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ESTIMATION OF A COMMON MEAN OF SEVERAL UNIVARIATE INVERSE GAUSSIAN POPULATIONS

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Abstract. The problem of estimating the common mean μ of k independent and univariate inverse Gaussian populations $IG(\mu, \lambda_i)$, $i = 1, \ldots, k$ with unknown and unequal λ's is considered. The difficulty with the maximum likelihood estimator of μ is pointed out, and a natural estimator $\tilde{\mu}$ of μ along the lines of Graybill and Deal is proposed. Various finite sample properties and some decision-theoretic properties of $\tilde{\mu}$ are discussed.

Key words and phrases: Inverse-Gaussian population, Graybill-Deal type estimate, squared error loss, equivariant estimator, admissibility.

1. Introduction

Although the importance of an inverse Gaussian distribution as a mathematical model for analyzing positively skewed data emerged from the pioneering work of Tweedi (1957*a*, 1957*b*), it received an enormous amount of attention since the publication of the review article of Folks and Chhikara (1978). Useful applications of the inverse Gaussian distributions have been demonstrated in the works of Sheppard (1962), Hasofer (1964), Lancaster (1972), Banerjee and Bhattacharya (1976) and Whitmore (1979, 1986*a*, 1986*b*). We refer to the recent book by Chhikara and Folks (1989) for various properties and applications of the inverse Gaussian distribution.

An inverse Gaussian distribution has striking similarities with and provocative departures from a normal distribution. For example, if $X \sim IG(\mu, \lambda)$, where $\mu = E(X)$ and $\mu^3/\lambda = V(X)$, and $X_1, \ldots, X_n$ are iid $\sim X$, then $\bar{X}$ and U, where $\bar{X} = n^{-1}\sum X_i$ and $U = \sum(X_i^{-1} - \bar{X}^{-1})$, are jointly minimal sufficient for μ and λ, complete and also independent. Moreover, $\bar{X} \sim IG(\mu, n\lambda)$ and $\lambda U \sim \chi^2_{n-1}$.

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These results are analogous to those for a normal distribution. However, unlike in the normal case, in general an arbitrary linear combination $\sum a_i X_i$ does not follow an inverse Gaussian distribution, and while μ and λ easily admit UMVUE's, the variance μ^3/λ does not (Korwar (1980), Iwase and Seto (1983)). This is a point of departure from the normal model. For some other similarities with and departures from the normal case, we refer to Letac *et al.* (1985), Pandey and Malik (1988), Bravo and MacGibbon (1988), Pal and Sinha (1989) and Hsieh *et al.* (1990).

In this paper we consider the problem of estimation of the common mean μ of several inverse Gaussian populations with unknown and unequal variances. It should be noted that the analogous problem of the estimation of common mean of several univariate normal populations with unknown and unequal variances has been extensively studied in the literature (Graybill and Deal (1959), Brown and Cohen (1974), Khatri and Shah (1974), Bhattacharya (1980) and Nair (1986)).

Let $\{X_{ij},\ j = 1, 2, \ldots, n_i\}$ be an iid sample from an inverse Gaussian distribution $IG(\mu, \lambda_i)$, $i = 1, 2, \ldots, k$ where $\mu > 0$ is an unknown common mean and $\lambda_1, \ldots, \lambda_k > 0$ are unknown and unequal. For the estimation of common mean μ when λ's are known, it can be easily shown that $\hat{\mu} = \sum \phi_i \bar{X}_i$ is the UMVUE of μ where $\phi_i = n_i\lambda_i / \sum n_j\lambda_j$, $i = 1, 2, \ldots, k$. We study this problem for the case of unknown λ's. The set of mutually independent statistics $\{(\bar{X}_i, U_i),\ i = 1, 2, \ldots, k\}$ are jointly minimal sufficient, where

$$\bar{X}_i = \frac{1}{n_i}\sum_j X_{ij}, \qquad U_i = \sum_j (X_{ij}^{-1} - \bar{X}_i^{-1}), \qquad i = 1, \ldots, k.$$

As noted before, $\bar{X}_i \sim IG(\mu, n_i\lambda_i)$, $\lambda_i U_i \sim \chi^2_{\nu_i}$, $\nu_i = n_i - 1$, $i = 1, 2, \ldots, k$. The joint distribution of $\bar{\boldsymbol{X}} = (\bar{X}_1, \bar{X}_2, \ldots, \bar{X}_k)$ and $\boldsymbol{U} = (U_1, U_2, \ldots, U_k)$ is not complete, and thus it does not permit the existence of a UMVUE of μ.

In Section 2 we discuss estimation of μ by the method of maximum likelihood, and point out some difficulties with this method. In Section 3 we consider the unbiased estimator $\tilde{\mu} = \sum \hat{\phi}_i \bar{X}_i$, where

$$\hat{\phi}_i = \hat{\phi}_i(U_1, \ldots, U_k) = \alpha_i U_i^{-1} / \sum_j \alpha_j U_j^{-1}, \qquad \alpha_i = n_i(n_i - 1), \qquad i = 1, \ldots, k.$$

This estimator is similar to the one proposed by Graybill and Deal (1959) for the estimation of a common mean of k normal populations. We refer to $\tilde{\mu}$ as the Graybill-Deal type estimator of μ and give two alternative expressions for its variance, one in the form of an infinite series (as in the normal case given by Khatri and Shah (1974) and Nair (1986)) and the other as a finite linear combination of incomplete beta integrals. The latter is more convenient for actual computation of the variance using the standard subroutines for integration. An unbiased estimator of the variance of $\tilde{\mu}$ based on $\bar{X}$ and $\boldsymbol{U}$ for the case $k \geq 3$ is also proposed. A difficulty for $k = 2$ is pointed out.

Certain necessary and sufficient conditions on the k sample sizes $n_1, n_2, \ldots, n_k$ which guarantee that $\tilde{\mu}$ has a smaller variance than each of $\bar{X}_i$, $i = 1, 2, \ldots, k$, are given in Section 4. In Section 5 we investigate some decision-theoretic properties of $\tilde{\mu}$ under squared error loss, and establish its admissibility within a suitable class. The inadmissibility of $\tilde{\mu}$ under some *a priori* information about $\lambda_1, \ldots, \lambda_k$ is pointed out in Section 6.

2. Maximum likelihood estimation of μ

The maximum likelihood estimator of μ is obtained by maximizing the likelihood function with respect to μ, $\lambda_1, \ldots, \lambda_k$. Direct maxmization of the likelihood function yields

$$(2.1)\qquad \hat{\mu} = \frac{\sum_{i=1}^{k} n_i \bar{x}_i \hat{\lambda}_i}{\sum_{i=1}^{k} n_i \hat{\lambda}_i}, \qquad \hat{\lambda}_i = \frac{n_i}{u_i + n_i \bar{x}_i \left(\dfrac{1}{\hat{\mu}} - \dfrac{1}{\bar{x}_i}\right)^2}, \qquad i = 1, \ldots, k.$$

thus resulting in the following equation for $\hat{\mu}$:

$$(2.2)\qquad \sum_{i=1}^{k} \frac{n_i^2 \bar{x}_i^3}{u_i \hat{\mu}^2 \bar{x}_i^2 + n_i \bar{x}_i (\bar{x}_i - \hat{\mu})^2} = \hat{\mu} \sum_{i=1}^{k} \frac{n_i^2 \bar{x}_i^2}{u_i \hat{\mu}^2 \bar{x}_i^2 + n_i \bar{x}_i (\bar{x}_i - \hat{\mu})^2}.$$

In general, the above leads to an equation involving $(2k-1)$-th degree polynomial in $\hat{\mu}$, and hence, computation of the MLE of μ is quite complicated. For $k = 2$, (2.2) yields the following cubic equation in $\hat{\mu}$:

$$(2.3)\qquad g(\hat{\mu}) \equiv a\hat{\mu}^3 - b\hat{\mu}^2 + c\hat{\mu} - d = 0$$

where

$$\begin{aligned} a &= n_1 n_2 (n_1 \bar{x}_1 + n_2 \bar{x}_2) + \bar{x}_1 \bar{x}_2 (n_1^2 u_2 + n_2^2 u_1), \\ b &= 2 n_1 n_2 \bar{x}_1 \bar{x}_2 (n_1 + n_2) + n_1 n_2 (n_1 \bar{x}_1^2 + n_2 \bar{x}_2^2) + \bar{x}_1 \bar{x}_2 (n_1^2 \bar{x}_1 u_2 + n_2^2 \bar{x}_2 u_1), \\ c &= n_1 n_2 \bar{x}_1 \bar{x}_2 (n_1 + n_2)(\bar{x}_1 + \bar{x}_2) + n_1 n_2 \bar{x}_1 \bar{x}_2 (n_1 \bar{x}_1 + n_2 \bar{x}_2), \\ d &= n_1 n_2 \bar{x}_1^2 \bar{x}_2^2 (n_1 + n_2). \end{aligned}$$

It is not difficult to verify that for $\bar{x}_1 < \bar{x}_2$, $g(\bar{x}_1) < 0$ and $g(\bar{x}_2) > 0$ while for $\bar{x}_1 > \bar{x}_2$, $g(\bar{x}_1) > 0$ and $g(\bar{x}_2) < 0$. Hence the above equation has at least one root between $\bar{x}_1$ and $\bar{x}_2$. A sufficient condition for the uniqueness of the root of (2.3) is $b^2 - 3ac < 0$ (see Uspenski (1948), pp. 86–87). Note that this condition holds with probability one as n_1 or $n_2 \to \infty$.

Once $\hat{\mu}$ is obtained from (2.2), the MLE's of λ_i's are easily obtained from (2.1). Let $\theta = (\mu, \lambda_1, \ldots, \lambda_k)$ and $\hat{\theta}$ be the MLE of θ. It is straightforward to derive the information matrix $I(\theta)$ and thus conclude that the asymptotic covariance matrix of $\hat{\theta}$ is given by

$$(2.4)\qquad I^{-1}(\theta) = \operatorname{diag}\left[\frac{\mu^3}{\sum_{i=1}^{k} n_i \lambda_i}, \frac{2\lambda_1^2}{n_1}, \ldots, \frac{2\lambda_k^2}{n_k}\right].$$

Here $\hat{\mu}$, $\hat{\lambda}_1, \ldots, \hat{\lambda}_k$ are all mutually independent. Moreover, the MLE of μ has the same asymptotic variance as that of the estimator in (2.1) with λ_i's assumed known. A better limiting distribution of $\hat{\mu}$ compared to the normal approximation can be obtained by adhering to the fact that as $n_i \to \infty$,

$$\left| \frac{\sum_{i=1}^k n_i \lambda_i \bar{X}_i}{\sum_{i=1}^k n_i \lambda_i} - \frac{\sum_{i=1}^k n_i \hat{\lambda}_i \bar{X}_i}{\sum_{i=1}^k n_i \hat{\lambda}_i} \right| \to 0 \quad \text{a.s.}$$

Since $\sum_{i=1}^k n_i \lambda_i \bar{X}_i \,/\, \sum_{i=1}^k n_i \lambda_i$ has an exact IG distribution with parameters $\left(\mu, \sum_{i=1}^k n_i \lambda_i\right)$, the limiting distribution of $\hat{\mu}$ can be approximated by $IG\left(\mu, \sum_{i=1}^k n_i \lambda_i\right)$.

3. Graybill-Deal type unbiased estimator of μ and its variance

It is easy to verify that when the parameters λ's are known, $\hat{\mu} = \sum_{i=1}^k n_i \lambda_i \bar{X}_i \,/\, \sum_{i=1}^k n_i \lambda_i$ is the UMVUE of μ. It is also the MLE of μ (see (2.1)). Moreover $\hat{\mu}$ is a complete sufficient statistic for μ, and $\hat{\mu} \sim IG\left(\mu, \sum_{i=1}^k n_i \lambda_i\right)$. In what follows, when the λ's are unknown, we can replace them by their suitable estimators and consider the resultant $\tilde{\mu}$. Writing $\lambda_i = 1/\sigma_i^2$, $i = 1, 2, \ldots, k$, and noting that $U_i \sim \sigma_i^2 \chi^2_{(n_i-1)}$, $i = 1, 2, \ldots, k$, we propose the pooled estimator $\tilde{\mu}$ of μ given by

$$\tilde{\mu} = \left\{ \sum_{i=1}^k \frac{n_i(n_i-1)\bar{X}_i}{U_i} \right\} \left\{ \sum_{j=1}^k \frac{n_j(n_j-1)}{U_j} \right\}^{-1} = \left\{ \sum_{i=1}^k \frac{\alpha_i \bar{X}_i}{U_i} \right\} \left\{ \sum_{i=1}^k \frac{\alpha_i}{U_i} \right\}^{-1} \tag{3.1}$$

with $\alpha_i = n_i(n_i-1)$, $i = 1, 2, \ldots, k$.

Using the fact that $\bar{X}_i$'s and U_i's are independent for all $i = 1, 2, \ldots, k$, it follows upon using a standard conditional argument that $E[\tilde{\mu}] = \mu$ and

$$\operatorname{Var}(\tilde{\mu}) = \operatorname{Var}\{E(\tilde{\mu} \mid U_1, \ldots, U_k)\} + E\{\operatorname{Var}(\tilde{\mu} \mid U_1, \ldots, U_k)\} = \mu^3 E\left[\frac{\sum_{i=1}^k \alpha_i^2 \sigma_i^2 / n_i U_i^2}{\left\{\sum_{i=1}^k \alpha_i / U_i\right\}^2} \right]. \tag{3.2}$$

In the case $k = 2$, the variance of $\tilde{\mu}$ can be expressed as an explicit function of the parameters μ, $\sigma_i^2 = 1/\lambda_i$, and the sample size n_i, $i = 1, 2$, as in the case of two normal populations (see Khatri and Shah (1974) and Nair (1986)) and one gets

$$\operatorname{Var}(\tilde{\mu}) = \frac{\mu^3}{n\lambda_1} \left\{ \sum_{i=0}^{\infty} (i+1)(1-\rho)^i \frac{B\left(\frac{m_1}{2}+i, \frac{m_2}{2}+2\right)}{B\left(\frac{m_1}{2}, \frac{m_2}{2}\right)} + \rho \frac{m_2}{m_1} \sum_{i=0}^{\infty} (i+1)(1-\rho)^i \frac{B\left(\frac{m_1}{2}+i+2, \frac{m_2}{2}\right)}{B\left(\frac{m_1}{2}, \frac{m_2}{2}\right)} \right\}. \tag{3.3}$$

Here $m_i = n_i - 1$, $i = 1, 2$, $\rho = (n_1 m_1 \lambda_1)^{-1} n_2 m_2 \lambda_2$, and the expansion is valid for $0 < \rho \leq 1$. A similar expansion obtains for $\rho > 1$.

An alternative expression for $\mathrm{Var}(\tilde{\mu})$, as a weighted finite sum of incomplete beta integrals which may be more convenient for actual computation, can be derived when n_i, for $i = 1, 2$, is an integer of the form $2k_i + 3$ for some positive integer k_i. It can be easily shown under the same conditions, i.e., $0 < \rho \leq 1$, that

$$\mathrm{Var}(\tilde{\mu}) = \frac{\mu^3 \rho^{k_2 - 1}}{n_1 \lambda_1 (1-\rho)^{k_1 + k_2 - 1} B(k_1 + 1, k_2 + 1)} \cdot \left\{ \rho \sum_{j=0}^{k_2+2} \binom{k_2+2}{j} I_j + \frac{k_2+1}{k_1+1} \sum_{j=0}^{k_2} \binom{k_2}{j} I_j \right\} \tag{3.4}$$

where

$$I_j = (-1)^{k_2 - j + 2} \left(\frac{1-\rho}{\rho} \right)^{j-2} \int_\rho^1 y^{j-2} (1-y)^{k_1 + k_2 + 2 - j} dy.$$

A similar integral representation of the variance of $\tilde{\mu}$ exists for $\rho > 1$.

We now proceed to obtain an unbiased estimator of $\mathrm{Var}(\tilde{\mu})$ for $k \geq 3$. Noting that $E[\bar{X}_i] = \mu$, $i = 1, 2, \ldots, k$ and that $\bar{X}_1, \ldots, \bar{X}_k$ are independent, it follows that an unbiased estimator of μ^3 based on $\bar{X}_1, \ldots, \bar{X}_k$ is given by

$$\hat{\mu}^3 = \left[\binom{k}{3} \right]^{-1} \sum_{i<j<l} \bar{X}_i \bar{X}_j \bar{X}_l. \tag{3.5}$$

Now an independent unbiased estimator of

$$\theta = E \left[\frac{\sum_{i=1}^k \alpha_i^2 \sigma_i^2 / n_i U_i^2}{\left\{ \sum_{i=1}^k \alpha_i / U_i \right\}^2} \right] \tag{3.6}$$

can be obtained following the ideas of Sinha (1985). This is based on an application of a powerful identity for the chi-square distribution (see Haff (1979)).

Towards this end we write

$$\theta = \sum_{i=1}^k \frac{\alpha_i^2}{n_i} E \left[\frac{\sigma_i^2 / U_i^2}{\left\{ \sum_{i=1}^k \alpha_i / U_i \right\}^2} \right] \tag{3.7}$$

and seek a function $\Psi_i(U_1, \ldots, U_k)$ such that

$$E[\Psi_i(U_1, \ldots, U_k)] = E \left[\frac{\sigma_i^2 / U_i^2}{\left\{ \sum_{i=1}^k \alpha_i / U_i \right\}^2} \right], \qquad i = 1, 2, \ldots, k. \tag{3.8}$$

Clearly then an unbiased estimator of $\mathrm{Var}(\tilde{\mu})$, given in (3.2), is provided by

$$\tilde{\mathrm{V}}\mathrm{ar}(\tilde{\mu}) = \left\{ \left[\binom{k}{3} \right]^{-1} \sum_{i<j<l} \bar{X}_i \bar{X}_j \bar{X}_l \right\} \left\{ \sum_{i=1}^{k} \frac{\alpha_i^2}{n_i} \Psi_i(U_1, \ldots, U_k) \right\}. \tag{3.9}$$

From Sinha (1985), it follows that

$$\Psi_i(U_1, \ldots, U_k) = \sum_{l=0}^{\infty} a_l \tag{3.10}$$

where

$$a_l = \frac{A_{(-i)}^l U_i^{2(l+1)} 2^l (l+1)!}{(n_i+1)^{[l]} (n_i + A_{(-i)} U_i)^{l+2}} \tag{3.11}$$

with

$$A_{(-i)} = \sum_{j \neq i} \frac{n_j}{U_j}, \qquad (n+1)^{[l]} = \begin{cases} (n+1)(n+3)\cdots(n+2l-1) \\ 1 \quad \text{for} \quad l = 0. \end{cases}$$

It is interesting to observe that the above unbiased estimate of the variance of $\tilde{\mu}$ is always positive. Moreover, for computational purposes, if one uses

$$\tilde{\mathrm{V}}\mathrm{ar}(\tilde{\mu})_{(m)} = \left[\binom{k}{3} \right]^{-1} \left\{ \sum_{i<j<l} \bar{X}_i \bar{X}_j \bar{X}_l \right\} \left\{ \sum_{i=1}^{k} \frac{\alpha_i^2}{n_i} \Psi_{i(m)} \right\} \tag{3.12}$$

with $\Psi_{i(m)} = \Psi_{i(m)}(U_1, \ldots, U_k) = \sum_{l=0}^{m-1} a_l$ then it again follows from Sinha (1985) that

$$|E\{\tilde{\mathrm{V}}\mathrm{ar}(\tilde{\mu})_{(m)}\} - \mathrm{Var}(\tilde{\mu})| = O\left(\sum_{i=1}^{k} \frac{\sigma_i^2}{n_i^{m+1}} \right).$$

Hence, for moderately large values of the n_i's, using m as low as 2 or 3 one would get a good approximation to the unbiased estimator of the variance.

For $k = 2$, the above argument cannot be used because we do not readily have an unbiased estimator of μ^3 based on just $\bar{X}_1$ and $\bar{X}_2$. However, noting that

$$E(\bar{X}_1 - \bar{X}_2)^2 = \mu^3 \left(\frac{1}{n_1 \lambda_1} + \frac{1}{n_2 \lambda_2} \right) = \mu^3 \left(\frac{\sigma_1^2}{n_1} + \frac{\sigma_2^2}{n_2} \right),$$

one can write $\mathrm{Var}(\tilde{\mu})$ as

$$\mathrm{Var}(\tilde{\mu}) = \{E(\bar{X}_1 - \bar{X}_2)^2\} E \left\{ \frac{\sum_{i=1}^{2} \alpha_i^2 \sigma_i^2 / n_i U_i^2}{\left(\frac{\sigma_1^2}{n_1} + \frac{\sigma_2^2}{n_2} \right) \left(\sum_{i=1}^{2} \alpha_i / U_i \right)^2} \right\}. \tag{3.13}$$

An unbiased estimator of $\mathrm{Var}(\tilde{\mu})$ can therefore be obtained once we have derived functions $\Psi_i(U_1, U_2)$, $i = 1, 2$ such that

$$E\left\{\frac{\sigma_i^2/n_iU_i^2}{\left(\frac{\sigma_1^2}{n_1}+\frac{\sigma_2^2}{n_2}\right)\left(\sum_{i=1}^2 \alpha_i/U_i\right)^2}\right\} = E\{\Psi_i(U_1, U_2)\}, \qquad i = 1, 2. \tag{3.14}$$

Unfortunately, however, it turns out that such Ψ_i's do not exist. This is primarily because, based on two independent chi-square variables $V_1 \sim \sigma_1^2\chi_{m_1}^2$ and $V_2 \sim \sigma_2^2\chi_{m_2}^2$, there does not exist an unbiased estimator of $\sigma_1^2/(\sigma_1^2+\sigma_2^2)$. A nearly unbiased estimator of the above expression is obtained by replacing the factor $(\sigma_1^2/n_1)(\sigma_1^2/n_1+\sigma_2^2/n_2)^{-1}$ by some constant δ, $0 < \delta < 1$. We therefore propose

$$\tilde{\mathrm{V}}\mathrm{ar}(\tilde{\mu}) = (\bar{X}_1 - \bar{X}_2)^2\left\{\frac{\delta(\alpha_1^2/U_1) + (1-\delta)(\alpha_2^2/U_2)}{\left(\sum_{i=1}^2 \alpha_i/U_i\right)^2}\right\} \tag{3.15}$$

as an approximate unbiased estimator of $\mathrm{Var}(\tilde{\mu})$. Typically, if $n_1 = n_2$, $\delta = 1/2$ may be a reasonable choice.

4. Necessary and sufficient conditions under which $\tilde{\mu}$ is better than $\bar{X}_i$ for all i

The following general result essentially follows from Norwood and Hinkelmann (1977). Let $Y_1, \ldots, Y_k, V_1, V_2, \ldots, V_k$ be independent random variables where Y_i's have a common mean μ and $\mathrm{Var}(Y_i) = c\tau_i^2$ for some constant $c > 0$. Furthermore, $m_iV_i/\tau_i^2 \sim \chi_{m_i}^2$, $i = 1, \ldots, k$. It is well known that when τ_i's are known, the best linear unbiased estimator of μ is given by

$$\tilde{\mu}^* = \sum(Y_i/\tau_i^2)\left[\sum(1/\tau_i^2)\right]^{-1}. \tag{4.1}$$

For unknown τ_i's, we consider a natural estimator of μ given by

$$\tilde{\mu} = \sum(Y_i/V_i)\left[\sum(1/V_i)\right]^{-1}. \tag{4.2}$$

Clearly the estimator $\tilde{\mu}$ is unbiased for μ. We now seek conditions under which $\tilde{\mu}$ has a smaller variance compared to each Y_i, $i = 1, \ldots, k$. This is contained in the following theorem.

THEOREM 4.1. *The estimator $\tilde{\mu}$ is better than any Y_i in the sense that* $\mathrm{Var}(\tilde{\mu}) < c\tau_i^2$, $i = 1, 2, \ldots, k$, *if either* (a) $m_i > 9$, $i = 1, 2, \ldots, k$ *or* (b) $m_i = 9$ *for some i and $m_j > 17$, $j = 1, 2, \ldots, k$; $j \neq i$.*

Taking $Y_i = \bar{X}_i$, $m_i = n_i - 1$, $c = \mu^3$, $\tau_i^2 = 1/(n_i\lambda_i)$, $V_i = U_i/m_in_i$, $i = 1, \ldots, k$, and applying the above theorem, it follows that for estimating the common mean μ of k inverse Gaussian populations $\tilde{\mu}$ is better than the individual sample means if and only if either $n_i \geq 10$ for all i or $n_i = 10$ for some i and $n_j \geq 18$ for $j \neq i$.

5. Decision-theoretic properties of $\tilde{\mu}$

In this section we discuss some decision-theoretic properties of $\tilde{\mu}$ under the squared error loss function $L(\delta, \mu) = (\delta - \mu)^2$. Towards this end we first consider a meaningful class of estimators of μ of the form

$$\tilde{\mu}_\Phi = \sum_{i=1}^{k} \bar{X}_i \phi_i(U_1, \dots, U_k) \tag{5.1}$$

where $\phi_i(U_1, \dots, U_k)$'s are nonnegative real-valued functions of $U_1, \dots, U_k$, satisfying $\sum_{i=1}^{k} \phi_i(U_1, \dots, U_k) = 1$ with probability one. Clearly, any estimator of the form $\tilde{\mu}_\Phi$ such that $E\{\phi_i(U_1, \dots, U_k)\} < \infty$, $i = 1, \dots, k$, is unbiased for μ. Moreover, using independence of $\bar{X}_i$'s and U_i's and the conditional argument as in (3.2), it follows that

$$\text{risk of } \tilde{\mu}_\Phi = E\left[\mu^3 \left\{\sum_{i=1}^{k} \phi_i^2(U_1, \dots, U_k)/n_i\lambda_i\right\}\right]. \tag{5.2}$$

Now if we choose a prior density of the form $h_1(\mu)h_2(\lambda_1, \dots, \lambda_k)$, the Bayes risk of $\tilde{\mu}_\Phi$ can be written as

$$\text{Bayes risk of } \tilde{\mu}_\Phi = [E(\mu^3)] \left[E\left\{E\left(\sum_{i=1}^{k} \phi_i^2(U_1, \dots, U_k)/n_i\lambda_i\right)\right\}\right] \tag{5.3}$$

where the innermost expectation is with respect to the U_i's for fixed $\lambda_1, \dots, \lambda_k$, and the second expectation is with respect to the λ's under $h_2(\lambda_1, \dots, \lambda_k)$. Assuming that $E(\mu^3) < \infty$ and $E(\lambda_i^{-1}) < \infty$, $i = 1, 2, \dots, k$, it follows by Fubini's theorem that the unique Bayes estimator of μ is given by

$$\tilde{\mu}_{\text{Bayes}} = \sum_{i=1}^{k} \bar{X}_i \phi_{i\text{Bayes}}(U_1, \dots, U_k) \tag{5.4}$$

where

$$\phi_{i\text{Bayes}}(U_1, \dots, U_k) = \left\{\frac{1}{E_{\boldsymbol{\lambda}|\boldsymbol{U}}(1/n_i\lambda_i)}\right\} \Bigg/ \left\{\sum_{j=1}^{k} \frac{1}{E_{\boldsymbol{\lambda}|\boldsymbol{U}}(1/n_j\lambda_j)}\right\}, \quad i = 1, \dots, k \tag{5.5}$$

and $E_{\boldsymbol{\lambda}|\boldsymbol{U}}(\cdot)$ above denotes expectation with respect to the posterior distribution of $\boldsymbol{\lambda} = (\lambda_1, \dots, \lambda_k)$, given $\boldsymbol{U} = (U_1, \dots, U_k)$.

The equation (5.4) provides a class of admissible estimators of μ in the class $\tilde{\mu}_\Phi$. The treatment here is similar to Zacks (1966) for the normal distribution. It is not difficult to show that the Graybill-Deal type unbiased estimator $\tilde{\mu}$ is a generalized Bayes estimator since it results from a choice of the prior of the form

$h_2(\boldsymbol{\lambda})\alpha\prod_{i=1}^{k}\lambda_i^{-1/2-1}$ and $h_1(\mu)$ satisfying $E(\mu^3) < \infty$. Unfortunately, however, such a density of $\boldsymbol{\lambda}$ is improper and the admissibility of $\tilde{\mu}$ does not readily follow. On the other hand, if we restrict our attention to a subclass of $\tilde{\mu}_\Phi$ of the type

$$\tilde{\tilde{\mu}}_\Phi = \sum_{i=1}^{k} \bar{X}_i\phi_i\left(\frac{U_1}{U_k}, \dots, \frac{U_{k-1}}{U_k}\right), \tag{5.6}$$

it follows from Sinha and Mouqadem (1982) that for $k = 2$, $\tilde{\mu}$ is admissible within this class. It may be noted that the normality of $\bar{X}_i$'s in Sinha and Mouqadem (1982) is not crucial for this purpose in view of the representation (5.2) and the same distribution of U_i's in the inverse Gaussian case as in the normal case.

We now turn our attention to a discussion of equivariant estimators of μ. Unlike in the normal case, in the context of several inverse Gaussian distributions, an estimator $\hat{\mu}(\bar{X}_1, \dots, \bar{X}_k;\ U_1, \dots, U_k)$ is scale equivariant if $\hat{\mu}(\cdot)$ is scale preserving, i.e. $\hat{\mu}(\cdot)$ satisfies

$$\hat{\mu}\left(\alpha\bar{X}_1, \dots, \alpha\bar{X}_k;\ \frac{U_1}{\alpha}, \dots, \frac{U_k}{\alpha}\right) = \alpha\hat{\mu}(\bar{X}_1, \dots, \bar{X}_k;\ U_1, \dots, U_k), \tag{5.7}$$

for all $\alpha > 0$. A characterization of $\hat{\mu}(\cdot)$ subject to (5.7) is provided by

$$\hat{\mu}(X_1, \dots, \bar{X}_k;\ U_1, \dots, U_k) = \sum_{i=1}^{k} \bar{X}_i\phi_i(R_1, \dots, R_{k-1};\ T_1, \dots, T_k) \tag{5.8}$$

where $R_i = \bar{X}_i/\bar{X}_k$, $i = 1, \dots, k-1$, $T_i = \bar{X}_iU_i$, $i = 1, \dots, k$, and $\phi_i(\cdot)$'s are real-valued functions adding up to one.

It may be noted that the estimators $\tilde{\mu}_\Phi$ described in (5.1) are equivariant if and only if $\phi_1, \dots, \phi_k$ are scale invariant, i.e., ϕ's depend on the U's only through the ratios U_i/U_j's. In general an equivariant estimator of μ is not unbiased unless it is of the form $\tilde{\mu}_\Phi$. See Hirano and Iwase (1989*a*, 1989*b*) and Zacks (1970) for some related results.

An attempt to derive the Bayes equivariant estimator of μ in general under a prior distribution of μ and $\lambda_1, \dots, \lambda_k$ has been unsuccessful even for $k = 2$. This is because although the conditional moments of $\bar{X}_1$ and $\bar{X}_2$, given T_1, T_2 and R_1, can be explicitly computed involving Bessel functions of the second kind, the computation of the expectations of the resultant expressions with respect to any meaningful prior turns out to be a formidable task.

6. Inadmissibility of $\tilde{\mu}$

In this section we mention the inadmissibility of $\tilde{\mu}$ under squared error loss when some *a priori* information about λ's is available by actually constructing an improved estimator of μ. Details are omitted since the technique is similar to what Sinha (1979) used in the case of normal population. Without loss of generality let us assume that $\lambda_1/\lambda_{i_0} \geq c_0$ for some $i_0 \geq 2$, and consider estimators of μ of the form

$$\Psi(\bar{X}_1, \ldots, \bar{X}_k, U_1, \ldots, U_k) = \bar{X}_1 \left(1 - \sum_{j=2}^{k} \phi_j\right) + \sum_{j=2}^{k} \phi_j \bar{X}_j = \tilde{\mu}_\Phi \qquad \text{(say)}$$

where

$$\Phi = (\phi_1, \ldots, \phi_k), \qquad \phi_1 = 1 - \sum_{j=2}^{k} \phi_j, \qquad \phi_j \equiv \phi_j(\boldsymbol{U}), \qquad j = 2, \ldots, k.$$

It then follows from Sinha (1979) that $\tilde{\mu}_{\Phi^*}$, where $\Phi^* = (\phi_1^*, \ldots, \phi_k^*)$ with $\phi_j^* = \phi_j$ for $j \neq i_0$ and $\phi_{i_0} = \min\{\phi_{i_0}, (1 + c_0 n_1/n_0)^{-1}\}$, provides uniform improvement over $\tilde{\mu}_\Phi$ provided the conditions $\sum_{i \neq i_0} \phi_i \geq 0$ and $\phi_{i_0} > (1 + c_0 n_1/n_0)^{-1}$ hold with a positive probability. This would in turn imply that one can improve upon $\tilde{\mu}$ since it corresponds to $\tilde{\mu}_{\Phi_0}$ with $\Phi_0 = (\phi_{01}, \ldots, \phi_{0k})$ where

$$\phi_{0i} = \frac{\alpha_i U_1}{\alpha_1 U_i} \Bigg/ \left\{1 + \sum_{j=2}^{k} \alpha_j U_1/\alpha_1 U_j\right\},$$

$i = 1, \ldots, k$. The improved estimator is given by $\tilde{\mu}_{\Phi_0^*}$ because ϕ_{0i}'s do satisfy the above condition with a positive probability. It should be noted that $\tilde{\mu}_{\Phi_0^*}$ can be interpreted as a testimator.

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