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ESTIMATION OF DIRECT AND CROSS POWER SPECTRAL DENSITY OF DISCRETE DATA USING THE FAST FOURIER TRANSFORM

By

A. C. Jolly

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SUMMARY

The use of the averaged periodogram of a discrete, random function in estimating the direct (i.e., from a single random function) and cross power spectral density functions is described. Application of the algorithm known as the "fast Fourier transform" is suggested as the method of calculating Fourier coefficients and details of the numerical techniques involved in its derivation and use are given. The advisability of modifying the Fourier coefficients before converting to periodograms and averaging is pointed out and details of a suitable modifying function, expressed in both the time and frequency domains, are given. The method of converting the periodogram to power spectral density such that the integral of the latter function equals the mean square of the random function is derived, showing the requirement for an empirically determined factor. Finally, a digital filter is specified which is suitable for use in a filtering and decimation process that is applied to the calculation of power spectral density with a varying frequency resolution over the total frequency range of interest.

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LIST OF SYMBOLS

Symbol	Definition
A	Finite Fourier transform of discrete function X
A'	Finite Fourier transform of X after smoothing with weights derived from a spectral window other than the Dirac comb
a(k)	Real part of A(k)
a'(k)	Real part of A'(k)
a	Constant used in the apodizing function applied to the discrete filter of Section 6. This constant determines the number of side lobes of the filter response in the time domain
B	Finite Fourier transform of the complex time function formed from sections of one or two real valued discrete functions of time
BW	Bandwidth of the spectral window
b(k)	Imaginary part of A(k)
b'(k)	Imagainary part of A'(k)
C	Finite Fourier transform of discrete function Y
c(k)	Real part of C(k)
D	Real part of cross power spectral density
d(k)	Imaginary part of C(k)
E{ }	Expected value
E(j)	Imaginary part of inverse transform of cross power spectral density
EDF	Equivalent degrees of freedom of power spectral density; obtained from the number of degrees of freedom of a chi-square distribution with the same coefficient of variation
e	Base of natural logarithms; also written exp
e(x)	Special function derived from e and equal to exp $(2\pi i x)$

LIST OF SYMBOLS (Continued)

Symbol	Definition
$F(t)$	Apodizing function applied to digital filter in the time domain in Section 6
$F(k)$	Frequency response function, defined in Section 8
f	Circular frequency in cycles/second
f_c	Cut-off frequency in low-pass filter
$G(f)$	Low pass digital filter frequency response
$g(t)$	Inverse Fourier transform of $G(f)$ giving the filter function in the time domain
$g'(k)$	Filter weights of low pass filter with cut-off at one half the frequency range
$H(f)$	Frequency window
h	Data sampling rate for discrete functions
$I(k)$	Periodogram of a discrete function of time
$\overline{I(k)}$	Averaged periodogram. Subscripts x, y and xy refer to the time functions from which the periodograms are obtained
i	$\sqrt{-1}$
$J(f)$	Spectral window $\left(= \frac{1}{T_m} \lvert H(f) \rvert^2\right)$
j	Index of discrete functions of time such that the time corresponds to $j\,\Delta t$
$j_0, j_1 \cdots, j_{p-1}$	Variables in the decomposition of j used in the fast Fourier transform algorithm
K	Degrees of freedom in a chi-square distribution
k	Index of discrete functions of frequency such that the frequency corresponds to $k\,\Delta f$

LIST OF SYMBOLS (Continued)

Symbol	Definition
$k_0, k_1, \ldots k_{p-1}$	Variables in the decomposition of k used in the fast Fourier transform algorithm
ℓ	Variable of integration in the convolution of power spectral density and frequency window
m	Number of discrete data used for each Fourier transform when calculating the averaged periodogram from subsections of a function of time
N	Total number of discrete data used in calculating power spectral density
n	2n + 1 is the number of data points used in the digital filter of Section 6
P(k)	Power spectral density estimate
$P_x(k)$	Power spectral density of X
$P_y(k)$	Power spectral density of Y
$P_{xy}(k)$	Cross power spectral density of X with Y
$P_{yx}(k)$	Cross power spectral density of Y with X
$\hat{P}(k)$	True value of power spectral density corresponding to estimated value P(k)
p	Number of subsections of data of length m formed from the total number of data N. Also, in Section 7, the number of factors into which the transform size may be decomposed.
Q	Imaginary part of cross power spectral density
$R_x(j)$	Autocorrelation function of X(j)
$R_{xy}(j)$	Cross correlation function of X(j) with Y(j)
$R_{yx}(j)$	Cross correlation function of Y(j) with X(j)
r	Index which specifies the subsection number for the p subsections of data
$r_0, r_1, \ldots r_p$	Factors of the transform size in the fast Fourier transform algorithm

LIST OF SYMBOLS (Continued)

Symbol	Definition
S(j)	Real part of the inverse discrete Fourier transform of the cross power spectral density
s	Index which specifies the number of the subsection pair for the p/2 subsection pairs of data formed from the N values of a time function
T_m	Time interval represented by the m discrete values of a time function
t	Time
u(k)	Real part of B(k)
v(k)	Imaginary part of B(k)
W(t)	Data window, which modifies the function of time for which the power spectral density is required
X(j)	Discrete function of time, obtained at j Δt values of time
$X_f(t)$	Filtered function of time obtained from X(t)
$X_f(j)$	Discrete form of $X_f(t)$
Y(j)	As X(j)
Z(j)	Complex discrete function of time formed by treating pairs of subsections of X(j) as real and imaginary parts, or similarly treating subsections of X(j) and Y(j)
β	Spatial correlation function
γ	Coherence function
Δt	Time interval between discrete data of functions of time
Δf	Frequency interval between discrete values of functions of frequency
π	Ratio of circumference to diameter of a circle
τ	Variable of integration in the convolution integral

1.0 INTRODUCTION

The technique of power spectral density analysis has received considerable attention in recent years because of its suitability in the analysis of stationary random functions. A random process in time is stationary when its statistical properties obtained at a fixed time from an ensemble of sample functions are independent of time. It is generally sufficient for purposes of spectral density analysis for only the first two statistical moments to be independent of time, in which case the process is said to be weakly stationary. For a Gaussian process, of course, the independence of time of the first two statistical moments implies strong stationarity. If, further, the statistical properties of each sample record are dependent only on time differences and not on absolute time the process is ergodic and the properties of the process may be determined from one sample record.

Random or stochastic, processes occur in many branches of engineering including communications, vibrations, acoustics, turbulent fluid flow and also in various aspects of bio-dynamics, oceanography, meteorology and seismology; however, the majority of random functions derived from physical processes are rarely more than approximately stationary and ergodic and can be treated as exactly so for only limited periods of time.

Among uses of the power spectral density function are the investigation of physical mechanisms underlying random processes, the response of linear dynamic systems to random excitations, study of radio propagation phenomena and the simulation of random time series. The joint properties of pairs of random functions are also important and give rise to cross power spectral density analysis. The prefix "direct" is given here to the power spectral density function derived from a single random function where it is necessary to distinguish it from the cross power spectral density of two random functions.

The most usual and convenient method of expressing the power spectral density function of a random function of time is in terms of the Fourier transformation of the autocorrelation function. This definition has the advantage of not being troubled mathematically by difficulties and restrictions because of integral convergence properties, and leads to an efficient method of computing estimates of the spectral density function. Full details may be obtained from Reference 1. In the case of the cross power spectral density function the cross correlation function replaces the autocorrelation function. An alternative method of defining the power spectral density function is in terms of the square of the Fourier integral of the time function (or, in the case of cross power spectral density, the product of the Fourier integral of the first random function and the complex conjugate of the Fourier integral of the second). For the case of a random function defined for all time, the existence of its Fourier integral requires that restrictions are placed on the form of the time function; however, for the finite time for which the output of a real stochastic process is available and from which the power spectral density must be estimated the Fourier integral restrictions are satisfied and the Fourier transform may be used in the estimation process. The development of

improved numerical techniques in evaluating the discrete form of the Fourier transform has made the use of Fourier methods practicable and this report outlines the application of these techniques to the economical estimation of direct and cross power spectral density functions.

Because of the finite amount of data available from a real random process the problem of estimating its spectral density is essentially a statistical problem of estimation based on a sample drawn from a large population, with the inherent errors associated with this technique. A further source of error in the estimation process, also due to the finite data sample, is introduced through the Fourier transformation process and is equivalent to viewing the frequency domain through a filter with non-zero bandwidth, resulting in a distorted spectral function. In the design of a power spectral density estimation procedure the major considerations are the minimizing of these errors consistent with economy of computational effort or within the confines of the limited amount of data available. These points are discussed further in subsequent sections of this report.

2.0 POWER SPECTRAL DENSITY CALCULATION BY DIRECT FOURIER TRANSFORMATION

2.1 The Modified Periodogram

The starting point in calculating power spectral density of discrete random data by the direct method is the periodogram defined as

$$I(k) = |A(k)|^2 \tag{2.1}$$

where

$$A(k) = \frac{1}{N} \sum_{j=0}^{N-1} X(j) \exp\left(-2\pi i \frac{kj}{N}\right) \tag{2.2}$$

$$k = 0,1,2, \ldots, N-1$$

$$i = \sqrt{-1}$$

and X(j) is the time function defined at N points in time, Δt apart. A(k) is the discrete Fourier transform of the time function and is equivalent to a harmonic analysis of X assuming a period of $1/(N\Delta t)$ since

$$X(j) = \sum_{k=0}^{N-1} A(k) \exp\left(2\pi i \frac{kj}{N}\right) \tag{2.3}$$

$$j = 0,1,2, \ldots, N-1$$

The above definition of the Fourier transform of X(j) is equivalent to multiplying the infinite time function, of which X(j) is a sample, by a finite Dirac comb which has the value 1 at points on the time axis given by $j\Delta t$ and is zero elsewhere. Multiplication of two functions in the time domain is equivalent to convolving them in the frequency domain and thus

$$A(f_k) = \int_{-\infty}^{\infty} H(f_k - f)\, A_\infty(f)\, df \tag{2.4}$$

where $A_\infty(f)$ is the Fourier transform of the infinite time function and H(f) is the Fourier transform of the Dirac comb. This effect may be regarded as viewing $A(f_k)$ through a "window" whose transmissibility is represented by H(f); the name spectral

window has been given in Reference 1 to $|H(f)|^2$ and H(f) is known as a kernel (of the integral given in (2.3)) or frequency window. The spectral window is analogous to the frequency response of an electrical filter and the same criteria concerning desirable characteristics apply. Equation (2.4) shows that the calculated value of A(f) at frequency f_k is affected by the value of A(f) at all other frequencies f, resulting in a lack of definition of the true value of A(f) and thence in the true value of spectral density. The degree of lack of resolution is dependent on the form of H(f), and may be minimized by a choice of $H(f_k - f)$ which has its largest value at f_k and decreases rapidly as $|f_k - f|$ increases.

The spectral window corresponding to the Dirac comb function described above is

$$J_D(f) = T_m \frac{\sin \pi f T_m}{\pi f T_m} \tag{2.5}$$

where $T_m = N \Delta t$, and the graphical representation of $J_D(f)$ is shown in Figure 1. The characteristics of this window are observed to be far from desirable and many better windows have been proposed. Section 4 gives the details of a suggested window which may be applied in either time or spectral domain. A periodogram obtained from data to which a window other than the Dirac comb has been applied is known as a modified periodogram.

The process of modifying a periodogram is obviously identical to the attempt to eliminate the Gibbs Phenomenon, using the Fejer or Lanczos methods, when approximating an expansion of a periodic function with discontinuous derivatives by a truncated Fourier series. Details of these techniques are given in Reference 2.

2.2 From Periodogram to Spectral Density

It is well known that the modified periodogram is not a good estimate of the power spectral density of a stationary, random function of time, the reason being that while the mean periodogram for large N tends to the spectral density the variance of I(k) about the true value of spectral density does not decrease to zero as $N \to \infty$, that is, the spectral density estimator I(k) is not consistent.

In Reference 3, however, it is shown how it is possible to obtain spectral estimates from the periodogram. The method is to divide the N values of X(j) into p sections each of length m, sum the I(k) for each k and divide by p, i.e.,

$$\overline{I(k)} = \frac{1}{p} \sum_{r=1}^{p} I_r(k) \tag{2.6}$$

where now $k = 0, 1, 2, \ldots, m - 1$ and the total number of values of X(j) which have been used is mp. The $I_r(k)$ are obtained using m instead of N in Equation (2.2).

In converting an averaged, modified periodogram to power spectral density several conversion factors must be applied. To derive these the useful relation between the data in the time domain and the Fourier coefficients known as Parseval's equation may be employed. This is

$$\frac{1}{m}\sum_{j=0}^{m-1} X^2(j) = \sum_{k=0}^{m-1} (a^2(k) + b^2(k)) \qquad (2.7)$$

which expresses the mean square of the time function as the sum of the Fourier coefficients squared; the relationship is easily proved from the definition of a(k) and b(k). The integral of the power spectral density function with respect to frequency must also equal the mean square of X(j) so the periodogram must be divided by the frequency interval. The coefficients a(k) and b(k) are symmetrical about k = m/2 (see Section 5.2), and the power spectral density is defined only for m/2 + 1 frequencies, requiring a factor of 2 to be applied to the periodogram for the restricted range of k. The application of a spectral window is equivalent to weighting the X(j) with weights W(j), and the weighted mean square is

$$\frac{\sum_{j=0}^{m-1} X^2(j)\, W^2(j)}{\sum_{j=0}^{m-1} W^2(j)} = \frac{2m}{\sum_{j=0}^{m-1} W^2(j)} \sum_{k=0}^{m/2} \left\{(a'(k))^2 + (b'(k))^2\right\} \qquad (2.8)$$

where a'(k) and b'(k) represent the modified Fourier coefficients. For the spectral window defined in Section 4.0 the value of $1/m \sum W^2(j)$ is 3/8.

It is desirable, however, that the integral of the power spectral density be equal to the unweighted mean square of X(j), which requires a further multiplying factor to be used in Equation (2.8). This factor has been determined empirically for the spectral window of Section 4.1, to a good approximation, as 6.0. Obviously this value depends on the form of W(j) and will change for other spectral windows. Combining these conversion factors, and noting that Δf the frequency interval between coefficients is $1/(m\,\Delta t)$ (or h/m, where h is the data sampling rate) the power spectral density is given by

$$P(k) = \frac{32\,m}{h}\,\overline{I(k)} \qquad (2.9)$$

assuming the use of the data window described in Section 4.0, and where k = 0,1,2, ..., m/2.

2.3 Numerical Techniques

The evaluation of A(k) from Equation (2.2) is facilitated by the use of a computational method which expands the single transform of size N into a multi-dimensional transform of dimensions which are factors of N. The most effective use of this method has been found to arise when N is chosen as an integral power of 2. Details of this computational technique, known as the "fast Fourier transform," are given in Reference 4 and a particular implementation is discussed further in Section 7 of this report.

Equation (2.2) applies equally to real and complex values of X(j), but since the time series is always real the linearity property of Fourier transforms may be used to calculate a pair of $I_r(k)$ results simultaneously, as follows. For the p sets of data form

$$Z_s(j) = X_{2s-1}(j) + i\, X_{2s}(j) \tag{2.10}$$

where

$j = 0,1,2,\ldots,m-1$

$r = 1,2,3,\ldots,p$

and $s = 1,2,3,\ldots,p/2$ assuming p is even.

If p is odd, there remains one set of data to be treated singly in Equation (2.2) with zero imaginary part. Substituting $Z_s(j)$ in Equation (2.2) in place of X(j) the discrete Fourier transform $B_s(k)$ is obtained, and from this

$$A_{2s-1}(k) = \frac{1}{2}\left\{B_s(k) + B_s^*(m-k)\right\} \tag{2-11a}$$

$$A_{2s}(k) = \frac{1}{2i}\left\{B_s(k) - B_s^*(m-k)\right\} \tag{2-11b}$$

Defining

$$A_r(k) = a_r(k) + i\, b_r(k) \tag{2.12}$$

and

$$B_s(k) = u_s(k) + i\, v_s(k) \tag{2.13}$$

then

$$a_{2s-1}(k) = \frac{1}{2}\left\{u_s(k) + u_s(m-k)\right\} \tag{2.14}$$

$$b_{2s-1}(k) = \frac{1}{2}\left\{v_s(k) - v_s(m-k)\right\} \tag{2.15}$$

$$a_{2s}(k) = \frac{1}{2}\left\{v_s(k) + v_s(m-k)\right\} \tag{2.16}$$

$$b_{2s}(k) = \frac{1}{2}\left\{-v_s(k) + v_s(m-k)\right\} \tag{2.17}$$

where $k = 0,1,2, \ldots, m/2$.

The spectral window, details of which are given in Section 4, is then applied to the a and b coefficients by summing products of triplets of coefficients and the weights $-1/4$, $1/2$, $-1/4$, i.e.,

$$a_r'(k) = -\frac{1}{4}a_r(k-1) + \frac{1}{2}a_r(k) - \frac{1}{4}a_r(k+1) \tag{2.18}$$

and similarly for the $b_r'(k)$. At $k = 0$, $a_r'(0)$ is given by the weighted sum of $a_r(0)$ and $a_r(1)$ with weights $1/2$, $-1/2$ and at $k = m/2$ the last two values of $a_r(k)$ are summed with similar weights; $b_r'(0)$ and $b_r'(m/2)$ are obtained in the same manner.

For economy of computation time it is advantageous in calculating $I_r(k)$ to take out the common factors $1/2$ in Equations (2.14) to (2.17) and (2.18), and also to take out the factor $1/m$ in Equation (2.2) and calculate the coefficients in the from $4m\,a_r'(k)$. From Equations (2.6) and (2.9) the final expression in computing the power spectral density is then

$$P(k) = \frac{2}{hN}\sum_{r=1}^{p}\left\{(4m\,a_r'(k))^2 + (4m\,b_r'(k))^2\right\} \tag{2.19}$$

after substituting $p = N/m$.

A computer program which uses the above equations to calculate power spectral density functions of stationary, discrete, random data is described in Reference 5. Also included in Reference 5 is the description of a computer program which uses the equations of Section 3 to calculate the direct power spectral density of each of two discrete, random functions and the cross power spectral density between them. The result of an example calculation of power spectral density using the equations given above is shown in Figure 9 and, for comparison, the psd for the same data obtained using conventional analog equipment is shown in Figure 10.

3.0 CROSS POWER SPECTRAL DENSITY CALCULATION BY DIRECT FOURIER TRANSFORMATION

3.1 The Modified Cross Periodogram and Cross Spectral Density

For the case of cross power spectral density between time varying signals X(j) and Y(j), defined at increments of time Δt, the cross periodogram may be defined as

$$I_{xy}(k) = A(k)\, C^*(k) \tag{3.1}$$

where A(k) is defined by Equation (2.2) and

$$C(k) = \frac{1}{N} \sum_{j=0}^{N-1} Y(j) \exp\left(-2\pi i \frac{kj}{N}\right) \tag{3.2}$$

* means complex conjugate.

The remarks concerning data and spectral windows given in Section 2.1 apply equally to the cross periodogram, which needs to be modified in the same manner as the periodogram for a single function of time.

Statistical stability is obtained as before by averaging over a series of cross periodograms calculated from adjacent sets of time data, each of length $m\,\Delta t$. That is,

$$\overline{I_{xy}(k)} = \frac{1}{p} \sum_{r=1}^{p} (I_{xy}(k))_r \tag{3.3}$$

where $k = 0,1,2, \ldots, m-1$.

3.2 Numerical Techniques

A(k) and C(k) can be evaluated using the FFT technique as mentioned in Section 2.3. Sets of X(j) and Y(j) may be transformed simultaneously by treating them as the real and imaginary parts of complex data. Let

$$Z_r(j) = X_r(j) + i\, Y_r(j) \tag{3.4}$$

where $j = 0,1,2, \ldots, m-1$ and $r = 1,2,3, \ldots, o$. Substitute $Z_r(j)$ for X(j) in Equation (2.2) to give the discrete Fourier transform $B_r(k)$. Let $A_r(k)$ and $B_r(k)$ have real and imaginary parts as defined in Equations (2.10) and (2.11) and let

$$C_r(k) = c_r(k) + i\, d_r(k) \tag{3.5}$$

then

$$a_r(k) = \frac{1}{2}\left\{u_r(k) + u_r(m-k)\right\} \tag{3.6}$$

$$b_r(k) = \frac{1}{2}\left\{v_r(k) - v_r(m-k)\right\} \tag{3.7}$$

$$c_r(k) = \frac{1}{2}\left\{v_r(k) + v_r(m-k)\right\} \tag{3.8}$$

$$d_r(k) = \frac{1}{2}\left\{-u_r(k) + u_r(m-k)\right\} \tag{3.9}$$

A spectral window is then applied to the above coefficients by summing weighted triplets as described in Equation (2.18). It is apparent that the power spectral densities of the functions X(j) and Y(j) are also readily available from the above coefficients. Denoting these spectral densities by $P_x(k)$ and $P_y(k)$, respectively, and calculating the coefficients with the same multipliers as used in Equation (2.19) gives

$$P_x(k) = \frac{2}{hN}\sum_{r=1}^{p}\left\{(4m\,a_r'(k))^2 + (4m\,b_r'(k))^2\right\} \tag{3.10}$$

$$P_y(k) = \frac{2}{hN}\sum_{r=1}^{p}\left\{(4m\,c_r'(k))^2 + (4m\,b_r'(k))^2\right\} \tag{3.11}$$

$$P_{xy}(k) = \frac{2}{hN}\sum_{r=1}^{p} 16\,m^2\left\{a_r'(k)\,c_r'(k) + b_r'(k)\,d_r'(k) + i\,(b_r'(k)\,c_r'(k) - a_r'(k)\,d_r'(k))\right\} \tag{3.12}$$

The cross power spectral density is seen to be a complex function in contrast to the direct spectral densities which are real. Also, writing

$$P_{xy}(k) = D(k) - i\,Q(k) \tag{3.13}$$

then it is easily shown that

$$P_{yx}(k) = D(k) + i\,Q(k) \tag{3.14}$$

4.0 A SPECTRAL WINDOW

The need for weighting the time data, or alternatively smoothing the periodogram, has been pointed out in Section 2.1. The weighting function in the time domain has been named a data window and the square of its Fourier transform is called a spectral window.

A window which embodies reasonably satisfactory characteristics is one based on the Hanning lag window; the latter is commonly advocated when calculating power spectral density by the indirect method using auto- or cross-correlation functions and is defined in the correlation, or time lag, domain. Details of the Hanning lag window, and the general class of windows from which it derives, are given in Reference 1.

The data window proposed here is

$$W(t) = \frac{1}{2}\left(1 - \cos\frac{2\pi t}{T_m}\right) \qquad 0 \leq t \leq T_m$$
$$W(t) = 0 \qquad T_m < t < 0 \tag{4.1}$$

where $T_m = N\Delta t$. Figure 2 shows a curve of W(t) as a function of time; it is seen that the effect of weighting a time function with this window is to reduce the amplitude of the data samples near $t = 0$ and $t = T_m$ relative to the center of the range, and to remove the discontinuities in the derivatives at each end of the data sample.

The frequency window, defined as the Fourier transform of W(t), is required for convolving with the Fourier coefficients obtained from Equations (2.12) to (2.15) and (3.6) to (3.9). Thus,

$$H(f) = \int_{-\infty}^{\infty} W(t)\, e^{-2\pi i f t}\, dt \tag{4.2}$$

Substituting for W(t) and integrating gives

$$H(f) = \frac{1}{2} Q_0(f) - \frac{1}{4} Q_0\left(f + \frac{1}{T_m}\right) - \frac{1}{4} Q_0\left(f - \frac{1}{T_m}\right) \tag{4.3}$$

where

$$Q_0(f) = \frac{\sin 2\pi f T_m}{2\pi f} + i\,\frac{\sin^2 \pi f T_m}{\pi f} \tag{4.4}$$

Substituting for $Q_0(f)$, $Q_0(f + 1/T_m)$ and $Q_0(f - 1/T_m)$ in Equation (4.3) and rearranging

$$H(f) = \frac{T_m}{2}\left(\frac{1}{1 - f^2 T_m^2}\right)\left\{\frac{\sin^2 \pi f T_m}{2\pi f T_m} + i\,\frac{\sin^2 \pi f T_m}{\pi f T_m}\right\} \tag{4.5}$$

and the spectral window is given by

$$J(f) = \frac{1}{T_m}\,|H(f)|^2 \tag{4.6}$$

Thus,

$$J(f) = \frac{T_m}{4(1 - f^2 T_m^2)^2}\left\{\frac{\sin \pi f T_m}{\pi f T_m}\right\}^2 \tag{4.7}$$

This is similar to the Bartlett spectral window described in References 1 and 3, but with the factor $(2\,(1 - f^2 T_m^2))^{-2}$ included here.

Where $f T_m = \pm 1$ Equation (4.7) may be expanded to find the value of $J\left(\left|1/T_m\right|\right)$ as

$$J(f) = \frac{T_m}{4 f T_m}\left\{\frac{\sin \pi f T_m}{\pi (1 - f T_m)(1 + f T_m)}\right\}^2 \tag{4.8}$$

That is, when $f T_m = \pm 1$

$$J\left(\left|\frac{1}{T_m}\right|\right) = \frac{T_m}{16} \tag{4.9}$$

The variation of $J(f)/T_m$ with $f T_m$, normalized to the value 1 at $f T_m = 0$, is shown in Figure 3. For comparison the Hanning spectral window is reproduced in Figure 4.

The window may be applied in the time domain by multiplying X(t) by W(t), or in the frequency domain by forming the convolution of A(f) and H(f), where A(f) is the Fourier transform of X(t). In the frequency domain the convolution function is

$$A'(f) = \int_{-\infty}^{\infty} H(\ell)\, A(f - \ell)\, d\ell \qquad (4.10)$$

However, in calculating A(f) from discrete data, equation (4.10) must be written as

$$A'(k) = \frac{1}{m} \sum_{\ell=-\infty}^{\infty} H(\ell)\, A(k - \ell) \qquad (4.11)$$

where ℓ is now restricted to integer values. Defining A'(k) in terms of real and imaginary parts as in Equation (2.10) then

$$a'(k) = \frac{1}{2} \sum_{\ell=-\infty}^{\infty} a(k-\ell) \left\{ \frac{\sin 2\pi\ell}{2\pi\ell} - \frac{1}{2} \frac{\sin 2\pi(\ell+1)}{2\pi(\ell+1)} - \frac{1}{2} \frac{\sin 2\pi(\ell-1)}{2\pi(\ell-1)} \right\}$$

$$- \frac{1}{2} \sum_{\ell=-\infty}^{\infty} b(k-\ell) \left\{ \frac{\sin^2 \pi\ell}{\pi\ell} - \frac{1}{2} \frac{\sin^2 \pi(\ell+1)}{\pi(\ell+1)} - \frac{1}{2} \frac{\sin^2 \pi(\ell-1)}{\pi(\ell-1)} \right\} \qquad (4.12)$$

In the first summation on the right hand side of Equation (4.12) any value of ℓ other than 0 or ± 1 makes all the trigonometric terms zero, and in the second summation the trigonometric terms are all zero for all ℓ. Thus,

$$a'(k) = -\frac{1}{4} a(k-1) + \frac{1}{2} a(k) - \frac{1}{4} a(k+1) \qquad (4.13)$$

and similarly for the imaginary part of A'(k), leading to

$$b'(k) = -\frac{1}{4} b(k-1) + \frac{1}{2} b(k) - \frac{1}{4} b(k+1) \qquad (4.14)$$

giving the smoothing weights used in Sections 2.3 and 3.2. When k = 0 the coefficients are given by

$$a'(0) = \frac{1}{2} a(0) - \frac{1}{2} a(1) \qquad (4.15)$$

and when k = m/2

$$a'(m/2) = \frac{1}{2} a(m/2) - \frac{1}{2} a\left(\frac{m}{2} - 1\right) \qquad (4.16)$$

since the transform is symmetrical about k = 0 and m/2.

5.0 ANALYSIS BANDWIDTH, FREQUENCY INTERVAL AND STATISTICAL STABILITY

5.1 Analysis Bandwidth

The bandwidth of the spectral window is analogous to the bandwidth of the filter used in the analog method of calculating power spectral density. It may be defined as the interval between the frequencies at which the spectral window has fallen to one half its maximum value. Equating the right hand side of Equation (4.7) to half its maximum value, T_m 8, the analysis bandwidth is given by twice the value of f satisfying the resulting equation. The equation is

$$\left(\frac{\sin \pi f T_m}{\pi f T_m}\right)^2 = \frac{1}{2}(1 - f^2 T_m^2)^2 \tag{5.1}$$

and the bandwidth is given by

$$BW = \frac{2 f'}{T_m} \tag{5.2}$$

where f' is a solution of Equation (5.1) with T_m as the variable. The approximate value of f' is 0.73 and for discrete data $T_m = m \Delta t = m/h$ where h is the data sampling rate. Thus, the bandwidth is given by

$$BW = 1.46 \frac{h}{m} \tag{5.3}$$

which is approximately 3h/2m, and may be compared with the analysis bandwidth of h/m for the Hanning spectral window applied to power spectral density calculated by the method of Fourier transforming the correlation function of the time signal(s). It should be noted, however, that m lags specified for the calculation of the correlation function is equivalent to 2m points used in calculating the periodogram since the correlation function is obtained for m positive and m negative lags, giving 2m values of the correlation function; in the case of autocorrelation the evenness of the function requires only m explicit values to be calculated. The difference in bandwidth for the two spectral windows is clearly shown in Figures 3 and 4.

5.2 Frequency Interval

The finite Fourier transform defined in Equation (2.2) produces m pairs of coefficients for m values of the time function. The Fourier coefficients are spaced at frequency intervals of h/m cycles/second giving a total frequency range of h cycles/second. However, it is known from information theory (Reference 7, for instance) that the highest recognizable frequency in a discrete signal sampled h times per second is

h/2 cycles/second. Further, it can be shown that a discrete Fourier transform of a set of real data yields complex results, the real part of which is symmetrical about the m/2th point and the imaginary part of which is antisymmetrical about the mid-point. In Reference 8 the Fourier coefficients are said to be "aliased" about the h/2 point of the frequency scale. The coefficients for frequencies from h/2 to h may be considered as the coefficients of frequencies in the range 0 to -h/2; thus, only m/2 + 1 values of power spectral density are obtained for positive frequencies and, as pointed out in Section 2, the power spectral density must be multiplied by 2 to permit the total power to be distributed over positive frequencies only.

5.3 Statistical Stability

The statistical stability of each estimate of power spectral density can be expressed by the variance of the estimate. In order to determine the variance in general terms it is usually necessary to assume that the time function concerned is a sample from a Gaussian process and that the true power spectral density is relatively flat for frequency increments equal to the "equivalent width" of the spectral window (see Reference 1, page 19). The variance of each estimate may then be derived, using the covariance function of adjacent power spectral density estimates and the properties of the spectral window, and expressed in terms of the coefficient of variation (standard deviation/mean).

For a chi-square distribution, which is a distribution of positive terms derived from a Gaussian distribution, it is known that the coefficient of variation equals $(2/K)^{1/2}$ where K is the number of degrees of freedom of the chi-square distribution. By analogy, it is possible to express the stability of the power spectral density function in terms of the number of degrees of freedom of a chi-square distribution with the same coefficient of variation. These are generally known as "equivalent degrees of freedom" of the spectral density function. For the method of calculating power spectral density described in Sections 2 and 3 the variance of the spectral estimates is shown in Reference 3 to be given approximately by

$$\text{Variance of } P(k) = \frac{\hat{P}^2(k)}{p} \tag{5.4}$$

where $\hat{P}(k)$ is the true power spectral density corresponding to the calculated value $P(k)$, and $p = N/m$. The expected, or average, value of $P^2(k)$ is

$$E\left\{P(k)\right\}^2 = \hat{P}^2(k) \tag{5.5}$$

and thus

$$\frac{\text{Variance of } P(k)}{E\left\{P(k)\right\}^2} = (\text{Coefficient of Variation})^2 = \frac{1}{p} \tag{5.6}$$

and, using the relationship between coefficient of variation and K for a chi-square distribution, the equivalent degrees of freedom are given by

$$EDF = \frac{2N}{m} \tag{5.7}$$

This applies strictly only to Gaussian signals without sharp peaks in the spectrum; but although few real signals are exactly Gaussian a large number of natural random processes approximate to Gaussian distributions due to central limit effects and EDF is thus a useful and meaningful expression of the statistical stability of the power spectral density. It is shown in Reference 1, page 24 that in the vicinity of sharp peaks the equivalent degrees of freedom become equal to 2.

6.0 DATA FILTERING AND DECIMATION

6.1 Object of the Filtering and Decimation

In the previous section it is seen that the frequency interval and analysis bandwidth are functions of the data sampling rate h and the number of transform points m. For a fixed value of h the bandwidth is proportional to $1/m$, and thus the narrower the bandwidth the larger the value of m; but from Section 7 it will be observed that m must be an integral power of 2 to use the FFT implementation specified there, which means that as the required m gets larger available values become more restricted. A more important point is that the time required to compute the power spectral density is proportional to m. It is desirable, therefore, to choose m as small as possible consistent with the requirement that the bandwidth permits adequate detail to be shown in the power spectral density results.

The power spectral density results are usually converted to a logarithmic form, and the bandwidth requirements are then determined by the resolution required at low frequencies. However, in this case the frequency resolution becomes unnecessarily high at middle and low frequencies at the expense of computation time. One solution to this problem is to divide the total frequency range into a series of bands (on a logarithmic basis) and compute results with a different resolution in each band. A convenient method of doing this is to vary the sampling rate h within each band, but because h determines the recognizable frequency content in the data it is necessary to filter the data before reducing the sampling rate.

If the frequency bands are octaves then the appropriate filter is low-pass with a cut-off frequency one half of the sampling rate. The sampling rate is reduced by calculating only alternate filtered data points, which effectively discards one half of the points in the series, resulting in a doubling of the time interval between data points. This process has been called "data decimation" in Reference 1. If there are n frequency bands the sampling rates are h, h/2, h/4, ..., $h/2^{(n-1)}$, respectively, starting with the highest band and descending. It will be noted that since m is constant the bandwidth doubles in each ascending octave band. If it is desired to keep the equivalent statistical degrees of freedom constant in each band then the same number of data points m must be used for the analysis in each band, which implies that the total number of data points required is $2^{n-1} \cdot m$ for n frequency bands.

6.2 A Numerical Low Pass Filter

An ideal low pass filter has the following characteristics:

$$G(f) = 1 \qquad f < f_c$$
$$G(f) = 0 \qquad f > f_c \tag{6.1}$$

where f_c is the cut-off frequency in cycles second and G(f) is the frequency response of the filter. In the time domain the filter output is the convolution of the inverse Fourier transform of G(f) and the input signal. This is,

$$X_f(t) = \int_{-\infty}^{\infty} g(\tau)\, X(t - \tau)\, d\tau \qquad (6.2)$$

where g(t) is the inverse Fourier transform of G(f) and X(t) is the time function which is to be filtered to give $X_f(t)$.

The Fourier transform of G(f) is

$$g(t) = \frac{1}{\pi} \frac{\sin 2\pi f_c t}{t} \qquad (6.3)$$

However, to obtain the properties of the ideal filter specified by the transform pair defined by Equations (6.1) and (6.3) the time function X(t) must be defined for all time. In practice, of course, the time function for real data is always truncated to a finite range and this becomes equivalent to truncating the filter response in the time domain. It is shown in Reference 6 that the resulting filter is far from ideal and that an improvement can be obtained by modifying the truncation process to give a sharper cut-off and eliminate the oscillations in the frequency response of the filter. The modification is to multiply the filter response in the time domain given in Equation (6.3) by a function similar to the data window specified in Section 4, which forces g(t) to approach zero smoothly before being truncated. In Reference 6 modifying functions of this type are given the name apodizing functions because of an analogy with a process in optics. From Reference 6, the most effective apodizing function is

$$F(t) = \frac{1}{2}\left(1 + \cos \frac{2\pi f_c t}{a}\right) \qquad |t| \leq \frac{a}{2 f_c}$$

$$F(t) = 0 \qquad |t| > \frac{a}{2 f_c} \qquad (6.4)$$

where a is a constant which controls the number of zero crossings of the product g(t) F(t) for a given f_c. The filter is given by the product g(t) F(t) and its frequency response for a = 5 is given in Figure 5 and for a = 10 in Figure 6; the corresponding responses in the time domain are shown in Figures 7 and 8, respectively. The cut-off frequency f_c is taken to be 500 cycles/second.

For he case of discrete data, where X(t) is known at intervals of time Δt, Equation (6.2) must be converted to discrete form, that is,

$$X_f(j) = \sum_{k=-n}^{n} g'(k)\, X(j-k)\, \Delta t \qquad (6.5)$$

where g'(k) is defined for 2n + 1 discrete points and is the filter function g(t) F(t) in discrete form. Substituting for g'(k) gives

$$X_f(j) = \frac{1}{2\pi} \sum_{k=-n}^{n} X(j-k) \left(1 + \frac{\cos 2\pi f_c\, k\, \Delta t}{a}\right) \frac{\sin 2\pi f_c\, k\, \Delta t}{k} \qquad (6.6)$$

where $n = a/(2 f_c\, \Delta t)$.

6.3 Cut-Off at One-Half the Frequency Range

For the particular case mentioned in Section 6.1, where the signal X(j) is filtered to remove one half of its frequency content, $f_c = 1/(4\,\Delta t)$. Substituting in Equation (6.6):

$$X_f(j) = \frac{1}{2\pi} \sum_{k=-2a}^{2a} X(j-k) \left(1 + \cos \frac{\pi k}{2a}\right) \frac{\sin (k\pi/2)}{k} \qquad (6.7)$$

The discrete filter weights in this case are

$$g'(k) = \frac{1}{2k\pi} \left(1 + \cos \frac{k\pi}{2a}\right) \sin \frac{k\pi}{2} \qquad (6.8)$$

which are zero for even values of k, except at k = 0, and the non-zero terms alternate in sign. In numerical calculations using the filter, advantage may be taken of the fact that the filter is an even function and the number of multiplications can be reduced by a factor of 2 by summing pairs of X(j) which have the same weights before multiplying. Thus, for a = 5 giving 21 values of g'(k), 10 of the weights are zero and there are 6 distinct non-zero weights which means that for each filtered point 6 multiplications and 11 additions are required.

6.4 Octave and One-Third Octave Analysis

The technique of carrying out power spectral density analysis in the manner described in Section 6.1 has particular application in the case of octave and one-third octave analysis, or for any other percentage frequency bandwidth analysis. Results are obtained in a series of octave bands, with a constant frequency resolution in each band, and converted to the required bandwidth by an averaging process, interpolating between frequencies where required. Applying this technique, Figure 11 shows the results of Figure 9 obtained using four frequency bands, and Figure 12 presents a third octave form of the same results.

7.0 THE FAST FOURIER TRANSFORM

The "fast Fourier transform" is the name given to an algorithm which permits the rapid evaluation of the discrete Fourier transform of a set of data, compared with the straightforward method of calculating the Fourier coefficients. The algorithm in its present form was first published in Reference 4 and a minor variation has since been produced (Reference 10); the historical background in the discovery of the algorithm is covered in Reference 11. The basic idea of the algorithm will be summarized here, and an attempt made to illustrate the method for a particular case.

Define a function

$$e\left(\frac{m}{N}\right) = e^{-2\pi i \left(\frac{m}{N}\right)} \tag{7.1}$$

which has the value 1 for all integral values of m/N and has the property $e(a+b) = e(a)\,e(b)$. The discrete Fourier transform of $X(j)$ is

$$A(k) = \frac{1}{N} \sum_{j=0}^{N=1} X(j)\, e\left(\frac{kj}{N}\right) \tag{7.2}$$

If N has p factors such that $N = r_1 r_2 \ldots . r_p$ then j and k may be expressed as

$$j = j_{p-1}\, r_1 r_2 \ldots . r_{p-1} + j_{p-2}\, r_1 r_2 \ldots . r_{p-2} + \ldots . + j_2\, r_1 r_2 + j_1\, r_1 + j_0 \tag{7.3}$$

$$k = k_{p-1}\, r_p r_{p-1} \ldots . r_2 + k_{p-2}\, r_p r_{p-1} \ldots . r_3 + \ldots . + k_2\, r_p\, r_{p-1} + k_1\, r_p + k_0 \tag{7.4}$$

where

$$j_{p-1}, k_0 = 0,1,2, \ldots . , r_{p-1} - 1$$

$$j_{p-2}, k_1 = 0,1,2, \ldots . , r_{p-2} - 1$$

$$\vdots$$

$$j_0, k_{p-1} = 0,1,2, \ldots . , r_1 - 1$$

Substituting in $e(kj/N)$ and eliminating all integer terms in the argument gives

$$e\left(\frac{kj}{N}\right) = e\left\{j_0\left(\frac{k_{p-1}}{r_1} + \frac{k_{p-2}}{r_1 r_2} + \cdots + \frac{k_0}{N}\right) + j_1\left(\frac{k_{p-2}}{r_2} + \frac{k_{p-3}}{r_2 r_3} + \cdots + \frac{k_0}{r_2 r_3 \cdots r_p}\right)\right.$$

$$\left. + \cdots + \frac{j_{p-1} k_0}{r_p}\right\} \qquad (7.5)$$

and the discrete Fourier transform can be expressed as

$$A(k_0, k_1, k_2, \ldots, k_{p-1}) = \frac{1}{N} \sum_{j_0=0}^{r_1-1} e\left(\frac{j_0 k_{p-1}}{r_1}\right) e\left(j_0\left[\frac{k_{p-2}}{r_1 r_2} + \cdots + \frac{k_0}{N}\right]\right)$$

$$\times \sum_{j_1=0}^{r_2-1} e\left(\frac{j_1 k_{p-2}}{r_2}\right) e\left(j_1\left[\frac{k_{p-3}}{r_1 r_2} + \cdots + \frac{k_0}{r_2 r_3 \cdots r_p}\right]\right)$$

$$\times \sum_{j_2=0}^{r_3-1} \cdots e\left(\frac{j_{p-2} k_0}{r_{p-1} r_p}\right) \sum_{j_{p-1}=0}^{r_p-1} X(j_0, j_1, j_2, \ldots, j_{p-1})\, e\left(\frac{j_{p-1} k_0}{r_p}\right) \qquad (7.6)$$

It can be seen that the single dimension transform of size N has been converted to a p dimensional transform of sizes $r_1, r_2, \ldots, r_p$ plus the multiplication of each transform except the outer one by an e function which has been called a "twiddle factor" by Gentleman and Sande (Reference 10). Cooley and Tukey in Reference 4 show that the ratio of the number of arithmetic operations, and hence computation time, in the straightforward method to the FFT (fast Fourier Transform) is $N/\log_r N$ (when the p factors of N are all equal to r). Experience with the algorithm, applied to automatic digital computers, has shown these speed improvements to be attainable and has brought hitherto impracticable uses of the discrete Fourier transform within the bounds of practicability.

The variation introduced by Reference 10 is to rearrange the e terms in Equations (7.5) and (7.6) in the following manner:

$$e\left(\frac{kj}{N}\right) = e\left\{\frac{k_{p-1}j_0}{r_1} + \frac{k_{p-2}}{r_1 r_2}\left(j_0 + j_1 r_1\right) + \frac{k_{p-3}}{r_1 r_2 r_3}\left(j_0 + j_1 r_1 + j_2 r_1 r_2\right)\right.$$

$$\left. + \cdots + \frac{k_0}{r_1 r_2 r_3 \cdots r_p}\left(j_0 + j_1 r_1 + j_2 r_1 r_2 + \cdots + j_{p-1} r_1 r_2 \cdots r_{p-1}\right)\right\} \quad (7.7)$$

and Equation (7.6) is written

$$A\left(k_0, k_1, k_2, \ldots, k_{p-1}\right) = \frac{1}{N}\sum_{j_0=0}^{r_1-1} e\left(\frac{j_0 k_{p-1}}{r_1}\right) e\left(\frac{j_0 k_{p-2}}{r_1 r_2}\right)$$

$$\times \sum_{j_0=0}^{r_2-1} e\left(\frac{j_1 k_{p-2}}{r_2}\right) e\left(\frac{k_{p-3}}{r_1 r_2 r_3}\left(j_0 + j_1 r_1\right)\right) \sum_{j_2=0}^{r_3-1} \cdots$$

$$\times \sum_{j_{p-1}=0}^{r_p-1} X\left(j_0, j_1, j_2, \ldots, j_{p-1}\right) e\left(\frac{j_{p-1} k_0}{r_p}\right) \quad (7.8)$$

Equation (7.8) has a slight advantage over (7.6) when being programmed for evaluation on a digital computer.

The use of constant factors $r = 2$, or $r = 4$, leads to the best compromise between calculation speed and convenience, and the algorithm has been programmed extensively in this form; in order to use the algorithm under these conditions N must be an integral power of 2, but in many cases this is not a severe restriction. The implementation documented in Reference 5 for calculating power spectral density from Equations (2.19) and (3.12) uses factors of $r = 4$, plus, if $\log_2 N$ is odd, one factor of 2.

As an illustration to aid in the understanding of the workings of the algorithm consider the case of $N = 64$, giving $r_1 = r_2 = r_3 = 4$, in Equation (7.8). Let the data X have subscripts 0 to 63, and

$$j = j_0 + 4j_1 + 16j_2 \tag{7.9}$$

$$k = k_0 + 4k_1 + 16k_2 \tag{7.10}$$

Equation (7.8) becomes

$$A(k_0 + 4k_1 + 16k_2) = \frac{1}{N} \sum_{j_0=0}^{3} e\left(\frac{j_0 k_2}{4}\right) e\left(\frac{j_0 k_1}{16}\right)$$

$$\times \sum_{j_1=0}^{3} e\left(\frac{j_1 k_1}{4}\right) e\left(\frac{k_0}{64}\left(j_0 + 4j_1\right)\right)$$

$$\times \sum_{j_2=0}^{3} X\left(j_0 + 4j_1 + 16j_2\right) e\left(\frac{j_2 k_0}{4}\right) \tag{7.11}$$

All the subscripted indices take the values of 0, 1, 2, 3. In Equation (7.11) all the e functions with a factor 4 in the denominator of the argument have real and imaginary parts selected from 0 or ± 1, since they are exponentials with arguments as multiples of $\pi/2$, and only additions and subtractions are involved when using them. The complex matrix formed by the e function of the form $e(j_s k_{p-s-1}/r_{s+1})$ for all values of j_s and k_{p-s-1} is

$$\left[Z\right] = \begin{bmatrix} 1,0 & 1,0 & 1,0 & 1,0 \\ 1,0 & 0,1 & -1,0 & 0,-1 \\ 1,0 & -1,0 & 1,0 & -1,0 \\ 1,0 & 0,-1 & -1,0 & 0,1 \end{bmatrix} \tag{7.12}$$

in which the real and imaginary parts are combined in the form a, b.

A further point in applying the FFT in a digital computer is that the transform of a set of data may be calculated without the use of scratch storage if the original data are not required after the transform has been obtained. The results of the transform may be stored in place of the input data. The steps in evaluating the transform may be written in terms of matrix operations provided the definition of a non-standard matrix operation, which is given below, is accepted. In the following it is recognized that all arithmetic is of complex form:

(i) for $k_0 = 0, 1, 2, 3$ carry out the inner multiplication and summation over j_2 for all values of j_0 and j_1, and store the four results of each summation in place of each datum involved in the summation. Thus, for $j_0 = j_1 = 0$

$$\left[X'_0, X'_{16}, X'_{32}, X'_{48}\right] = \left[X_0, X_{16}, X_{32}, X_{48}\right]\left[Z\right] \tag{7.13}$$

and for $j_0 = 0, j_1 = 1$

$$\left[X'_4, X'_{20}, X'_{36}, X'_{52}\right] = \left[X_4, X_{20}, X_{36}, X_{52}\right]\left[Z\right] \tag{7.14}$$

and so on for all j_0 and j_1 combinations. The array X' may now be subscripted by $(j_0 + 4j_1 + 16k_0)$.

(ii) the matrix of "twiddle factors" in the middle summation is indexed by $j_1 = 0, 1, 2, 3$, for given j_0 and k_0 and can be represented as a 4x1 matrix $\left[T(j_1)\right]$. The middle summation is thus, for $j_0 = k_0 = 0$.

$$\left[X''_0, X''_4, X''_8, X''_{12}\right] = \left[X'_0, X'_4, X'_8, X'_{12}\right]\left\{\left[Z\right] * \left[T(j_1)\right]\right\} \tag{7.15}$$

where the operation denoted by * means that each element of each column of $[Z]$ is multiplied by the corresponding element of the column matrix $[T]$; for $j_0 = 0$ and $k_0 = 1$.

$$[X''_{16}, X''_{20}, X''_{24}, X_{28}] = [X'_{16}, X'_{20}, X'_{24}, X'_{28}] \left\{[Z] * [T(j_1)]\right\} \qquad (7.16)$$

and so on for all combinations of j_0 and k_0. The new array may now be subscripted $(j_0 + 4k_1 + 16\,k_0)$.

(iii) the T matrix for the outer summation is indexed by j_0 and the summation is carried out for all values of k_0 and k_1. Thus, for $k_0 = k_1 = 0$

$$[X'''_0, X'''_1, X'''_2, X'''_3] = [X''_0, X''_1, X''_2, X''_3] \left\{[Z] * [T(j_0)]\right\} \qquad (7.17)$$

and for $k_0 = 0$, $k_1 = 1$

$$[X'''_4, X'''_5, X'''_6, X'''_7] = [X''_4, X''_5, X''_6, X''_7] \left\{[Z] * [T(j_0)]\right\} \qquad (7.18)$$

and so on for all combinations of k_0 and k_1.

The final values of A are subscripted $(k_2 + 4k_1 + 16\,k_0)$ which is not the required final order, and the results must be rearranged. It can be seen that if the subscript is given a binary representation then the order in which A is obtained is equivalent to reversing the digits of the subscript in the required order, and this fact may be of use in the procedure for reordering the computed results.

8.0 EXTENSIONS TO RELATED FUNCTIONS

Direct and cross-power spectral densities are used in the calculation of several useful functions which express the properties of random processes or the response and excitation of linear systems. These functions include:

(i) coherence function

$$\gamma(k) = \frac{|P_{xy}(k)|^2}{P_x(k)\, P_y(k)} \tag{8.1}$$

(ii) spatial correlation function for relating the random variation of a property at two points in space x and y

$$\beta(k) = \frac{D_{xy}(k)}{\sqrt{P_x(k)\, P_y(k)}} \tag{8.2}$$

where $D_{xy}(k)$ is the real part of $P_{xy}(k)$.

(iii) frequency response function

$$F(k) = \frac{P_{xy}(k)}{P_x(k)} \tag{8.3}$$

where $P_y(k)$ is the output spectral density of a system due to the input function whose spectral density is P_x and $P_{xy}(k)$ is the cross power spectral density between them. The phase angle ϕ of the frequency response function is equal to the phase angle of the cross-power spectral density $P_{xy}(k)$.

(iv) autocorrelation function, which may be obtained by the inverse cosine transform of the power spectral density, i.e.,

$$R_x(j) = h \sum_{k=0}^{m/2} P_x(k) \cos\left(\frac{2\pi k j}{m}\right) \tag{8.4}$$

where, if $P_x(k)$ has been obtained using a spectral window, $R_x(j)$ is the product of the autocorrelation function and the lag window corresponding to the particular spectral window used.

(v) cross correlation function, obtained by the inverse discrete Fourier transform of the cross power spectral density

$$R_{xy}(j) \quad S(j) + F(j) \tag{8.5}$$

$$R_{yx}(j) = S(j) - E(j) \tag{8.6}$$

where

$$S(j) + i\,E(j) = h \sum_{k=0}^{m/2} P_{xy}(k) \exp\left(\frac{2\pi i k j}{m}\right) \tag{8.7}$$

The remarks concerning spectral and lag window apply here also.

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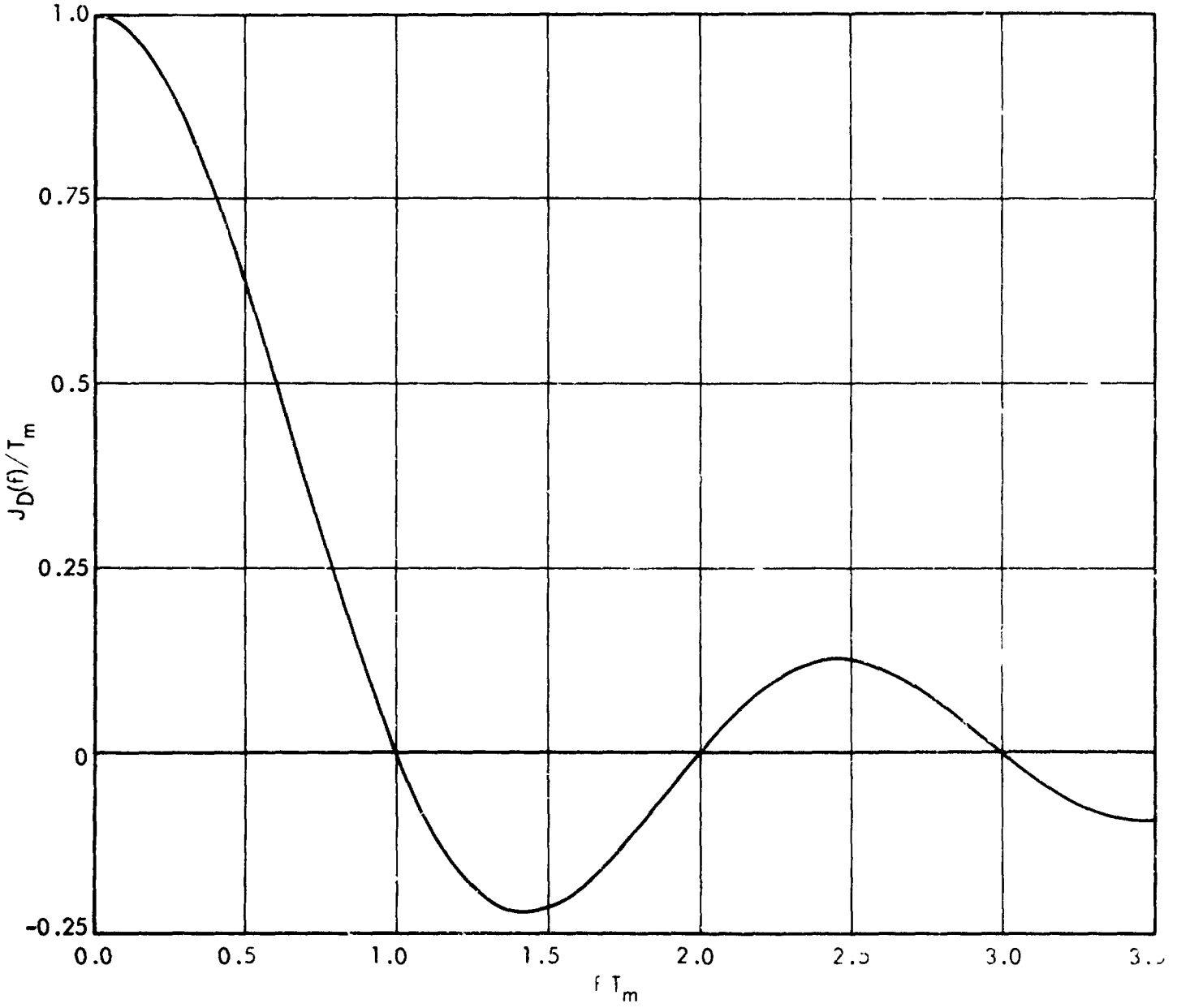

Figure 1. Spectral Window Corresponding to the Dirac Comb

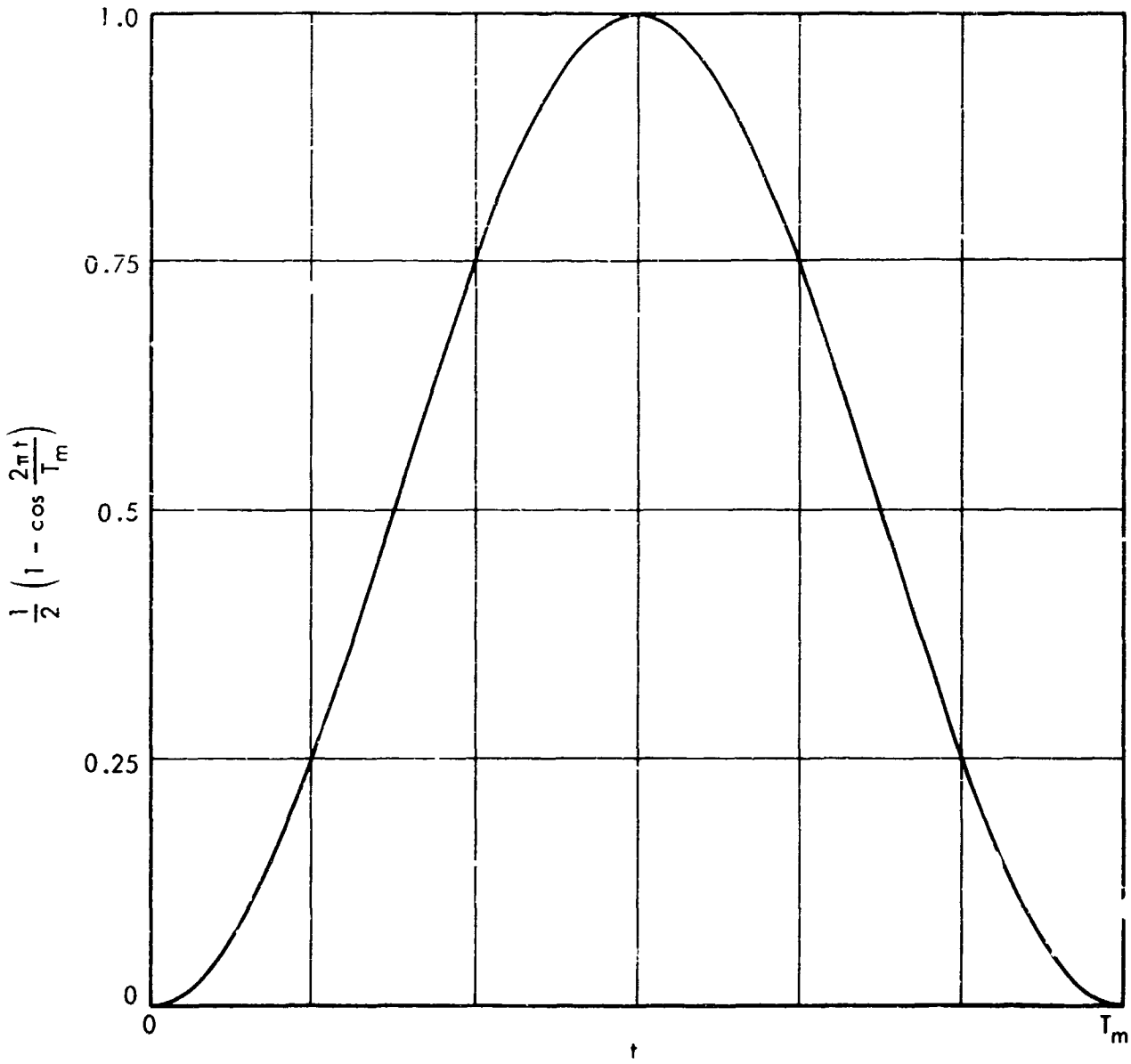

Figure 2. Data Window $\frac{1}{2}\left(1 - \cos\frac{2\pi t}{T_m}\right)$

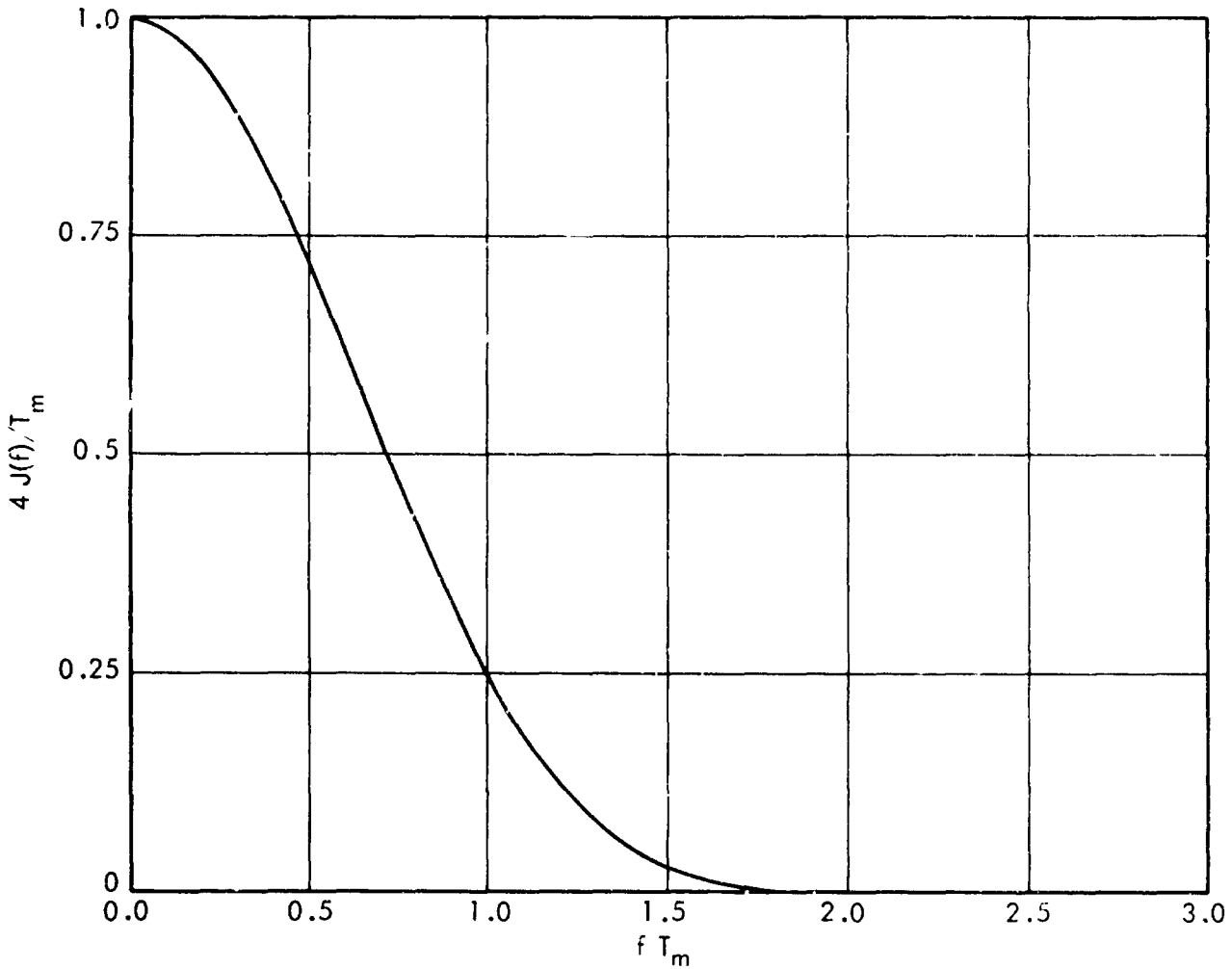

Figure 3. Spectral Window Corresponding to Data Window $\frac{1}{2}\left(1 - \cos\frac{2\pi t}{T_m}\right)$

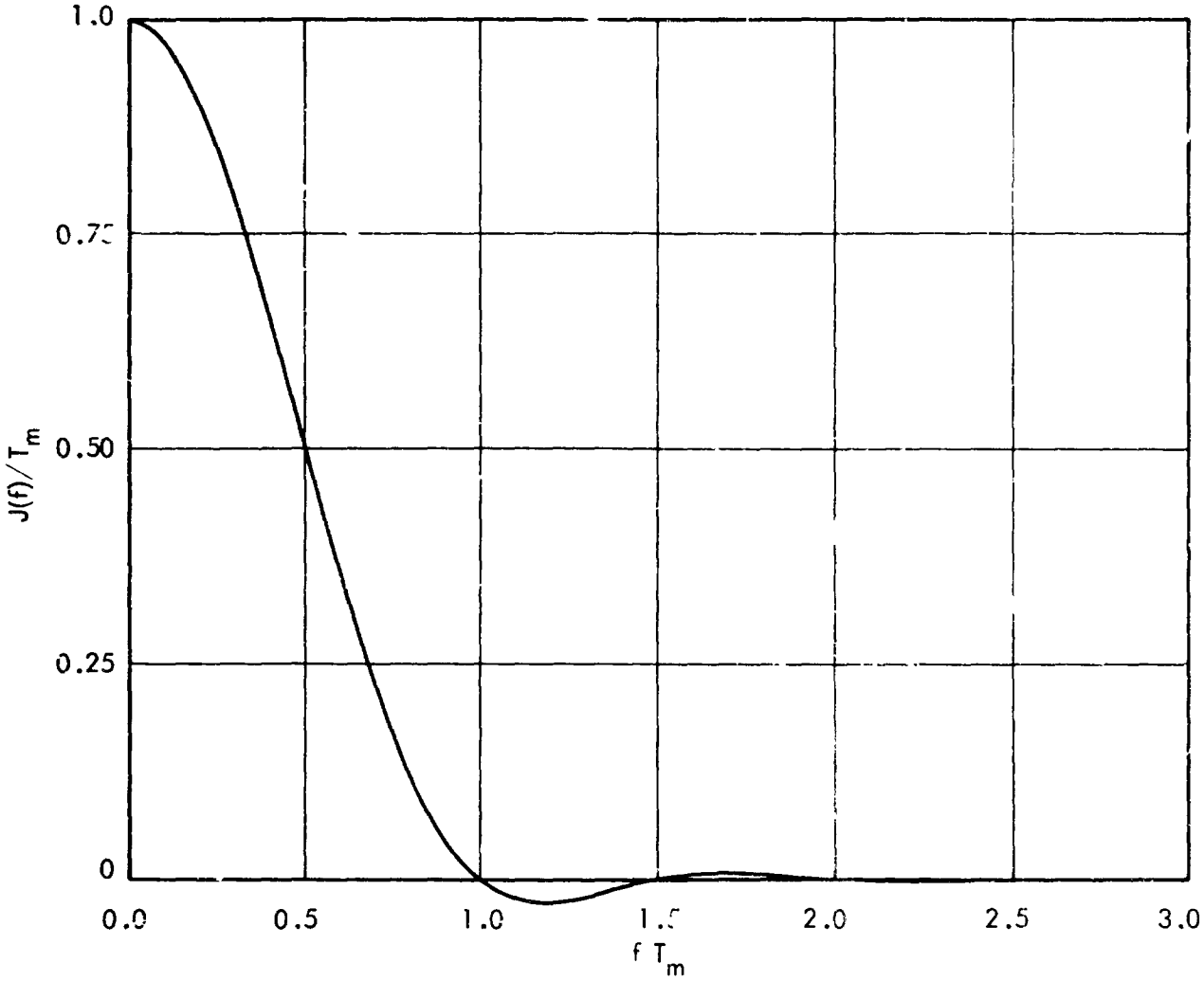

Figure 4. Hanning Spectral Window

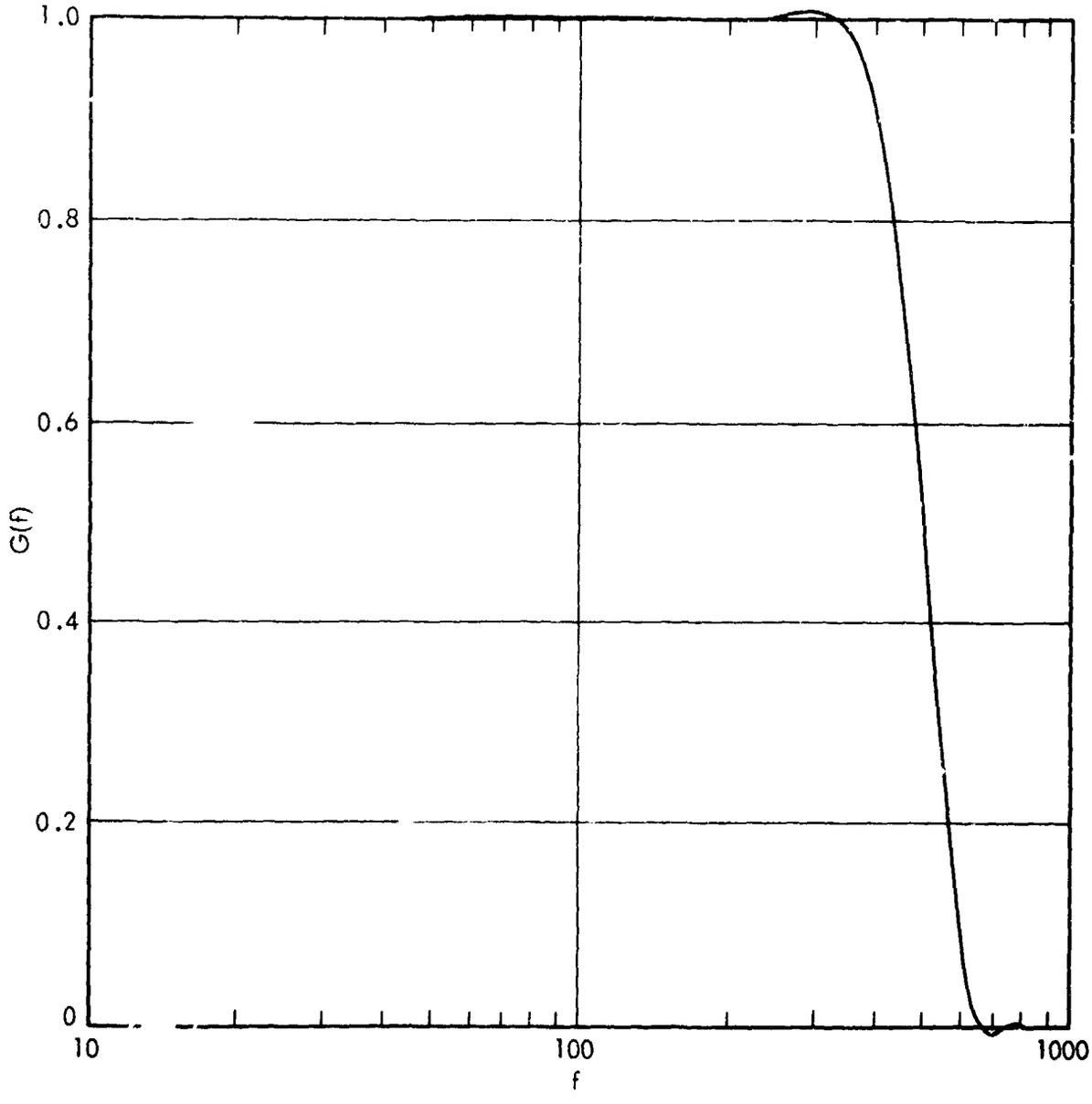

Figure 5. Filter Frequency Response: $a = 5$, $f_c = 500$

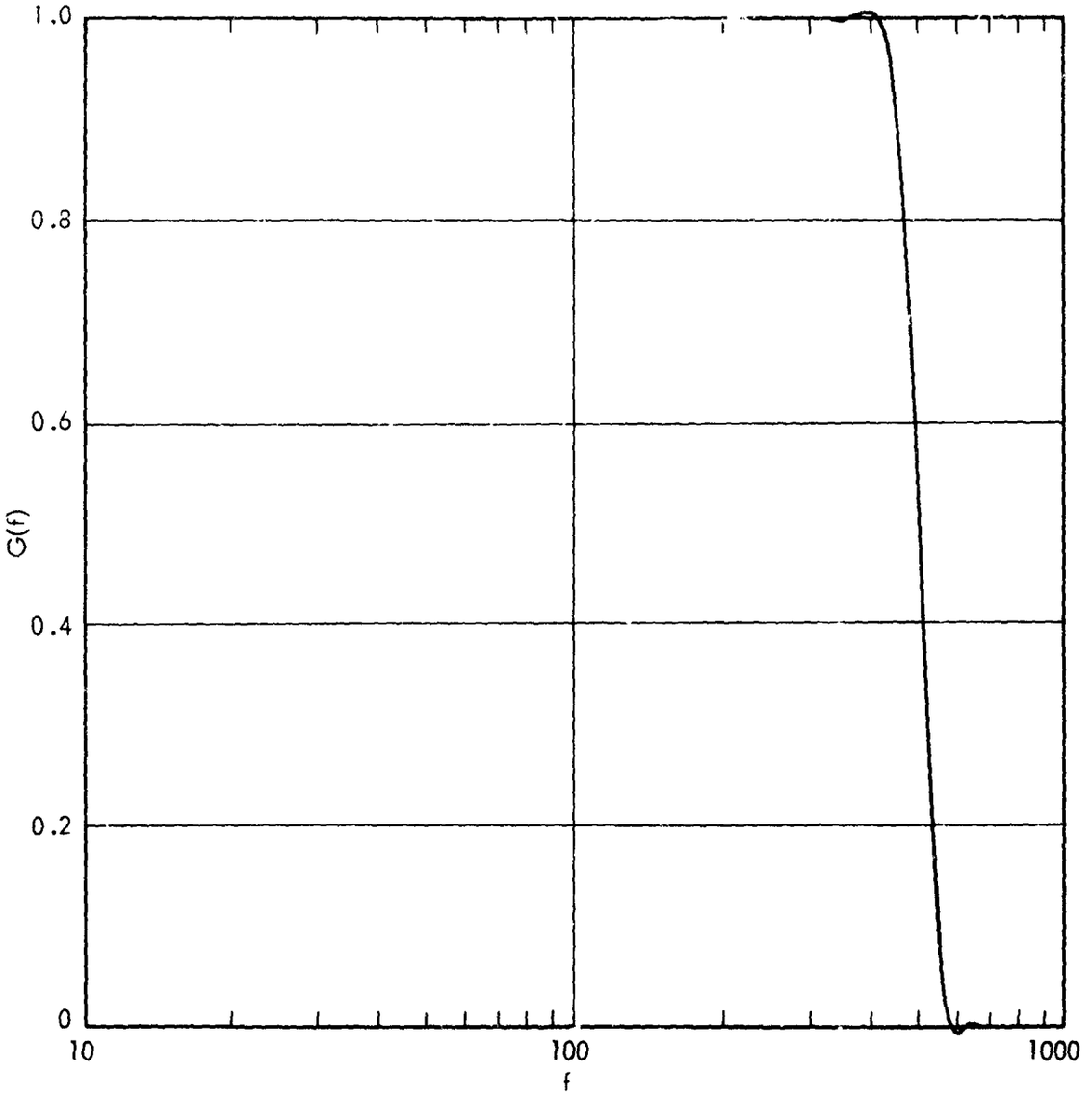

Figure 6. Filter Frequency Response: $a = 10$, $f_c = 500$

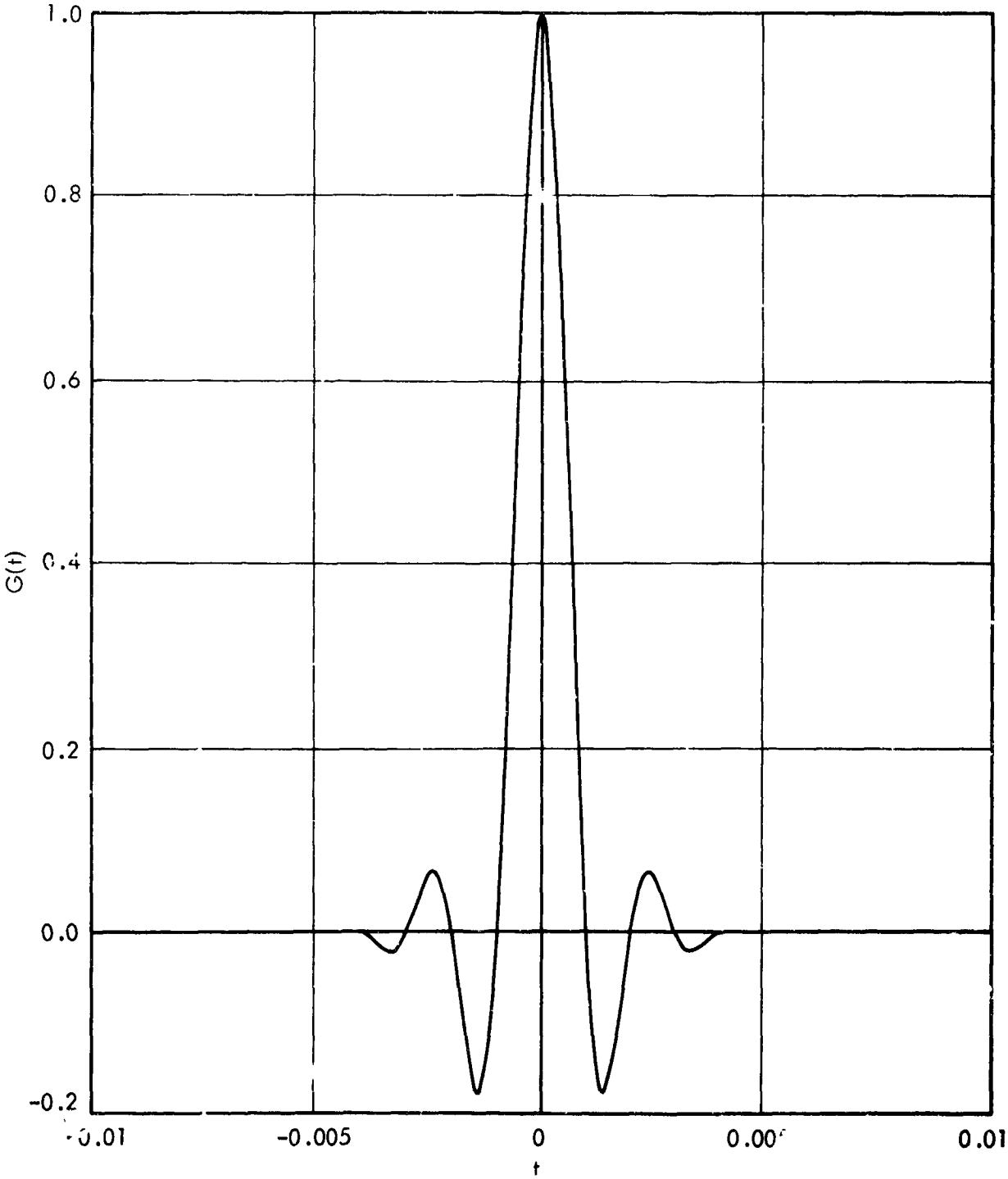

Figure 7. Filter in Time Domain for a = 5

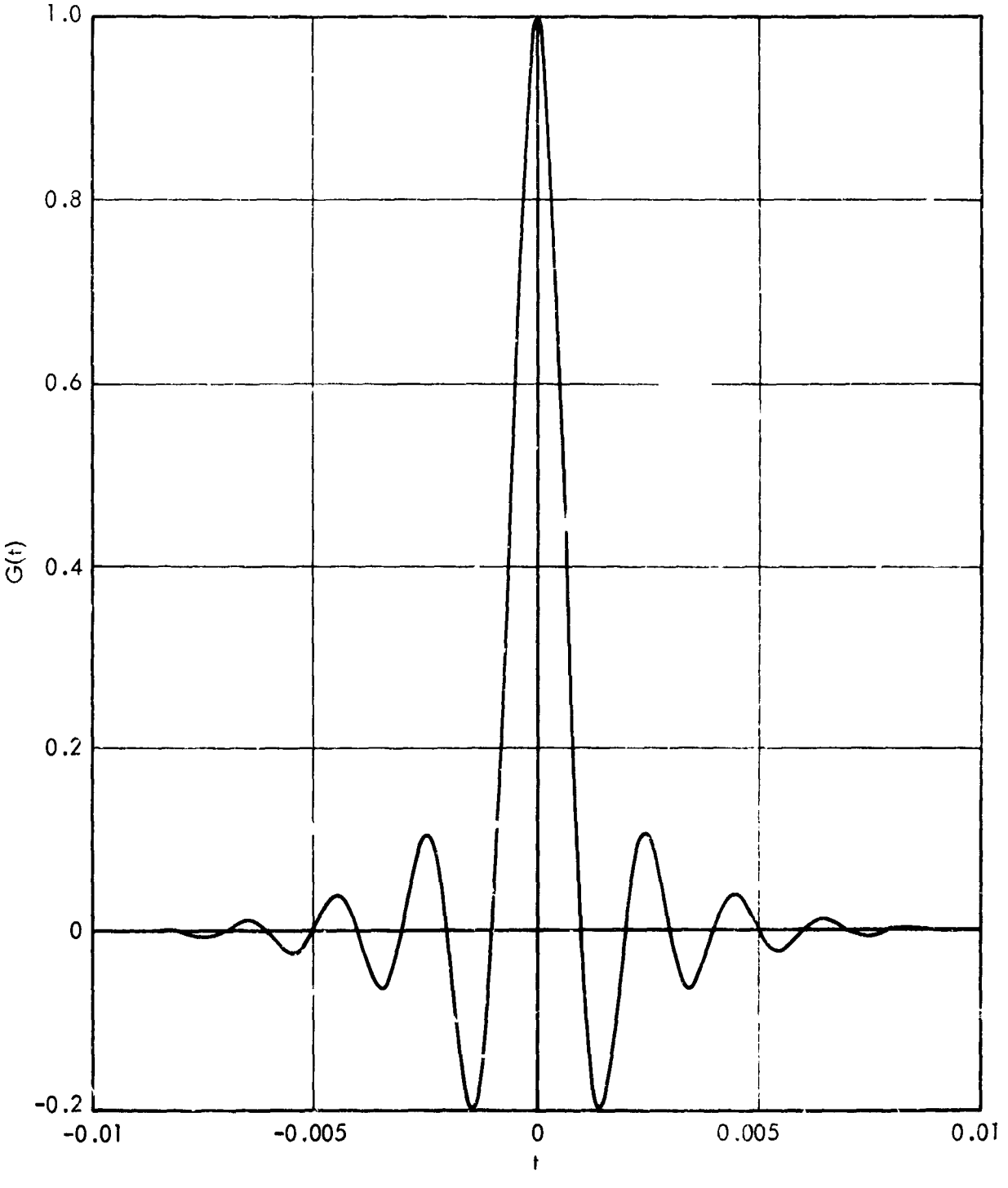

Figure 8. Filter in Time Domain for a = 10

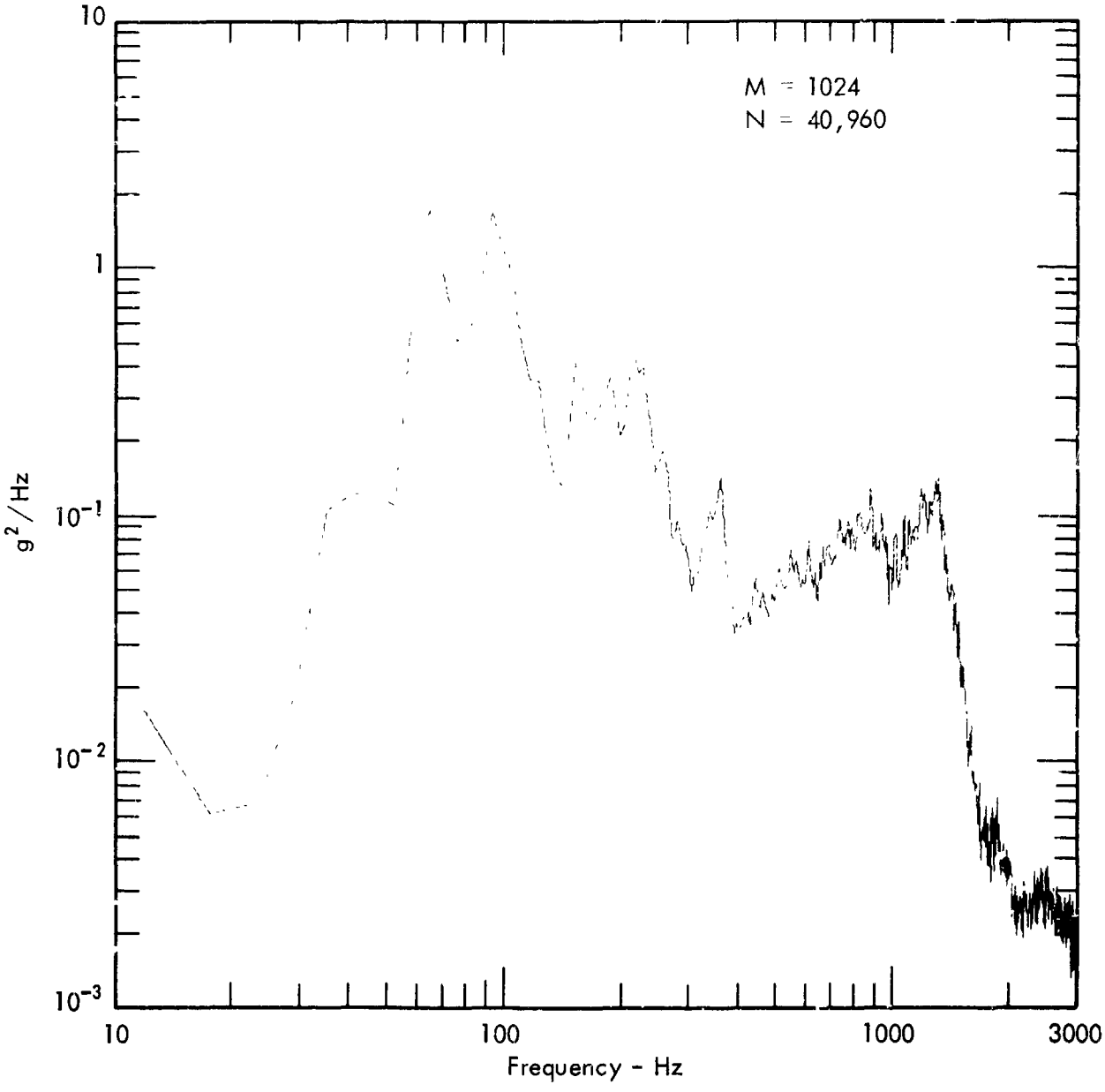

Figure 9. Example Power Spectral Density Function

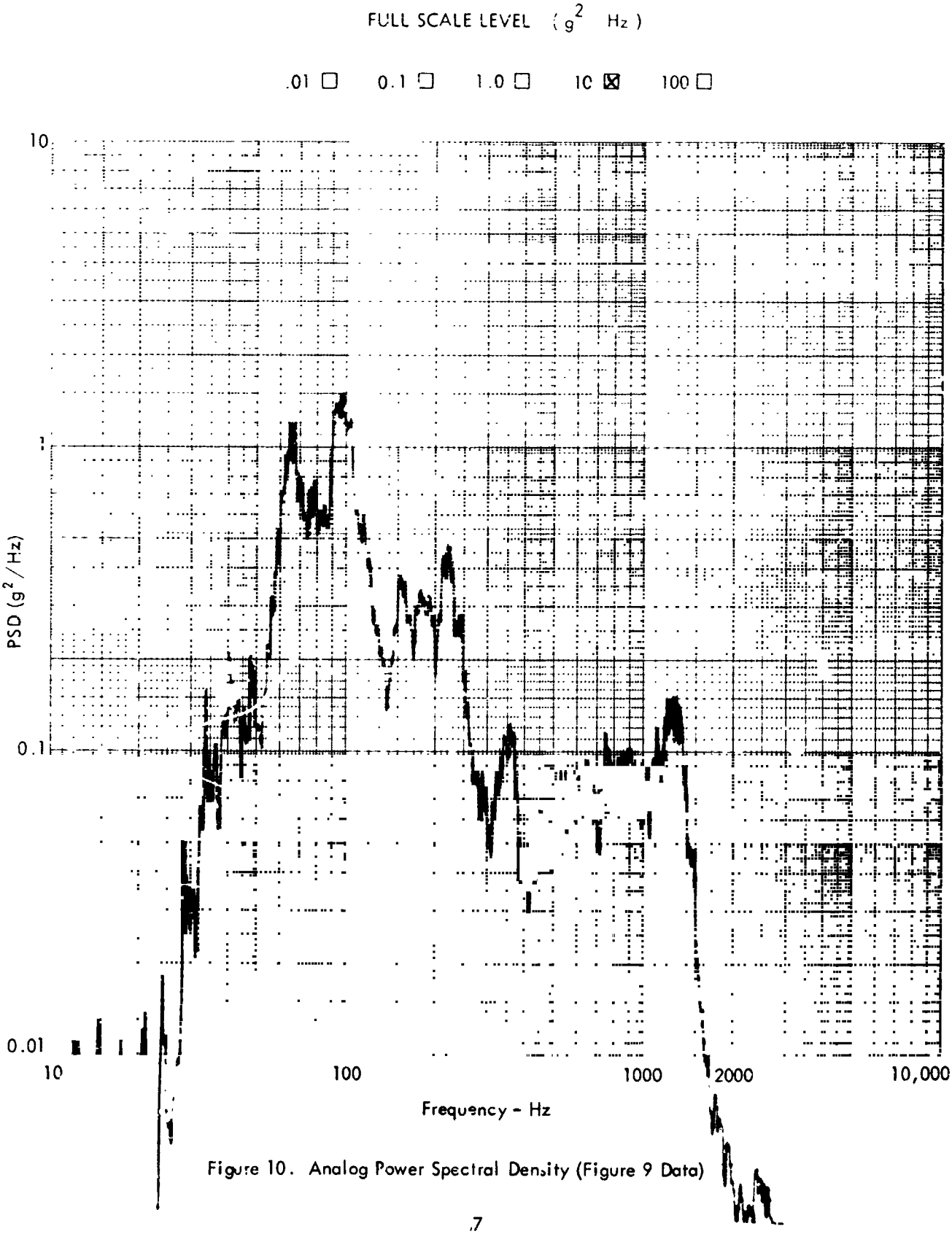

Figure 10. Analog Power Spectral Density (Figure 9 Data)

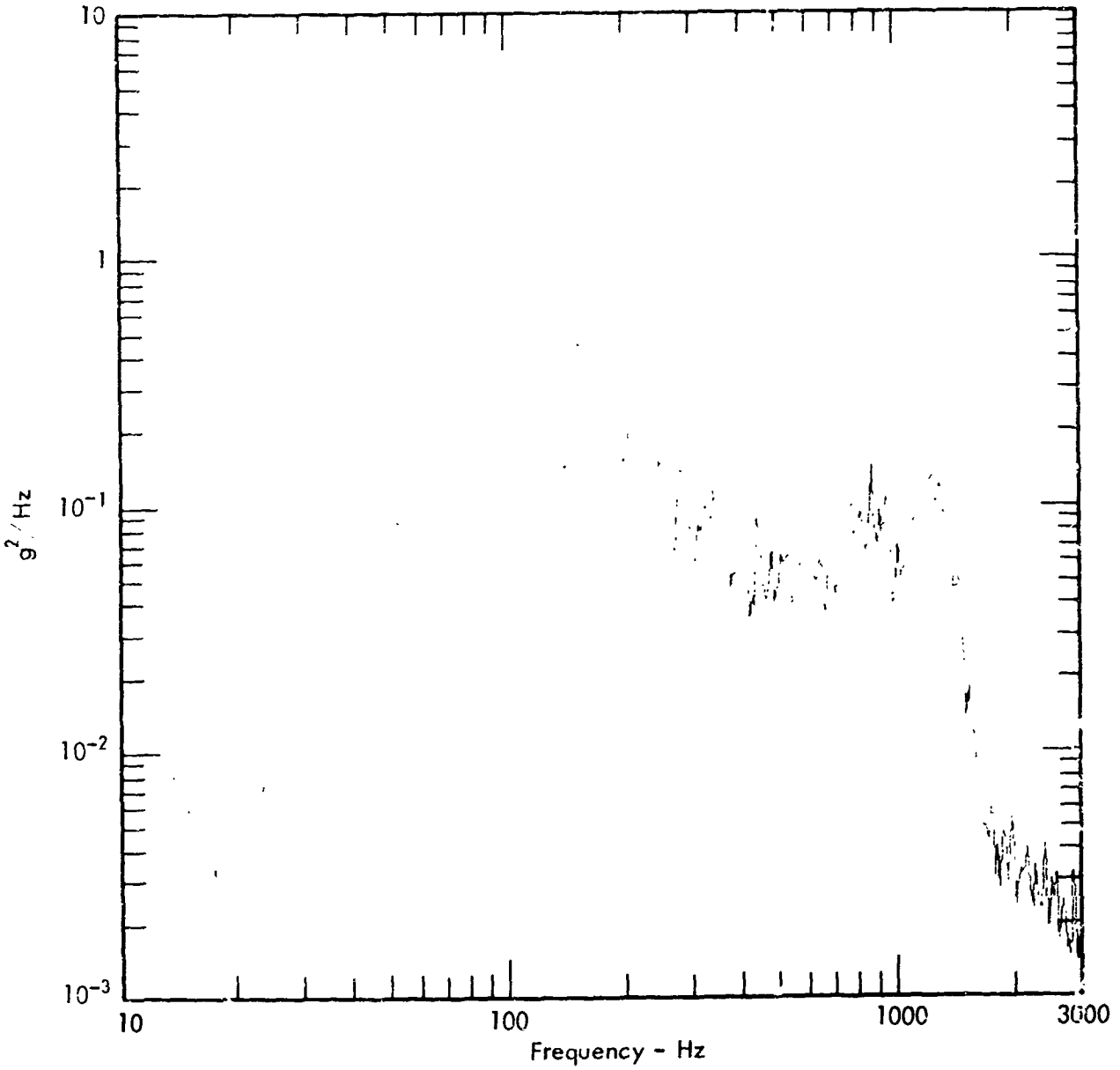

Figure 11. Power Spectral Density Obtained from Filtered and Decimated Data (Figure 9 Data)

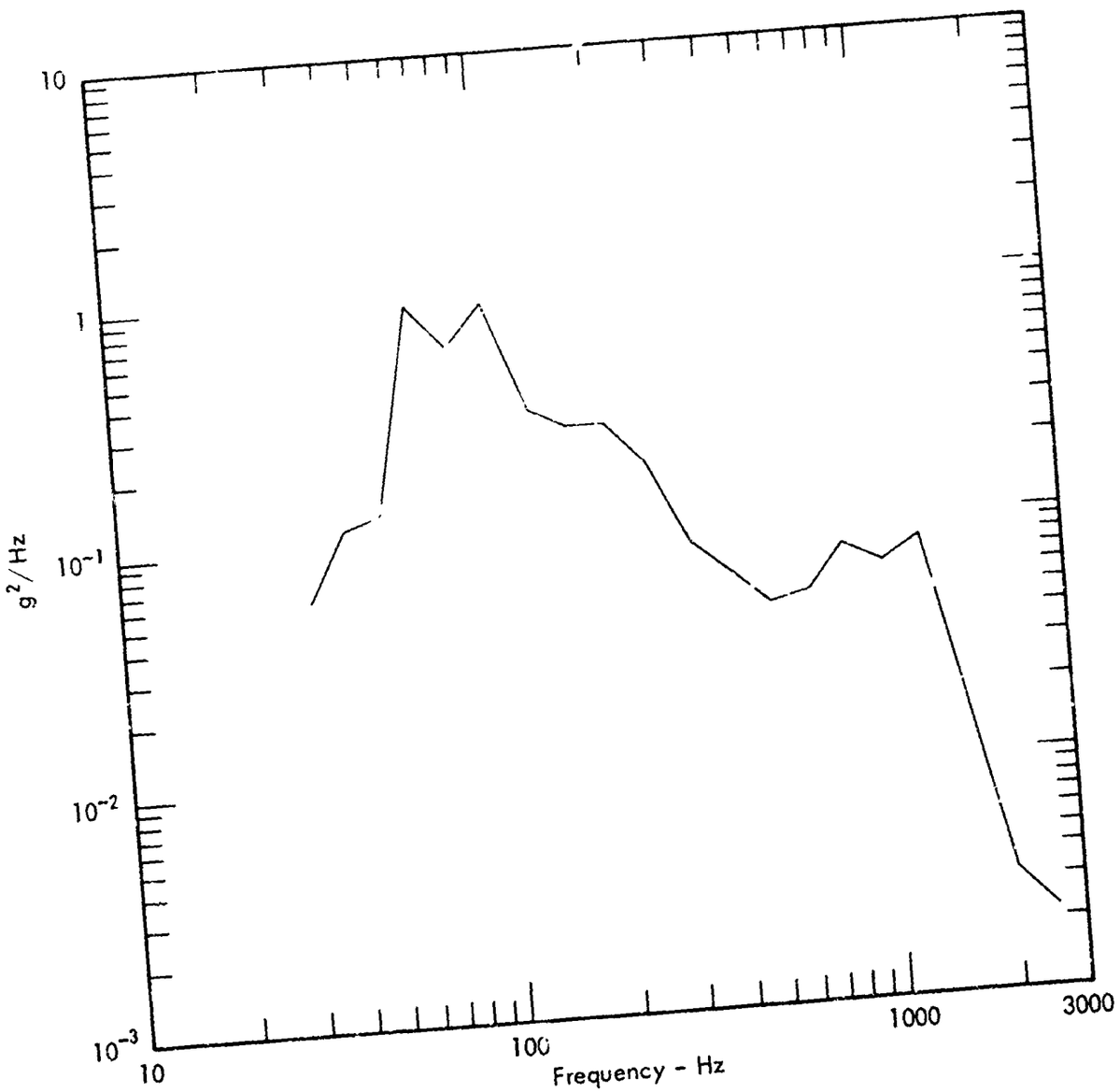

Figure 12. One-Third Octave Power Spectral Density (Figure 9 Data)