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Estimation of Finite Population Ratio When Other Auxiliary Variables are Available in the Study

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Abstract

The estimation of the population total t_y, by using one or more auxiliary variables, and the population ratio $\theta_{xy} = t_y/t_x$, t_x is the population total for the variable X, for a finite population are heavily discussed in the literature. In this paper, the idea of estimating the finite population ratio θ_{xy} is extended to use the availability of auxiliary variable Z in the study. The availability of such variable can be used to increase the precision of estimating the population ratio θ_{xy}. Our idea is supported by the fact that the variable Z may be more correlated with the variable Y than the correlation between the variables X and Y. To our knowledge, this idea is not discussed in the literature, and may be extended to use the availability of p auxiliary variables.

The bias, variance and the mean squares error are given for our approach. Simulation from real data set, the empirical relative bias and the empirical relative mean squares error are computed for our approach and for different estimators proposed in the literature for estimating the population ratio θ_{xy}. Analytically and the simulation results show that, by suitable choices, our approach gives negligible bias and has less mean squares error.

Further, under simple random sampling without replacement, the population variances of the estimators that are used in this paper are computed. Based on the random samples, that are used for estimating the population ratio θ_{xy}, the sample variances for the different estimators that are used in our approach are compared with the population variances for each estimators i.e. the empirical mean, the empirical relative bias, and the empirical relative mean squares error for the sample variances are reported. As a result of this simulation study, our approach is more efficient than other estimators proposed in the literature.

Keywords: population ratio, auxiliary variables, bias, mean squared error, general sampling design, mean, variance.

1. Introduction

Consider a finite population U of N units indexed by the set $\{1, 2, \cdots, N\}$. For the ith unit, let y_i, and x_i be the values of the variables Y and X, respectively. One of the main interest in survey sampling is to estimate the population ratio $\theta_{yx} = t_y/t_x$, where $t_y = \sum_{i \in U} y_i$ be the population total for the variable Y, and $t_x = \sum_{i \in U} x_i$ be the population total for the variable X. In the literature, there are different ideas for estimating the population ratio θ_{yx}. To our knowledge, none of them used the availability of another auxiliary variable Z in the study.

The availability of such auxiliary variable can be used to improve the precision of estimating θ_{yx}. Our idea is to use the auxiliary variable Z to improve the precision of the estimator of θ_{yx}.

Under simple random sampling without replacement (srs) design, Hartley and Ross (1954) proposed an exactly unbiased estimator for θ_{yx}. The proposed estimator is given by

$$\hat{\theta}_{HR} = \bar{r}_s + \frac{n(N-1)}{N(n-1)\bar{x}_u}(\bar{y}_s - \bar{r}_s\bar{x}_s), \tag{1}$$

where, $\bar{y}_s = \sum_{i\in s} y_i/n$, $\bar{r}_s = \sum_{i\in s} r_i/n$, $r_i = y_i/x_i$, $\bar{x}_s = \sum_{i\in s} x_i/n$, and $\bar{x}_u = t_x/N$. This estimator can be rewritten under general sampling design $p(\cdot)$. In this case, this estimator is no longer unbiased but still with negligible bias (Al-Jararha 2012).

Under general sampling design, Al-Jararha and Al-Haj Ebrahem (2012) proposed an estimator for estimating the population ratio θ_{yx}. This estimator, has negligible relative bias especially for small sample sizes n and approaches zero with increasing n. Under srs, and based on simulation results, the performance of this estimator is better than Hartley and Ross (1954) estimator. Their estimator is defined by

$$\hat{\theta}_{JM} = \bar{r}_s + \frac{1}{\bar{x}_u}(\bar{y}_s - \bar{r}_s\bar{x}_s). \tag{2}$$

Under General sampling design, Al-Jararha (2012) obtained an exactly unbiased estimator for the population ratio θ. This estimator, under srs design, gives the Hartley and Ross (1954) estimator. Further, the variance and an unbiased estimator of the variance of such estimator were obtained. This estimator, works well in stratified sampling designs.

Define π_i, the first order inclusion probability, by

$$\pi_i = Pr\left(i^{th} \text{ element} \in s\right) = \sum_{s\ni i} p(s).$$

For $i \neq j$, the second order inclusion probability is defined by

$$\pi_{ij} = Pr\left(i^{th} \text{ and } j^{th} \text{ elements} \in s\right) = \sum_{s\ni i,\, j} p(s).$$

The Horvitz and Thompson (1952) estimator of the population total $t_y = \sum_{i\in U} y_i$ is defined by

$$\hat{t}_{y\pi} = \sum_{i\in U} y_i \frac{I_{\{i\in s\}}}{\pi_i},$$

where $I_{\{i\in s\}}$ is one if $i \in s$ and zero otherwise. Further,

$$\bar{y}_s = \frac{1}{N}\hat{t}_{y\pi},$$

can be used to estimate the population mean $\bar{y}_u = \frac{1}{N}t_y$. It can be noted that $\hat{t}_{y\pi}$ and $\bar{y}_s$ are unbiased estimators for t_y, and $\bar{y}_u$ respectively. However, $\hat{t}_{y\pi}$ and $\bar{y}_s$ do not use the availability of auxiliary variables in the study. In similar way,

$$\bar{x}_s = \frac{1}{N}\hat{t}_{x\pi}, \quad \text{and} \quad \bar{r}_s = \frac{1}{N}\hat{t}_{r\pi}$$

are unbiased estimators for $\bar{x}_u$ and $\bar{r}_U$ respectively.

The availability of more than one auxiliary variable is used in literature for estimating the finite population total t_y, or finite population mean $\overline{y}_u$.

Under srs, Olkin (1958) was the first one who deals with the problem of estimating the population mean using more than one auxiliary variables. His estimator is given by

$$\hat{\bar{y}}_u = \sum_{i=1}^{p} w_i \bar{x}_{iu} \hat{\theta}_{yx_i},$$

where p is the number of the auxiliary variables, $\hat{\theta}_{yx_i} = \bar{y}_s/\bar{x}_{is}$, w_i is the weight of the ith auxiliary variable such that $\sum_{i=1}^{p} w_i = 1$, $\bar{y}_s$ is the sample mean of Y and $\bar{x}_{iu}$, $\bar{x}_{is}$ are the population mean and the sample mean of X_i, respectively, for $i = 1, \ldots, p$.

Singh and Chaudhary (1986) proposed the following estimator

$$\hat{\bar{y}}_u = \bar{y}_s \left(w_1 \frac{\bar{x}_{1u}}{\bar{x}_{1s}} + w_2 \frac{\bar{x}_{2u}}{\bar{x}_{2s}} \right)$$

for estimating the population mean $\bar{y}_u$, where $w_1 + w_2 = 1$.

Abu-Dayyeh, Ahmad, Ahmad, and Hassen (2003) studied the general form of Singh and Chaudhary (1986) estimator. They proposed two classes of estimators using two auxiliary variables to estimate the population mean for the variable of interest Y.

Kadilar and Cingi (2004) suggested a new multivariate ratio estimator using the regression estimator instead of $\bar{y}_s$ which used in Singh and Chaudhary (1986) estimator. Their estimator is given by

$$\bar{y}_{pr} = \sum_{i=1}^{2} w_i \frac{\bar{y}_s + b_i\left(\bar{x}_{iu} - \bar{x}_{is}\right)}{\bar{x}_{is}} \bar{x}_{iu},$$

where b_i, $i = 1, 2$ are the regression coefficients. Based on the mean squares error (MSE), they found that their estimator is more efficient than Singh and Chaudhary (1986) estimator when

$$MSE\left(\bar{y}_{pr}\right) < MSE\left(\bar{y}_u\right),$$

where $MSE\left(\bar{y}_{pr}\right)$, and $MSE\left(\bar{y}_u\right)$ are defined by Equations (2.4), and (1.2) of Kadilar and Cingi (2004), respectively.

Other authors are using different ideas for estimating the population mean $\bar{y}_u$. On the other side, there are different ideas for estimating θ_{yx}, to our knowledge, none of them discussed the idea of using the availability of other auxiliary variable Z for estimating the population ratio θ_{yx}. In this article, under general sampling design, a family of estimators is adopted for estimating the population ratio θ_{yx}. For such family, the bias, variance, MSE are given. Based on simulation from real data set, we will compare between given estimators for θ_{yx}, proposed in the literature and our approach.

2. Proposed Family

The existence of one or more auxiliary variables can be used to improve the estimate of θ_{yx}. In our approach, for the ith unit, let y_i, x_i and z_i be the values of the variable of interest Y, and the auxiliary variables X, and Z respectively. Our goal is to estimate the population ratio $\theta_{yx} = t_y/t_x$ when the auxiliary variable Z is available in the study.

Our approach is summarized by rewriting the definition of θ_{yx} as

$$\theta_{yx} = \lambda\theta_{yx} + (1-\lambda)\,\theta_{zx}\theta_{yz}, \tag{3}$$

for given λ and $\theta_{zx} = t_z/t_x$. Usually, t_x and t_z are assumed to be known; therefore, we assume θ_{zx} to be known. Based on this, estimate θ_{yx} by

$$\tilde{\theta}_{yx} = \lambda\hat{\theta}_{yx} + (1-\lambda)\,\theta_{zx}\breve{\theta}_{yz}. \tag{4}$$

Remark 2.1. *The estimators $\hat{\theta}_{yx}$, and $\breve{\theta}_{yz}$ can be computed from proposed estimators for the population ratio in the literature. Both, $\hat{\theta}_{yx}$, $\breve{\theta}_{yz}$ can be computed from the same estimator of the population ratio, or from different estimators.*

From Equation(4), take the expectation of $\tilde{\theta}_{yx}$, we have

$$E\left(\tilde{\theta}_{yx}\right) = \lambda E\left(\hat{\theta}_{yx}\right) + (1-\lambda)\,\theta_{zx} E\left(\breve{\theta}_{yz}\right). \tag{5}$$

Therefore,

$$bias\left(\tilde{\theta}_{yx}\right) = \lambda bias\left(\hat{\theta}_{yx}\right) + (1-\lambda)\,\theta_{zx} bias\left(\breve{\theta}_{yz}\right). \tag{6}$$

Remark 2.2. *From Equation(6), $\tilde{\theta}_{yx}$ is unbiased or asymptotically unbiased is achieved by choosing $\hat{\theta}_{yx}$, and $\breve{\theta}_{yz}$ to be unbiased or asymptotically unbiased.*

From Equation(4), the variance of $\tilde{\theta}_{yx}$ is

$$var\left(\tilde{\theta}_{yx}\right) = \lambda^2 var\left(\hat{\theta}_{yx}\right) + (1-\lambda)^2\,\theta_{zx}^2 var\left(\breve{\theta}_{yz}\right) + 2\lambda\,(1-\lambda)\,\theta_{zx} cov\left(\hat{\theta}_{yx}, \breve{\theta}_{yz}\right). \tag{7}$$

From Equations (6), and (7), the MSE of $\tilde{\theta}_{yx}$ is

$$MSE\left(\tilde{\theta}_{yx}\right) = var\left(\tilde{\theta}_{yx}\right) + bias^2\left(\tilde{\theta}_{yx}\right). \tag{8}$$

Assume that $\tilde{\theta}_{yx}$ to be unbiased or asymptotically unbiased, by choosing $\hat{\theta}_{yx}$, and $\breve{\theta}_{yz}$ to be unbiased or asymptotically unbiased. In this case, $MSE\left(\tilde{\theta}_{yx}\right) = var\left(\tilde{\theta}_{yx}\right)$. The optimal value of λ, can be obtained by differentiating the right hand side of Equation(8) with respect to λ, equate to zero, and solve for λ we have

$$\lambda_{opt} = \frac{1}{1+\lambda^*}, \tag{9}$$

where

$$\lambda^* = \frac{var\left(\hat{\theta}_{yx}\right) - \theta_{zx} cov\left(\hat{\theta}_{yx}, \breve{\theta}_{yz}\right)}{\theta_{zx}^2 var\left(\breve{\theta}_{yz}\right) - \theta_{zx} cov\left(\hat{\theta}_{yx}, \breve{\theta}_{yz}\right)}. \tag{10}$$

From Equations (4) and (9) the optimal estimator for θ_{yx} is

$$\tilde{\theta}_{yx} = \lambda_{opt}\hat{\theta}_{yx} + (1-\lambda_{opt})\,\theta_{zx}\breve{\theta}_{yz}. \tag{11}$$

Remark 2.3. *In general, the transformation given by Equation (11) is not a convex transformation. However, the transformation is a convex transformation when $0 \le \lambda_{opt} \le 1$, this condition holds if $\lambda^* \ge 0$. In this case, the numerator and the denominator of λ^* should be positive; equivalently, from Equation (10), if*

$$\rho\left(\hat{\theta}_{yx}, \breve{\theta}_{yz}\right) \le min\left\{\frac{1}{\theta_{zx}} \cdot \sqrt{\frac{var\left(\hat{\theta}_{yx}\right)}{var\left(\breve{\theta}_{yz}\right)}}, \theta_{zx} \cdot \sqrt{\frac{var\left(\breve{\theta}_{yz}\right)}{var\left(\hat{\theta}_{yx}\right)}}\right\} \quad for \quad \theta_{zx} > 0,$$

where $\rho\left(\hat{\theta}_{yx}, \breve{\theta}_{yz}\right)$ is the correlation between $\hat{\theta}_{yx}$ and $\breve{\theta}_{yz}$.

In real applications, λ_{opt} is unknown; however, λ_{opt} can be estimated from random sample. Under general sampling design $p(\cdot)$, draw the random sample S, estimate λ_{opt} by

$$\hat{\lambda}_{opt} = \frac{1}{1+\hat{\lambda}^*}, \tag{12}$$

where

$$\hat{\lambda}^* = \frac{\widehat{var}\left(\hat{\theta}_{yx}\right) - \theta_{zx}\widehat{cov}\left(\hat{\theta}_{yx}, \breve{\theta}_{yz}\right)}{\theta_{zx}^2\widehat{var}\left(\breve{\theta}_{yz}\right) - \theta_{zx}\widehat{cov}\left(\hat{\theta}_{yx}, \breve{\theta}_{yz}\right)}. \tag{13}$$

From Equation (11), $\tilde{\theta}_{yx}$ is computed from

$$\tilde{\theta}_{yx} = \hat{\lambda}_{opt}\hat{\theta}_{yx} + \left(1 - \hat{\lambda}_{opt}\right)\theta_{zx}\breve{\theta}_{yz}. \tag{14}$$

In the next section, we describe how we can apply our approach. In most applicable cases, t_x and t_z are known from previous studies or from a pilot study. However, the worst scenario happens when $\theta_{zx} = t_z/t_x$ is unknown. In this case, estimate θ_{yx} by

$$\tilde{\theta}_{yx} = \lambda\hat{\theta}_{yx} + (1-\lambda)\acute{\theta}_{zx}\breve{\theta}_{yz}, \tag{15}$$

where $\acute{\theta}_{zx}$ is an estimate for θ_{zx}. Our goal is to find the bias, variance, and the MSE of $\tilde{\theta}_{yx}$. As it is clear from Equation (15), $\tilde{\theta}_{yx}$ is not a linear function in $\acute{\theta}_{zx}$, and $\breve{\theta}_{yz}$, and to avoid the 3rd and 4th order inclusion probabilities, to first order and by using Taylor expansion, expand the right hand side of Equation(15), we have

$$\tilde{\theta}_{yx} \cong \lambda\hat{\theta}_{yx} + (1-\lambda)\left\{\theta_{yx} + \theta_{zx}\left(\breve{\theta}_{yz} - \theta_{yz}\right) + \theta_{yz}\left(\acute{\theta}_{zx} - \theta_{zx}\right)\right\}. \tag{16}$$

Remark 2.4. *The first order linearization is widely used in survey practice, but that in general it is very difficult to evaluate the quality of approximation analytically. Therefore, simulations are presented that show reasonable results at least in the particular case described.*

From Equation(16), the bias of $\tilde{\theta}_{yx}$ is

$$\text{bias}\left(\tilde{\theta}_{yx}\right) \cong \lambda\text{bias}\left(\hat{\theta}_{yx}\right) + (1-\lambda)\left\{\theta_{zx}\text{bias}\left(\breve{\theta}_{yz}\right) + \theta_{yz}\text{bias}\left(\acute{\theta}_{zx}\right)\right\}. \tag{17}$$

The variance of $\tilde{\theta}_{yx}$ is

$$\begin{aligned}\text{var}\left(\tilde{\theta}_{yx}\right) &\cong \lambda^2\text{var}\left(\hat{\theta}_{yx}\right) + (1-\lambda)^2\left\{\theta_{zx}^2\text{var}\left(\breve{\theta}_{yz}\right) + \theta_{yz}^2\text{var}\left(\acute{\theta}_{zx}\right)\right.\\ &+ \left.2\theta_{yx}\text{cov}\left(\breve{\theta}_{yz}, \acute{\theta}_{zx}\right)\right\} + 2\lambda(1-\lambda)\left\{\theta_{zx}\text{cov}\left(\hat{\theta}_{yx}, \breve{\theta}_{yz}\right)\right.\\ &+ \left.\theta_{yz}\text{cov}\left(\hat{\theta}_{yx}, \acute{\theta}_{zx}\right)\right\},\end{aligned} \tag{18}$$

From Equations (17) and (18), the MSE of $\tilde{\theta}_{yx}$ is

$$\text{MSE}\left(\tilde{\theta}_{yx}\right) = \text{var}\left(\tilde{\theta}_{yx}\right) + \text{bias}\left(\tilde{\theta}_{yx}\right)^2. \tag{19}$$

Remark 2.5. *From the right hand side of Equation(17), it is clear that the need of using unbiased or asymptotically unbiased estimators for estimating θ_{yx}, θ_{zx}, and θ_{yz}. In this case, bias$\left(\tilde{\theta}_{yx}\right)$ is zero or asymptotically zero i.e. $\tilde{\theta}_{yx}$ is unbiased or asymptotically unbiased estimator for θ_{yx}. As a result of this,*

$$MSE\left(\tilde{\theta}_{yx}\right) \cong var\left(\tilde{\theta}_{yx}\right). \tag{20}$$

Under the assumption $\hat{\theta}_{yx}$, $\acute{\theta}_{zx}$, and $\breve{\theta}_{yz}$ are unbiased (or asymptotically unbiased) estimator for θ_{yx}, θ_{zx}, and θ_{yz}, respectively. The optimum value of λ which is minimizing the right hand side of Equation(19) is

$$\lambda_{opt} = \frac{\text{var}\left(\theta_{zx}\breve{\theta}_{yz} + \theta_{yz}\acute{\theta}_{zx}\right) - \theta_{zx}\text{cov}\left(\hat{\theta}_{yx}, \breve{\theta}_{yz}\right) - \theta_{yz}\text{cov}\left(\hat{\theta}_{yx}, \acute{\theta}_{zx}\right)}{\text{var}\left(\theta_{zx}\breve{\theta}_{yz} + \theta_{yz}\acute{\theta}_{zx} - \hat{\theta}_{yx}\right)} \tag{21}$$

In real applications, λ_{opt} needs to be estimated from random sample. In this case, the estimate value of λ_{opt} is

$$\hat{\lambda}_{opt} = \frac{4\widehat{\text{var}}\left(\breve{\theta}_{yz}\acute{\theta}_{zx}\right) - \acute{\theta}_{zx}\widehat{\text{cov}}\left(\hat{\theta}_{yx}, \breve{\theta}_{yz}\right) - \breve{\theta}_{yz}\widehat{\text{cov}}\left(\hat{\theta}_{yx}, \acute{\theta}_{zx}\right)}{\widehat{\text{var}}\left(2\acute{\theta}_{zx}\breve{\theta}_{yz} - \hat{\theta}_{yx}\right)} \tag{22}$$

Remark 2.6. *Insert $\hat{\lambda}_{opt}$ into Equation(15), we have the optimal choice of estimating θ_{yx} i.e. estimate θ_{yx} by*

$$\tilde{\theta}_{yx} = \hat{\lambda}_{opt}\hat{\theta}_{yx} + \left(1 - \hat{\lambda}_{opt}\right)\acute{\theta}_{zx}\breve{\theta}_{yz}. \tag{23}$$

In real application, the first case, $\theta_{zx} = t_z/t_x$ is known, is more applicable than the second case, $\theta_{zx} = t_z/t_x$ is unknown. Therefore, in the next section, we will describe how we can apply the first approach. However, the second approach can be used in similar way as the first one.

3. Applying Our Approach

In this section, we will apply the first case, $\theta_{zx} = t_z/t_x$ is known. However, the second approach, $\theta_{zx} = t_z/t_x$ is unknown, can be used in similar way as the first one. Based on Remark(2.2), we restrict ourselves to the estimation of θ_{yx}, and θ_{yz}, by unbiased or asymptotically unbiased estimators from the literature. In this paper, we will use the classical ratio estimator, and the estimators given by Equations (1) and (2).

3.1. Classical Ratio Estimator

In this subsection, we will compute $\hat{\theta}_{yx}$ and $\breve{\theta}_{yz}$ from the usual classical ratio estimator, i.e. $\hat{\theta}_{yx}$, and $\breve{\theta}_{yz}$ are computed from

$$\hat{\theta}_{yx} = \frac{\hat{t}_{y\pi}}{\hat{t}_{x\pi}} \tag{24}$$

and

$$\breve{\theta}_{yz} = \frac{\hat{t}_{y\pi}}{\hat{t}_{z\pi}}, \tag{25}$$

respectively. In this case,

$$\widehat{var}\left(\hat{\theta}_{yx}\right) = \sum_{ij\in S} \frac{\hat{w}_i}{\pi_i}\frac{\hat{w}_j}{\pi_j}\frac{\Delta_{ij}}{\pi_{ij}}, \tag{26}$$

$$\widehat{var}\left(\breve{\theta}_{yz}\right) = \sum_{ij\in S} \frac{\breve{w}_i}{\pi_i}\frac{\breve{w}_j}{\pi_j}\frac{\Delta_{ij}}{\pi_{ij}}, \tag{27}$$

$$\widehat{cov}\left(\hat{\theta}_{yx}, \breve{\theta}_{yz}\right) = \sum_{ij\in S} \frac{\hat{w}_i}{\pi_i}\frac{\breve{w}_j}{\pi_j}\frac{\Delta_{ij}}{\pi_{ij}}, \tag{28}$$

respectively. Where

$$\hat{w}_i = \left(y_i - \hat{\theta}_{yx}x_i\right)/N\bar{x}_u, \tag{29}$$

and

$$\breve{w}_i = \left(y_i - \breve{\theta}_{yz}z_i\right)/N\bar{z}_u. \tag{30}$$

For more details, see Al-Jararha and Al-Haj Ebrahem (2012).

In order to use Equation (14), insert the estimators in Equations (26), (27), and (28) into Equation (13) to compute $\hat{\lambda}^*$, use the result in Equation (12). Now Equation (14) is ready to be used.

3.2. Hartley and Ross Estimator

Under srs sampling design, Hartley and Ross (1954) proposed an exactly an unbiased estimator for estimating the population ratio. This estimator can be rewritten under general sampling design (Al-Jararha 2012). In this case, $\hat{\theta}_{yx}$ and $\breve{\theta}_{yz}$ are computed from

$$\hat{\theta}_{yx} = \bar{r}_{yxs} + \frac{n(N-1)}{N(n-1)\bar{x}_u}\left(\bar{y}_s - \bar{r}_{yxs}\bar{x}_s\right) \tag{31}$$

and

$$\breve{\theta}_{yz} = \bar{r}_{yzs} + \frac{n(N-1)}{N(n-1)\bar{z}_u}\left(\bar{y}_s - \bar{r}_{yzs}\bar{z}_s\right), \tag{32}$$

respectively. To compute $\widehat{var}\left(\hat{\theta}_{yx}\right)$, $\widehat{var}\left(\breve{\theta}_{yz}\right)$, and $\widehat{cov}\left(\hat{\theta}_{yx}, \breve{\theta}_{yz}\right)$ reuse Equations (26), (27), and (28) but with the following definitions

$$\hat{w}_i = \frac{n(N-1)}{N^2(n-1)\bar{x}_u}\left(y_i - \bar{r}_{yxs}x_i\right) - \frac{N-n}{N^2(n-1)}r_{i_{yx}}, \tag{33}$$

and

$$\breve{w}_i = \frac{n(N-1)}{N^2(n-1)\bar{z}_u}\left(y_i - \bar{r}_{yzs}z_i\right) - \frac{N-n}{N^2(n-1)}r_{i_{yz}}. \tag{34}$$

For more details, see Al-Jararha and Al-Haj Ebrahem (2012).

3.3. Al-Jararha and Al-Haj Ebrahem Estimator

Under general sampling design, Al-Jararha and Al-Haj Ebrahem (2012) proposed an asymptotic unbiased estimator for estimating the population ratio. This estimator is working better than Hartley and Ross (1954). In this case, $\hat{\theta}_{yx}$ and $\breve{\theta}_{yz}$ are computed from

$$\hat{\theta}_{yx} = \bar{r}_{yxs} + \frac{1}{\bar{x}_u}\left(\bar{y}_s - \bar{r}_{yxs}\bar{x}_s\right) \tag{35}$$

and

$$\breve{\theta}_{yz} = \bar{r}_{yzs} + \frac{1}{\bar{z}_u}\left(\bar{y}_s - \bar{r}_{yzs}\bar{z}_s\right), \tag{36}$$

respectively. To compute $\widehat{var}\left(\hat{\theta}_{yx}\right)$, $\widehat{var}\left(\breve{\theta}_{yz}\right)$, and $\widehat{cov}\left(\hat{\theta}_{yx}, \breve{\theta}_{yz}\right)$ reuse Equations (26), (27), and (28) but with the following definitions

$$\hat{w}_i = (y_i - \bar{r}_{yxs} x_i) / N\bar{x}_u, \tag{37}$$

and

$$\breve{w}_i = (y_i - \bar{r}_{yzs} z_i) / N\bar{z}_u. \tag{38}$$

For more details, see Al-Jararha and Al-Haj Ebrahem (2012).

Remark 3.1. *In order to compute the* $\widehat{cov}\left(\hat{\theta}_{yx}, \breve{\theta}_{yz}\right)$ *when* $\hat{\theta}_{yx}$ *and* $\breve{\theta}_{yz}$ *are to be computed from different estimators, for example,* $\hat{\theta}_{yx}$ *is computed from Equation (24), and* $\breve{\theta}_{yz}$ *is computed from Equation (32); in this case, use Equation (28) with the definition of* $\hat{w}_i$ *as given in Equation (29), and* $\breve{w}_i$ *as given in Equation (34).*

4. Simulation Studies and Conclusions

4.1. Estimation the Population Ratio θ_{yx}

Consider the real data set FEV: Forced Expiratory Volume. FEV is an index of pulmonary function that measures the volume of air expelled after one second of constant effort. This data is downloaded from http://www.amstat.org/publications/jse/datasets/fev.dat.txt. The FEV data set was taken from a study conducted in East Boston, Massachusetts, 1980, on 654 children aged from 3 to 19 years who were seen in the childhood respiratory disease (CRD). The variable of interest is Y: Forced expiratory volume, and the auxiliary variables are X: Children age, from 3-19 years age, and Z: Children height in inches. For this data set, $t_y = 1724$, $t_x = 6495$, and $t_z = 39988$. In this section, we will assume that $t_x = 6495$, and $t_z = 39988$ are known.

In this section, our main goal is to estimate the population ratio $\theta = t_y/t_x = 0.2655$ by using our approach i.e. by using Equation (14) and the three estimators given by Equations (24), (31), and (35) under different sampling designs i.e. under srs, probability proportional to size and without replacement πps; in this case, the size variable will be the age, and stratified sampling design; in this case, the FEV data set will be divided into $H = 2$ non-overlapping strata according to the variable sex.

The empirical mean (EM) of the estimator $\tilde{\theta}$ of θ is defined by

$$\text{EM}\left(\tilde{\theta}\right) = \frac{1}{m}\sum_{i=1}^{m} \tilde{\theta}_i, \tag{39}$$

where $\tilde{\theta}_i$ is the estimate of θ based on the i^{th} simulated random sample, and m is the number of simulated random samples under different random sampling designs. The empirical relative bias (ERB) of $\tilde{\theta}$ is defined by

$$\text{ERB}\left(\tilde{\theta}\right) = \frac{\frac{1}{m}\sum_{i=1}^{m} \tilde{\theta}_i - \theta}{\theta} \times 100\%. \tag{40}$$

The empirical mean squares error (EMSE) of $\tilde{\theta}$ is defined by

$$\text{EMSE}\left(\tilde{\theta}\right) = \frac{1}{m}\sum_{i=1}^{m} \left(\tilde{\theta}_i - \theta\right)^2, \tag{41}$$

and the empirical relative mean squares error (RE) of the estimator $\tilde{\theta}$ is defined by

$$\text{RE}\left(\tilde{\theta}\right) = \frac{\frac{1}{m}\sum_{i=1}^{m}\left(\tilde{\theta}_i - \theta\right)^2}{\frac{1}{m}\sum_{i=1}^{m}\left(\hat{\theta}_i - \theta\right)^2} = \frac{\text{EMSE}\left(\tilde{\theta}\right)}{\text{EMSE}\left(\hat{\theta}\right)}, \tag{42}$$

where $\hat{\theta}$ is another estimator for θ.

From Equation (14), recall our approach,

$$\tilde{\theta}_{yx} = \hat{\lambda}_{opt}\hat{\theta}_{yx} + \left(1 - \hat{\lambda}_{opt}\right)\theta_{zx}\breve{\theta}_{yz}, \tag{43}$$

to make the notations clear, consider the following

	$\hat{\theta}_{yx}$ is computed from	$\breve{\theta}_{yz}$ is computed from Eq(25)	$\breve{\theta}_{yz}$ is computed from Eq(32)	$\breve{\theta}_{yz}$ is computed from Eq(36)
group I	Eq(24)	$\tilde{\theta}_{yx.RR}$	$\tilde{\theta}_{yx.RH}$	$\tilde{\theta}_{yx.RJ}$
group II	Eq(31)	$\tilde{\theta}_{yx.HR}$	$\tilde{\theta}_{yx.HH}$	$\tilde{\theta}_{yx.HJ}$
group III	Eq(35)	$\tilde{\theta}_{yx.JR}$	$\tilde{\theta}_{yx.JH}$	$\tilde{\theta}_{yx.JJ}$

Further, for group I, compute $\hat{\theta}_{RR}$ from Equation(24), for group II, compute $\hat{\theta}_{HH}$ from Equation(31), and for group III, compute $\hat{\theta}_{JJ}$ from Equation(35). We can see that the computation of $\hat{\theta}_{RR}$, $\hat{\theta}_{HH}$, and $\hat{\theta}_{JJ}$ depend on the variable of interest Y and the auxiliary variable X only. In order to use Equation (42), and for the ith group, compute EMSE $\left(\tilde{\theta}\right)$ for the estimators in this group and the EMSE $\left(\hat{\theta}\right)$ for its corresponding group.

From the described population, simulate $m = 3,000$ samples under different sampling designs i.e. srs, πps, and stratified sampling design, when the sample size $n = 20$, 30, 40, 50 and 60. Sampling from the population will be achieved by using procedure `surveyselect` of SAS Institute, and the computations are computed by using a macro written in SAS. For a given sample of size n, and based on each sample, compute the estimators $\tilde{\theta}_{yx}$, and $\hat{\theta}_{ww}$, $w = R, H, J$, as they described above.

4.2. Variance Estimation of the $\tilde{\theta}_{yx}$

In this section, under srs, our main goal is to compute the population variances for the 12 estimators described in the Subsection (4.1). Further, we will compute the empirical sample mean, relative bias, and the MSE for the sample variances computed from the random samples simulated in the Subsection (4.1).

Recall that $\hat{t}_{y\pi} = \sum_{i\in U} y_i \frac{I_{\{i\in s\}}}{\pi_i}$, the Horvitz and Thompson (1952) estimator of the population total $t_y = \sum_{i\in U} y_i$. Under srs (Särndal, Swensson, and Wretman 1992),

$$var_{srs}\left(\hat{t}_{y\pi}\right) = N^2\frac{1-f}{n}S^2_{yu}, \tag{44}$$

and

$$\widehat{var}_{srs}\left(\hat{t}_{y\pi}\right) = N^2\frac{1-f}{n}s^2_{ys}, \tag{45}$$

where

$$S^2_{yu} = \frac{1}{N-1}\sum_{i=1}^{N}\left(y_i - \bar{y}_u\right)^2,$$

$$s^2_{ys} = \frac{1}{n-1}\sum_{i=1}^{n}\left(y_i - \bar{y}_s\right)^2,$$

and $f = n/N$. Similarly, the covariance between $\hat{t}_{y\pi}$ and $\hat{t}_{z\pi}$ is computed from

$$cov_{srs}\left(\hat{t}_{y\pi}, \hat{t}_{z\pi}\right) = N^2 \frac{1-f}{n} S_{yzu}, \tag{46}$$

which is estimated by

$$\widehat{cov}_{srs}\left(\hat{t}_{y\pi}, \hat{t}_{z\pi}\right) = N^2 \frac{1-f}{n} s_{yzs}, \tag{47}$$

where

$$S_{yzu} = \frac{1}{N-1} \sum_{i=1}^{N} \left(y_i - \bar{y}_u\right)\left(z_i - \bar{z}_u\right),$$

and

$$s_{yzs} = \frac{1}{n-1} \sum_{i=1}^{n} \left(y_i - \bar{y}_s\right)\left(z_i - \bar{z}_s\right)$$

Remark 4.1. *Since the 12 estimators discussed in the Subsection (4.1) are linearized to first order Taylor expansion (Al-Jararha and Al-Haj Ebrahem 2012), Equations (44)-(47) are ready to be used for such estimators. The computations in this part are similar to the computations as in Subsection (4.1), but for variances.*

The empirical mean (MV) of the $\widehat{var}_{srs}\left(\tilde{\theta}\right)$ of $var_{srs}\left(\tilde{\theta}\right)$ is

$$\text{MV}\left(\tilde{\theta}\right) = \frac{1}{m} \sum_{i=1}^{m} \widehat{var}_{srs}\left(\tilde{\theta}\right)_i, \tag{48}$$

where $\widehat{var}_{srs}\left(\tilde{\theta}\right)_i$ is computed from the ith simulated random sample. The empirical relative bias (RBV) of $\widehat{var}_{srs}\left(\tilde{\theta}\right)$ is

$$\text{RBV}\left(\tilde{\theta}\right) = \frac{\frac{1}{m}\sum_{i=1}^{m} \widehat{var}_{srs}\left(\tilde{\theta}\right)_i - var_{srs}\left(\tilde{\theta}\right)}{var_{srs}\left(\tilde{\theta}\right)} \times 100\%. \tag{49}$$

The empirical mean squares error (MSEV) of $\widehat{var}_{srs}\left(\tilde{\theta}\right)$ is

$$\text{MSEV}\left(\tilde{\theta}\right) = \frac{1}{m} \sum_{i=1}^{m} \left(\widehat{var}_{srs}\left(\tilde{\theta}\right)_i - var_{srs}\left(\tilde{\theta}\right)\right)^2, \tag{50}$$

and the empirical relative mean squares error (REV) of the estimator $\widehat{var}_{srs}\left(\tilde{\theta}\right)$ is

$$\text{REV}\left(\tilde{\theta}\right) = \frac{\frac{1}{m}\sum_{i=1}^{m}\left(\widehat{var}_{srs}\left(\tilde{\theta}\right)_i - var_{srs}\left(\tilde{\theta}\right)\right)^2}{\frac{1}{m}\sum_{i=1}^{m}\left(\widehat{var}_{srs}\left(\hat{\theta}\right)_i - var_{srs}\left(\hat{\theta}\right)\right)^2} = \frac{\text{MSEV}\left(\widehat{var}_{srs}\left(\tilde{\theta}\right)\right)}{\text{MSEV}\left(\widehat{var}_{srs}\left(\hat{\theta}\right)\right)}, \tag{51}$$

where $\widehat{var}_{srs}\left(\hat{\theta}\right)$ is another estimator for $var_{srs}\left(\hat{\theta}\right)$.

Under srs, population variances are computed for every estimator mentioned in Subsection (4.1). Further, based on every simulated sample used for estimating such estimators is also used to compute the sample variances for the 12 estimators. Results are reported in Table (5).

This Subsection is restricted to srs sampling design since there are difficulties to use other sampling designs. For example, under πps, procedure `surveyselect` gives the first and second order inclusion probabilities for the sample only. Even though, the computations under srs are not an easy task!

4.3. Results and Conclusions

The nine estimators, $\underbrace{\tilde{\theta}_{yx.RR},\, \tilde{\theta}_{yx.RH},\, \tilde{\theta}_{yx.RJ}}_{\text{group I}},\ \underbrace{\tilde{\theta}_{yx.HR},\, \tilde{\theta}_{yx.HH},\, \tilde{\theta}_{yx.HJ}}_{\text{group II}}$, and $\underbrace{\tilde{\theta}_{yx.JR},\, \tilde{\theta}_{yx.JH},\, \tilde{\theta}_{yx.JJ}}_{\text{group III}}$, are used to estimate θ_{yx} based on our approach i.e. the estimators $\tilde{\theta}_{yx.wv}$, for $w,\ v = R,\ H,\ J$, are using the availability of another auxiliary variable Z in the study . However, the three estimators, $\hat{\theta}_{ww}$, for $w = R,\ H,\ J$, are not using the availability of Z.

From Tables (1), (2), (3), and (4), we can conclude the following:

1. The nine estimators, $\tilde{\theta}_{yx.wv}$, for $w,\ v = R,\ H,\ J$, have negligible empirical relative biased regardless the sample size n, and the group. This comes from the behavior of the estimators that are used in each group described above. In general, from Equation (6), the bias of $\tilde{\theta}_{yx}$ depends on the behavior of $\hat{\theta}_{yx}$ and $\breve{\theta}_{yz}$; the estimators $\hat{\theta}_{yx}$ and $\breve{\theta}_{yz}$ must be unbiased or asymptotically unbiased for θ_{yx} and θ_{yz}, respectively.

2. The use of the estimators, $\tilde{\theta}_{yx.wv}$, for $w,\ v = R,\ H,\ J$, perform much better than the estimators $\hat{\theta}_{ww}$, for $w = R,\ H,\ J$, from the empirical relative mean squares error point of view. In other words, the availability of auxiliary variable can be used to improve the precision of the estimation the population ratio θ_{xy}.

Population variances , the empirical means , relative bias, and relative mean squares error of the sample variances for the estimators discussed in the Subsection (4.1) are reported in Table (5). From this Table, we can see that all the discussed estimators have negligible relative biased. Further, in the meaning of the relative efficiency , the estimators based on our approach, $\tilde{\theta}_{yx.wv}$, for $w,\ v = R,\ H,\ J$, are more efficient than the proposed estimators $\hat{\theta}_{ww}$, for $w = R,\ H,\ J$. These results are true regardless the sample size n.

The absolute differences between the EV from Table(1), and the MV from Table(5) are summarized in Table (6). From Table (6), we can see that all the absolute differences are negligible regardless the sample size.

As a final remark, our approach can be adopted if we carefully choose the estimators $\hat{\theta}_{yx}$ and $\breve{\theta}_{yz}$ to be unbiased or asymptotically unbiased for θ_{yx} and θ_{yz}, respectively. In this case, our approach can be used to improve the precision of the estimation the population ratio θ_{xy}. Further, in similar steps our ideas can be extended to use more than one auxiliary variable.

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n		$\hat{\theta}_{RR}$	$\tilde{\theta}_{yx.RR}$	$\tilde{\theta}_{yx.RH}$	$\tilde{\theta}_{yx.RJ}$	$\hat{\theta}_{HH}$	$\tilde{\theta}_{yx.HR}$	$\tilde{\theta}_{yx.HH}$	$\tilde{\theta}_{yx.HJ}$	$\hat{\theta}_{JJ}$	$\tilde{\theta}_{yx.JR}$	$\tilde{\theta}_{yx.JH}$	$\tilde{\theta}_{yx.JJ}$
	EM	0.266	0.266	0.266	0.266	0.266	0.265	0.266	0.265	0.266	0.266	0.266	0.266
20	ERB	0.076	0.043	0.073	0.042	0.006	-0.003	0.027	-0.004	0.073	0.041	0.071	0.040
	EV	1.65E-4	1.42E-4	1.43E-4	1.43E-4	1.68E-4	1.43E-4	1.43E-4	1.43E-4	1.66E-4	1.42E-4	1.42E-4	1.42E-4
	RE	1.000	0.863	0.867	0.866	1.000	0.850	0.854	0.852	1.000	0.853	0.857	0.855
	EM	0.266	0.266	0.266	0.266	0.266	0.266	0.266	0.266	0.266	0.266	0.266	0.266
30	ERB	0.064	0.052	0.072	0.052	0.018	0.021	0.041	0.021	0.061	0.050	0.070	0.050
	EV	1.13E-4	9.61E-5	9.65E-5	9.64E-5	1.14E-4	9.64E-5	9.67E-5	9.66E-5	1.14E-4	9.59E-5	9.63E-5	9.62E-5
	RE	1.000	0.853	0.857	0.856	1.000	0.842	0.845	0.844	1.000	0.844	0.847	0.846
	EM	0.266	0.266	0.266	0.266	0.266	0.266	0.266	0.266	0.266	0.266	0.266	0.266
40	ERB	0.107	0.071	0.086	0.071	0.071	0.048	0.063	0.048	0.104	0.070	0.084	0.069
	EV	7.97E-5	6.86E-5	6.88E-5	6.88E-5	8.08E-5	6.86E-5	6.88E-5	6.87E-5	8.04E-5	6.83E-5	6.86E-5	6.85E-5
	RE	1.000	0.861	0.864	0.863	1.000	0.848	0.851	0.851	1.000	0.850	0.853	0.852
	EM	0.266	0.266	0.266	0.266	0.265	0.265	0.265	0.265	0.266	0.266	0.266	0.266
50	ERB	0.024	0.000	0.011	0.000	-0.002	-0.017	-0.006	-0.017	0.023	0.000	0.011	0.000
	EV	6.18E-5	5.40E-5	5.42E-5	5.42E-5	6.26E-5	5.39E-5	5.41E-5	5.41E-5	6.23E-5	5.38E-5	5.39E-5	5.39E-5
	RE	1.000	0.873	0.877	0.876	1.000	0.861	0.864	0.864	1.000	0.863	0.866	0.865
	EM	0.266	0.266	0.266	0.266	0.266	0.266	0.266	0.266	0.266	0.266	0.266	0.266
60	ERB	0.039	0.033	0.042	0.033	0.017	0.018	0.028	0.018	0.038	0.032	0.042	0.032
	EV	4.98E-5	4.41E-5	4.42E-5	4.42E-5	5.04E-5	4.40E-5	4.41E-5	4.41E-5	5.02E-5	4.39E-5	4.40E-5	4.40E-5
	RE	1.000	0.885	0.888	0.888	1.000	0.873	0.876	0.875	1.000	0.874	0.877	0.876

Table 1: Under srs: Comparisons between different estimators. $\mathrm{EV}\left(\tilde{\theta}\right) = \sum_{i=1}^{m}\left(\tilde{\theta}_i - \bar{\tilde{\theta}}\right)^2 / (m-1)$.

n		$\hat{\theta}_{RR}$	$\tilde{\theta}_{yx.RR}$	$\tilde{\theta}_{yx.RH}$	$\tilde{\theta}_{yx.RJ}$	$\hat{\theta}_{HH}$	$\tilde{\theta}_{yx.HR}$	$\tilde{\theta}_{yx.HH}$	$\tilde{\theta}_{yx.HJ}$	$\hat{\theta}_{JJ}$	$\tilde{\theta}_{yx.JR}$	$\tilde{\theta}_{yx.JH}$	$\tilde{\theta}_{yx.JJ}$
	EM	0.265	0.266	0.265	0.265	0.265	0.266	0.265	0.265	0.265	0.266	0.265	0.265
20	ERB	-0.010	0.095	-0.013	-0.044	-0.075	0.052	-0.055	-0.086	-0.010	0.095	-0.013	-0.044
	RE	1.000	0.864	0.879	0.867	1.000	0.918	0.935	0.920	1.000	0.864	0.879	0.867
	EM	0.265	0.266	0.265	0.265	0.265	0.266	0.265	0.265	0.265	0.266	0.265	0.265
30	ERB	-0.027	0.045	-0.026	-0.046	-0.070	0.017	-0.054	-0.074	-0.027	0.045	-0.026	-0.046
	RE	1.000	0.826	0.835	0.830	1.000	0.861	0.870	0.864	1.000	0.826	0.835	0.830
	EM	0.265	0.266	0.265	0.265	0.265	0.266	0.265	0.265	0.265	0.266	0.265	0.265
40	ERB	-0.057	0.012	-0.062	-0.076	-0.088	-0.009	-0.082	-0.097	-0.057	0.012	-0.062	-0.076
	RE	1.000	0.791	0.802	0.799	1.000	0.820	0.830	0.826	1.000	0.791	0.802	0.799
	EM	0.266	0.266	0.266	0.265	0.266	0.266	0.265	0.265	0.266	0.266	0.266	0.265
50	ERB	0.032	0.105	0.008	-0.003	0.007	0.089	-0.008	-0.020	0.032	0.105	0.008	-0.003
	RE	1.000	0.695	0.717	0.716	1.000	0.722	0.741	0.739	1.000	0.695	0.717	0.716
	EM	0.265	0.266	0.265	0.265	0.265	0.265	0.265	0.265	0.265	0.266	0.265	0.265
60	ERB	-0.063	0.011	-0.067	-0.076	-0.083	-0.002	-0.080	-0.089	-0.063	0.011	-0.067	-0.076
	RE	1.000	0.696	0.711	0.710	1.000	0.718	0.732	0.730	1.000	0.696	0.711	0.710

Table 2: Under πps: Comparisons between different estimators.

n		$\hat{\theta}_{RR}$	$\tilde{\theta}_{yx.RR}$	$\tilde{\theta}_{yx.RH}$	$\tilde{\theta}_{yx.RJ}$	$\hat{\theta}_{HH}$	$\tilde{\theta}_{yx.HR}$	$\tilde{\theta}_{yx.HH}$	$\tilde{\theta}_{yx.HJ}$	$\hat{\theta}_{JJ}$	$\tilde{\theta}_{yx.JR}$	$\tilde{\theta}_{yx.JH}$	$\tilde{\theta}_{yx.JJ}$
	EM	0.266	0.266	0.266	0.266	0.266	0.265	0.266	0.265	0.266	0.266	0.266	0.266
20	ERB	0.089	0.027	0.057	0.027	0.020	-0.018	0.012	-0.019	0.086	0.025	0.055	0.025
	RE	1.000	0.851	0.855	0.853	1.000	0.837	0.840	0.839	1.000	0.840	0.843	0.842
	EM	0.266	0.266	0.266	0.266	0.266	0.265	0.266	0.265	0.266	0.266	0.266	0.266
30	ERB	0.062	0.022	0.041	0.021	0.017	-0.008	0.012	-0.008	0.061	0.021	0.041	0.021
	RE	1.000	0.867	0.871	0.870	1.000	0.855	0.858	0.857	1.000	0.857	0.860	0.859
	EM	0.266	0.266	0.266	0.266	0.266	0.266	0.266	0.266	0.266	0.266	0.266	0.266
40	ERB	0.060	0.036	0.051	0.036	0.027	0.014	0.029	0.014	0.059	0.035	0.050	0.035
	RE	1.000	0.857	0.860	0.859	1.000	0.845	0.848	0.847	1.000	0.846	0.849	0.848
	EM	0.266	0.266	0.266	0.266	0.266	0.266	0.266	0.266	0.266	0.266	0.266	0.266
50	ERB	0.070	0.038	0.049	0.038	0.044	0.021	0.032	0.021	0.069	0.038	0.049	0.037
	RE	1.000	0.879	0.882	0.881	1.000	0.866	0.869	0.868	1.000	0.867	0.870	0.869
	EM	0.266	0.266	0.266	0.266	0.266	0.266	0.266	0.266	0.266	0.266	0.266	0.266
60	ERB	0.049	0.035	0.044	0.035	0.026	0.020	0.029	0.020	0.047	0.034	0.043	0.034
	RE	1.000	0.851	0.854	0.853	1.000	0.839	0.842	0.841	1.000	0.840	0.843	0.842

Table 3: Stratified sampling design: Under srs, draw random sample of size n_h from each stratum and combined samples into one sample of size n.

n		$\hat{\theta}_{RR}$	$\tilde{\theta}_{yx.RR}$	$\tilde{\theta}_{yx.RH}$	$\tilde{\theta}_{yx.RJ}$	$\hat{\theta}_{HH}$	$\tilde{\theta}_{yx.HR}$	$\tilde{\theta}_{yx.HH}$	$\tilde{\theta}_{yx.HJ}$	$\hat{\theta}_{JJ}$	$\tilde{\theta}_{yx.JR}$	$\tilde{\theta}_{yx.JH}$	$\tilde{\theta}_{yx.JJ}$
	EM	0.265	0.265	0.265	0.265	0.265	0.265	0.265	0.265	0.265	0.265	0.265	0.265
20	ERB	-0.125	-0.181	-0.282	-0.298	-0.192	-0.203	-0.304	-0.320	-0.125	-0.181	-0.282	-0.298
	RE	1.000	0.954	0.980	0.961	1.000	0.997	1.028	1.005	1.000	0.954	0.980	0.961
	EM	0.265	0.265	0.265	0.265	0.265	0.265	0.265	0.265	0.265	0.265	0.265	0.265
30	ERB	-0.086	-0.141	-0.213	-0.225	-0.128	-0.159	-0.231	-0.244	-0.086	-0.141	-0.213	-0.225
	RE	1.000	0.905	0.925	0.916	1.000	0.936	0.958	0.947	1.000	0.905	0.925	0.916
	EM	0.266	0.265	0.265	0.265	0.265	0.265	0.265	0.265	0.266	0.265	0.265	0.265
40	ERB	0.028	-0.044	-0.099	-0.108	-0.003	-0.057	-0.112	-0.122	0.028	-0.044	-0.099	-0.108
	RE	1.000	0.870	0.884	0.878	1.000	0.896	0.910	0.904	1.000	0.870	0.884	0.878
	EM	0.265	0.265	0.265	0.265	0.265	0.265	0.265	0.265	0.265	0.265	0.265	0.265
50	ERB	-0.050	-0.082	-0.133	-0.142	-0.075	-0.095	-0.146	-0.155	-0.050	-0.082	-0.133	-0.142
	RE	1.000	0.778	0.791	0.789	1.000	0.797	0.810	0.807	1.000	0.778	0.791	0.789
	EM	0.265	0.265	0.265	0.265	0.265	0.265	0.265	0.265	0.265	0.265	0.265	0.265
60	ERB	-0.143	-0.168	-0.213	-0.220	-0.164	-0.178	-0.223	-0.231	-0.143	-0.168	-0.213	-0.220
	RE	1.000	0.795	0.806	0.804	1.000	0.812	0.823	0.821	1.000	0.795	0.806	0.804

Table 4: Stratified sampling design: Under πps, draw random sample of size n_h from each stratum and combined samples into one sample of size n.

n		$\hat{\theta}_{RR}$	$\tilde{\theta}_{.RR}$	$\tilde{\theta}_{.RH}$	$\tilde{\theta}_{.RJ}$	$\hat{\theta}_{HH}$	$\tilde{\theta}_{.HR}$	$\tilde{\theta}_{.HH}$	$\tilde{\theta}_{.HJ}$	$\hat{\theta}_{JJ}$	$\tilde{\theta}_{.JR}$	$\tilde{\theta}_{.JH}$	$\tilde{\theta}_{.JJ}$
	var	1.66E-4	1.69E-4	1.68E-4	1.43E-4	1.43E-4	1.43E-4	1.43E-4	1.44E-4	1.43E-4	1.42E-4	1.43E-4	1.42E-4
20	MV	1.65E-4	1.70E-4	1.68E-4	1.34E-4	1.34E-4	1.34E-4	1.34E-4	1.35E-4	1.35E-4	1.34E-4	1.34E-4	1.34E-4
	RBV	-7.51E-3	1.08E-3	1.76E-3	-6.15E-2	-6.21E-2	-6.17E-2	-6.12E-2	-6.18E-2	-6.14E-2	-6.00E-2	-6.06E-2	-6.02E-2
	REV	1.00	7.20E-1	7.30E-1	7.25E-1	1.00	6.75E-1	6.84E-1	6.79E-1	1.00	6.91E-1	7.00E-1	6.96E-1
	var	1.09E-4	1.11E-4	1.10E-4	9.36E-5	9.40E-5	9.39E-5	9.36E-5	9.40E-5	9.39E-5	9.32E-5	9.36E-5	9.35E-5
30	MV	1.09E-4	1.11E-4	1.10E-4	8.99E-5	9.03E-5	9.02E-5	9.00E-5	9.04E-5	9.03E-5	8.97E-5	9.00E-5	8.99E-5
	RBV	-3.96E-3	2.09E-3	2.20E-3	-3.92E-2	-3.95E-2	-3.94E-2	-3.85E-2	-3.89E-2	-3.87E-2	-3.81E-2	-3.85E-2	-3.83E-2
	REV	1.00	7.29E-1	7.37E-1	7.34E-1	1.00	6.89E-1	6.97E-1	6.94E-1	1.00	7.00E-1	7.07E-1	7.05E-1
	var	8.05E-5	8.16E-5	8.11E-5	6.91E-5	6.93E-5	6.93E-5	6.90E-5	6.93E-5	6.92E-5	6.88E-5	6.91E-5	6.90E-5
40	MV	7.96E-5	8.11E-5	8.07E-5	6.65E-5	6.67E-5	6.67E-5	6.65E-5	6.67E-5	6.67E-5	6.63E-5	6.65E-5	6.65E-5
	RBV	-1.04E-2	-5.86E-3	-5.68E-3	-3.72E-2	-3.74E-2	-3.73E-2	-3.68E-2	-3.70E-2	-3.69E-2	-3.64E-2	-3.67E-2	-3.66E-2
	REV	1.00	7.09E-1	7.17E-1	7.15E-1	1.00	6.74E-1	6.81E-1	6.79E-1	1.00	6.82E-1	6.89E-1	6.87E-1
	var	6.33E-5	6.41E-5	6.39E-5	5.44E-5	5.45E-5	5.45E-5	5.43E-5	5.45E-5	5.44E-5	5.41E-5	5.43E-5	5.43E-5
50	MV	6.26E-5	6.36E-5	6.34E-5	5.25E-5	5.27E-5	5.27E-5	5.25E-5	5.27E-5	5.26E-5	5.24E-5	5.26E-5	5.25E-5
	RBV	-1.16E-2	-7.85E-3	-7.73E-3	-3.33E-2	-3.34E-2	-3.34E-2	-3.29E-2	-3.31E-2	-3.30E-2	-3.27E-2	-3.29E-2	-3.28E-2
	REV	1.00	7.32E-1	7.39E-1	7.38E-1	1.00	7.00E-1	7.06E-1	7.04E-1	1.00	7.06E-1	7.12E-1	7.11E-1
	var	5.19E-5	5.25E-5	5.23E-5	4.45E-5	4.47E-5	4.47E-5	4.45E-5	4.46E-5	4.46E-5	4.44E-5	4.45E-5	4.45E-5
20	MV	5.17E-5	5.24E-5	5.23E-5	4.36E-5	4.37E-5	4.37E-5	4.35E-5	4.37E-5	4.37E-5	4.34E-5	4.36E-5	4.36E-5
	RBV	-4.08E-3	-1.15E-3	-1.07E-3	-2.12E-2	-2.14E-2	-2.13E-2	-2.09E-2	-2.10E-2	-2.10E-2	-2.08E-2	-2.09E-2	-2.09E-2
	REV	1.00	7.23E-1	7.30E-1	7.28E-1	1.00	6.94E-1	7.00E-1	6.99E-1	1.00	6.99E-1	7.05E-1	7.04E-1

Table 5: Under srs: Comparisons between the variances for the different estimators. var := population variance of the estimator.

n	$\hat{\theta}_{RR}$	$\tilde{\theta}_{.RR}$	$\tilde{\theta}_{.RH}$	$\tilde{\theta}_{.RJ}$	$\hat{\theta}_{HH}$	$\tilde{\theta}_{.HR}$	$\tilde{\theta}_{.HH}$	$\tilde{\theta}_{.HJ}$	$\hat{\theta}_{JJ}$	$\tilde{\theta}_{.JR}$	$\tilde{\theta}_{.JH}$	$\tilde{\theta}_{.JJ}$
20	1.34E-7	2.74E-5	2.50E-5	8.78E-6	3.36E-5	8.59E-6	9.17E-6	8.35E-6	3.15E-5	8.22E-6	8.29E-6	8.23E-6
30	4.5E-6	1.48E-5	1.37E-5	6.46E-6	2.42E-5	6.21E-6	6.72E-6	6.26E-6	2.34E-5	6.27E-6	6.29E-6	6.26E-6
40	4.85E-8	1.25E-5	1.18E-5	2.29E-6	1.41E-5	1.88E-6	2.33E-6	2.03E-6	1.37E-5	2.04E-6	2.04E-6	2.04E-6
50	7.50E-7	9.60E-6	9.16E-6	1.62E-6	9.86E-6	1.21E-6	1.60E-6	1.38E-6	9.68E-6	1.39E-6	1.39E-6	1.39E-6
60	1.86E-6	8.36E-6	8.04E-6	6.22E-7	6.64E-6	2.44E-7	5.90E-7	4.19E-7	6.57E-6	4.34E-7	4.36E-7	4.37E-7

Table 6: Under srs: Numbers in this Table are the absolute differences between EV, Table(1), and MV, Table(5).

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