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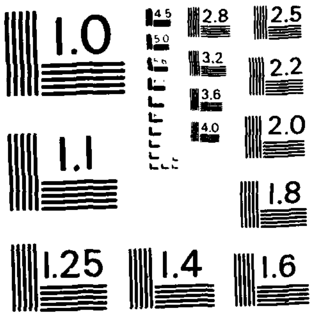

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ESTIMATION OF NOISY TELEGRAPH PROCESSES: NONLINEAR FILTERING VS. NONLINEAR SMOOTHING

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Estimation of Noisy Telegraph Processes: Nonlinear Filtering vs. Nonlinear Smoothing

Abstract

In the estimation problem of a two-state stationary Markov process with Gaussian white noise added, the optimal smoother is a two-filter smoother. In a special case, we compare analytically the optimal nonlinear filter and smoother and find that the latter is significantly better than the former when either the noise intensity or the rate of jump of the states is low.

Key words: Nonlinear filtering, nonlinear smoothing, telegraph process

AMS 1980 subject classification: Primary 93E14; Secondary 62M05,93E11

1. Introduction

In a certain class of linear dynamic Gaussian systems, the optimal smoothing estimator of the states may be regarded as a two-filter smoother. See Wall, et al (1981) for a complete discussion. Several authors, e.g. Rauch, et al (1965) and Mehra and Bryson (1968) compared the performance of filters and smoothers in linear dynamic systems and found that in certain cases smoothers may be more preferable even at the expense of time delay.

In this paper, we consider the system of a telegraph process in the presence of additive Gaussian white noise, and study the relationship between the optimal filtering and smoothing estimators of the states. To be specific, define the signal process $\{\mu_t; -\infty < t < \infty\}$ to be a stationary two-state Markov process such that

$$Pr(\mu_t = 1) = p = 1 - Pr(\mu_t = -1)$$

$$(1.1)\quad Pr(\mu_{t+h} = 1 \mid \mu_t = 1) = 1 - \nu h + o(h)$$

$$Pr(\mu_{t+h} = -1 \mid \mu_t = -1) = 1 - \nu' h + o(h)$$

where $p\nu = (1-p)\nu'$. Let the observed process

$$(1.2)\quad Z_t \equiv \int_0^t \mu_s \, ds + \alpha W_t, \quad t \in (-\infty, \infty)$$

where $\{W_t\}$ is a standard Wiener process $(W_0 = 0)$

independent of $\{\nu_t\}$ and α is a positive constant to represent the intensity of the noise.

In the next section, we show that the optimal smoothing estimator of ν_t is a function of the forward and backward filtering estimators of ν_t. In Section 3, we compare analytically the performance of the optimal nonlinear filter and smoother in a special case.

2. Optimal Nonlinear Smoother as a Two-Filter Smoother

In this section, we assume the process $\{Z_t\}$ is observed from $t = 0$ on. Denote by $\hat{\nu}_t^L$, $\hat{\nu}_t^R$ and $\hat{\nu}_t$ the left-sided, right-sided and two-sided conditional expectations of ν_t, i.e. $E(\nu_t|Z_0^t)$, $E(\nu_t|Z_t^T)$ and $E(\nu_t|Z_0^T)$, respectively, where T is a fixed time span (possibly $+\infty$), $0 < t < T$, $Z_s^t \equiv \{Z_u;\ s \leq u \leq t\}$ and $Z_t^T \equiv \{Z_s - Z_t;\ t \leq s \leq T\}$. Wonham (1965) showed that $\hat{\nu}_t^L$ satisfies a stochastic differential equation. We can easily see that $\hat{\nu}_t^R$ must satisfy a reversed-time stochastic differential equation of the same type. Now, the following proposition tells us that the smoothing estimate $\hat{\nu}_t$ of ν_t can be easily computed from knowing $\hat{\nu}_t^L$ and $\hat{\nu}_t^R$.

Proposition 2.1

$$\hat{\nu}_t = \frac{(1+\hat{\nu}_t^L)(1+\hat{\nu}_t^R)-2p(1+\hat{\nu}_t^L\hat{\nu}_t^R)}{(1+\hat{\nu}_t^L)(1+\hat{\nu}_t^R)-2p(\hat{\nu}_t^L+\hat{\nu}_t^R)}$$

i.e. $$\tanh^{-1}\hat{\nu}_t = \tanh^{-1}\hat{\nu}_t^L + \tanh^{-1}\hat{\nu}_t^R + \frac{1}{2}\log\frac{1-p}{p}$$

In particular, when $p = 1/2$,

$$\hat{\nu}_t = (\hat{\nu}_t^L + \hat{\nu}_t^R)/(1 + \hat{\nu}_t^L\hat{\nu}_t^R)$$

i.e. $$\tanh^{-1}\hat{\nu}_t = \tanh^{-1}\hat{\nu}_t^L + \tanh^{-1}\hat{\nu}_t^R$$

Proof of Proposition 2.1

Denote by $f_{Z_0^T}(\cdot)$ (or $f_{Z_0^T}(\cdot|\nu_t=\pm 1)$) the Radon-Nikodym derivative of the measure on $C[0,T]$ induced by Z_0^T *(or Z_0^T conditional on $\nu_t = \pm 1$, respectively)* with respect to the Wiener measure on $C[0,T]$. The existence of the derivatives is a consequence of Girsanov's Theorem (see [1], Theorem 7.2). Let $b = 1$ or -1. Using the independence of Z_0^t and Z_t^T given ν_t,

$$\begin{aligned}\Pr(\nu_t = b|Z_0^T=z_0^T) &= f_{Z_0^T}(z_0^T|\nu_t=b)\Pr(\nu_t=b)/f_{Z_0^T}(z_0^T)\\ &= f_{Z_0^T}(z_0^T|\nu_t=b)\Pr(\nu_t=b)\\ &= f_{Z_0^t}(z_0^t|\nu_t=b)f_{Z_t^T}(z_t^T|\nu_t=b)\Pr(\nu_t=b)\\ &= \Pr(\nu_t=b|Z_0^t=z_0^t)\Pr(\nu_t=b|Z_t^T=z_t^T)/\Pr(\nu_t=b)\end{aligned}$$

So,

$$\frac{\Pr(\mu_t=1|Z_0^T=z_0^T)}{\Pr(\mu_t=-1|Z_0^T=z_0^T)} = \frac{\Pr(\mu_t=1|Z_0^t=z_0^t)}{\Pr(\mu_t=-1|Z_0^t=z_0^t)} \; \frac{\Pr(\mu_t=1|Z_t^T=z_t^T)}{\Pr(\mu_t=-1|Z_t^T=z_t^T)} \; \frac{\Pr(\mu_t=-1)}{\Pr(\mu_t=1)}$$

Since $\hat{\mu}_t^L = \Pr(\mu_t=1|Z_0^t=z_0^t) - \Pr(\mu_t=-1|Z_0^t=z_0^t)$, etc.

$$\frac{1+\hat{\mu}_t}{1-\hat{\mu}_t} = \frac{1+\hat{\mu}_t^L}{1-\hat{\mu}_t^L} \; \frac{1+\hat{\mu}_t^R}{1-\hat{\mu}_t^R} \; \frac{1-p}{p}$$

Observing $2 \tanh^{-1} x = \log(1+x)(1-x)^{-1}$, we complete the proof. □

3. Comparison Between Filtering and Smoothing

In this section, we study the performance of the optimal nonlinear filter and smoother. In order to derive some analytic results, we consider estimates based on an infinite time span and $p = 1/2$. Denote by q_t^L, q_t^R and q_t the left-sided, right-sided and two-sided conditional expectations of μ_t, i.e. $E(\mu_t|Z_{-\infty}^t)$, $E(\mu_t|\ Z_t^{\infty})$ and $E(\mu_t|Z_{-\infty}^{\infty})$, respectively. Define

$$(3.1)\quad MSE(q_t^L) \equiv E[E(\mu_t|Z_{-\infty}^t)-\mu_t]^2 \equiv E(q_t^L-\mu_t)^2$$

and

$$(3.2)\quad MSE(q_t) \equiv E[E(\mu_t|Z_{-\infty}^{\infty})-\mu_t]^2 \equiv E(q_t-\mu_t)^2$$

It should be noted that $MSE(q_t)$ is constant for $t \in (-\infty,\infty)$ but $MSE(q_t^L) < MSE(q_u^L) = MSE(q_v^L)$ for $t<0 \le u < v$, for when $t < 0$

$$(3.3)\quad \mathcal{L}(q_t^L|\mu_t) = \mathcal{L}(E[\mu_t|Z_t, Z_s - Z_t, s \le t]|\mu_t)$$

$$= \mathcal{L}(E[\mu_0|Z_{-\infty}^0, Z_{-t}]|\mu_0)$$

where $\mathcal{L}(Y)$ is the distribution of random variable Y and $\mathcal{L}(Y|X)$ is the conditional distribution of Y given X.

In the following, we only consider $t \ge 0$.

We can readily modify the proof of Proposition 2.1 to show

Proposition 3.1

$$q_t = \frac{q_t^L + q_t^R}{1+q_t^L\, q_t^R}$$

Proposition 3.2 (Wonham (1965))

(A)

$$\Pr(q_t^L \in [q,q+dq)|\mu_t=\pm 1) = c(\gamma)(1\pm q)(1-q^2)^{-2}\exp[-2\gamma(1-q^2)^{-1}]dq$$

where

$$c(\gamma) = \left[\int_1^\infty z^{1/2}(z-1)^{-1/2} e^{-2\gamma z}\,dz\right]^{-1}$$

and

$$\gamma = \alpha^2 \nu$$

(B)

$$MSE(q_t^L) = \int_0^\infty z^{-1/2}(z+1)^{-1/2} e^{-2\gamma z}\,dz \Big/ \int_0^\infty z^{-1/2}(z+1)^{1/2} e^{-2\gamma z}\,dz$$

$$= \begin{cases} -2\gamma\log\gamma + o(\gamma\log\gamma) & (\gamma \to 0^+) \\ 1 - (4\gamma)^{-1} + O(\gamma^{-2}) & (\gamma \to \infty) \end{cases}$$

Proposition 3.3

$$MSE(q_t) = c(\gamma)^2 \int_{-1}^{1}\int_{-1}^{1} \left[\frac{x+y}{1+xy} - 1\right]^2 (1+x)(1-x^2)^{-2}(1+y)(1-y^2)^{-2}$$

$$\cdot \exp(-2\gamma[(1-x^2)^{-1}+(1-y^2)^{-1}])\,dx\,dy$$

$$= \begin{cases} 2\gamma + o(\gamma) & (\gamma \to 0^+) \\ 1 - \frac{1}{2}\gamma^{-1} + O(\gamma^{-2}) & (\gamma \to \infty) \end{cases}$$

Proof of Proposition 3.3

Since $\{\nu_t\}$ is time reversible, $L(q_t^R|\nu_t) = L(q_t^L|\nu_t)$.

Also, q_t^R is independent of q_t^L given ν_t. Therefore, by applying Propositions 3.1 and 3.2 (A), we can readily derive the formula for $MSE(q_t)$. The computation of its asymptotic behavior is given in the Appendix. □

We may also compute the MSE for the Wiener filtering estimate of ν_t and the best linear estimate based on $Z_{-\infty}^{\infty}$.

$MSE_W \equiv$ MSE for the Wiener filtering estimate of ν_t

$$(3.4) \qquad = 2\gamma[(1+\gamma^{-1})^{1/2}-1]$$

$$= \begin{cases} 2\gamma^{1/2} + o(\gamma^{1/2}) & (\gamma \to 0^+) \\ 1 - (4\gamma)^{-1} + O(\gamma^{-2}) & (\gamma \to \infty) \end{cases}$$

$MSE_{BL} \equiv$ MSE for the best linear estimate of ν_t based on $Z_{-\infty}^{\infty}$

$$(3.5) \qquad = \left(\frac{\gamma}{1+\gamma}\right)^{1/2}$$

$$= \begin{cases} \gamma^{1/2} + o(\gamma^{1/2}) & (\gamma \to 0^+) \\ 1 - \frac{1}{2}\gamma^{-1} + O(\gamma^{-2}) & (\gamma \to \infty) \end{cases}$$

Now we are ready to compare the performance of the various estimates. See Table 3.1 for the summary of the asymptotic results on their MSE.

<u>Remark 1</u>: As $\gamma \to 0^+$, the linear estimates are not efficient. It seems that in non-Gaussian systems linear estimates are rather inflexible and therefore can not perform well.

<u>Remark 2</u>: As $\gamma \to 0^+$, the optimal smoother is more efficient than the optimal filter by a factor $-\log \gamma$. This factor is about 6.9 when $\gamma = 0.001$ (e.g. $\alpha = \nu = 0.1$). Therefore, when either the noise intensity or the rate of jump of the states is low, the optimal smoother is *significantly* better than the optimal filter.

<u>Remark 3</u>: In finite-state processes, error probability is also an interesting criterion. In the following, we present the error probabilities for several optimal decision procedures. We consider decisions on whether u_0 is 1 or -1.

(i) Based on $Z^0_{-\infty}$:

An optimal decision d_L is:

$$d_L(Z^0_{-\infty}) = 1 \quad \text{iff} \quad \Pr(u_0 = 1 \mid Z^0_{-\infty}) \geq \tfrac{1}{2}.$$

The error probability is

$$e_L = c(\gamma) \int_{-1}^{0} (1+q)(1-q^2)^{-2} e^{-2\gamma(1-q^2)^{-1}} \, dq$$

$$= \tfrac{1}{2}\left(1 - \frac{c(\gamma)}{2\gamma} e^{-2\gamma}\right)$$

$$(3.6) \qquad = \begin{cases} -\tfrac{1}{2}\gamma\log\gamma + o(\gamma\log\gamma) & (\gamma \to 0^+) \\ \tfrac{1}{2}\left(1 - \frac{1}{\sqrt{2\pi\gamma}} + o(\gamma^{-1/2})\right) & (\gamma \to \infty) \end{cases}$$

(ii) Based on $Z^{\infty}_{-\infty}$:

An optimal decision d is

$$d(Z^{\infty}_{-\infty}) = 1 \quad \text{iff} \quad \Pr(u_0 = 1 \mid Z^{\infty}_{-\infty}) \geq \tfrac{1}{2}$$

The error probability is

$$e = c(\gamma)^2 \int_{-1}^{1} \int_{-y}^{1} (1-x)(1-x^2)^{-2} e^{-2\gamma(1-x^2)^{-1}} dx \, (1-y)(1-y^2)^{-2} \cdot e^{-2\gamma(1-y^2)^{-1}} dy$$

$$(3.7) \qquad = c(\gamma)^2 \left\{ \tfrac{1}{2} c(\gamma)^{-1}\gamma^{-1} e^{-2\gamma} \int_0^{\infty} \left(\left(1 + \tfrac{\gamma}{v}\right)^{1/2} - 1\right) e^{-2v} dv + \tfrac{1}{2}\gamma^{-2} e^{-4\gamma} \int_0^{\infty} \left(1 + \tfrac{\gamma}{v}\right)^{\frac{1}{2}} e^{-4v} dv + \tfrac{1}{4}\gamma^{-2} e^{-4\gamma} \int_0^{\infty} \left(1 + \tfrac{\gamma}{v}\right)^{1/2} \cdot e^{-2v} dv \right\}$$

$$= \begin{cases} \gamma\log 2 + o(\gamma) & (\gamma \to 0^+) \\ \tfrac{1}{2}\left(1 - \frac{1}{\sqrt{\pi\gamma}} + o(\gamma^{-\frac{1}{2}})\right) & (\gamma \to \infty) \end{cases}$$

Table 3.1 Asymptotic Behavior of Mean Squared Errors for Various Estimates for the Noisy Two-State Markov Process

Estimate / MSE	Linear		Nonlinear	
	Filtering	Smoothing	Filtering	Smoothing
	MSE_W	MSE_{BL}	$MSE(q_t^L)$	$MSE(q_t)$
$\gamma \to 0+$	$2\gamma^{1/2}$	$\gamma^{1/2}$	$-2\gamma \log \gamma$	2γ
$\gamma \to \infty$	$1-(4\gamma)^{-1}$	$1-(2\gamma)^{-1}$	$1-(4\gamma)^{-1}$	$1-(2\gamma)^{-1}$

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Appendix

Proof of the Asymptotic Expansions for $MSE(q_t)$ in Proposition 3.3

In the following, $c(\gamma)$ is abbreviated to c. We use the notation $A \sim B\ (\gamma \to \gamma_0)$ to mean $\lim_{\gamma \to \gamma_0} \frac{A}{B} = 1$.

(I) The case of $\gamma \to 0^+$.

$$c^{-1} = \int_1^\infty z^{1/2}(z-1)^{-1/2} e^{-2\gamma z} dz$$

$$= \gamma^{-1} \int_\gamma^\infty u^{1/2}(u-\gamma)^{-1/2} e^{-2u} du \quad (\gamma z = u)$$

$$= \gamma^{-1} \int_0^\infty (v+\gamma)^{1/2} v^{-1/2} e^{-2(v+\gamma)} dv \quad (u = v+\gamma)$$

$$\sim \gamma^{-1} \int_0^\infty v^{1/2} v^{-1/2} e^{-2v}\, dv$$

$$\text{(A.1)} \qquad = \frac{1}{2}\gamma^{-1}$$

$$\int_{-1}^{1} \int_{-1}^{1} \left(\frac{x+y}{1+xy} - 1\right)^2 \frac{(1+x)(1+y)}{(1-x^2)^2(1-y^2)^2} \exp\left(\frac{-2\gamma}{1-x^2} + \frac{-2\gamma}{1-y^2}\right) dxdy$$

$$= \int_{-1}^{1} \int_{-1}^{1} \frac{1}{(1+xy)^2(1+x)(1+y)} \exp\left(\frac{-2\gamma}{1-x^2} + \frac{-2\gamma}{1-y^2}\right) dxdy$$

$$\text{(A.2)} \quad \sim \left\{ \int_{-1}^{-1+\epsilon} \int_{-1}^{-1+\epsilon} + 2 \int_{-1}^{-1+\epsilon} \int_{-\epsilon}^{1} \right\} \frac{1}{(1+xy)^2(1+x)(1+y)} \cdot \exp\left(\frac{-2\gamma}{1-x^2} + \frac{-2\gamma}{1-y^2}\right) dxdy \quad (\gamma \to 0^+)$$

Here ϵ is a small positive number.

$$\int_{-1}^{-1+\epsilon} \int_{-1}^{-1+\epsilon} \frac{1}{(1+xy)^2(1+x)(1+y)} \exp\left(\frac{-2\gamma}{1-x^2} + \frac{-2\gamma}{1-y^2}\right) dxdy$$

$$\sim \int_{-1}^{-1+\epsilon} \int_{-1}^{-1+\epsilon} \frac{1}{4(1+x)(1+y)} \exp\left(\frac{-2\gamma}{1-x^2} + \frac{-2\gamma}{1-y^2}\right) dxdy \quad (\gamma \to 0^+)$$

$$= \frac{1}{4} \left[\int_{-1}^{-1+\epsilon} \frac{1}{1+x} \exp\left(\frac{-2\gamma}{1-x^2}\right) dx \right]^2$$

$$= \frac{1}{4} \left[\int_0^{\epsilon\gamma^{-1}} \frac{1}{u} \exp\left[\frac{-2}{u(2-\gamma u)}\right] du \right]^2 \quad (1 + x = \gamma u)$$

$$\sim \frac{1}{4} \left[\int_M^{\epsilon\gamma^{-1}} \frac{1}{u} \exp\left[\frac{-2}{u(2-\gamma u)}\right] du \right]^2 \quad (M \text{ is a large number})$$

$$\sim \frac{1}{4}\left[\int_{M}^{\epsilon\gamma^{-1}} \frac{1}{u}\, du\right]^2$$

$$\text{(A.3)} \quad \sim \frac{1}{4}(\log\gamma)^2$$

$$\int_{-1}^{-1+\epsilon}\int_{1-\epsilon}^{1} \frac{1}{(1+xy)^2(1+x)(1+y)} \exp\left(\frac{-2\gamma}{1-x^2}+\frac{-2\gamma}{1-y^2}\right) dxdy$$

$$= \int_0^{\epsilon}\int_0^{\epsilon} \frac{1}{(u+v-uv)^2(2-u)v} \exp\left(\frac{-2\gamma}{u(2-u)}+\frac{-2\gamma}{v(2-v)}\right) dudv$$

$(u=1-x,\ v=1+y)$

$$\sim \frac{1}{2}\int_0^{\epsilon}\int_0^{\epsilon} \frac{1}{(u+v-uv)^2 v} \exp\left(\frac{-2\gamma}{u(2-u)}+\frac{-2\gamma}{v(2-v)}\right) dudv$$

$$\sim \frac{1}{2}\int_0^{\epsilon}\int_0^{\epsilon} \frac{1}{(u+v)^2 v} \exp\left(\frac{-2\gamma}{u(2-u)}+\frac{-2\gamma}{v(2-v)}\right) dudv$$

$$= \frac{1}{2\gamma}\int_0^{\epsilon/\gamma}\int_0^{\epsilon/\gamma} \frac{1}{(x+y)^2 y} \exp\left(\frac{-2}{x(2-\gamma x)}+\frac{-2}{y(2-\gamma y)}\right) dxdy$$

$(u=\gamma x,\ v=\gamma y)$

$$= \frac{1}{4\gamma}\int_0^{\epsilon/\gamma}\int_0^{\epsilon/\gamma} \left[\frac{1}{(x+y)^2 y}+\frac{1}{(x+y)^2 x}\right] \exp\left[\frac{-2}{x(2-\gamma x)}+\frac{-2}{y(2-\gamma y)}\right] dxdy$$

$$\sim \frac{1}{4\gamma}\int_0^{\infty}\int_0^{\infty} \frac{1}{xy(x+y)} \exp\left(-\frac{1}{x}-\frac{1}{y}\right) dxdy$$

$$= \frac{1}{4\gamma}\int_0^{\frac{\pi}{2}}\int_0^{\infty} \frac{1}{\cos\theta+\sin\theta} \exp\left[-r(\cos\theta+\sin\theta)\right] drd\theta$$

$(x^{-1} = r\cos\theta,\ y^{-1} = r\sin\theta)$

$$\text{(A.4)} \quad = \frac{1}{4\gamma}$$

Therefore, from (A.1), (A.2), (A.3), (A.4),

$$MSE(q_t) \sim 2\gamma \quad (\gamma \to 0^+)$$

(II) The case of $\gamma \to \infty$.

$$c^{-1} = \int_1^{\infty} z^{1/2}(z-1)^{-1/2} e^{-2\gamma z}\, dz$$

$$\sim \int_1^{1+\epsilon} (1+(z-1))^{1/2}(z-1)^{-1/2} e^{-2\gamma z} dz \quad (\epsilon \text{ is small positive})$$

$$= \int_1^{1+c} \left[1 + \tfrac{1}{2}(z-1) + O((z-1)^2)\right] (z-1)^{-1/2} e^{-2\gamma z}\, dz$$

$$\text{(A.5)} \quad = e^{-2\gamma}\left[\left(\tfrac{\pi}{2}\right)^{1/2} \gamma^{-1/2} + 2^{-7/2}\pi^{1/2}\gamma^{-3/2} + o(\gamma^{-3/2})\right]$$

$$\int_{-1}^{1}\int_{-1}^{1}\left(\frac{x+y}{1+xy} - 1\right)^2 \frac{(1+x)(1+y)}{(1-x^2)^2(1-y^2)^2} \exp\left(\frac{-2\gamma}{1-x^2} + \frac{-2\gamma}{1-y^2}\right) dxdy$$

$$= \int_{-1}^{1}\int_{-1}^{1} \frac{1}{(1+xy)^2(1+x)(1+y)} \exp\left(\frac{-2\gamma}{1-x^2} + \frac{-2\gamma}{1-y^2}\right) dxdy$$

$$= \int_{-1}^{1}\int_{-1}^{1} (1-x-y-xy + x^2 + y^2 + o(x^2+y^2)) \exp\left(\frac{-2\gamma}{1-x^2} + \frac{-2\gamma}{1-y^2}\right) dxdy$$

$$\text{(A.6)} \quad = \left[\int_{-1}^{1} \exp\left(\frac{-2\gamma}{1-x^2}\right) dx\right]^2 + 2\left[\int_{-1}^{1} x^2 \exp\left(\frac{-2\gamma}{1-x^2}\right) dx\right]$$

$$\cdot\left[\int_{-1}^{1} \exp\left(\frac{-2\gamma}{1-y^2}\right) dy\right] (1 + o(1))$$

$$\int_{-1}^{1} \exp\left(\frac{-2\gamma}{1-x^2}\right) dx = 2\int_0^1 \exp\left(\frac{-2\gamma}{1-x^2}\right) dx$$

$$= 2e^{-2\gamma}\int_0^\infty e^{-\gamma s} s^{-1/2} (2+s)^{-3/2} ds \quad \left(s = \frac{2x^2}{1-x^2}\right)$$

$$= 2e^{-2\gamma}\int_0^\infty e^{-\gamma s} s^{-1/2}(2^{-3/2} - 3\cdot 2^{-7/2}s + o(s))ds$$

$$\text{(A.7)} \quad = 2e^{-2\gamma}\left(2^{-3/2}\pi^{1/2}\gamma^{-1/2} - 3\cdot 2^{-9/2}\pi^{1/2}\gamma^{-3/2} + o(\gamma^{-3/2})\right)$$

$$\int_{-1}^{1} x^2 \exp\left(\frac{-2\gamma}{1-x^2}\right) dx = 2\int_0^1 x^2 \exp\left(\frac{-2\gamma}{1-x^2}\right) dx$$

$$= 2e^{-2\gamma}\int_0^\infty e^{-\gamma s} s^{1/2}(2+s)^{-5/2} ds \quad \left(s = \frac{2x^2}{1-x^2}\right)$$

$$= 2e^{-2\gamma}\int_0^\infty e^{-\gamma s} s^{1/2}(2^{-5/2} + O(s))\, ds$$

(A.8) $= 2\, e^{-2\gamma}\left(2^{-7/2}\pi^{1/2}\gamma^{-3/2} + o\,(\gamma^{-3/2})\right)$

From (A.6), (A.7), (A.8),

$$\int_{-1}^{1}\int_{-1}^{1}\left(\frac{x+y}{1+xy} - 1\right)^2 \frac{(1+x)(1+y)}{(1-x^2)^2(1-y^2)^2}\exp\left(\frac{-2\gamma}{1-x^2} + \frac{-2\gamma}{1-y^2}\right) dxdy$$

(A.9) $= e^{-4\gamma}\left(2^{-1}\pi\gamma^{-1} - 2^{-3}\pi\gamma^{-2} + o(\gamma^{-2})\right)$

So, from (A.5) and (A.9),

$$MSE(q_t) = \frac{2^{-1}\pi\gamma^{-1} - 2^{-3}\pi\gamma^{-2} + o(\gamma^{-2})}{\left[(\frac{\pi}{2})^{1/2}\gamma^{-1/2} + 2^{-7/2}\pi^{1/2}\gamma^{-3/2} + o(\gamma^{-3/2})\right]^2}$$

$$= 1 - \frac{1}{2}\gamma^{-1} + o(\gamma^{-1})$$

This completes the proof. □

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