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Estimation of concurrence for multipartite mixed states

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Abstract. We investigate the lower bound of concurrence for multipartite quantum mixed states. Analytical lower bounds are derived for some multipartite systems, by establishing functional relations between concurrence and the generalized partial transpositions.

1 Introduction

Quantum entanglement plays a crucial role in the rapidly developing theory of quantum information [1], since they constitute the most important resource for quantum information processing. An important theoretical challenge in the theory of quantum entanglement is to give a proper description and quantification of quantum entanglement of multipartite quantum systems. Entanglement of formation (EoF) [2,3] and concurrence [4] are two well defined quantitative measures of entanglement. For two-qubit case EoF is a monotonically increasing function of concurrence and an elegant formula of concurrence was derived analytically by Wootters in [5], which plays an essential role in describing quantum phase transition in various interacting quantum many-body systems [6,7] and can be experimentally measured [8].

For higher dimensional case, due to the extremizations involved in the calculation, only a few of explicit analytic formulae for EoF and concurrence have been found for some special symmetric states [9,10]. Therefore some nice algorithms and progresses have been concentrated on possible lower bounds of the EoF and concurrence for qubit-qudit systems [11,12] and for bipartite systems in arbitrary dimensions [13,14] from numerical optimization over a large number of free parameters. In [15,16] analytical lower bounds of EoF and concurrence for any dimensional mixed bipartite quantum states have been presented, which are further shown to be exact for some special classes of states and detect many bound entangled states. In [17] another lower bound of EoF for bipartite states has been presented from a new separability criterion.

Although the entanglement of formation is only well defined for bipartite systems, the concurrence is well defined even for multipartite states. The lower bound of concurrence for tripartite states has been studied in [18]. In this note we summarize the results related to the lower bound of concurrence for bipartite, tripartite systems, and generalize them to arbitrary multipartite systems.

2 Lower bounds of concurrence for bipartite and tripartite systems

Let $\mathcal{H}_1$, $\mathcal{H}_2$, …, $\mathcal{H}_M$ be $M(\geq 2)$ $N_1, N_2, \ldots, N_M$-dimensional Hilbert spaces respectively. The concurrence for a general pure multipartite state $|\psi\rangle \in \mathcal{H}_1 \otimes \mathcal{H}_2 \otimes \cdots \otimes \mathcal{H}_M$ is defined by

$$C(|\psi\rangle) = \sqrt{d - \sum_{\alpha=1}^{d} \mathrm{Tr}\rho_\alpha^2}\,, \tag{1}$$

where $d = 2^{M-1} - 1$ is the number of all possible bipartite separations of an M-partite system, the reduced density matrix ρ_α, $\alpha = 1, \ldots, d$, is obtained by tracing over one part of the subsystems associated with the α-th bipartite separation.

For bipartite case ($M = 2$), the concurrence of a pure state $|\psi\rangle$ is given by $C(|\psi\rangle) = \sqrt{2(1 - \mathrm{Tr}\rho_1^2)}$, where the reduced density matrix ρ_1 is obtained by tracing over the second subsystem. Suppose ($N_1 \leq N_2$), in this case $|\psi\rangle$ has a standard Schmidt form

$$|\psi\rangle = \sum_i \sqrt{\mu_i}|a_i b_i\rangle, \tag{2}$$

where $\sqrt{\mu_i}$, $i = 1, \ldots, N_1$, are the Schmidt coefficients, $|a_i\rangle$ and $|b_i\rangle$ are orthonormal basis in $\mathcal{H}_1$ and $\mathcal{H}_2$, respectively. It follows

$$C(|\psi\rangle) = 2\sqrt{\sum_{i<j} \mu_i \mu_j}\,, \tag{3}$$

which varies smoothly from 0 for separable states to $2(N_1 - 1)/N_1$ for maximally entangled states.

Let $\|G\|$ denote the trace norm of a matrix G defined by $\|G\| = Tr(GG^\dagger)^{1/2}$. Set $\rho = |\psi\rangle\,\langle\psi|$. One has

$$\|\rho^{T_1}\| = \|\mathcal{R}(\rho)\| = \left(\sum_i \sqrt{\mu_i}\right)^2, \tag{4}$$

where ρ^{T_1} is the partial transposed matrix of ρ with respect to the first subsystem, $\mathcal{R}(\rho)$ is realigned matrix of ρ defined by $\mathcal{R}(\rho)_{ij,kl} = \rho_{ik,jl}$, where i and j are the row and column indices for the first subsystem respectively, while k and l are such indices for the second subsystem [19–21].

Assume that one has already found an optimal decomposition $\sum_i p_i \rho^i$ for ρ to achieve the infimum of $C(\rho)$, where ρ^i are pure state density matrices. Then $C(\rho) = \sum_i p_i C(\rho^i)$ by definition. From (3, 4) and the convex property of the trace norm, $\|\rho^{T_1}\| \leq \sum_i p_i \|(\rho^i)^{T_1}\|$, $\|\mathcal{R}(\rho)\| \leq \sum_i p_i \|\mathcal{R}(\rho^i)\|$, one can prove that for any $N_1 \otimes N_2$ ($N_1 \leq N_2$) mixed quantum state ρ, the concurrence $C(\rho)$ satisfies

$$C(\rho) \geq \sqrt{\frac{2}{N_1(N_1-1)}}(\max(\|\rho^{T_1}\|, \|\mathcal{R}(\rho)\|) - 1). \tag{5}$$

For the $U \otimes U^*$ invariant mixed Isotropic states with $N_1 = N_2 = N$ [22,23], The bound (5) gives the exact value of the concurrence derived in [10].

For multipartite case, we do not have a Schmidt expression like (2). To get a lower bound of the multipartite concurrence, we need the operations of generalized partial transpose and realignment. We first recall some notations used in various matrix operations [24,25]. A generic matrix G can be always written as $G = \sum_{i,j} a_{ij}\,\langle j| \otimes |i\rangle$, where $|i\rangle\,, |j\rangle$ are vectors of a suitably selected normalized real orthogonal basis. We define the operations $\mathcal{T}_r$ (resp. $\mathcal{T}_c$) to be the row transposition (resp. column transposition) of G which transposes the second (resp. first) vector in the above tensor product expression of G:

$$\mathcal{T}_r(G) = \sum_{i,j} a_{ij}\,\langle j| \otimes \langle i|\,, \qquad \mathcal{T}_c(G) = \sum_{i,j} a_{ij}\,|j\rangle \otimes |i\rangle\,. \tag{6}$$

It is easily verified that $\mathcal{T}_c\mathcal{T}_r(G) = \mathcal{T}_r\mathcal{T}_c(G) = G^T$, where T denotes matrix transposition.

In the following we define $\mathcal{T}_{r_k}$ (resp. $\mathcal{T}_{c_k}$) to be the row (resp. column) transpositions with respect to the subsystem k. For instance, $\mathcal{T}_{r_{12}}$ stands for the row transpositions with respect to the subsystems 1 and 2. Let $\mathcal{Y} = \{x_1, x_2, \ldots\}$ be a set of such operations on a density matrix ρ, we denote $\rho^{\mathcal{T}_\mathcal{Y}} = \mathcal{T}_\mathcal{Y}(\rho) = \mathcal{T}_{x_1}\mathcal{T}_{x_2}\ldots(\rho)$, e.g. $\rho^{\mathcal{T}_{\{c_1,r_2,r_3\}}} \equiv \mathcal{T}_{\{c_1\}}\mathcal{T}_{\{r_2\}}\mathcal{T}_{\{r_3\}}(\rho)$.

We first consider the tripartite case. The concurrence for a general pure tripartite state $|\psi\rangle \in \mathcal{H}_1 \otimes \mathcal{H}_2 \otimes \mathcal{H}_3$ is defined by

$$C(|\psi\rangle) = \sqrt{3 - \mathrm{Tr}(\rho_1^2 + \rho_2^2 + \rho_3^2)}, \tag{7}$$

where the reduced density matrix ρ_1 (resp. ρ_2, ρ_3) is obtained by tracing over the subsystems 2 and 3 (resp. 1 and 3, 1 and 2). We discuss a special class of $\mathcal{Y}$: $\mathcal{Y}_i = \{c_i, r_i\}$, $i = 1, 2, 3$, $\mathcal{Y}_4 = \{c_1, r_{23}\}$, $\mathcal{Y}_5 = \{c_{12}, r_3\}$, $\mathcal{Y}_6 = \{c_{13}, r_2\}$. As $\rho^{\mathcal{T}_{\mathcal{Y}_i}} = \rho^{T_i}$, $i = 1, 2, 3$, where T_i stands for the partial transposition with respect to the subsystem i, the operations $\mathcal{Y}_1$, $\mathcal{Y}_2$ and $\mathcal{Y}_3$ correspond to the partial transpositions of ρ.

For the most simple tripartite system, the three qubits case, a state $|\Psi\rangle$ can be written in terms of the generalized Schmidt decomposition [26],

$$|\Psi\rangle = \lambda_0|000\rangle + \lambda_1 e^{i\phi}|100\rangle + \lambda_2|101\rangle + \lambda_3|110\rangle + \lambda_4|111\rangle \tag{8}$$

with normalization condition $\lambda_i \geq 0$, $0 \leq \phi \leq \pi$, $\sum_i \lambda_i^2 = 1$. The corresponding density matrix $\rho = |\Psi\rangle\langle\Psi|$ has the following properties:

$$Tr\rho_1^2 = 1 - 2\mu_0(1 - \mu_0 - \mu_1),$$

$$Tr\rho_2^2 = 1 - 2\mu_0(1 - \mu_0 - \mu_1 - \mu_2) - 2\Delta,$$

$$Tr\rho_3^2 = 1 - 2\mu_0(1 - \mu_0 - \mu_1 - \mu_3) - 2\Delta,$$

where $\Delta \equiv |\lambda_1\lambda_4 e^{i\phi} - \lambda_2\lambda_3|^2$, $\mu_i = \lambda_i^2$, $i = 0, 1, \ldots, 4$. Therefore from (7) we have

$$C^2(\rho) = 2\mu_0(3 - 3\mu_0 - 3\mu_1 - \mu_2 - \mu_3) + 4\Delta, \tag{9}$$

which varies smoothly from 0, for pure product states, to 3/2 for maximally entangled pure states.

On the other hand, under the operations of $\mathcal{Y}_i$, $i = 1, 2, 3$, one gets

$$\begin{aligned} \|\rho^{\mathcal{T}_{\mathcal{Y}_1}}\| &= 1 + 2\sqrt{\mu_0(\mu_2 + \mu_3 + \mu_4)}, \\ \|\rho^{\mathcal{T}_{\mathcal{Y}_2}}\| &= 1 + 2\sqrt{\Delta + \mu_0(\mu_3 + \mu_4)}, \\ \|\rho^{\mathcal{T}_{\mathcal{Y}_3}}\| &= 1 + 2\sqrt{\Delta + \mu_0(\mu_2 + \mu_4)}. \end{aligned} \tag{10}$$

Combining (9) and (10) we have

$$C(\rho) \geq (\|\rho^{\mathcal{T}_{\mathcal{Y}_j}}\| - 1), \quad j = 1, 2, 3. \tag{11}$$

A three qubits ($2 \otimes 2 \otimes 2$) system can be viewed as three different bipartite ($2 \otimes 4$ or $4 \otimes 2$) systems. From the results for bipartite systems [15], these three bipartite separations give rise to, respectively

$$1 - Tr((\rho_1)^2) \geq \frac{1}{2}(\|\rho^{\mathcal{T}_{\{c_1, r_{23}\}}}\| - 1)^2,$$

$$1 - Tr((\rho_2)^2) \geq \frac{1}{2}(\|\rho^{\mathcal{T}_{\{c_{13}, r_2\}}}\| - 1)^2,$$

$$1 - Tr((\rho_3)^2) \geq \frac{1}{2}(\|\rho^{\mathcal{T}_{\{c_{12}, r_3\}}}\| - 1)^2.$$

Therefore

$$C(\rho) = \sqrt{3 - Tr(\rho_1^2) - Tr(\rho_2^2) - Tr(\rho_3^2)} \geq \frac{1}{\sqrt{2}} max\left\{\|\rho^{\mathcal{T}_{\mathcal{Y}_j}}\| - 1\right\}, \quad j = 4, 5, 6. \tag{12}$$

Hence if we assume that $\sum_i p_i\rho^i$ is the optimal decomposition of ρ such that $C(\rho) = \sum_i p_i C(\rho^i)$, where ρ^i are pure state density matrices. Accounting to that $\|\rho^{\mathcal{T}_{\mathcal{Y}}}\| \leq \sum_i p_i \|(\rho^i)^{\mathcal{T}_{\mathcal{Y}}}\|$, from (11) and (12) one gets a lower bound for the concurrence of three-qubit states:

For any three-qubit mixed quantum state ρ, the concurrence $C(\rho)$ satisfies

$$C(\rho) \geq max\left\{\|\rho^{\mathcal{T}_{\mathcal{Y}_i}}\| - 1,\ \frac{1}{\sqrt{2}}(\|\rho^{\mathcal{T}_{\mathcal{Y}_j}}\| - 1)\right\}, \tag{13}$$

where $i = 1, 2, 3,\ j = 4, 5, 6$.

For higher dimensional tripartite systems, we do not have an expression of (8). The related lower bound of concurrence for this will be studied in the next section for arbitrary multipartite systems.

3 Lower bounds of concurrence for multipartite systems

Concerning multipartite ($M > 3$) systems, we first study a special kind of states. Let us consider an M-qubit state, the generalized Greenberger-Horne-Zeilinger (GHZ) state,

$$|\Phi\rangle = \cos\theta|00\cdots 0\rangle + \sin\theta|11\cdots 1\rangle. \tag{14}$$

For $\rho = |\Phi\rangle\langle\Phi|$, we get $\rho_i = Tr_{\{1,\ldots,i-1,i+1,\ldots,M\}}\rho = \cos^2\theta|0\rangle\langle 0| + \sin^2\theta|1\rangle\langle 1|$. Therefore $Tr\rho_i^2 = \cos^4\theta + \sin^4\theta = 1 - 2\sin^2\theta\cos^2\theta$, $i = 1, 2, \ldots, M$. In fact, one can prove that $Tr\rho^2_{i_1 i_2\cdots i_m} = 1 - 2\sin^2\theta\cos^2\theta$ for all $i_1 \neq i_2 \neq \cdots \neq i_m \in \{1, 2, \ldots, M\}$, $1 \leq m \leq M$. Hence we have by definition

$$C(\rho) = \sqrt{2d\sin^2\theta\cos^2\theta}. \tag{15}$$

On the other hand, the partial transpose of ρ with respect to the ith qubit space gives rise to

$$\begin{aligned}\rho^{T_i} &= \cos^2\theta|0\cdots 0_i\cdots 0\rangle\langle 0\cdots 0_i\cdots 0| + \cos\theta\sin\theta|0\cdots 1_i\cdots 0\rangle\langle 1\cdots 0_i\cdots 1|\\ &\quad + \cos\theta\sin\theta|1\cdots 0_i\cdots 1\rangle\langle 0\cdots 1_i\cdots 0| + \sin^2\theta|1\cdots 1_i\cdots 1\rangle\langle 1\cdots 1_i\cdots 1|,\end{aligned}$$

$i = 1, 2, \ldots, M$. As ρ^{T_i} is Hermitian, its singular values are simply given by the square root of the eigenvalues of $(\rho^{T_i})^2$. The trace norm of ρ^{T_i} takes the form $\|\rho^{T_i}\| = 1 + 2\sqrt{\sin^2\theta\cos^2\theta}$. The trace norms of partial transposed ρ with respect to the other sub-qubit spaces can be similarly calculated. All together we have

$$\|\rho^{T_{i_1 i_2\cdots i_m}}\| = 1 + 2\sqrt{\sin^2\theta\cos^2\theta}, \tag{16}$$

where $i_1 \neq i_2 \neq \cdots \neq i_m \in \{1, 2, \ldots, M\}$, $1 \leq m \leq M$.

We consider now the norm of ρ under bipartite realignment. If we make a bipartite realignment with respect to the subsystems i and j, $1 \leq i \neq j \leq M$, while leaving the other subsystems untouched, we have

$$\begin{aligned}\mathcal{R}_{i|j}(\rho) = &\cos^2\theta|0\cdots 0_i\cdots 0_j\cdots 0\rangle\langle 0\cdots 0_i\cdots 0_j\cdots 0|\\ &+ \cos\theta\sin\theta|0\cdots 0_i\cdots 1_j\cdots 0\rangle\langle 1\cdots 0_i\cdots\ 1_j\cdots 1|\\ &+ \cos\theta\sin\theta|1\cdots 1_i\cdots 0_j\cdots 1\rangle\langle 0\cdots 1_i\cdots 0_j\cdots 0|\\ &+ \sin^2\theta|1\cdots 1_i\cdots 1_j\cdots 1\rangle\langle 1\cdots 1_i\cdots 1_j\cdots 1|.\end{aligned}$$

Hence the sum of its singular values gives the norm, $\|\mathcal{R}_{i|j}(\rho)\| = 1 + 2\sqrt{\sin^2\theta\cos^2\theta}$. Let Θ_1 and Θ_2 be two different subsystems. One can similarly verify that

$$\|\mathcal{R}_{\Theta_1|\Theta_2}(\rho)\| = 1 + 2\sqrt{\sin^2\theta\cos^2\theta}. \tag{17}$$

From (15), (16) and (17) we have:

For any M-qubit mixed state with decomposition $\rho = \sum_i p_i|\Psi_i\rangle\langle\Psi_i|$, if $|\Psi_i\rangle$ can be written in the form (14) for all i, then the concurrence $C(\rho)$ satisfies

$$C(\rho) \geq max\{\|\rho^{T_\Theta}\|, \|\mathcal{R}_{\Theta_1|\Theta_2}(\rho)\|\} - 1, \tag{18}$$

where $\Theta, \Theta_1, \Theta_2$ are subsets of the indices $\{1, 2, \ldots, M\}$, $\Theta_1 \bigcap \Theta_2 = \emptyset$.

We remark that once a density matrix has a decomposition with all the pure states of the form (14), then its all other possible decompositions will also have the form (14), since other decompositions can be obtained from the unitarily linear combinations of this decomposition, and any linear combinations of the type (14) still have the form (14).

Second, let us consider another M-qubit state, the generalized W state,

$$|\Psi\rangle = a_1|10\cdots 0\rangle + a_2|01\cdots 0\rangle + \cdots + a_M|00\cdots 1\rangle. \tag{19}$$

For $\rho = |\Psi\rangle\langle\Psi|$, we get $\rho_i = Tr_{\{1,\ldots,i-1,i+1,\ldots,M\}}\rho = |a_i|^2|1\rangle\langle 1| + (\sum_{j\neq i}|a_j|^2)|0\rangle\langle 0|$. Therefore $Tr\rho_i^2 = |a_i|^4 + (\sum_{j\neq i}|a_j|^2)^2$, $i = 1, 2, \ldots, M$. Generally one can prove that $Tr\rho^2_{i_1 i_2 \ldots i_m} = (|a_{i_1}|^2 + a_{i_2}|^2 + \ldots + a_{i_m}|^2)^2 + (\sum_{k\neq\{i_1,i_2,\ldots,i_m\}}|a_k|^2)^2$ for all $i_1 \neq i_2 \neq \cdots \neq i_m \in \{1, 2, \ldots, M\}$, $1 \leq m \leq M$. Hence we have by definition

$$C(\rho) = \sqrt{2^{M-1}\sum_{i<j}|a_i a_j|^2}. \tag{20}$$

From direct calculation, the trace norm of the partial transposed matrix ρ^{T_i} of ρ with respect to the ith qubit space is given by $\|\rho^{T_i}\| = 1 + 2\sqrt{\sum_{j\neq i}|a_i a_j|^2}$. The trace norms of partial transposed ρ with respect to the other sub-qubit spaces can be also similarly calculated,

$$\|\rho^{T_{i_1 i_2\cdots i_m}}\| = 1 + 2\sqrt{\sum_{l\neq\{i_1,i_2,\cdots,i_m\}}(|a_{i_1}a_l|^2 + |a_{i_2}a_l|^2 + \cdots + |a_{i_m}a_l|^2)}, \tag{21}$$

where $i_1 \neq i_2 \neq \cdots \neq i_m \in \{1, 2, \ldots, M\}$, $1 \leq m \leq M$.

An M-qubit W state can be viewed as d different bipartite systems. Let $\Gamma^1_\alpha, \Gamma^2_\alpha$ denote two subsets of the indices $\{1, 2, \ldots, M\}$, $\Gamma^1_\alpha \bigcap \Gamma^2_\alpha = \emptyset$, $\Gamma^1_\alpha \bigcup \Gamma^2_\alpha = \{1, 2, \ldots, M\}$, $\alpha = 1, \ldots, d$. From the results for bipartite systems [15], these d bipartite separations give rise to, respectively

$$1 - Tr((\rho_{\Gamma^1_\alpha})^2) \geq \frac{1}{2}(\|\mathcal{R}_{\Gamma^1_\alpha|\Gamma^2_\alpha}(\rho)\| - 1)^2, \quad \alpha = 1, \ldots, d.$$

Hence

$$C(\rho) = \sqrt{d - \sum_{\alpha=1}^{d}\mathrm{Tr}(\rho^2_{\Gamma^1_\alpha})} \geq \frac{1}{\sqrt{2}}max\left\{\|\mathcal{R}_{\Gamma^1_\alpha|\Gamma^2_\alpha}(\rho)\| - 1, \quad \alpha = 1, \ldots, d\right\}.$$

Therefore for any M-qubit mixed state with decomposition on the generalized W states, $\rho = \sum_i p_i|\Psi_i\rangle\langle\Psi_i|$, such that $|\Psi_i\rangle$ can be written in the form (19) for all i, the concurrence $C(\rho)$ satisfies

$$C(\rho) \geq max\left\{\|\rho^{T_{\Gamma^1_\alpha}}\| - 1, \frac{1}{\sqrt{2}}(\|\mathcal{R}_{\Gamma^1_\alpha|\Gamma^2_\alpha}(\rho)\| - 1), \quad \alpha = 1, \ldots, d\right\}. \tag{22}$$

From (18) and (22), we see that the lower bound for the class of mixed states with decompositions on the generalized GHZ states is better than the one for the class of mixed states with decompositions on the generalized W states, in the sense that in (18) the realignment is associated with two arbitrary subsystems Θ_1 and Θ_2 such that $\Theta_1 \bigcap \Theta_2 = \emptyset$, but not necessary $\Theta_1 \bigcup \Theta_2 = \{1, 2, \ldots, M\}$. While in (22) we simply treat the realignment associated with bipartite separations, so that the two subsystems Γ^1_α and Γ^2_α satisfy both $\Gamma^1_\alpha \bigcap \Gamma^2_\alpha = \emptyset$ and $\Gamma^1_\alpha \bigcup \Gamma^2_\alpha = \{1, 2, \ldots, M\}$.

For general multipartite systems, we can still consider a general M-partite pure state as bipartite separations, which give rise to, respectively

$$1 - Tr((\rho_{\Gamma_\alpha^1})^2) \geq \frac{1}{D_\alpha(D_\alpha - 1)} max\left\{(\|\rho^{T_{\Gamma_\alpha^1}}\| - 1)^2, (\|\mathcal{R}_{\Gamma_\alpha^1|\Gamma_\alpha^2}(\rho)\| - 1)^2\right\}, \quad \alpha = 1, \ldots, d,$$

where $D_\alpha = min(dim(\Gamma_\alpha^1), dim(\Gamma_\alpha^2))$, $dim(\Gamma_\alpha^1)$ (resp. $dim(\Gamma_\alpha^2)$) is the dimension associated with the subsystems contained in Γ_α^1 (resp. Γ_α^2).

Therefore from the definition of $C(\rho)$, we have:

For any $N_1 \otimes N_2 \otimes \cdots \otimes N_M$ M-partite mixed quantum state ρ, the concurrence $C(\rho)$ satisfies

$$C(\rho) \geq max\left\{K\left(\|\rho^{T_{\Gamma_\alpha^1}}\| - 1\right),\ K\left(\|\mathcal{R}_{\Gamma_\alpha^1|\Gamma_\alpha^2}(\rho)\| - 1\right),\ \ \alpha = 1, \ldots, d\right\}, \tag{23}$$

where $K = 1/\sqrt{D_\alpha(D_\alpha - 1)}$.

Here for general mixed states, it is difficult to find the relation between the concurrence of a pure state and the corresponding norm of the partial transposed state with respect to certain subsystems, like the one between (20) and (21). The bound (23) is obtained by bipartite separations of the system, and there is an extra factor K, which makes this bound weaker than (22), when it is applied to the special class of mixed states with decompositions on the generalized W states.

4 Summary and conclusions

By making a novel connection with the generalized partial transpositions, we have provided an entirely analytical formula for lower bound of concurrence for bipartite, tripartite and multipartite systems. One only needs to calculate the trace norm of certain matrices, which avoids complicated optimization procedure over a large number of free parameters in numerical approaches. The results could be used to indicate possible quantum phase transitions in condensed matter systems, and to analyze finite size or scaling behavior of entanglement in various interacting quantum many-body systems.

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