(\frac{\nu}{2}\right)\right]^{-1}\Gamma\left(\frac{\nu-1}{2}\right)\sqrt{\nu-2}\left(\xi-\frac{1}{\xi}\right) \\ s &= \sqrt{\xi^2+\xi^{-2}-1-m^2}. \end{aligned}$$

The skewness and kurtosis of z equal its third and fourth uncentered moment:

$$E(z^q) = \int_{-\infty}^{\infty} z^q t_{\nu,\xi}(z)dz,$$

which we compute by numerical integration. Lambert and Laurent (2001) show that the quantile function of the standardized skewed Student t equals:

$$T_{\nu,\xi}^{-1}(\alpha) = \begin{cases} \left[(\sigma\xi)^{-1}T_{\nu}^{-1}\left(\frac{\alpha}{2}(1+\xi^2)\right)-m\right]/s & \text{if} \quad \alpha < 1/(1+\xi^2) \\ \left[-(\xi/\sigma)T_{\nu}^{-1}\left(\frac{1-\alpha}{2}(1+\xi^{-2})\right)-m\right]/s & \text{if} \quad \alpha \geq 1/(1+\xi^2). \end{cases}$$

E Percentage contribution MSR portfolio

The Maximum Sharpe Ratio (MSR) portfolio maximizes the ratio between the portfolio mean μ_p and the portfolio standard deviation σ_p. This implies the first order condition

$$\partial_i(\mu_p/\sigma_p) = 0 \Rightarrow \partial_i\mu_p/\partial_i\sigma_p = \mu_p/\sigma_p.$$

It follows that:

$$\begin{aligned} \frac{\partial_i \text{GVaR}}{\text{GVaR}} &= \frac{-\partial_i\mu_p - \Phi^{-1}(\alpha)\partial_i\sigma_p}{-\mu_p - \Phi^{-1}(\alpha)\sigma_p} = \frac{\partial_i\sigma_p}{\sigma_p}\left(\frac{-\partial_i\mu_p/\partial_i\sigma_p - \Phi^{-1}(\alpha)}{-\mu_p/\sigma_p - \Phi^{-1}(\alpha)}\right) = \frac{\partial_i\sigma_p}{\sigma_p} \\ &= \frac{\partial_i\sigma_p}{\sigma_p}\left(\frac{-\partial_i\mu_p/\partial_i\sigma_p + (1/\alpha)\phi[\Phi^{-1}(\alpha)]}{-\mu_p/\sigma_p + (1/\alpha)\phi[\Phi^{-1}(\alpha)]}\right) = \frac{\partial_i \text{GES}}{\text{GES}}. \end{aligned}$$

Hence, the percentage risk contributions of portfolio standard deviation, GVaR and GES coincide for the MSR portfolio.

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