ESTIMATION OF LOCATION AND SCALE PARAMETERS BY ORDER STATISTICS FROM SINGLY AND DOUBLY CENSORED SAMPLES

Part I. The Normal Distribution up to Samples of Size 10

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1. Introduction. Type II censored samples [3] are considered, whereby the total number of the sample elements is known but the observations for some of the extreme elements are missing. Singly censored samples are those in which only the smallest r_1 observations or the largest r_2 observations are missing, whereas samples having both r_1 smallest and r_2 largest observations missing are called doubly censored samples. This general case of estimation includes, as special cases, those estimates obtained from singly censored samples as well as those obtained by taking all the sample elements (i.e., r_1 or $r_2 = 0$ and $r_1 = r_2 = 0$).

The approach to the general case in censoring is of value not only for its numerical results. It enables the drawing of inferences concerning interesting and important patterns for the coefficients, variances, and the relative efficiencies of the estimates. These features could not be and were not revealed in the earlier, less general studies. These conclusions will be considered in Section (5).

In Part I, estimation of the mean and standard deviation from singly and doubly censored samples drawn from the normal distribution will be considered for samples $n \leqq 10$. A generalization of an alternative estimate previously given by Gupta is also obtained. In future work, it is planned to extend the tables up to samples of size $n \leqq 20$ and to include the two- and one-parameter single-exponential distributions.

Estimates of the parameters using the best linear systematic statistics are obtained by arranging the known sample elements in ascending order (i.e., $y_{(1)} \leqq y_{(2)} \leqq \cdots \leqq y_{(n)}$) and applying the method of least squares to get the best linear combination of them. The coefficients provided for these linear estimates of the ordered observations make them unbiased with minimum variance. The method used in calculation is identical with that given by Gupta in [3] with slight modifications.

2. The normal distribution. To advance the study of order statistics for the normal distribution, Hastings *et al.* [4] calculated the means, variances, and covariances of the order statistics up to samples of size 10. Godwin [1] calculated these quantities more accurately as well as extending them to more decimal places. From his tables, he was able to calculate the best linear systematic statistic of the standard deviation [2] using all the sample elements for samples of

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TABLE I

Variances and covariances of order statistics in samples of sizes up to 20 from a normal population

n	i	j	Value	n	i	j	Value	n	i	j	Value
2	1	1	.6816901139	7				9			
		2	.3183098861		3	3	.2197215626		4	4	.1705588454
	2	2	.6816901139			4	.1655598429			5	.1369913669
3	1	1	.5594672038			5	.1296048425			6	.1126671842
		2	.2756644477		4	4	.2104468615		5	5	.1661012814
		3	.1648683485	8	1	1	.3728971434	10	1	1	.3443438233
	2	2	.4486711046			2	.1863073997			2	.1712629030
4	1	1	.4917152369			3	.1259660300			3	.1162590989
		2	.2455926930			4	.0947230277			4	.0882494247
		3	.1580080701			5	.0747650242			5	.0707413677
		4	.1046840000			6	.0602075169			6	.0583987134
	2	2	.3604553434			7	.0482985508			7	.0489206279
		3	.2359438935			8	.0368353073			8	.0410844589
5	1	1	.4475340691		2	2	.2394010458			9	.0340406470
		2	.2243309596			3	.1631958727			10	.0266989351
		3	.1481477252			4	.1232633317		2	2	.2145241430
		4	.1057719776			5	.0975647193			3	.1466226180
		5	.0742152685			6	.0787224662			4	.1117015961
	2	2	.3115189521			7	.0632466118			5	.0897428245
		3	.2084354440		3	3	.2007687900			6	.0741995414
		4	.1499426668			4	.1523584312			7	.0622278486
	3	3	.2868336616			5	.1209637555			8	.0523067222
6	1	1	.4159271090			6	.0978171355			9	.0433711561
		2	.2085030023		4	4	.1871862195		3	3	.1750032834
		3	.1394352565			5	.1491754908			4	.1338022448
		4	.1024293940	9	1	1	.3573533264			5	.1077445336
		5	.0773637839			2	.1781434240			6	.0892254012
		6	.0563414544			3	.1207454442			7	.0749183943
	2	2	.2795777392			4	.0913071400			8	.0630332449
		3	.1889859560			5	.0727422354		4	4	.1579389144
		4	.1396640604			6	.0594831125			5	.1275089295
		5	.1059054582			7	.0490764061			6	.1057858169
	3	3	.2462125354			8	.0400936927			7	.0889462026
		4	.1832727978			9	.0310552188		5	5	.1510539039
7	1	1	.3919177761		2	2	.2256968778			6	.1255989678
		2	.1961990246			3	.1541163526	11	1	1	.3332474428
		3	.1321155811			4	.1170056918			2	.1653647712
		4	.0984868607			5	.0934477394			3	.1123584351
		5	.0765598346			6	.0765461431			4	.0855170596
		6	.0599187124			7	.0632354695			5	.0688483064
		7	.0448022105			8	.0517146091			6	.0572007586
	2	2	.2567328862		3	3	.1863826133			7	.0483754063
		3	.1744833274			4	.1420779776			8	.0412423472
		4	.1307298656			5	.1137680176			9	.0351103357
		5	.1019550089			6	.0933625386			10	.0294198503
		6	.0799811748			7	.0772351806			11	.0233152868

TABLE I—*Continued*

n	i	j	Value	n	i	j	Value	n	i	j	Value
11				12	3			13	3		
	2	2	.2051975798			4	.1212063211			8	.0589221432
		3	.1403096511			5	.0982605602			9	.0514460445
		4	.1071492595			6	.0822228461			10	.0449637542
		5	.0864430257			7	.0701213964			11	.0390643799
		6	.0719305024			8	.0604384621		4	4	.1330111820
		7	.0608869662			9	.0522825611			5	.1082512667
		8	.0519504506			10	.0450357615			6	.0909855605
		9	.0442549455		4	4	.1398109405			7	.0780173339
		10	.0371029977			5	.1135687821			8	.0677217143
	3	3	.1657242880			6	.0951645279			9	.0591628729
		4	.1269672925			7	.0812419810			10	.0517328050
		5	.1026407291			8	.0700795832		5	5	.1232503256
		6	.0855178832			9	.0606620874			6	.1037367701
		7	.0724741050		5	5	.1306137359			7	.0890434754
		8	.0618873278			6	.1096212247			8	.0773552864
		9	.0527550069			7	.0936951520			9	.0676230994
	4	4	.1479546565			8	.0808972960		6	6	.1183175325
		5	.1198752861		6	6	.1266377911			7	.1016824204
		6	.1000346585			7	.1083945831			8	.0884194610
		7	.0848765182	13	1	1	.3152053842		7	7	.1167989950
		8	.0725451434			2	.1557272904	14	1	1	.3077301026
	5	5	.1396410804			3	.1058908842			2	.1517203662
		6	.1167449805			4	.0808649736			3	.1031719531
		7	.0991935960			5	.0654634499			4	.0788715916
	6	6	.1371624335			6	.0548221797			5	.0639657428
12	1	1	.3236363870			7	.0468833088			6	.0537064714
		2	.1602373762			8	.0406132548			7	.0460899189
		3	.1089309641			9	.0354226462			8	.0401141688
		4	.0830686767			10	.0309322744			9	.0352141760
		5	.0670884464			11	.0268537250			10	.0310371163
		6	.0559933694			12	.0228858068			11	.0273362865
		7	.0476620974			13	.0184348220			12	.0239061001
		8	.0410208554		2	2	.1904130721			13	.0205080257
		9	.0354439060			3	.1302055829			14	.0166279801
		10	.0305012591			4	.0997262696		2	2	.1844200252
		11	.0257945392			5	.0808785938			3	.1260791989
		12	.0206221233			6	.0678145832			4	.0966524633
	2	2	.1972646039			7	.0580457285			5	.0785202981
		3	.1349020328			8	.0503167946			6	.0660028340
		4	.1031959206			9	.0439095087			7	.0566896715
		5	.0835045822			10	.0383601798			8	.0493708148
		6	.0697859658			11	.0333147765			9	.0433617156
		7	.0594590652			12	.0284018130			10	.0382337404
		8	.0512113198		3	3	.1513917013			11	.0336863221
		9	.0442747124			4	.1162698131			12	.0294681314
		10	.0381191478			5	.0944566603			13	.0252863928
		11	.0322507340			6	.0792922993		3	3	.1457045665
	3	3	.1579786877			7	.0679282354			4	.1119816877

TABLE I—*Continued*

n	i	j	Value	n	i	j	Value	n	i	j	Value
14	3			15	2			16	1		
		5	.0911181271			6	.0643390895			3	.0985009764
		6	.0766754957			7	.0554074400			4	.0754040023
		7	.0659084825			8	.0484238833			5	.0613086724
		8	.0574341188			9	.0427294113			6	.0516624963
		9	.0504677802			10	.0379177516			7	.0445503705
		10	.0445169192			11	.0337151721			8	.0390194716
		11	.0392352316			12	.0299152347			9	.0345378158
		12	.0343322071			13	.0263303885			10	.0307810093
	4	4	.1272273070			14	.0227213594			11	.0275353612
		5	.1036931108		3	3	.1407322502			12	.0246479007
		6	.0873562483			4	.1082138452			13	.0219956755
		7	.0751519909			5	.0881605755			14	.0194585037
		8	.0655310936			6	.0743268436			15	.0168710289
		9	.0576120957			7	.0640558183			16	.0138287378
		10	.0508402240			8	.0560136122		2	2	.1743940788
		11	.0448243469			9	.0494485109			3	.1191409287
	5	5	.1171012461			10	.0438960670			4	.0914359918
		6	.0987747550			11	.0390426915			5	.0744591145
		7	.0850536546			12	.0346513382			6	.0628093909
		8	.0742181416			13	.0305060359			7	.0542033941
		9	.0652867776		4	4	.1222328270			8	.0475009769
		10	.0576401464			5	.0997323941			9	.0420638230
	6	6	.1115324579			6	.0841705696			10	.0375018250
		7	.0961405595			7	.0725946869			11	.0335574912
		8	.0839617110			8	.0635175907			12	.0300461298
		9	.0739069221			9	.0560990511			13	.0268189579
	7	7	.1090269480			10	.0498187836			14	.0237301562
		8	.0953087256			11	.0443247452			15	.0205785433
15	1	1	.3010415703			12	.0393501820		3	3	.1363385612
		2	.1481297708		5	5	.1118698986			4	.1048706756
		3	.1007223449			6	.0945206004			5	.0855189036
		4	.0770594060			7	.0815891122			6	.0722075087
		5	.0625845851			8	.0714331681			7	.0623568515
		6	.0526530129			9	.0631224388			8	.0546749107
		7	.0453078886			10	.0560795065			9	.0484366096
		8	.0395736673			11	.0499127743			10	.0431979377
		9	.0349035905		6	6	.1058666366			11	.0386652995
		10	.0309614122			7	.0914683204			12	.0346277256
		11	.0275211039			8	.0801407559			13	.0309149135
		12	.0244126313			9	.0708582099			14	.0273595378
		13	.0214819828			10	.0629824402		4	4	.1178657554
		14	.0185333263		7	7	.1026916923			5	.0962513413
		15	.0151137071			8	.0900499964			6	.0813480448
	2	2	.1791215291			9	.0796738323			7	.0703000911
		3	.1224176953		8	8	.1016946521			8	.0616728990
		4	.0939067144	16	1	1	.2950098090			9	.0546595026
		5	.0763912337			2	.1448881689			10	.0487647746

TABLE I—*Continued*

n	i	j	Value	n	i	j	Value	n	i	j	Value
16	4			17	2			17	6		
		11	.0436607328			9	.0413928192			12	.0478122599
		12	.0391112669			10	.0370349110		7	7	.0929031780
		13	.0349253749			11	.0332940892			8	.0818194607
	5	5	.1073517089			12	.0299982825			9	.0728154074
		6	.0908232622			13	.0270170379			10	.0652667274
		7	.0785480532			14	.0242386812			11	.0587626219
		8	.0689488802			15	.0215459396		8	8	.0907361650
		9	.0611364182			16	.0187658306			9	.0808000267
		10	.0545638941		3	3	.1324207975			10	.0724599963
		11	.0488684327			4	.1018792434		9	9	.0900465814
		12	.0437882959			5	.0831421716	18	1	1	.2845301297
	6	6	.1010461906			6	.0702850403			2	.1392501620
		7	.0874627156			7	.0607964413			3	.0946172637
		8	.0768239668			8	.0534208202			4	.0724851730
		9	.0681545540			9	.0474555487			5	.0590304274
		10	.0608534805			10	.0424726884			6	.0498600635
		11	.0545210724			11	.0381925587			7	.0431302310
	7	7	.0974026613			12	.0344194567			8	.0379260195
		8	.0856181916			13	.0310047771			9	.0337388141
		9	.0760015577			14	.0278210708			10	.0302610667
		10	.0678931922			15	.0247342095			11	.0272938041
	8	8	.0957213007		4	4	.1140068197			12	.0247002471
		9	.0850291218			5	.0931620339			13	.0223801573
17	1	1	.2895330037			6	.0788266621			14	.0202537421
		2	.1419424629			7	.0682298909			15	.0182488619
		3	.0964748737			8	.0599826092			16	.0162850441
		4	.0738849615			9	.0533057575			17	.0142368875
		5	.0601272302			10	.0477239973			18	.0117719054
		6	.0507326948			11	.0429261816		2	2	.1662929294
		7	.0438236491			12	.0386942630			3	.1135058132
		8	.0384672834			13	.0348624030			4	.0871597604
		9	.0341441055			14	.0312881041			5	.0710825990
		10	.0305389548		5	5	.1034004377			6	.0600975754
		11	.0274465527			6	.0875729930			7	.0520217423
		12	.0247237144			7	.0758534534			8	.0457683625
		13	.0222620771			8	.0667204245			9	.0407317967
		14	.0199690651			9	.0593187706			10	.0365451034
		15	.0177476891			10	.0531257771			11	.0329704894
		16	.0154552071			11	.0477987292			12	.0298442464
		17	.0127264751			12	.0430970793			13	.0270462261
	2	2	.1701426762			13	.0388375657			14	.0244806359
		3	.1161866734		6	6	.0968824669			15	.0220607111
		4	.0891982557			7	.0839811738			16	.0196894667
		5	.0726970385			8	.0739130260			17	.0172154925
		6	.0613998459			9	.0657442736		3	3	.1288998943
		7	.0530761573			10	.0589030403			4	.0991828539
		8	.0466140918			11	.0530137275			5	.0809899792

TABLE I—*Continued*

n	i	j	Value	n	i	j	Value	n	i	j	Value
18	3			18				19	3		
		6	.0685324700		8	8	.0864960639			8	.0511541418
		7	.0593598602			9	.0771762286			9	.0456228816
		8	.0522488413			10	.0693891332			10	.0410365629
		9	.0465162123			11	.0627116906			11	.0371346427
		10	.0417473296		9	9	.0853127880			12	.0337391171
		11	.0376730987			10	.0767442321			13	.0307215918
		12	.0341080171	19	1	1	.2799358050			14	.0279835020
		13	.0309157650			2	.1367768168			15	.0254424108
		14	.0279875014			3	.0929061763			16	.0230195063
		15	.0252244786			4	.0711902425			17	.0206214645
		16	.0225161109			5	.0580094835		4	4	.1074740839
	4	4	.1105660331			6	.0490405678			5	.0879051965
		5	.0903973787			7	.0424705246			6	.0745033878
		6	.0765579277			8	.0374006329			7	.0646406188
		7	.0663522086			9	.0333319395			8	.0570032284
		8	.0584310521			10	.0299634144			9	.0508572608
		9	.0520394281			11	.0271011338			10	.0457576598
		10	.0467183404			12	.0246129452			11	.0414165091
		11	.0421694861			13	.0224037540			12	.0376368753
		12	.0381869632			14	.0204007370			13	.0342765540
		13	.0346192645			15	.0185431530			14	.0312262549
		14	.0313452497			16	.0167731147			15	.0283944527
		15	.0282548286			17	.0150223067			16	.0256935148
	5	5	.0999084321			18	.0131789994		5	5	.0967944745
		6	.0846879168			19	.0109382527			6	.0821055695
		7	.0734460811		2	2	.1627856651			7	.0712796742
		8	.0647101858			3	.1110590145			8	.0628870095
		9	.0576543520			4	.0852931053			9	.0561272025
		10	.0517756675			5	.0695970759			10	.0505141639
		11	.0467468133			6	.0588910196			11	.0457330144
		12	.0423415563			7	.0510351093			12	.0415681234
		13	.0383932046			8	.0449652247			13	.0378636088
		14	.0347682770			9	.0400891754			14	.0344995261
	6	6	.0932407331			10	.0360490040			15	.0313752928
		7	.0809202644			11	.0326137544		6	6	.0900218693
		8	.0713338046			12	.0296258236			7	.0782029063
		9	.0635829688			13	.0269716592			8	.0690294360
		10	.0571197288			14	.0245641909			9	.0616336896
		11	.0515868552			15	.0223306885			10	.0554877905
		12	.0467370896			16	.0202017247			11	.0502493169
		13	.0423879846			17	.0180952193			12	.0456834841
	7	7	.0890167025			18	.0158767294			13	.0416203596
		8	.0785179677		3	3	.1257138904			14	.0379290224
		9	.0700199026			4	.0967367097		7	7	.0856172981
		10	.0629269074			5	.0790298792			8	.0756153413
		11	.0568501034			6	.0669273696			9	.0675433161
		12	.0515199092			7	.0580336124			10	.0608297030

TABLE I—*Concluded*

n	i	j	Value
19	7		
		11	.0551032224
		12	.0501089625
		13	.0456621835
	8	8	.0828339961
		9	.0740273546
		10	.0666958229
		11	.0604372723
		12	.0549752083
	9	9	.0812876330
		10	.0732703911
		11	.0664202898
	10	10	.0807909751
20	1	1	.2756966156
		2	.1344941714
		3	.0913234064
		4	.0699879991
		5	.0570566384
		6	.0482701093
		7	.0418437826
		8	.0368937058
		9	.0329296302
		10	.0296562523
		11	.0268838808
		12	.0244839567
		13	.0223649803
		14	.0204584277
		15	.0187096782
		16	.0170711408
		17	.0154951854
		18	.0139227072
		19	.0122530117
		20	.0102047204
	2	2	.1595731636
		3	.1088143707
		4	.0835758044
		5	.0682247554
		6	.0577699656
		7	.0501109523
		8	.0442041191
		9	.0394693443
		10	.0355565554

n	i	j	Value
20	2		
		11	.0322405467
		12	.0293684960
		13	.0268315105
		14	.0245479493
		15	.0224526609
		16	.0204888032
		17	.0185994024
		18	.0167136502
		19	.0147107671
	3	3	.1228134687
		4	.0945049010
		5	.0772355098
		6	.0654510179
		7	.0568056677
		8	.0501310269
		9	.0447763202
		10	.0403482354
		11	.0365934287
		12	.0333397949
		13	.0304645792
		14	.0278756579
		15	.0254994381
		16	.0232716371
		17	.0211277373
		18	.0189874448
	4	4	.1046766243
		5	.0856442356
		6	.0726321560
		7	.0630731775
		8	.0556855081
		9	.0497539273
		10	.0448455403
		11	.0406811669
		12	.0370709493
		13	.0338793392
		14	.0310045146
		15	.0283650517
		16	.0258897454
		17	.0235070343
	5	5	.0939960007
		6	.0797773755

n	i	j	Value
20	5		
		7	.0693175756
		8	.0612251429
		9	.0547222526
		10	.0493374275
		11	.0447662310
		12	.0408014074
		13	.0372948400
		14	.0341351571
		15	.0312332040
		16	.0285109200
	6	6	.0871511254
		7	.0757703360
		8	.0669555789
		9	.0598659769
		10	.0539910639
		11	.0490008080
		12	.0446702771
		13	.0408385549
		14	.0373845194
		15	.0342111024
	7	7	.0826123955
		8	.0730383676
		9	.0653307665
		10	.0589387428
		11	.0535056766
		12	.0487882257
		13	.0446121090
		14	.0408459989
	8	8	.0796309757
		9	.0712591607
		10	.0643103375
		11	.0583997310
		12	.0532644495
		13	.0487159834
	9	9	.0778118317
		10	.0702526464
		11	.0638176734
		12	.0582229133
	10	10	.0769474356
		11	.0699266198

TABLE II

The coefficients of the most efficient linear systematic statistics of the mean and standard deviation in censored samples of sizes ≦10 from a normal population

n	r_1	r_2	$y_{(1)}$	$y_{(2)}$	$y_{(3)}$	$y_{(4)}$	$y_{(5)}$	$y_{(6)}$	$y_{(7)}$	$y_{(8)}$	$y_{(9)}$	$y_{(10)}$
2	0	0	.50000000 −.88622693	.50000000 .88622693								
3	0	0	.33333333 −.59081795	.33333333 .00000000	.33333333 .59081795							
	0	1	.00000000 −1.18163590	1.00000000 1.18163590								
4	0	0	.25000000 −.45394040	.25000000 −.11018073	.25000000 .11018073	.25000000 .45394040						
	0	1	.11606577 −.69713303	.24083805 −.12681665	.64309618 .82394968							
	0	2	−.40555159 −1.36544125	1.40555159 1.36544125								
	1	1		.50000000 −1.68343717	.50000000 1.68343717							
5	0	0	.20000000 −.37238157	.20000000 −.13521392	.20000000 .00000000	.20000000 .13521392	.20000000 .37238157					
	0	1	.12515679 −.51173274	.18304590 −.16678091	.21471643 .02740065	.47708089 .65111300						
	0	2	−.06377484 −.76958387	.14982836 −.21211572	.91394649 .98169958							
	0	3	−.74110683 −1.49712813	1.74110683 1.49712813								
	1	1		.38929103 1.01006230	.22141794 .00000000	.38929103 1.01006230						

5	1	2		.00000000 −2.02012460	1.00000000 2.02012460							
6	0	0	.16666667 −.31752484	.16666667 −.13855961	.16666667 −.04321165	.16666667 .04321165	.16666667 .13855961	.16666667 .31752484				
	0	1	.11828773 −.40969394	.15097353 −.16845737	.16803141 −.04061162	.18280232 .07395248	.37990501 .54481044					
	0	2	.01848367 −.55281996	.12260668 −.20913743	.17614875 −.02897078	.68276091 .79092817						
	0	3	−.21591800 −.82435700	.06485167 −.27604235	1.15106633 1.10039935							
	0	4	−1.02606712 −1.59884545	2.02606712 1.59884545								
	1	1		.31977001 −.75309024	.18022999 −.08286186	.18022999 .08286186	.31977001 .75309024					
	1	2		.15385831 −1.14382882	.17811699 −.08783818	.66802470 1.23166700						
	1	3		−.45784434 −2.27165234	1.45784434 2.27165234							
	2	2			.50000000 −2.48081297	.50000000 2.48081297						
7	0	0	.14285714 −.27781036	.14285714 −.13509780	.14285714 −.06246312	.14285714 .00000000	.14285714 .06246312	.14285714 .13509780	.14285714 .27781036			
	0	1	.10882014 −.34400143	.12954538 −.16098444	.13997050 −.06807697	.14873929 .01143886	.15705206 .09006788	.31587262 .47155609				
	0	2	.04654966 −.43696302	.10721153 −.19432593	.13748095 −.07179355	.16260139 .03213315	.54615647 .67094936					
	0	3	−.07380239 −.58481466	.06771901 −.24284221	.13752310 −.07174176	.86856027 .89939863						

TABLE II—*Continued*

n	r_1	r_2	$y_{(1)}$	$y_{(2)}$	$y_{(3)}$	$y_{(4)}$	$y_{(5)}$	$y_{(6)}$	$y_{(7)}$	$y_{(8)}$	$y_{(9)}$	$y_{(10)}$
7	0	4	−.34744564 −.86817366	−.01345544 −.32689877	1.36090107 1.19507242							
	0	5	−1.27331716 −1.68122579	2.27331716 1.68122579								
	1	1		.27183155 −.61077842	.15198954 −.10607146	.15235782 .00000000	.15198954 .10607146	.27183155 .61077842				
	1	2		.17480153 −.82879521	.14322942 −.12575459	.16338504 .02477706	.51858402 .92977274					
	1	3		−.05916442 −1.24827431	.12704486 −.15477199	.93211956 1.40304629						
	1	4		−.87159736 −2.47116575	1.87159736 2.47116575							
	2	2			.41569461 −1.41760741	.16861078 .00000000	.41569461 1.41760741					
	2	3			.00000000 −2.83521483	1.00000000 2.83521483						
8	0	0	.12500000 −.24758623	.12500000 −.12944776	.12500000 −.07130849	.12500000 −.02295726	.12500000 .02295726	.12500000 .07130849	.12500000 .12944776	.12500000 .24758623		
	0	1	.09966946 −.29775817	.11386201 −.15150866	.12079601 −.07963530	.12649213 −.02000181	.13176484 .03635632	.13698715 .09505131	.27042840 .41749631			
	0	2	.05691876 −.36375811	.09621316 −.17875554	.11531993 −.08808945	.13090112 −.01319507	.14512418 .05698091	.45552284 .58681726				
	0	3	−.01672011 −.45862177	.06765099 −.21555013	.10840499 −.09699747	.14131770 .00022386	.69934643 .77094550					
	0	4	−.15491146 −.61096114	.01760383 −.27072110	.10013416 −.10611506	1.03717346 .98779730						

	0	5	−.46316724 −.90454196	−.08553809 −.36895282	1.54870533 1.27349478						
	0	6	−1.49153218 −1.75016272	2.49153218 1.75016272							
	1	1		.23666261 −.51837013	.13147170 −.11152847	.13186568 −.03605506	.13186568 .03605506	.13147170 .11152847	.23666261 .51837013		
	1	2		.17163950 −.66079246	.12217411 −.13189331	.13381130 −.03179351	.14417854 .06302434	.42819655 .76145495			
	1	3		.04308423 −.88940019	.10609003 −.16049536	.14058990 −.01973924	.71023584 1.06963479				
	1	4		−.25191114 −1.33367137	.07414879 −.20859974	1.17776234 1.54227111					
	1	5		−1.24622969 −2.63572419	2.24622969 2.63572419						
	2	2			.35694901 −1.03574984	.14305099 −.06736528	.14305099 .06736528	.35694901 1.03574984			
	2	3			.17417957 −1.56608726	.14291598 −.06775706	.68290445 1.63384432				
	2	4			−.47614906 −3.12199414	1.47614906 3.12199414					
	3	3				.50000000 −3.27837896	.50000000 3.27837896				
9	0	0	.11111111 −.22373410	.11111111 −.12326850	.11111111 −.07509922	.11111111 −.03596908	.11111111 .00000000	.11111111 .03596908	.11111111 .07509922	.11111111 .12326850	.11111111 .22373410
	0	1	.09148453 −.26325227	.10175129 −.14211324	.10665020 −.08407936	.11060016 −.03699545	.11419180 .00618347	.11765844 .04915546	.12118875 .09539434	.23647484 .37570705	
	0	2	.06023974 −.31289447	.08756442 −.16465459	.10056358 −.09375143	.11099937 −.03636313	.12038626 .01604052	.12937290 .06776492	.39087375 .52385821		

TABLE II—*Continued*

n	r_1	r_2	$y_{(1)}$	$y_{(2)}$	$y_{(3)}$	$y_{(4)}$	$y_{(5)}$	$y_{(6)}$	$y_{(7)}$	$y_{(8)}$	$y_{(9)}$	$y_{(10)}$
9	0	3	.01040388 −.37968567	.06597348 −.19359128	.09230224 −.10482346	.11331998 −.03325300	.13204377 .03166420	.58595666 .67968924				
	0	4	−.07313367 −.47658635	.03155502 −.23351551	.08087391 −.11807995	.11994592 −.02556713	.84075879 .85374893					
	0	5	−.22717960 −.63301232	−.02842070 −.29441786	.06443680 −.13477100	1.19116347 1.06220121						
	0	6	−.56642662 −.93553052	−.15208218 −.40469106	1.71850879 1.34022157							
	0	7	−1.68675768 −1.80924841	2.68675768 1.80924841								
	1	1		.20970437 −.45274773	.11588689 −.11065463	.11624355 −.05323076	.11633036 .00000000	.11624355 .05323076	.11588689 .11065463	.20970437 .45274773		
	1	2		.16260603 −.55443214	.10736149 −.12906081	.11483307 −.05627598	.12137684 .01089528	.12749454 .07752140	.36632802 .65135225			
	1	3		.07989097 −.70150427	.09363515 −.15346705	.11399182 −.05777177	.13209087 .02994545	.58039119 .88279764				
	1	4		−.07684209 −.93990137	.06990479 −.18956185	.11531222 −.05576340	.89162509 1.18522662					
	1	5		−.42723635 −1.40567629	.02184604 −.25344578	1.40539031 1.65912206						
	1	6		−1.58736731 −2.77525943	2.58736731 2.77525943							
	2	2			.31343459 −.83170114	.12427928 −.08848435	.12457225 .00000000	.12427928 .08848435	.31343459 .83170114			
	2	3			.20395900 −1.12219549	.11906520 −.10231994	.13296497 .02227016	.54401083 1.20224528				

9	2	4			− .05271887 −1.68944478	.10983900 −.12270952	.94287987 1.81215430					
	2	5			− .92294725 −3.36196762	1.92294725 3.36196762						
	3	3				.43149371 −1.82132165	.13701259 .00000000	.43149371 1.82132165				
	3	4				.00000000 −3.64264330	1.00000000 3.64264330					
10	0	0	.10000000 −.20438349	.10000000 −.11719379	.10000000 −.07625921	.10000000 −.04358325	.10000000 −.01421667	.10000000 .01421667	.10000000 .04358325	.10000000 .07625921	.10000000 .11719379	.10000000 .20438349
	0	1	.08432557 −.23641943	.09206394 −.13341377	.09568489 −.08507858	.09856130 −.04652372	.10112450 −.01191838	.10356728 .02150761	.10600993 .05586656	.10852533 .09368358	.21013724 .34229613	
	0	2	.06045239 −.27530689	.08044709 −.15233666	.08978431 −.09469015	.09718132 −.04877160	.10374300 −.00765306	.10994657 .03189894	.11605505 .07222921	.34239029 .47463020		
	0	3	.02442821 −.32524451	.06355637 −.17575100	.08178041 −.10578536	.09616708 −.05017756	.10885704 −.00056384	.12073778 .04685799	.50447311 .61066429			
	0	4	−.03158434 −.39304766	.03829283 −.20633250	.07072145 −.11917222	.09620404 −.05013282	.11851783 .01113054	.70784819 .75755466				
	0	5	−.12396225 −.49191253	−.00163103 −.24905990	.05485447 −.13615341	.09896448 −.04717854	.97177434 .92430437					
	0	6	−.29230821 −.65203499	−.07092971 −.31497342	.03053696 −.15928304	1.33270096 1.12629145						
	0	7	−.65962405 −.96246073	−.21376659 −.43568764	1.87339063 1.39814837							
	0	8	−1.86335148 −1.86082625	2.86335148 1.86082625								
	1	1		.18835246 −.40337309	.10363971 −.10738113	.10395180 −.06163683	.10405602 −.02013735	.10405602 .02013735	.10395180 .06163683	.10363971 .10738113	.18835246 .40337309	
	1	2		.15245229 −.48025640	.09611128 −.12350392	.10126825 −.06738392	.10570764 −.01660026	.10981910 .03247948	.11376585 .08265445	.32087559 .57261056		

TABLE II—*Concluded*

n	r_1	r_2	$y_{(1)}$	$y_{(2)}$	$y_{(3)}$	$y_{(4)}$	$y_{(5)}$	$y_{(6)}$	$y_{(7)}$	$y_{(8)}$	$y_{(9)}$	$y_{(10)}$
10	1	3		.09423044 −.58415474	.08464767 −.14396103	.09790957 −.07337756	.10954691 −.00974899	.12039960 .05136065	.49326580 .75988167			
	1	4		−.00426039 −.73588101	.06648054 −.17194771	.09383130 −.07966018	.11787039 .00307342	.72607816 .98441548				
	1	5		−.18661330 −.98311476	.03511935 −.21446715	.08919507 −.08594598	1.06229888 1.28352789					
	1	6		− .58772619 −1.46776140	−.02885426 −.29176358	1.61658046 1.75952498						
	1	7		−1.89997978 −2.89604971	2.89997978 2.89604971							
	2	2			.27977552 −.70208441	.10994078 −.09470422	.11028371 −.03101584	.11028371 .03101584	.10994078 .09470422	.27977552 .70208441		
	2	3			.20496319 −.88982266	.10382533 −.11005067	.11220127 −.02620385	.11982080 .05494874	.45918942 .97112842			
	2	4			.06055627 −1.19522527	.09352928 −.13182552	.11776906 −.01442865	.72814539 1.34147944				
	2	5			− .26483993 −1.79471038	.07347216 −.16877724	1.19136777 1.96348761					
	2	6			−1.34060718 −3.56767731	2.34060718 3.56767731						
	3	3				.38067840 −1.28320587	.11932160 −.05586351	.11932160 .05586351	.38067840 1.28320587			
	3	4				.18711959 −1.97905791	.11984110 −.05532554	.69303931 2.03438345				
	3	5				− .48466706 −3.95105520	1.48466706 3.95105520					
	4	4					.50000000 −4.07605089	.50000000 4.07605089				

sizes $\leqq 10$. Owing to the fact that the sum of the rows of the variance matrix of the order statistics (and by symmetry the sum of the columns) is equal for any sample size, the best linear systematic statistic of the population mean is, of course, the sample mean [5].

Later, Gupta [3] used Godwin's tables of means, variances, and covariances to find the best linear systematic statistic of the population mean and standard deviation from singly censored samples. Although Gupta considered Godwin's results as being a special case of his own, the more general case of estimation from doubly censored samples considered here includes the other two cases (r_1 or $r_2 = 0$ and $r_1 = r_2 = 0$).

Recently, Rosser [6], at the National Bureau of Standard's Numerical Research Analysis in Los Angeles, tabulated up to nineteen decimal places the expected values of the standardized ith order statistics X_i, where $y_i = \mu + \sigma X_i$. Following this, Teichroew [7], under the same sponsorship, calculated the expected values of the product of the ith and the jth order statistics in samples of sizes $\leqq 20$, drawn from a normal population. These valuable accomplishments make possible an advance in the study of systematic statistics.

Using these tables, the variances and covariances of the order statistics in samples from a normal distribution up to ten decimal places were calculated. The variances and the covariances for order statistics of sample sizes up to 20, to ten decimal places are given in Table I. The missing entries may be obtained by

$$\text{Cov}[X_{(i)}X_{(j)}] = \text{Cov}[X_{(n-i+1)}X_{(n-j+1)}].$$

In comparing the variances and covariances for $n \leqq 10$ with those tabulated (to five decimals) by Godwin [1], it can be seen that several of the latter values are in error by more than one unit in the fifth decimal place, and one value Cov $[X_{(1)}X_{(10)}]$ when $n = 10$ is in error by as much as 5 units.

Table II gives the coefficients α_{1i} and α_{2i} for the best linear systematic statistics of the mean and standard deviation, respectively, from singly and doubly censored samples of sizes $\leqq 10$ drawn from a normal population. For the case when $r_1 = r_2 = 0$, these tabulated results are more accurate than Godwin's [2]. For the singly censored sample, i.e., r_1 or $r_2 = 0$, the tabulated results are more accurate than those of Gupta because the latter based his calculations on the earlier tabulations of Godwin. The tabulated values in Table II are correct except for the last place, which may be in error by one or two units.

In comparing these tabulated values with those of the cited references, the results of Gupta [3] can be compared directly, whereas Godwin's [2] values are obtained in terms of rank differences. Consequently, Godwin's first coefficient should be exactly equivalent, except for sign, to that herein, and the remainder can be obtained by subtraction.

If the coefficients of an estimate are sought for a value of r_1 not given in the table, these can be obtained by interchanging the values of r_1 and r_2 and rearranging the observations in descending order. In such an event, the coefficients

TABLE III

Variances and covariances of the estimates of the mean (μ^*) and standard deviation (σ^*) for censored samples up to size 10 from a normal population

n	r_1		r_2 = 0	1	2	3	4	5	6	7	8
2	0	$V(\mu^*)$	.50000000								
		$V(\sigma^*)$	.57079633								
		$Cov(\mu^*, \sigma^*)$	0								
3	0	$V(\mu^*)$	.33333333	.44867111							
		$V(\sigma^*)$	.27548197	.63782627							
		$Cov(\mu^*, \sigma^*)$	0	.20443088							
4	0	$V(\mu^*)$	.25000000	.28701097	.51299280						
		$V(\sigma^*)$	.18005021	.30207500	.67303155						
		$Cov(\mu^*, \sigma^*)$	0	.06720309	.35673623						
	1	$V(\mu^*)$		.29819962							
		$V(\sigma^*)$		.70572111							
		$Cov(\mu^*, \sigma^*)$		0							
5	0	$V(\mu^*)$	.20000000	.21772341	.28393206	.61122918					
		$V(\sigma^*)$	.13332124	.19476301	.31808583	.69570833					
		$Cov(\mu^*, \sigma^*)$	0	.03299936	.12335995	.47492042					
	1	$V(\mu^*)$		.22579422	.28683366						
		$V(\sigma^*)$		.32968861	.74060939						
		$Cov(\mu^*, \sigma^*)$		0	.15837417						
6	0	$V(\mu^*)$	.16666667	.17688040	.20682717	.29985334	.71864005				
		$V(\sigma^*)$	.10570264	.14277434	.20436158	.32919818	.71192848				
		$Cov(\mu^*, \sigma^*)$	0	.01945869	.06240445	.17016841	.57052121				
	1	$V(\mu^*)$		.18249856	.20698549	.32960016					
		$V(\sigma^*)$		.21017023	.34598687	.76280252					
		$Cov(\mu^*, \sigma^*)$		0	.05766917	.28373932					

6	2	V(μ*)			.21474267						
		V(σ*)			.77471679						
		Cov(μ*, σ*)			0						
7	0	V(μ*)	.14285714	.14942352	.16600214	.20713842	.32479943	.82636223			
		V(σ*)	.08749856	.11233100	.14927902	.21136168	.33752616	.72430317			
		Cov(μ*, σ*)	0	.01276945	.03751914	.08805478	.20989335	.65033968			
	1	V(μ*)		.15346296	.16680538	.20946061	.39543079				
		V(σ*)		.15269771	.22005773	.35717357	.77852463				
		Cov(μ*, σ*)		0	.02997909	.10645593	.38638223				
	2	V(μ*)			.17312828	.21044686					
		V(σ*)			.36219907	.79619765					
		Cov(μ*, σ*)			0	.12726434					
8	0	V(μ*)	.12500000	.12954025	.13988098	.16225105	.21375841	.35409951	.93099884		
		V(σ*)	.07461121	.09242321	.11706951	.15419334	.21678723	.34408411	.73416689		
		Cov(μ*, σ*)	0	.00899283	.02495718	.05377493	.11055562	.24421558	.71859780		
	1	V(μ*)		.13255044	.14091823	.16234808	.22331839	.47112520			
		V(σ*)		.11928888	.15943388	.22720134	.36548926	.79042118			
		Cov(μ*, σ*)		0	.01832825	.05643660	.14825957	.47276076			
	2	V(μ*)			.14567896	.16269076	.23922424				
		V(σ*)			.22999555	.37323003	.81130952				
		Cov(μ*, σ*)			0	.04936270	.23246854				
	3	V(μ*)				.16818086					
		V(σ*)				.81706103					
		Cov(μ*, σ*)				0					

TABLE III—*Continued*

n	r_1		r_2 = 0	1	2	3	4	5	6	7	8
9	0	$V(\mu^*)$	.11111111	.11441700	.12139976	.13515042	.16291237	.22407238	.38537601	1.03129529	
		$V(\sigma^*)$	.06501502	.07842444	.09604682	.12074575	.15810000	.22116450	.34943152	.74228259	
		$Cov(\mu^*, \sigma^*)$	0	.00665933	.01775128	.03618024	.06838314	.13048810	.27432806	.77806418	
	1	$V(\mu^*)$		.11673259	.12244352	.13518326	.16466906	.24349912	.55048580		
		$V(\sigma^*)$		.09758705	.12420684	.16448335	.23270051	.37199378	.79983467		
		$Cov(\mu^*, \sigma^*)$		0	.01232979	.03498177	.07983082	.18461869	.54702931		
	2	$V(\mu^*)$			.12605965	.13612535	.16566283	.28513252			
		$V(\sigma^*)$			.16624730	.23712095	.38138044	.82268191			
		$Cov(\mu^*, \sigma^*)$			0	.02670942	.09198622	.32159927			
	3	$V(\mu^*)$				.14097979	.16610128				
		$V(\sigma^*)$				.38407790	.83165694				
		$Cov(\mu^*, \sigma^*)$				0	.10603703				
10	0	$V(\mu^*)$	.10000000	.10250531	.10749284	.11666340	.13359472	.16641008	.23658560	.41735056	1.12690295
		$V(\sigma^*)$	.05759553	.06806087	.08129462	.09891696	.12372658	.16131247	.22479946	.35390665	.74912245
		$Cov(\mu^*, \sigma^*)$	0	.00512045	.01324471	.02595717	.04645253	.08157225	.14831978	.30108776	.83064069
	1	$V(\mu^*)$		.10433167	.10845984	.11682744	.13388367	.17122372	.26722758	.63041997	
		$V(\sigma^*)$		.08241682	.10135027	.12799713	.16847458	.23711263	.37726676	.80752873	
		$Cov(\mu^*, \sigma^*)$		0	.00884085	.02377304	.05004837	.10067393	.21667108	.61197840	
	2	$V(\mu^*)$			.11126628	.11795477	.13388616	.17665047	.34008123		
		$V(\sigma^*)$			.12920079	.17132071	.24257700	.38772575	.83163934		
		$Cov(\mu^*, \sigma^*)$			0	.01678448	.05047738	.12926308	.39861255		
	3	$V(\mu^*)$				.12182093	.13426986	.18655569			
		$V(\sigma^*)$				.24418850	.39205377	.84259431			
		$Cov(\mu^*, \sigma^*)$				0	.04290412	.19638665			
	4	$V(\mu^*)$					.13832644				
		$V(\sigma^*)$					.84582636				
		$Cov(\mu^*, \sigma^*)$					0				

for the best linear systematic statistic of the mean will be identical with those given in the table, whereas those for the standard deviation will be numerically the same but with opposite sign.

The variances of the estimates and their covariances are given in Table III in terms of σ^2. These values (for $r_1 = 0$) may also be compared with those of Gupta. Their percentage efficiencies relative to the best linear systematic statistic based on the complete sample are tabulated in Table IV.

3. Alternative estimates. Gupta suggested alternative linear estimates for the mean and standard deviation of a normal population to be used for samples of size >10 because of the tediousness of calculating the exact estimate and because the variances and covariances of the order statistics for larger samples were unavailable. He calculated the variances and relative efficiencies of these alternative estimates for a sample of size 10. Gupta's alternative linear estimates were intended for the special case of singly censored samples. In the present work, this has been extended to permit its use in the general case of doubly censored samples.

This alternative estimate is based on the assumption that the variance matrix of the order statistics is a unit matrix. Therefore, the alternative estimates for doubly censored samples with r_1 and r_2 missing observations will be

$$\mu^{*\prime} = \sum_{i=r_1+1}^{n-r_2} b_i y_{(i)}, \tag{3.1}$$

where $\mu^{*\prime}$ is the alternative estimate of the population mean, and

$$\sigma^{*\prime} = \sum_{i=r_1+1}^{n-r_2} c_i y_{(i)}, \tag{3.2}$$

where $\sigma^{*\prime}$ is the alternative estimate of the population standard deviation. The values of b_i and c_i in $\mu^{*\prime}$ and $\sigma^{*\prime}$ are determined by

$$b_i = \frac{1}{n - r_1 - r_2} - \frac{\bar{u}_k(u_i - \bar{u}_k)}{\sum_{j=r_1+1}^{n-r_2} (u_j - \bar{u}_k)^2} \tag{3.3}$$

and

$$c_i = \frac{(u_i - \bar{u}_k)}{\sum_{j=r_1+1}^{n-r_2} (u_j - \bar{u}_k)^2}, \tag{3.4}$$

where

$$\bar{u}_k = \frac{1}{n - r_1 - r_2} \sum_{j=r_1+1}^{n-r_2} u_j,$$

or the arithmetic mean of the expected values of the uncensored sample elements.

The estimates (3.1) and (3.2) are unbiased estimates of the mean and stand-

TABLE IV

Percentage efficiencies of estimates of the mean (μ^*) and standard deviation (σ^*) for censored samples relative to uncensored samples in a normal population up to size 10

n	r_1	r_2								
		0	1	2	3	4	5	6	7	8
2	0	100.00								
		100.00								
3	0	100.00	74.29							
		100.00	43.19							
4	0	100.00	87.10	48.73						
		100.00	59.60	26.75						
	1		83.84							
			25.51							
5	0	100.00	91.86	70.44	32.72					
		100.00	68.45	41.91	19.16					
	1		88.58	69.73						
			40.44	18.00						
6	0	100.00	94.23	80.58	55.58	23.19				
		100.00	74.03	51.72	32.11	14.85				
	1		91.32	80.52	50.57					
			50.29	30.55	13.86					
	2			77.61						
				13.64						
7	0	100.00	95.61	86.06	68.97	43.98	17.29			
		100.00	77.89	58.61	41.40	25.92	12.08			
	1		93.09	85.64	68.20	36.13				
			57.30	39.76	24.50	11.24				
	2			82.52	67.88					
				24.16	10.99					
8	0	100.00	96.50	89.36	77.04	58.48	35.30	13.43		
		100.00	80.73	63.73	48.39	34.42	21.68	10.16		
	1		94.30	88.70	77.00	55.97	26.53			
			62.55	46.80	32.84	20.41	9.44			
	2			85.81	76.83	52.25				
				32.44	19.99	9.20				
	3				74.32					
					9.13					

TABLE IV—*Concluded*

n	r_1	r_2								
		0	1	2	3	4	5	6	7	8
9	0	100.00	97.11	91.52	82.21	68.20	49.59	28.83	10.77	
		100.00	82.90	67.69	53.84	41.12	29.40	18.61	8.76	
	1		95.18	90.74	82.19	67.48	45.63	20.18		
			66.62	52.34	39.53	27.94	17.48	8.13		
	2			88.14	81.62	67.07	38.97			
				39.11	27.42	17.05	7.90			
	3				78.81	66.89				
					16.93	7.82				
10	0	100.00	97.56	93.03	85.72	74.85	60.09	42.27	23.96	8.87
		100.00	84.62	70.85	58.23	46.55	35.70	25.62	16.27	7.69
	1		95.85	92.20	85.60	74.69	58.40	37.42	15.86	
			69.88	56.83	45.00	34.19	24.29	15.27	7.13	
	2			89.87	84.78	74.69	56.61	29.40		
				44.58	33.62	23.74	14.85	6.93		
	3				82.09	74.48	53.60			
					23.59	14.69	6.84			
	4					72.29				
						6.81				

ard deviation, respectively. In order to compare these estimates with the best linear systematic statistics of the mean and standard deviation, the coefficients, variances, and relative efficiencies of the alternative estimates were calculated for all samples of size up to and including 10 and for all values of r_1 and r_2. The variances and relative efficiencies of the alternative estimate are provided only for samples of size 10 in Table V. The variance matrix of the estimates is obtained from $(A'A)^{-1}A'VA(A'A)^{-1}\sigma^2$, where V is the variance matrix of the order statistics and A is the coefficient matrix when their means are expressed in terms of the parameters. The percentage efficiencies are relative to the corresponding best linear systematic statistics.

Examination of Table V shows that for $n = 10$, and for all the values of r_1 and r_2, the alternative estimates are highly efficient and can replace the best linear systematic statistics without great loss of precision. In fact, use of these estimates for samples greater than 10 may be preferable to awaiting tabulation of coefficients for the most efficient estimates which involve the inversion of matrices of large order. The most efficient linear systematic statistics for the mean and standard deviation of a normal population from censored samples of sizes $\leqq 20$ will be given in a sequel to this paper (Part II). The authors feel,

TABLE V

Variances and relative efficiencies of alternative estimates of the mean ($\mu^{*\prime}$) and standard deviation ($\sigma^{*\prime}$) for censored samples of size 10 from a normal population*

n	r_1		r_2 = 0	1	2	3	4	5	6	7	8
10	0	$\mu^{*\prime}$ V.	.10000000	.10308928	.10961456	.12148953	.14281760	.18261677	.26336721	.45614307	1.12690295
		$\mu^{*\prime}$ R.E.	100.00	99.43	98.06	96.03	93.54	91.13	89.83	91.50	100.00
		$\sigma^{*\prime}$ V.	.05766121	.07022636	.08642325	.10748774	.13638670	.17888915	.24796349	.38065951	.74912245
		$\sigma^{*\prime}$ R.E.	99.87	96.92	94.07	92.03	90.72	90.17	90.66	92.97	100.00
	1	$\mu^{*\prime}$ V.		.10534509	.11034725	.12008840	.13904835	.17855378	.27531177	.63041997	
		$\mu^{*\prime}$ R.E.		99.04	98.29	97.28	96.29	95.89	97.06	100.00	
		$\sigma^{*\prime}$ V.		.08489780	.10545474	.13383112	.17586018	.24533968	.38411228	.80752873	
		$\sigma^{*\prime}$ R.E.		97.08	96.11	95.64	95.80	96.65	98.22	100.00	
	2	$\mu^{*\prime}$ V.			.11330806	.12042277	.13682291	.17922164	.34008123		
		$\mu^{*\prime}$ R.E.			98.20	97.95	97.85	98.57	100.00		
		$\sigma^{*\prime}$ V.			.13414020	.17683334	.24748758	.38944383	.83163934		
		$\sigma^{*\prime}$ R.E.			96.32	96.88	98.02	99.56	100.00		
	3	$\mu^{*\prime}$ V.				.12376594	.13558068	.18655569			
		$\mu^{*\prime}$ R.E.				98.43	99.03	100.00			
		$\sigma^{*\prime}$ V.				.24837948	.39221654	.84259430			
		$\sigma^{*\prime}$ R.E.				96.16	99.96	100.00			
	4	$\mu^{*\prime}$ V.					.13832644				
		$\mu^{*\prime}$ R.E.					100.00				
		$\sigma^{*\prime}$ V.					.84582633				
		$\sigma^{*\prime}$ R.E.					100.00				

* The percentage efficiency is calculated relative to the corresponding best linear systematic statistic.

however, that for samples of sizes >20, the alternative estimate should be used because of its high relative efficiency to the exact estimate.

4. Numerical example. Ten medical students were learning to measure systolic blood pressure and practicing by taking readings upon each other. Poor technique was present in all observations but the skill of the observers in using the measuring device was limited initially to the central portion of the range. Owing to the relatively larger measurement error known to exist at the extremes, observations believed to be less than 105 mm. and greater than 125 mm. were censored.[1] This practice resulted in censoring two observations on the left and three on the right in the sample of ten.

The data, when arranged in an array, appear as the first column in the following tabulation:

Ordered observations	Coefficients			
	Exact estimate		Alternate estimative	
	μ^*	σ^*	$\mu^{*\prime}$	$\sigma^{*\prime}$
1. —	0	0	0	0
2. —	0	0	0	0
3. 108	.20496319	−.88982266	.09515275	−.79906860
4. 111	.10382533	−.11005067	.15114637	−.37232645
5. 119	.11220127	−.02620385	.20170682	.01300816
6. 121	.11982080	.05494874	.25071680	.38652614
7. 125	.45918942	.97112842	.30127725	.77186075
8. —	0	0	0	0
9. —	0	0	0	0
10. —	0	0	0	0
Estimate of parameter by linear systematic statistics	118.9	16.61	119.1	17.17

If one assumes that the sample was drawn from a normal universe, the exact coefficients (α_{1i} and α_{2i}) for the best linear estimate, obtained from Table II for the case $n = 10, r_1 = 2, r_2 = 3$, are those shown in the second and third columns. If the alternative estimate is desired, the coefficients are obtainable from (3.3) and (3.4) in combination with a table for the values of u_i.

The exact and alternative estimates of the mean and standard deviation are provided. They are similar, as they should be according to the relative efficiencies

[1] The derivation of coefficients given in Table II for estimating the mean and standard deviation was based upon the assumption that censoring would occur by fixed percentages of the sample. The use of these same coefficients in the present example where censoring is performed according to fixed points on the abscissa raises a question of possible bias. Based upon the results in a sampling investigation, it is felt that this possible bias is probably not an appreciable one and of little concern in practical considerations.

given in Table V, viz., 97.95 percent for the mean and 96.88 percent for the standard deviation.

5. Conclusions. Certain characteristic features can be gleaned from the tables presented herein. These are as follows:

(1). *Table* II (*Coefficients for the exact estimates*)

(a) By censoring a sample, the coefficients for the estimate of the mean and standard deviation undergoing maximum change are those which are associated with the extreme known sample elements.

(b) For a fixed sample size and for r_1 fixed, as r_2 increases the coefficient of the largest known element increases, whereas that of the smallest known element decreases.

(c) If the sample size is odd, and all the sample elements are censored except the middle one and its neighbor on either side, the central observation will have all the weight in estimating the mean (i.e., the other observation is of no value). If the sample size is even under the same circumstances of censoring, each middle observation has one-half of the weight in estimating the mean.

(2). *Table* IV (*Relative efficiency of exact estimates*)

(a) Comparing the relative efficiencies of the estimate for σ with the corresponding efficiency for μ in censored samples, the efficiency of the estimate of the former drops more rapidly than that of μ.

(b) Reading the entries for σ^* in diagonal fashion reveals that, for fixed n and fixed uncensored sample size ($r_1 + r_2$ = constant), the efficiency of the "best" estimate of σ is remarkably constant independently of r_1 and r_2. That is, it does not matter whether the missing observations are at one end or the other of the sample or divided in any way between the two ends.

(c) Owing to the previous relationship, an interesting simple table is given showing how the efficiency in estimating σ varies with the number of known values for each sample size. This is useful for practical work in censoring.

Rough guide for assessing approximate efficiency (percent) of estimates of σ*

Sample size, n	Number of uncensored observations in sample, or $k = n - r_1 - r_2$								
	2	3	4	5	6	7	8	9	10
2	100								
3	45	100							
4	25	60	100						
5	20	40	70	100					
6	15	30	50	75	100				
7	11	25	40	60	75	100			
8	9	20	35	45	60	80	100		
9	8	17	30	40	50	65	80	100	
10	7	15	25	35	45	55	70	85	100

* These values are only 2 or 3 percent off (or less) in almost all cases.

(d) A simple fact is also evident for the estimate of the mean, μ. Its relative efficiency holds up—about *70 percent or better*—so long as the sample median value (the two middle values, if size n is even) remains known, no matter how many values below (or above) this one are missing. Thus, for $n = 10$, even if only the two values $y_{(5)}$ and $y_{(6)}$ are known, with the other 8 missing ($r_1 = r_2 = 4$), the efficiency is still 72 percent. But if one of these mid-values, e.g., $y_{(6)}$, is lost, then all of the other values (to one side) $y_{(1)}$, $y_{(2)}$, $y_{(3)}$, $y_{(4)}$, $y_{(5)}$ (i.e., $r_1 = 0$, $r_2 = 5$) cannot make up for it. They produce an efficiency no better than 60 percent. In other words, *a single central value is worth more than half the sample in estimating the mean.*

(e) In estimating the standard deviation, the foregoing situation for the mean is the direct opposite. In fact, hardly any censoring is tolerable even in the most favorable case. Thus, for $n = 10$, it can be seen from the above rough table that for as little as two missing observations ($r_1 + r_2 = 2$), whether at the same or opposite ends of the sample, the efficiency barely attains 70 percent. For more missing elements, the efficiency drops rapidly from a value under 60 to as little as 7 percent.

(3). *Table* V. (Relative efficiency and variances of alternative estimates, $n = 10$).

The alternative and the "best" estimate of either μ or σ are identical, as indicated by the 100 percent entry, when $k = n - r_1 - r_2 = 2$, i.e., when only two of the sample values are known. This is because there are then only two coefficients to be estimated, which makes the unbiased estimator unique and therefore the same as the (unbiased) alternative estimator, which is therefore a fortiori of minimum variance. The other case of identity is for a complete sample ($r_1 = r_2 = 0$), where the best estimate of μ is the sample mean, and this is also the alternative estimate. This is because the average, $\bar{u}_n = \sum_{i=1}^{n} u_i$, of all the means of the sample order statistics is equal to the population mean, which may be taken as zero without loss of generality; by (3.3) this shows that all the coefficients b_i must be equal, giving the sample mean.

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