Ann. Inst. Statist. Math.
Vol. 43, No. 4, 747–760 (1991)

ESTIMATION OF A SMOOTH QUANTILE FUNCTION UNDER THE PROPORTIONAL HAZARDS MODEL

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(Received April 16, 1990; revised September 20, 1990)

Abstract. The problem of estimating a smooth quantile function, $Q(\cdot)$, at a fixed point p, $0 < p < 1$, is treated under a nonparametric smoothness condition on Q. The asymptotic relative deficiency of the sample quantile based on the maximum likelihood estimate of the survival function under the proportional hazards model with respect to kernel type estimators of the quantile is evaluated. The comparison is based on the mean square errors of the estimators. It is shown that the relative deficiency tends to infinity as the sample size, n, tends to infinity.

Key words and phrases: Relative deficiency, mean square error, kernel type estimators, quantile function, right censored data, proportional hazards model.

1. Introduction

Let $(X_1, Y_1), \dots, (X_n, Y_n)$ be an iid sequence of pairs of nonnegative random variables. Assume that X_i and Y_i are independent random variables with distribution functions F and G respectively. Under the random censorship model, X_i and Y_i can not be observed separately. One only observes $Z_i = X_i \wedge Y_i$ and $\delta_i = I(X_i \le Y_i)$ for $i = 1, \dots, n$, where $I(A)$ denotes the indicator of the set A. Let $\bar{F}(t) = P(X > t)$, $\bar{G}(t) = P(Y > t)$ and $\bar{H}(t) = P(Z > t)$ denote the survival functions associated with X, Y and Z respectively. In the usual random censorship model one assumes only that the censored and the censoring sequence are independent and hence, the expected proportion of uncensored observations is given by

$$\alpha = P(\delta = 1) = \int \bar{G} dF \quad \text{and} \quad \bar{H}(t) = \bar{F}(t)\bar{G}(t). \tag{1.1}$$

However, in many situations, it can be assumed that $\bar{F}$ and $\bar{G}$ are related and $\bar{G}$ can be expressed as $\bar{G}(t) = (\bar{F}(t))^{\beta}$, for all $t > 0$, where $\beta > 0$ is some fixed unknown constant. This model is known as the proportional hazards model and was considered in its present form by Koziol and Green (1976) and subsequently by

Hollander and Proschan (1979), Csörgő and Horváth (1981), Abduskhurov (1984, 1987), Cheng and Lin (1987), Ghorai and Rejtő (1987) and Csörgő (1988, 1989). The assumption of the proportional hazards model of random censorship is not unrealistic. An example where such a model is appropriate is the Channing House data analyzed by Efron (1981) and by Csörgő (1989). Using various tests Csörgő (1989) has clearly demonstrated that this model fits the data very closely. Under this proportional hazards model of random censorship

$$\alpha = (1+\beta)^{-1} \quad \text{and} \quad \bar{F}(t) = (\bar{H}(t))^{\alpha}. \tag{1.2}$$

Abduskhurov (1984, 1987) and Cheng and Lin (1987) proposed and studied the large sample properties of the maximum likelihood estimate of $F(t)$. The maximum likelihood estimate, henceforth called ACL-estimator, is given by

$$\hat{F}_{\mathrm{ACL}}(t) = (1 - H_n(t))^{\alpha_n}, \tag{1.3}$$

where,

$$n\alpha_n = \sum_{i=1}^{n} \delta_i \quad \text{and} \quad nH_n(t) = \sum_{i=1}^{n} I(Z_i \le t). \tag{1.4}$$

We define the p-th quantiles of F and $\hat{F}_{\mathrm{ACL}}$ as

$$Q(p) = \inf\{t : F(t) \ge p\}, \tag{1.5}$$

$$Q_n(p) = \inf\{t : \hat{F}_{\mathrm{ACL}}(t) \ge p\}. \tag{1.6}$$

Since $1 - H_n(t) = \bar{H}_n(t)$ is a step function, the empirical quantile function, $Q_n(p)$, based on $\hat{F}_{\mathrm{ACL}}$ will also be a step function even if the true quantile function $Q(p)$ is continuous. In the case of no censoring Falk (1984) had defined kernel smoothed estimators of the quantile function. Falk (1984) has shown that certain kernel type estimators are better than the sample quantile in terms of relative deficiency. Large sample properties of the product limit quantile function have been studied by Sander (1975), Csörgő (1983), Cheng (1984), Aly *et al.* (1985), Lo and Singh (1986) and Gijbels and Veraverbeke (1988). A kernel type estimator for the smooth quantile function in the case of arbitrary right censored data was introduced by Padgett (1986). Its large sample properties were subsequently studied by Lio *et al.* (1986) and Lio and Padgett (1987). Lio and Padgett (1987) have shown that under certain conditions on the kernel function the mean square error of the kernel type estimator of the quantile function is less than that of the PL-quantile function. Ghorai and Rejtő (1989) have established a deficiency result similar to Falk (1984) for the product-limit quantile function. The goal of this paper is to establish a similar result for the ACL-quantile. A kernel type estimator of a smooth quantile function, based on $Q_n(\cdot)$ and a kernel function $k(\cdot)$ is defined as

$$\hat{Q}_n(p) = \int_0^1 Q_n(t)\frac{1}{h}k\left(\frac{p-t}{h}\right)dt. \tag{1.7}$$

In this paper we investigate the mean square errors of $Q_n(p)$ and $\hat{Q}_n(p)$ respectively and establish an asymptotic representation of the relative deficiency, $(i(n) - n)$, of $Q_n(p)$ with respect to $\hat{Q}_n(p)$, where $i(n)$ is defined as

$$i(n) = \min\{j : \mathrm{MSE}(Q_j(p)) \leq \mathrm{MSE}(\hat{Q}_n(p))\}. \tag{1.8}$$

In particular, we show that the relative deficiency, $(i(n) - n)$, tends on infinity as $n \to \infty$, if the kernel function satisfies some conditions. The question of relative deficiency of $Q_n(p)$ with respect to $\hat{Q}_n(p)$ was also raised by Csörgő (1989). Our results provide an answer to his query.

We now introduce some more notations and assumptions on the kernel function, $k(\cdot)$. The l-th derivative of any function g will be denoted by $g^{(l)}$. It will be assumed that the kernel function, $k(\cdot)$, is a bounded Borel measurable function with the following properties.

CONDITION K.

(i) $k(x) = 0$, for $|x| > 1$,
(ii) $\int_{-1}^{1} k(x)dx = 1$,
(iii) $\int_{-1}^{1} x^i k(x)dx = 0$ for $i = 1, \ldots, m$, and,
(iv) $M = \int_{-1}^{1} |x|^{m+1} |k(x)| dx < \infty$.

For future use define,

$$K(x) = \int_{-\infty}^{x} k(t)dt. \tag{1.9}$$

2. Main results and proofs

In our proofs we will make use of the asymptotic representation of $\hat{F}_{\mathrm{ACL}}$ due to Cheng and Lin (1987). For convenience we state their result in Lemma 2.1. Gijbels and Veraverbeke (1989) have studied the almost sure behavior of the Bahadur representation of the ACL-quantile. However, for fixed n, their result does not tell us anything about the order of magnitude of the MSE of the ACL-quantile. In Theorem 2.1 we derive the order of magnitude of the second moment of the error of the Bahadur representation of the ACL-quantile. This is then used to derive the MSE of the ACL-quantile. The result is stated in Theorem 2.2. The asymptotic expansion of the mean square error of the kernel type estimator, $\hat{Q}_n(p)$ is given in Theorem 2.3. Finally, the deficiency result is stated in Theorem 2.4. Define $T_F = \sup\{t : \bar{F}(t) > 0\}$.

LEMMA 2.1. (Cheng and Lin (1987)) *For $t < T_F$, we have*

$$\hat{\bar{F}}_{\mathrm{ACL}}(t) - \bar{F}(t) = \frac{1}{n}\sum_{i=1}^{n} \xi_i(t) + r_n(t), \tag{2.1}$$

where,

$$\xi_i(t) = \alpha(\bar{H}(t))^{\alpha-1}[I(Z_i > t) - \bar{H}(t)] + (\delta_i - \alpha)(\bar{H}(t))^{\alpha} \log \bar{H}(t), \tag{2.2}$$

and for $T < T_F$,

$$E \sup_{t \le T} r_n^2(t) = O(n^{-2}). \tag{2.3}$$

THEOREM 2.1. *Let p, $0 < p < 1$, be fixed and $f = F'$. Suppose F is twice differentiable in the neighborhood of $Q(p)$ with $f(Q(p)) > 0$ and $f'(t)$ is bounded in the neighborhood of $Q(p)$. If $E|Z_1|^{2+\eta} < \infty$ for some $\eta > 0$, then one can write*

$$Q_n(p) - Q(p) = \frac{1}{n} \sum_{i=1}^{n} \frac{\xi_i(Q(p))}{f(Q(p))} + R_n(p), \tag{2.4}$$

where $\xi_i(\cdot)$ is given in (2.2) *and*

$$ER_n^2(p) = O(n^{-3/2}). \tag{2.5}$$

Remark 2.1. In the case of no censoring Duttweiler (1973) has shown the same result under a slightly weaker condition.

PROOF. Define $\theta_n = 1 - (1-p)^{1/\alpha_n}$ and $\theta = 1 - (1-p)^{1/\alpha}$. Then it is easy to show that $Q(p) = F^{-1}(p) = H^{-1}(\theta)$ and $Q_n(p) = \hat{F}_{\mathrm{ACL}}^{-1}(p) = H_n^{-1}(\theta_n)$. For given θ_n, $H_n^{-1}(\theta_n)$ is an order statistic of $Z_1, \ldots, Z_n$. Suppose $H_n^{-1}(\theta_n) = Z_{(m)}$, m-th order statistic of $Z_1, \ldots, Z_n$. For $\epsilon_n = c\sqrt{(\log n)/n}$, define the event

$$A_n = \{|\alpha_n - \alpha| \le \epsilon_n\}. \tag{2.6}$$

We now decompose the difference, $Q_n(p) - Q(p)$, into various terms and extract the iid component.

$$Q_n(p) - Q(p) = Q_n(p)I(A_n^c) + (Q_n(p)I(A_n) - Q(p)) \tag{2.7}$$

where the second term can be expressed as

$$\begin{aligned} Q_n(p)I(A_n) - Q(p) &= H_n^{-1}(\theta_n)I(A_n) - H^{-1}(\theta) \\ &= (H_n^{-1}(\theta_n) - H^{-1}(\theta_n))I(A_n) \\ &\quad + (H^{-1}(\theta_n) - H^{-1}(\theta))I(A_n) - H^{-1}(\theta)I(A_n^c). \end{aligned} \tag{2.8}$$

The first term in (2.8) can be written as

$$\begin{aligned} &(H_n^{-1}(\theta_n) - H^{-1}(\theta_n))I(A_n) \\ &\quad = \left(Z_{(m)} - H^{-1}\left(\frac{m}{m+1}\right)\right) I(A_n) \\ &\quad\quad + \left(H^{-1}\left(\frac{m}{m+1}\right) - H^{-1}(\theta_n)\right) I(A_n) \\ &\quad = \left[\frac{\alpha(\bar{H}(Q(p)))^{\alpha-1}}{f(Q(p))}\right] (\bar{H}_n(Q(p)) - \bar{H}(Q(p)))I(A_n) + R_{n1} + R_{n2}, \end{aligned} \tag{2.9}$$

where

$$R_{n1} = \left[Z_{(m)} - H^{-1}\left(\frac{m}{m+1}\right) - \left(\frac{\alpha(\bar{H}(Q(p)))^{\alpha-1}}{f(Q(p))}\right)(\bar{H}_n(Q(p)) - \bar{H}(Q(p))) \right] I(A_n), \tag{2.10}$$

$$R_{n2} = \left(H^{-1}\left(\frac{m}{m+1}\right) - H^{-1}(\theta_n)\right) I(A_n). \tag{2.11}$$

The second term in (2.8) can be written as

$$\begin{aligned}(H^{-1}(\theta_n) - H^{-1}(\theta))I(A_n) &= [H^{-1}(1-(1-p)^{1/\alpha_n}) - H^{-1}(1-(1-p)^{1/\alpha})]I(A_n) \\ &= (\alpha_n - \alpha)\frac{d}{d\alpha}(H^{-1}(1-(1-p)^{1/\alpha}))I(A_n) \\ &\quad + \frac{1}{2}(\alpha_n-\alpha)^2 \frac{d^2}{d\alpha^2}(H^{-1}(1-(1-p)^{1/\alpha})) \mid_{\alpha=\alpha^*} I(A_n) \\ &= (\alpha_n - \alpha)\frac{(\bar{H}(Q(p)))^\alpha \log \bar{H}(Q(p))}{f(Q(p))} I(A_n) + R_{n3},\end{aligned} \tag{2.12}$$

where

$$R_{n3} = \frac{1}{2}(\alpha_n-\alpha)^2 \frac{d^2}{d\alpha^2}(H^{-1}(1-(1-p)^{1/\alpha})) \mid_{\alpha=\alpha^*} I(A_n), \tag{2.13}$$

and α^* is between α and α_n. Now using (2.9) and (2.12) in (2.8) we get

$$\begin{aligned}Q_n(p)I(A_n) - Q(p) &= \left[\frac{\alpha(\bar{H}(Q(p)))^{\alpha-1}}{f(Q(p))}\right](\bar{H}_n(Q(p)) - \bar{H}(Q(p)))I(A_n) \\ &\quad + (\alpha_n - \alpha)\frac{(\bar{H}(Q(p)))^\alpha \log \bar{H}(Q(p))}{f(Q(p))} I(A_n) \\ &\quad + R_{n1} + R_{n2} + R_{n3} \\ &= \frac{1}{n}\sum_{i=1}^{n} \frac{\xi_i(Q(p))}{f(Q(p))} I(A_n) + R_{n1} + R_{n2} + R_{n3}.\end{aligned} \tag{2.14}$$

Substituting (2.14) in (2.7) and writing $I(A_n) = 1 - I(A_n^c)$ we get

$$Q_n(p) - Q(p) = \frac{1}{n}\sum_{i=1}^{n} \frac{\xi(Q(p))}{f(Q(p))} + R_n(p) \tag{2.15}$$

where

$$R_n(p) = Q_n(p)I(A_n^c) - \frac{1}{n}\sum_{i=1}^{n} \frac{\xi_i(Q(p))}{f(Q(p))} I(A_n^c) + R_{n1} + R_{n2} + R_{n3}. \tag{2.16}$$

To complete the proof of the theorem it is enough to show that the expected value of the square of each term in (2.16) is $O(n^{-3/2})$. Since under the proportional hazards model $Z_1, \ldots, Z_n$ are independent of $\delta_1, \ldots, \delta_n$, it is not too difficult to show that $E(\xi_i(Q(p)))^2 = \sigma_{\mathrm{ML}}(Q(p), Q(p))$, where for $s \le t$ (see Cheng and Lin (1987)),

$$\begin{aligned}\sigma_{\mathrm{ML}}(s,t) = {} & \alpha^2(\bar{H}(s))^{\alpha-1}(\bar{H}(t))^{\alpha}(1-\bar{H}(s)) \\ & + \alpha(1-\alpha)(\bar{H}(s)\bar{H}(t))^{\alpha}\log\bar{H}(s)\log\bar{H}(t).\end{aligned} \tag{2.17}$$

Also using the Bernstein's inequality (see Serfling (1980), p. 59) we get

$$\begin{aligned}P(A_n^c) &= P(|\alpha_n - \alpha| > \epsilon_n) \le d_0 e^{-d_1 n \epsilon_n^2} \\ &= d_0 e^{-d_1 c^2 \log n} = O(n^{-d_1 c^2}).\end{aligned} \tag{2.18}$$

Since d_0 and d_1 are absolute constants, by choosing the constant c in the definition of ϵ_n appropriately we can get

$$P(A_n^c) = O(n^{-6}). \tag{2.19}$$

Using (2.17) and (2.19) we get

$$E\left[\frac{1}{n}\sum_{i=1}^{n}\frac{\xi_i(Q(p))}{f(Q(p))}I(A_n^c)\right]^2 = O(n^{-2}). \tag{2.20}$$

To handle the first term in (2.16), recall that $Q_n(p) = H_n^{-1}(\theta_n)$, where $\theta_n = 1-(1-p)^{1/\alpha_n}$. Since the random variables Z_i's are assumed to be nonnegative, clearly $0 \le Q_n(p) = H_n^{-1}(\theta_n) \le Z_{(n)}$. Hence by using the Hölder's inequality, for $\eta > 0$,

$$E(Q_n(p)I(A_n^c))^2 \le [E(Q_n(p))^{2+\eta}]^{2/(2+\eta)}(EI(A_n^c))^{\eta/(2+\eta)}, \tag{2.21}$$

where

$$\begin{aligned}E(Q_n(p))^{2+\eta} \le E(Z_{(n)})^{2+\eta} &= \int_0^\infty P((Z_{(n)})^{2+\eta} > t)dt \\ &= \int_0^\infty [1 - P(Z_{(n)} \le t^{1/(2+\eta)})]dt \\ &= \int_0^\infty [1 - (H(t^{1/(2+\eta)}))^n]dt \\ &\le n\int_0^\infty (1 - H(t^{1/(2+\eta)}))dt = nE(Z_1^{2+\eta}).\end{aligned} \tag{2.22}$$

Now using (2.22) and (2.18) in (2.21) we get

$$\begin{aligned}E(Q_n(p)I(A_n^c))^2 &= O(n^{-[d_1c^2\eta/(2+\eta)]+1}) \\ &= O(n^{-3/2}) \quad \text{if} \quad c \ge \sqrt{\frac{10+5\eta}{2d_1\eta}}.\end{aligned} \tag{2.23}$$

To get a bound on ER_{n3}^2 first observe that

$$\left(\frac{d^2}{d\alpha^2} H^{-1}(1-(1-p)^{1/\alpha}) \mid_{\alpha=\alpha^*}\right)^2 I(A_n)$$

can be bounded above by a fixed constant since α^* is between α_n and α and $|\alpha_n - \alpha| \le \epsilon_n$. Hence

$$ER_{n3}^2 = O(E(\alpha_n - \alpha)^4) = O(n^{-2}). \tag{2.24}$$

To get a bound on ER_{n2}^2, recall that m is such that $H_n^{-1}(\theta_n) = Z_{(m)}$. This implies that $(m-1)/n < \theta_n \le m/n$ or $|\theta_n - m/(n+1)| \le n^{-1}$. Using this and the mean value theorem we get

$$ER_{n2}^2 = E\left[\left(\theta_n - \frac{m}{n+1}\right)^2 \left(\frac{d}{dt} H^{-1}(t) \mid_{t=\theta_n^*}\right)^2 I(A_n)\right] \tag{2.25}$$

$$= O(n^{-2}). \tag{2.26}$$

In (2.25), $|\alpha_n - \alpha| \le \epsilon_n$ and hence θ_n^* is in the neighborhood of θ. Finally, to get a bound on ER_{n1}^2 again recall that $Z_1, \ldots, Z_n$ are independent of $\delta_1, \ldots, \delta_n$. Hence using Theorem 2 and Remark 1 of Duttweiler (1973) we get, for fixed α_n in A_n,

$$\begin{aligned} ER_{n1}^2 &= E[E(R_{n1}^2 \mid \alpha_n \in A_n)]I(A_n) \\ &= O(n^{-3/2}) \quad \text{if} \quad EZ_1^2 < \infty. \end{aligned} \tag{2.27}$$

Now using (2.20), (2.23), (2.24), (2.26) and (2.27) we get $ER_n^2(p) = O(n^{-3/2})$. This completes the proof of the theorem. □

THEOREM 2.2. *Let p, $0 < p < 1$, be fixed. Suppose F is twice differentiable in the neighborhood of $Q(p)$ with $f(Q(p)) > 0$ and f' is bounded in the neighborhood of $Q(p)$. If $E|Z_1|^{2+\eta} < \infty$ for some $\eta > 0$, then*

$$n\mathrm{MSE}(Q_n(p)) = \sigma^2(p) + O(n^{-1/4}), \tag{2.28}$$

where

$$\sigma^2(p) = \sigma_{\mathrm{ML}}(Q(p), Q(p))/(f(Q(p)))^2, \tag{2.29}$$

and $\sigma_{\mathrm{ML}}(s,t)$ is given in (2.17).

Remark 2.2. $\sigma_{\mathrm{ML}}(Q(p), Q(p))$ in (2.29) can be simplified as

$$\sigma_{\mathrm{ML}}(Q(p), Q(p)) = \alpha^2(1-p)^2[(1-p)^{-1/\alpha} - 1] + \frac{1-\alpha}{\alpha}((1-p)\log(1-p))^2. \tag{2.30}$$

PROOF. Using the representation given in (2.4) we get

$$\begin{aligned}
\mathrm{MSE}(Q_n(p)) &= E(Q_n(p) - Q(p))^2 \\
&= E\left(\frac{1}{n}\sum_{i=1}^{n}\frac{\xi_i(Q(p))}{f(Q(p))} + R_n(p)\right)^2 \\
&= E\left(\frac{1}{n}\sum_{i=1}^{n}\frac{\xi_i(Q(p))}{f(Q(p))}\right)^2 + ER_n^2(p) \\
&\quad + 2E\left[R_n(p)\frac{1}{n}\sum_{i=1}^{n}\frac{\xi_i(Q(p))}{f(Q(p))}\right] \\
&= \mathrm{I} + \mathrm{II} + \mathrm{III}.
\end{aligned}$$

It is easy to show that $n\mathrm{I} = \sigma^2(p)$. Also $ER_n^2(p) = O(n^{-3/2})$ and the CS-inequality together imply that $n\mathrm{III} = O(n^{-1/4})$. Hence

$$n\mathrm{MSE}(Q_n(p)) = \sigma^2(p) + O(n^{-1/4}). \qquad \square$$

Our next theorem gives an asymptotic expansion of the mean square error of the kernel type estimator $\hat{Q}_n(p)$.

THEOREM 2.3. *Let p, $0 < p < 1$, be fixed. Assume that* (i) $E|Z_1|^{2+\eta} < \infty$ *for some $\eta > 0$,* (ii) *for $m \geq 2$, F is $(m+1)$ times differentiable with $f(Q(p)) > 0$ and $|F^{(m+1)}|$ is bounded in the neighborhood of $Q(p)$. Then*

$$\left| n\mathrm{MSE}(\hat{Q}_n(p)) - \left[\sigma^2(p) - h\left(\frac{(1-p)^{1-1/\alpha}}{\alpha f^2(Q(p))}\right)\int ub(u)\,du\right]\right|$$
$$\leq O(nh^{2m+2}) + O(n^{-1/4}) + O(n^{1/4}h^{m+1}) + O(h^2),$$

where, $b(u) = 2k(u)K(u)$.

PROOF. Condition (ii) on F implies that there exist constants $a_0, \ldots, a_m$ and $b_0, \ldots, b_m$ and $A > 0$, $B > 0$ such that

$$\left|F(x) - \sum_{i=0}^{m}\frac{a_i}{i!}(x - Q(p))^i\right| \leq \frac{A|x - Q(p)|^{m+1}}{(m+1)!}, \tag{2.31}$$

in the neighborhood of $Q(p)$ and

$$\left|Q(t) - \sum_{i=0}^{m}(t-p)^i\right| \leq \frac{B|t-p|^{m+1}}{(m+1)!}, \tag{2.32}$$

in the neighborhood of p. Starting with the definition of $\hat{Q}_n(p)$, it is easy to show that

$$\begin{aligned}
\hat{Q}_n(p) - Q(p) &= \int_{p-h}^{p+h}(Q_n(t) - Q(t))\frac{1}{n}k\left(\frac{p-t}{h}\right)dt \\
&\quad + \int_{p-h}^{p+h}Q(t)\frac{1}{h}k\left(\frac{p-t}{h}\right)dt - Q(p) \\
&= A_n + B_n(p,h).
\end{aligned} \tag{2.33}$$

Hence,

$$\text{MSE}(\hat{Q}_n(p)) = EA_n^2 + B_n^2(p,h) + 2B_n(p,h)EA_n. \tag{2.34}$$

The second term in (2.34) can be handled in the usual way giving

$$|B_n(p,h)|^2 = O\left(\left(\frac{BMh^{m+1}}{(m+1)!}\right)^2\right). \tag{2.35}$$

Now writing A_n in (2.33) as

$$\begin{aligned} A_n &= \int_{p-h}^{p+h} (Q_n(t) - Q(t))\frac{1}{h}k\left(\frac{p-t}{h}\right)dt \\ &= \int_{p-h}^{p+h}\left[\frac{1}{n}\sum_{i=1}^{n}\frac{\xi_i(Q(t))}{f(Q(p))}\right]\frac{1}{h}k\left(\frac{p-t}{h}\right)dt \\ &\quad + \int_{p-h}^{p+h} R_n(t)\frac{1}{h}k\left(\frac{p-t}{h}\right)dt \\ &= A_{n1} + A_{n2}, \end{aligned} \tag{2.36}$$

we get,

$$EA_n^2 = EA_{n1}^2 + EA_{n2}^2 + 2EA_{n1}A_{n2}, \tag{2.37}$$

where,

$$nEA_{n2}^2 = O(n^{-1/2}) \tag{2.38}$$

and

$$\begin{aligned} nEA_{n1}^2 &= \int_{p-h}^{p+h}\int_{p-h}^{p+h}\frac{1}{h}k\left(\frac{p-s}{h}\right)\frac{1}{h}k\left(\frac{p-t}{h}\right) \\ &\quad\times (f(Q(s))f(Q(t)))^{-1}\sigma_{\text{ML}}(Q(s),Q(t))dsdt \\ &= 2\int_{p-h}^{p+h}\int_{s}^{p+h}\frac{1}{h}k\left(\frac{p-s}{h}\right)\frac{1}{h}k\left(\frac{p-t}{h}\right) \\ &\quad\times (f(Q(s))f(Q(t)))^{-1} \\ &\quad\times \Big[\alpha^2(1-s)^{1-1/\alpha}(1-t)(1-(1-s)^{1/\alpha}) \\ &\qquad + \frac{1-\alpha}{\alpha}(1-s)(1-t)\log(1-s)\log(1-t)\Big]dtds \\ &= \text{V} + \text{VI}. \end{aligned} \tag{2.39}$$

The first term in (2.39) can be simplified as

$$\text{V} = 2\alpha^2\int_{p-h}^{p+h}\frac{1}{h}k\left(\frac{p-s}{h}\right)(f(Q(s)))^{-1}(1-s)^{1-1/\alpha}(1-(1-s)^{1/\alpha}) \tag{2.40}$$

$$\times \int_s^{p+h} \frac{1}{h} k\left(\frac{p-t}{h}\right)\left(\frac{1-t}{f(Q(t))}\right) dt ds$$
$$= 2\alpha^2 \int_{p-h}^{p+h} \frac{1}{h} k\left(\frac{p-s}{h}\right)(f(Q(s)))^{-1}(1-s)^{1-1/\alpha}(1-(1-s)^{1/\alpha})$$
$$\times \left[K\left(\frac{p-s}{h}\right)\left(\frac{1-s}{f(Q(s))}\right) + \int_s^{p+h} K\left(\frac{p-t}{h}\right) d\left(\frac{1-t}{f(Q(t))}\right)\right] ds$$
$$= \mathrm{VII} + \mathrm{VIII}.$$

Now using the notation

$$w(s) = \left(\frac{1-s}{f(Q(s))}\right)^2 ((1-s)^{-1/\alpha} - 1), \tag{2.41}$$

the first term in (2.40) can be simplified as

$$\begin{aligned}\mathrm{VII} &= \alpha^2 \int_0^1 \frac{1}{h} b\left(\frac{p-s}{h}\right) w(s) ds \\ &= \alpha^2 \int_{-1}^1 b(u) w(p-hu) du \\ &= \alpha^2 \int_{-1}^1 b(u)\left[w(p) - huw^{(1)}(p) + \frac{h^2u^2}{2} w^{(2)}(p^*)\right] du \\ &= \alpha^2 w(p) - \alpha^2 h w^{(1)}(p) \int_{-1}^1 u b(u) du + O(h^2).\end{aligned} \tag{2.42}$$

Similarly denoting

$$C(s) = \left(\frac{1-s}{f(Q(s))}\right)((1-s)^{-1/\alpha} - 1) \quad \text{and} \quad R(t) = -\frac{d}{dt}\left(\frac{1-t}{f(Q(t))}\right) \tag{2.43}$$

the second term in (2.40) can be simplified as

$$\begin{aligned}\mathrm{VIII} &= 2\alpha^2 \int_{p-h}^{p+h} \frac{1}{h} k\left(\frac{p-s}{h}\right) C(s) \int_s^{p+h} K\left(\frac{p-t}{h}\right)(-R(t)) dt ds \\ &= -2h\alpha^2 \int_{p-h}^{p+h} \frac{1}{h} k\left(\frac{p-s}{h}\right) C(s) \int_{-1}^{(p-s)/h} K(u) R(p-hu) du ds \\ &= -2h\alpha^2 \int_{-1}^1 k(v) C(p-hv) \int_{-1}^v K(u) R(p-hu) du dv \\ &= -2h\alpha^2 \int_{-1}^1 k(v)[C(p) - hvC^{(1)}(p^*)] \\ &\quad \times \int_{-1}^v K(u)[R(p) - huR^{(1)}(p^{**})] du dv\end{aligned} \tag{2.44}$$

$$= -2hC(p)R(p)\alpha^2 \int_{-1}^{1} k(v) \int_{-1}^{v} K(u)dudv + O(h^2)$$
$$= -2hC(p)R(p)\alpha^2 \int_{-1}^{1} K(u) \int_{u}^{1} k(v)dvdu + O(h^2)$$
$$= -2h\alpha^2 C(p)R(p) \int_{-1}^{1} K(u)(1-K(u))du + O(h^2)$$
$$= -2h\alpha^2 C(p)R(p) \int_{-1}^{1} ub(u)du + O(h^2).$$

Hence using (2.42) and (2.44) in (2.40) we get

$$\begin{aligned} \text{(2.45)} \qquad \mathrm{V} &= \mathrm{VII} + \mathrm{VIII} \\ &= \alpha^2 w(p) - h\alpha^2[w^{(1)}(p) + 2C(p)R(p)] \int ub(u)du + O(h^2) \\ &= \alpha^2 w(p) - h\left[\frac{(1-p)^{1-1/\alpha}}{\alpha(f(Q(p)))^2}\right] \int ub(u)du + O(h^2). \end{aligned}$$

The second term in (2.39) can be handled easily. Define

$$g(s) = \frac{(1-s)\log(1-s)}{f(Q(s))}. \tag{2.46}$$

Now we can simplify VI in (2.39) as

$$\begin{aligned} \left(\frac{\alpha}{1-\alpha}\right)\mathrm{VI} &= \left(\int_{p-h}^{p+h} \frac{1}{h}k\left(\frac{p-s}{h}\right)g(s)ds\right)^2 \\ &= \left(\int_{-1}^{1} k(u)g(p-hu)du\right)^2 \\ &= (g(p) + O(h^{m+1}))^2 \\ &= g^2(p) + O(h^2). \end{aligned} \tag{2.47}$$

Using the expressions of $w(p)$ and $g(p)$, it is easy to see that

$$\sigma^2(p) = \alpha^2 w(p) + \left(\frac{1-\alpha}{\alpha}\right) g^2(p). \tag{2.48}$$

Now putting (2.45), (2.47) and (2.48) in (2.39) we get

$$nEA_{n1}^2 = \sigma^2(p) - h\left(\frac{(1-p)^{1-1/\alpha}}{\alpha f^2(Q(p))}\right) \int ub(u)du + O(n^{-1/4}) + O(h^2). \tag{2.49}$$

Hence

$$n|EA_{n1}A_{n2}| = O(n^{-1/4}) \quad \text{and} \tag{2.50}$$
$$nEA_n^2 = nEA_{n1}^2 + O(n^{-1/4}). \tag{2.51}$$

Finally, to get a bound on the third term in (2.34) observe that

$$|EA_n| = O(n^{-3/4}). \tag{2.52}$$

Hence

$$\begin{aligned} n|EA_nB_n(p,h)| &= n|B_n(p,h)EA_n| \\ &= n|B_n(p,h)||EA_n| = O(n^{1/4}h^{m+1}). \end{aligned} \tag{2.53}$$

The conclusion of the theorem now follows from (2.35), (2.49) and (2.53). □

Remark 2.3. The limiting value of nMSE of $Q_n(p)$ and $\hat{Q}_n(p)$ are equal to $\sigma^2(p)$ where $\sigma^2(p)$ is given in (2.29). The corresponding values from Theorem 3.1 and Theorem 4.1 of Lio and Padgett (1987) are given by $\sigma^2_{\mathrm{LP}}(p) = \alpha(1-p)^2[(1-p)^{-1/\alpha}-1]/(f(Q(p)))^2$ under the proportional hazards model. It is not too difficult to show that $\sigma^2_{\mathrm{LP}}(p) > \sigma^2(p)$.

We now state the main theorem of this section. It gives the relative deficiency result of Q_n with respect to $\hat{Q}_n$.

THEOREM 2.4. *Assume that the conditions of Theorem* 2.3 *hold. Further, assume that h is such that $nh^{2m+1} \to 0$ and $(nh^4)^{-1} \to 0$ as $n \to \infty$. Then* MSE(Q_n) *and* MSE$(\hat{Q}_n)$ *are finite for large n and*

$$\lim_n \left(\frac{i(n)-n}{nh}\right) = \frac{(1-p)^{1-1/\alpha}\int 2xk(x)K(x)dx}{\alpha f^2(Q(p))\sigma^2(p)}. \tag{2.54}$$

Remark 2.4. For $m \geq 2$, h could be taken as $h = o(n^{-1/(2m+1)})$. This choice of h is different from the optimal choice of $h = O(n^{-1/3})$ as mentioned in Azzalini (1981). For the choice of $h = O(n^{-1/3})$ the MSE of $\hat{Q}_n(p)$ may not be smaller than that of $Q_n(p)$. Azzalini (1981) did not study the problem of relative deficiency. In the case of no censoring Falk (1984) has established the relative deficiency of the sample quantile with respect to the kernel smoothed quantile under the similar conditions on h as the ones stated in this paper in Theorem 2.4.

Remark 2.5. In the case of no censoring, $\alpha = 1$ and hence the right hand side of (2.54) reduces to the expression given in Falk (1984).

Remark 2.6. If the kernel is such that $\int xk(x)K(x)dx > 0$, then the kernel type estimators of the quantile are better than the ACL-quantiles. In this case $(i(n)-n)$ tends to infinity as $n \to \infty$.

PROOF. It is clear from the definition of $i(n)$ that $i(n) \to \infty$ as $n \to \infty$. Also nMSE$(\hat{Q}_n) \to \sigma^2(p)$ as $n \to \infty$. Since MSE$(Q_{i(n)}(p)) \leq$ MSE$(\hat{Q}_n(p))$, it follows that

$$\mathrm{MSE}(\hat{Q}_n) \geq \frac{\sigma^2(p)}{i(n)} + \frac{O(i(n)^{-1/4})}{i(n)}. \tag{2.55}$$

Hence,

$$\frac{i(n)}{n} \geq \frac{\sigma^2(p) + O(i(n)^{-1/4})}{n\text{MSE}(\hat{Q}_n)} \to 1 \quad \text{as} \quad n \to \infty. \tag{2.56}$$

It also follows from the definition of $i(n)$ that if $n^* < i(n)$, then $\text{MSE}(Q_n^*(p)) \geq \text{MSE}(\hat{Q}_n)$. Hence using the above arguments we get

$$\frac{n^*}{n} \leq \frac{\sigma^2(p) + O((n^*)^{-1/4})}{n\text{MSE}(\hat{Q}_n(p))}. \tag{2.57}$$

In particular, taking $n^* = i(n) - 1$ and letting $n \to \infty$, we get $\limsup_n (i(n)/n) \leq 1$. Combining these we get

$$\lim_n \frac{i(n)}{n} = 1. \tag{2.58}$$

Using the asymptotic representations of $\text{MSE}(Q_n(p))$ and $\text{MSE}(\hat{Q}_n(p))$ and taking the difference we get

$$\begin{aligned}\sigma^2(p)\left(\frac{1}{n} - \frac{1}{i(n)}\right) &= h\left(\frac{(1-p)^{1-1/\alpha}\int 2xk(x)K(x)dx}{\alpha n f^2(Q(p))}\right) + O(n^{-5/4}) \\ &\quad + O(n^{-3/4}h^{m+1}) + O(i(n)^{-5/4}) \\ &\quad + O(n^{-1}h^2) + O(h^{2(m+1)}).\end{aligned} \tag{2.59}$$

Now multiplying both sides of (2.59) by $i(n)(h\sigma^2(p))^{-1}$ and taking limit, we get

$$\lim_n \left(\frac{i(n) - n}{nh}\right) = \frac{(1-p)^{1-1/\alpha}\int 2xk(x)K(x)dx}{\alpha f^2(Q(p))\sigma^2(p)}. \tag{2.60}$$

This completes the proof. □

Acknowledgements

The author is grateful to the referees for their suggestions which improved the readability of the paper to a great extent.

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