Estimation of Markov-Modulated Time-Series via EM Algorithm

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***Abstract*—In this letter, we consider the estimation of various Markov-modulated time series. We obtain maximum likelihood estimates of the time-series parameters including the Markov chain transition probabilities and the time-series coefficients using the expectation maximization (EM) algorithm. In addition, the recursive EM algorithm is used to obtain on-line parameter estimates. Simulation studies show that both algorithms yield satisfactory results.**

I. Introduction

Signal Model: Let s_k denote a N_s-state irreducible Markov chain with states $\{1, 2, \ldots, N_s\}$ with transition probability matrix $\Pi = (\pi_{mn})$, $\pi_{mn} = P(s_{k+1} = n \mid s_k = m)$ and initial state probability $\pi = (\pi_m)$, $\pi_m = P(s_1 = m)$. Define the Markov-modulated polynomials $A(z^{-1}, s_k)$, $B(z^{-1}, s_k)$, and $C(z^{-1}, s_k)$ as (where z^{-1} denotes the delay operator and k denotes discrete time)

$$A(z^{-1}, s_k) = 1 + \sum_{i=1}^{p} a_i(s_k) z^{-i},$$

$$B(z^{-1}, s_k) = 1 + \sum_{i=1}^{q} b_i(s_k) z^{-i},$$

$$C(z^{-1}, s_k) = 1 + \sum_{i=1}^{r} c_i(s_k) z^{-i}. \tag{1.1}$$

Let $A(m) \triangleq (a_1(m) \cdots a_p(m))'$, $B(m) \triangleq (b_1(m) \cdots b_q(m))'$, $C(m) \triangleq (c_1(m) \cdots c_r(m))'$, $m \in \{1, 2, \ldots, N_s\}$.

In this letter, we consider parameter estimation of any one of the following second-order stationary (see Remark 4 below) Markov-modulated time-series models:

$$\textbf{ARX}: A(z^{-1}, s_k) y_k = B(z^{-1}, s_k) u_k + w_k; \quad \phi = (A(m), B(m), \Pi, \sigma^2) \tag{1.2}$$

$$\textbf{MAX}: y_k = B(z^{-1}, s_k) u_k + C(z^{-1}, s_k) w_k; \quad \phi = (B(m), C(m), \Pi, \sigma^2) \tag{1.3}$$

$$\textbf{ARMA}: A(z^{-1}) y_k = C(z^{-1}, s_k) w_k; \quad \phi = (A, C(m), \Pi, \sigma^2) \tag{1.4}$$

where u_k and y_k are the measured input and output at time k, $w_k \sim$ white $N(0, \sigma^2)$ is independent of s_k, and ϕ is the parameter vector consisting of polynomial coefficients and Markov chain parameters. We assume u_k to be persistently exciting [4]. We also assume that $A(z^{-1}, s_k)$, $B(z^{-1}, s_k)$ and $C(z^{-1}, s_k)$ are coprime to each other for each m, $m \in \{1, 2, \ldots, N_s\}$.

Notations: $Y_k = (y_1 \ldots y_k)^T$, $U_k = (u_1 \cdots u_k)^T$, $S_k = (s_1 \cdots s_k)^T$, $Y_t^k = (y_t \cdots y_k)^T$, $U_t^k = (u_t \cdots u_k)^T$ and $Z_k = (Y_k, U_k)$, where superscript T denotes transpose.

Estimation Objectives: We use the expectation maximization (EM) algorithm [10] to obtain maximum likelihood (ML) estimates of ϕ, given Y_T and U_T (when appropriate) in Section II. In addition, based on the recursive EM algorithm [2], an on-line estimation scheme is presented in Section III.

Motivation and Applications: The models (1.2), (1.3), and (1.4) consist of parameter sets that are constant over segments with abrupt changes from segment to segment. The parameter sets are determined by the realization of a finite state Markov chain. Such so called "segmentation" models are used in econometrics, seismology, geology, and image analysis (see [5] and references therein).

In [5], the EM algorithm and a recursive EM algorithm are used to estimate Markov-modulated AR processes, which is a special case of our model (1.2) with $B = 0$. The three models we consider in this paper can be regarded as an extension of the work in [5].

Our model can be also viewed as a *random coefficient* time series. These are used to model the stochastic stability of short run market equilibrium under variations in supply (see [9] and references therein). Markov-modulated models also used in econometrics [6] and failure detection [7].

Remark 1: Models (1.2), (1.3), or (1.4) are special cases of the Markov-modulated ARMAX model

$$A(z^{-1}, s_k) y_k = B(z^{-1}, s_k) u_k + C(z^{-1}, s_k) w_k. \tag{1.5}$$

However, unlike (1.2), (1.3), and (1.4), ML estimation of (1.5) is computationally prohibitive since it requires computing probability density functions over all N_s^T realizations of a N_s state T-point Markov chain. We do not deal with estimating (1.5) in this letter. For similar reasons, we forbid $A(z^{-1})$ in (1.4) to be Markov modulated. Various suboptimal techniques for estimating Markov-modulated ARMAX models exist in the literature [11], [3].

Remark 2: As a more generalized model, one can have $A(z^{-1}, s_k)$, $B(z^{-1}, t_k)$ and $C(z^{-1}, r_k)$ modulated by three independent Markov chains s_k, t_k, r_k. In this letter, we consider

Manuscript received January 12, 1994; approved June 15, 1994. This work was supported by the Cooperative Research Centre for Robust and Adaptive Systems by the Australian Commonwealth Government under the Cooperative Research Centres Program.The associate editor coordinating the review of this letter and approving it for publication was Prof. T. S. Durrani.

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IEEE Log Number 9405625.

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the special case $s_k = t_k = r_k$ for notational simplicity, although our approach can be easily extended. Another obvious extension is to consider polynomials $A(z^{-1}, S_k)$, $B(z^{-1}, S_k)$, and $C(z^{-1}, S_k)$, where $S_k = (s_k, s_{k-1}, \ldots, s_{k-p})'$ is a vector state Markov chain.

Remark 3: State estimates of s_k are obtained from the E-step of the EM algorithm.

Remark 4: Deriving stationarity criteria for Markov-modulated time series is a difficult problem. For example, two switching, separately second-order AR stationary processes can result in an unstable system—whereas two individually unstable AR processes can be stabilized when allowed to switch according to a Markov regime. For sufficient conditions on the second-order stationarity of Markov-modulated time series, see [8] and [5].

II. ML Estimation via EM Algorithm

A. Markov-Modulated ARX Estimation

The EM algorithm for estimating ϕ in (1.2) involves two steps: **E-step** and **M-step**.

E Step: Following [3], the expectation of the log-likelihood function of a T-point "complete" data sequence $M_T = (Y_T, U_T, S_T)$ defined as $Q(\phi^{(l)}, \phi) \triangleq E\{\ln f(M_T \mid \phi) \mid Z_T, \phi^{(l)}\}$ can be written as

$$\begin{aligned} Q(\phi^{(l)}, \phi) = &-\frac{T}{2}\ln\sigma^2 - \frac{1}{2\sigma^2}\sum_{k=1}^{T-1}\sum_{m=1}^{N_s} \\ &\times \gamma_k(m)(A(z^{-1}, m)y_k - B(z^{-1}, m)u_k)^2 \\ &+ \sum_{k=1}^{T-1}\sum_{m=1}^{N_s}\sum_{n=1}^{N_s}\xi_k(m,n)\ln\pi_{mn} \\ &+ \sum_{m=1}^{N_s}\gamma_1(m)\ln\pi_m \end{aligned} \tag{2.1}$$

where $\xi_k(m,n) \triangleq f(s_k = m, s_{k+1} = n \mid Z_T, \phi^{(l)})$, and $\gamma_k(m) \triangleq f(s_k = m \mid Z_T, \phi^{(l)})$. $\gamma_k(m)$ is computed via the "forward backward" procedure [1] as $\alpha_k(m)\beta_k(m)/\sum_{m=1}^{N_s}\alpha_k(m)\beta_k(m)$, where $\alpha_k(m)$ and $\beta_k(m)$ are calculated recursively as $\alpha_k(m) = \sum_{n=1}^{N_s}\alpha_{k-1}(n)a_{nm}b_m(y_k)$, and $\beta_k(m) = \sum_{n=1}^{N_s}a_{mn}b_n(y_{k+1})\beta_{k+1}(n)$.

Here $b_m(y_k) \triangleq f(y_k \mid Y_{k-p}^k, U_{k-q}^k, s_k = m, \phi) = \frac{1}{\sqrt{2\pi\sigma^2}}\exp(-\frac{(A(z^{-1},m)y_k - B(z^{-1},m)u_k)^2}{2\sigma^2})$. $\phi^{(l)}$ is the estimate of the parameter vector at the lth iteration, assuming the iteration procedure starts with an initial estimate $\phi^{(0)}$.

M Step: This step involves computing $\text{argmax}_\phi Q(\phi^{(l)}, \phi)$. This yields

$$\begin{aligned} \pi_{mn} &= \frac{\sum_{k=1}^{T-1}\xi_k(m,n)}{\sum_{k=1}^{T-1}\gamma_k(m)}, \\ \sigma^2 &= \frac{1}{T-1}\sum_{k=1}^{T-1}\sum_{m=1}^{N_s}\gamma_k(m) \\ &\quad \times \left(A(z^{-1}, s_k)y_k - B(z^{-1}, s_k)u_k\right)^2 \qquad (2.2) \\ A(m) &= R_a(m)^{-1}v_a(m), \\ B(m) &= R_b(j)^{-1}v_b(m), \\ &\quad m \in \{1, 2, \ldots, N_s\} \end{aligned} \tag{2.3}$$

where $R_a(m) \in \Re^{p\times p}$ with elements $\sum_{k=1}^{T-1}\gamma_k(m)y(k-i)y(k-j)$, $i, j \in \{1, 2, \ldots, p\}$ $R_b(m) \in \Re^{q\times q}$ with elements $\sum_{k=1}^{T-1}\gamma_k(m)u(k-i)u(k-j)$, $i, j \in \{1, 2, \ldots, q\}$ $v_a(n) \in \Re^q$ with elements $\sum_{k=1}^{T-1}\gamma_k(n)(y_k - B(z^{-1}, s_k)u_k)y_{k-i}$, $i \in \{1, 2, \ldots, p\}$, and $v_b(n) \in \Re^q$ with elements $\sum_{k=1}^{T-1}\gamma_k(n)(A(z^{-1}, s_k)y_k - u_k)u_{k-i}$, $i \in \{1, 2, \ldots, q\}$.

B. Markov-Modulated MAX Estimation

The MAX model (1.3) can be written in equivalent ARX form as $A'(z^{-1}, s_k)y_k = B'(z^{-1}, s_k)u_k + e_k$, where $A'(z^{-1}, s_k)$ is "sufficiently" long enough (see Remark below) to ensure that e_k is almost white and $B'(z^{-1}, s_k) = A'(z^{-1}, s_k)B(z^{-1}, s_k)$. In addition, let $A'(m)$ and $B'(m)$ be the vectors containing the coefficients of $A'(z^{-1}, s_k)$ and $B'(z^{-1}, s_k)$, $m \in \{1, 2, \ldots, N_s\}$, respectively.

EM Algorithm: After estimating $A'(m)$ and $B'(m)$ by the above EM algorithm in Section II-A, $B(m)$ can be estimated by polynomial division. To estimate $C(m)$ in (1.3), a set of *inverse Yule-Walker equations* has to be solved (see p. 291 of [4]), which are

$$\theta_t(m) + \sum_{i=1}^{r}c_i\theta_{t-i}(m) = 0 \quad t \geq 1, \qquad m \in \{1, 2, \ldots, N_s\} \tag{2.4}$$

where $\theta_t(m)$, $t \geq 1$ can be estimated from the coefficients of $A'(z^{-1}, s_k)$ as $\theta_t(m) = \sum_{i=0}^{p'-t}a'_i(m)a'_{i+t}(m)$, where $A'(z^{-1}, s_k) = \sum_{i=0}^{p'}a'_i(m)z^{-i}$, $a'_0 = 1$.

Remark: The order p' of $A'(z^{-1}, s_k)$ has to be large enough to be a good approximation of $1/C(z^{-1}, s_k)$ in (1.3) (see p. 291 of [4]). For rigorous details, see Theorem 8.3.1 in p. 246 of [12], where it is proved that for weak consistency p' should be chosen as $O(T^{1/3})$.

C. Markov-Modulated ARMA Estimation

Since A in (1.4) is no longer Markov modulated, it can be estimated via the Yule-Walker equations

$$\eta_t + \sum_{i=1}^{p}a_i\eta_{t-i} = 0 \quad t \geq r+1 \tag{2.5}$$

where $\eta_t = E[y_k y_{k-t}]$ (see p. 289, [4]).

EM Algorithm: Rewrite (1.4) as $A(z^{-1})A'(z^{-1}, s_k)y_k = e_k$, where e_k and $A'(z^{-1}, s_k)$ are as defined in Section II-B. After obtaining an estimate of $A(z^{-1})A'(z^{-1}, s_k)$ via EM, dividing $A(z^{-1})$ by $A(z^{-1})A'(z^{-1}, s_k)$ gives an estimate of $C(z^{-1}, s_k)$, and hence, $C(m), m \in \{1, 2, \ldots, N_s\}$.

Remark: $C(z^{-1}, s_k)$ could be also estimated by solving the inverse Yule-Walker equations (2.4) for $t \geq p+1$. However, simulations show that the above technique yields better results.

III. ON-LINE ESTIMATION VIA THE RECURSIVE EM ALGORITHM

For brevity, we mention the relevant estimation equations only (for motivation and details of the recursive EM algorithm see [2] and the references therein).

Define ϕ_k as the estimate of the model ϕ at the k-th time instant and $\Phi_k = (\phi_1, \ldots, \phi_k)$. Following [2], our recursive EM algorithm based on maximizing the Kullback-Leibler information measure is

$$\phi_{k+1} = \phi_k + (I_{k+1}(\phi_k))^{-1} S(\phi_k, y_{k+1}) \tag{3.1}$$

where $I_{k+1}(\phi_k)$ is the Fisher information matrix (FIM) of the complete data M_k, given by $I_{k+1}(\phi_k) = -\partial^2 Q_{k+1}(\Phi_k, \phi)/\partial\phi^2|_{\phi=\phi_k}$, $Q_{k+1}(\Phi_k, \phi) \triangleq E\{\ln f(M_k \mid \phi) \mid Z_k, \Phi^{(k-1)}\}$, and $S(\phi_k, y_{k+1})$ is the score vector defined as $S(\phi_k, y_{k+1}) = \partial Q_{k+1}(\Phi_k, \phi)/\partial\phi|_{\phi=\phi_k}$.

Remarks:

1. Exponential forgetting can be used in updating the FIM as follows ($\lambda_F = 1$ means no forgetting):

$$I_{k+1} = \lambda_F I_k + V_{k+1}, 0 < \lambda_F \leq 1 \tag{3.2}$$

   where λ_F is the forgetting factor, and V_{k+1} is that part of the FIM computed at time $k+1$.
2. In the ARMA estimation problem, $A(z^{-1})$ is estimated by the recursive version of the Yule-Walker equations described in Section II with an appropriate forgetting factor λ_{yw}.

IV. SIMULATION STUDIES

We present two examples, with $N_s = 2$, $\pi_{11} = \pi_{22} = 0.9$, and u_k uniformly distributed in $(0, 1)$ (where applicable).

A. ML Estimation via EM Algorithm

Results: For 50 000 data-points, the following hold:

1. **MAX**: The true parameter vector $\phi_0 = (B(1)', C(1)', B(2)', C(2)', \pi_{11}\,\pi_{22}\,\sigma^2) = ((0.8\ 0.3), (0.5\ 0.3), (0.5\ 0.1), (-0.4\ 0.2), 0.9\ 0.9\ 0.25)$.

   After 50 passes, we obtained $\phi^{50} = ((0.7841\ 0.2882), (0.4756\ 0.2972), (0.4740\ 0.1189), (-0.3820\ 0.2043), (0.9066\ 0.9156\ 0.2586)$.
2. **ARMA**: The true parameter vector $\phi_0 = (A', C(1)', C(2)', \pi_{11}\,\pi_{22}\,\sigma^2) = ((-1.0\ 0.3), (0.5\ 0.3), (-0.4\ 0.2), 0.9\ 0.9\ 0.25)$.

   $\phi^{50} = ((-1.0037\ 0.3078), (0.4795\ 0.2892), (-0.3955\ 0.2162), 0.9099\ 0.9087\ 0.2525)$.

B. On-Line Estimation via Recursive EM Algorithm

Consider a jump time-varying 100 000-point Markov-modulated ARMA model with $\sigma^2 = 1$ and

$$\begin{aligned} A &= (0.8\ -0.5)', C(1) = (0.5\ 0.3)', \\ &C(2) = (-0.4\ 0.2)'\ t \leq 20000 \\ A &= (0.5\ -0.8)', C(1) = (0.7\ 0.5)', \\ &C(2) = (-0.2\ 0.5)'\ t > 20000. \end{aligned} \tag{4.1}$$

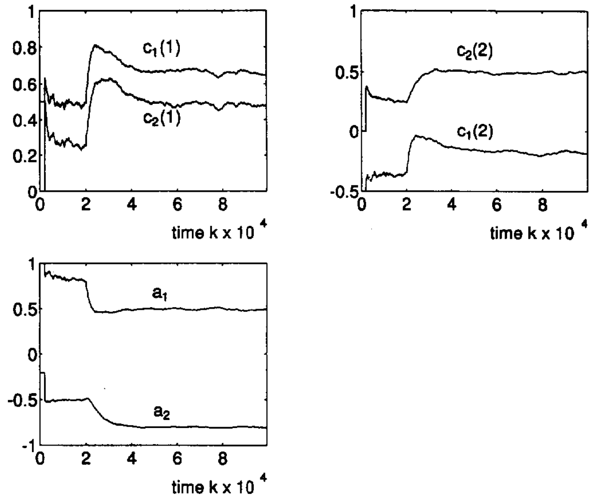

Fig. 1. Time evolution of ARMA parameter estimates.

In our simulation, $\lambda_F = 0.9999$, $\lambda_{yw} = 0.9999$, and $p' = 15$. We do not assume any *a priori* knowledge in the parameter values, and initial conditions may be arbitrary. To avoid the effect of the initial transients on the parameter estimates due to insufficient data, estimation starts after the first 2000 points, which is the period during which only the FIM is updated. Fig. 1 shows the time evolution of the estimates of the ARMA parameters.

Remark: The convergence proof of the recursive EM algorithm for HMM's is an open problem. Simulation studies show that the larger σ^2 is, the slower the convergence. It is seen from Fig. 1 that the estimates are close to the true values after a few thousand data points.

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